Affine Term Structure Models *

The term structure of interest rates is the yield-to-maturities on a set of bonds of different maturities, expressed as a function of the times-to-maturity. It is a simple, summary measure of the cross-section of bond prices measured at a point in time. An affine term structure model hypothesizes that the term structure – at any date – is a time-invariant linear function of a small set of common state variables (or factors). Once the dynamics of the state variables and their risk premiums are specified, then the dynamics of the term structure are determined.

Of course, in order for the term structure of interest rates (or yield curve) to be meaningful, the bonds that are being compared must have similar risk and payout characteristics. The literature that we examine in this chapter focuses on the term structure of default-risk free, nominal bonds that make a single payment at a pre-specified date in the future (so-called zero-coupon bonds). The models described below can be applied to other types of bonds, but this class of simple financial claims is important because it defines the market determined discount rates embedded in any more complicated claim that makes payments over time.

The literature on the term structure is large, and it reaches back to some of the giants of early twentieth century economics: Fisher, Hicks, and Keynes. The preeminent theoretical model of the term structure, prior to the advent of the explicit no-arbitrage approach to asset pricing, was the expectations hypothesis. Although it exists in a variety of forms (see Cox, Ingersoll, and Ross, 1981), we will follow Campbell (1986) and Campbell and Shiller (1991) by defining it as the hypothesis that term premiums on default-risk free zero-coupon bonds are constant through time. The other commonly espoused early term structure theories – the liquidity preference and preferred habitat theories – can be viewed as extensions of the expectation hypothesis that make additional predictions for the size of term premiums as a function of term-to-maturity.

Empirical tests of the expectations hypothesis are often based on the following

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regression:

$$y_{t+m}(n-m) - y_t(n) = k_{n,m} + \phi_{n,m}\left\{\frac{m}{n-m}\left[y_t(n) - y_t(m)\right]\right\} + \varepsilon_{t+m}(n,m), \quad (1)$$

where $y_t(n)$ is the yield to maturity on a long maturity (n period) bond and $y_t(m)$ is the yield-to-maturity on a shorter maturity (m period) bond such that n/m is an integer. $k_{n,m}$ is a maturity dependent constant. $\phi_{n,m}$ is the regression slope coefficient, and $\varepsilon_{t+m}(n,m)$ is a random shock observed at $t+m$ that may be maturity dependent. The left-hand side of (1) is the m-period change in an n-period bond between t and $t+m$, and the regressor is a maturity adjusted yield spread observed at time t. The empirical prediction of the expectations hypothesis is that $\phi_{n,m}$ is equal to one, for all choices of n and m.

This prediction has been rejected in the U.S. data for most choices of n and m. For example, Campbell and Shiller (1991) find that $\phi_{n,m}$ is negative and reliably different from one "... between almost any two maturities ..." ranging from two months to ten years. Fama and Bliss (1987) report similar results using a different data set and a slightly different specification of the expectations hypothesis regression (1). This overall rejection of the basic prediction of the expectations hypothesis has been replicated in a number of different studies using maturities from one month to ten years. Longstaff (2000) presents evidence that suggests that the prediction $\phi_{n,m} = 1$ is not rejected in data with extremely short maturities; i.e., days or weeks.

The failures of the expectations hypothesis imply that there are time-varying term premiums in the prices of default-risk free zero-coupon bonds. Explaining the dynamics of these term premiums is an important goal of affine term structure models. These models have two important strengths compared to the earlier theories of the term structure. They explicitly rule out arbitrage opportunities among the bonds being priced, and they simultaneously allow for flexible specifications of bond term premiums. Weaknesses of affine models include the fact that they are typically not easy to estimate, and there can be very limited intuition as to the interpretation of the fundamental factors. Chapman and Pearson (2001), Dai and Singleton (2003), and Piazzesi (2005) are all recent, more detailed, and more technical examinations of much of the material that follows.

Any affine term structure model starts from the assumption that there are no arbitrage opportunities in financial markets. This assumption implies that there exists a strictly positive stochastic process, Λ, that prices all assets.[1] This process is typically referred to as a state price deflator in continuous-time models of asset pricing and a stochastic discount factor in discrete-time models. In our discussion of affine pricing models, we follow the more common approach in the literature and develop the models in continuous-time. The existence of a state price deflator also implies that there exists a risk-neutral measure, $\mathbb{Q}$, which is distinct from the physical measure, $\mathbb{P}$, that generates observed variation in asset prices.

Independent of any specific model of bond prices, it is always possible to express the price at time t of a zero coupon bond that matures at time $t + \tau$ as

$$P_t(\tau) = E_t^{\mathbb{Q}}\left[\exp\left(-\int_0^{\tau} r_s ds\right)\right], \tag{2}$$

where $E_t^{\mathbb{Q}}[\cdot]$ denotes the expected value at time t under the risk-neutral measure and r is the instantaneous rate of interest (or short rate). The short rate can be defined as

$$r_t = \lim_{\tau \downarrow 0} P_t(\tau), \tag{3}$$

but it is also related to the expected value of the instantaneous rate of change of the state price deflator because

$$\frac{d\Lambda_t}{\Lambda_t} - r_t dt + \sigma_{\Lambda}(\Lambda_t, t)\, dW_t^{\mathbb{Q}}, \tag{4}$$

where $W_t^{\mathbb{Q}}$ is a Brownian motion under $\mathbb{Q}$, $\sigma_{\Lambda}(\cdot)$ is the time- (and possibly state-) varying instantaneous volatility of the state price deflator, and the second term in (4) is a common shorthand notation for an Itô stochastic integral.[2]

As equation (2) clearly shows, pricing zero-coupon default-risk free bonds comes down to specifying a model for the dynamics of the short rate under the risk neutral measure. In choosing models for r_t, there are two paramount considerations: (*i*) a

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[1]See Duffee (2001) for a textbook treatment of the implications of absence of arbitrage for asset pricing.

[2]See Duffie (2001) for a textbook treatment of continuous-time stochastic processes, including the definitions of Brownian motion and the Itô integral.

flexible specification that does a reasonable job of capturing the dynamics of proxies for the short rate (since r_t itself is unobservable), and (*ii*) a specification that yields a convenient form for the bond prices that are the ultimate objects of interest.

Short rate models, when developed in continuous time, are completely determined by the drift function, which defines the instantaneous expected value of the short rate, and the diffusion function, which determines the instantaneous volatility of the short rate. What is not clear from equation (2) is that in order to move from the theoretical risk-neutral measure, $\mathbb{Q}$, to the actual (or physical measure), $\mathbb{P}$, that generates the observed data, a term structure model must also specify a structure for the risk premium functions controling the transformation of bond prices between $\mathbb{Q}$ and $\mathbb{P}$.

Single-Factor Models

In a single-factor affine model, the determinant of bond prices is the short rate itself. The model is constructed by specifying a continuous-time process for the short rate and a form of the risk premium function. As Cox, Ingersoll, and Ross (1985) note, these choices must be mutually consistent in order to avoid accidentally introducing arbitrage opportunities into a (supposedly) arbitrage-free model. The fundamental building blocks of all affine models are the single-factor models due to Vasicek (1977) and Cox, Ingersoll, and Ross (1985) (hereafter CIR).

The Vasicek model assumes that the short rate evolves as an Ornstein-Uhlenbeck process under the risk neutral measure

$$dr_t = \kappa\left(\theta - r_t\right)dt + \sigma dW_t^{\mathbb{Q}}, \tag{5}$$

where $\kappa > 0$ determines the speed of reversion to the constant mean, $\theta > 0$, and σ is the unconditional instantaneous volatility of the process. The conditional and unconditional distributions of interest rate changes are Gaussian in this model. Accordingly, it is possible for the short rate to be negative. The risk premium function is a constant, λ_0, which means that the short rate is also Gaussian under the physical measure, $\mathbb{P}$. Solving the conditional expectation in (2) under these assumptions generates an explicit expression for the price of a default-risk free zero coupon bond

$$P_t\left(\tau\right) = \exp\left[a\left(\tau\right) + b\left(\tau\right)r_t\right], \tag{6}$$

where

$$a(\tau) = \left(\theta - \frac{\lambda_0}{\kappa} - \frac{1}{2}\frac{\sigma^2}{\kappa^2}\right)\left[\frac{1}{\kappa}(1 - \exp(-\kappa\tau)) - \tau\right] - \frac{\sigma^2}{4\kappa^3}[1 - \exp(-\kappa\tau)]^2 \tag{7}$$

and

$$b(\tau) = -\frac{1}{\kappa}[1 - \exp(-\kappa\tau)]. \tag{8}$$

Equation (6) is the first statement of an exponential-affine pricing function. It implies a simple structure where continuously compounded yields are Gaussian with constant volatility. The term structure of forward rates implied by this simple model can assume most (but not all) of the commonly observed shapes of the term structure. In particular, the term structure of forward rates can be upward sloping, downward sloping, or humped shaped, although the model cannot generate an inverted humped shape. Since prices at all maturities are driven by a single stochastic factor, this model implies that all yield levels are perfectly correlated. In the data, yield levels are very highly – but not perfectly – correlated.

In the single-factor CIR term structure model, the short rate evolves as

$$dr_t = \kappa(\theta - r_t)\,dt + \sigma\sqrt{r_t}dW_t^{\mathbb{Q}}, \tag{9}$$

where $\kappa > 0$ and $\theta > 0$ have the same interpretation as in the Vasicek case, but the short rate is no longer Gaussian. The parameter restriction $2\kappa\theta \geq \sigma^2$ is imposed in order to ensure that the short rate process does not get trapped at zero. r_t has a conditional noncentral chi-square distribution (and an unconditional Gamma distribution). The instantaneous conditional variance of the short rate is linear in the level of the rate. The risk premium specification that is consistent with no-arbitrage in the single-factor CIR specification is $\lambda(r_t) = \lambda_1 r_t$, and the no-arbitrage bond price is, again, of the form (6) with

$$a(\tau) = \frac{2\kappa\theta}{\sigma^2}\log\left[\frac{2\gamma\exp\left(\frac{1}{2}\tau(\kappa + \lambda_1 + \gamma)\right)}{(\kappa + \lambda_1 + \gamma)[\exp(\gamma\tau) - 1] + 2\gamma}\right] \tag{10}$$

and

$$b(\tau) = \frac{-2[\exp(\gamma\tau) - 1]}{(\kappa + \lambda_1 + \gamma)[\exp(\gamma\tau) - 1] + 2\gamma}, \tag{11}$$

where $\gamma \equiv \sqrt{(\kappa + \lambda_1)^2 + 2\sigma^2}$. Once again, the CIR model can generate the most common shapes of the term structure, but it still implies that all yield levels are perfectly correlated.

The Vasicek and CIR models are the most common forms of single-factor affine models, but Duffie and Kan (1996) provide the conditions on the drift, diffusion, and risk premium functions of a short rate specification, like (5) or (9), that ensure that the bond pricing function is exponential-affine under the risk neutral measure. In particular, a pricing function of the form of (6) will follow if

$$\mu(r_t) - \lambda(r_t) = \rho_0 + \rho_1 r_t \tag{12}$$

and

$$\sigma(r_t) = \sqrt{\beta_0 + \beta_1 r_t} \tag{13}$$

hold, where $\mu(r_t)$ is a general expression for the drift of the short rate and $\sigma(r_t)$ is a general expression for the instantaneous volatility of the short rate. For example, in the CIR case, $\rho_0 = \kappa\theta$, $\rho_1 = -(\kappa + \lambda_1)$, $\beta_0 = 0$, and $\beta_1 = \sigma^2$. In this more general case, the $a(\tau)$ and $b(\tau)$ functions will not generally have explicit closed-form expressions. Rather, they will be defined as the solutions to a pair of ordinary differential equations.

The empirical evidence clearly shows that a single-factor specification is not sufficient to describe the dynamics of the default-risk free term structure. As such, empirical analysis of simple specifications, like (5) and (9), have shifted away from attempting to completely characterize yields on all maturities and, instead, have concentrated on explaining the dynamics of a proxy for the unobservable short rate. Chan, Karolyi, Longstaff, and Sanders (1992) pioneered this approach, using a simple generalized method of moments estimation scheme. Durham (2003) is the natural evolution of this literature using state-of-the-art approximate maximum likelihood estimation. The conclusions of this literature are: (*i*) The evidence of mean reversion in the short rate is weak, at best, but (*ii*) there is little consistent evidence of nonlinear mean reversion. (*iii*) There are complicated volatility dynamics that are not consistent with either constant volatility (Vasicek) or instantaneous conditional variances that are linear in the short rate (CIR).

Multifactor Models

If single-factor models are insufficient to explain the observed term structure, then how many factors are needed and what are the dynamics of these factors? The common answer to the first question was provided by the analysis in Litterman and Scheinkman (1991). Using a simple principal components approach, they argue that three factors – extracted from yields themselves – can explain well over 95% of the variation in weekly changes to U.S. Treasury bond prices, for maturities of up to 18 years. The answer to the second question – in the most general form consistent with an exponential-affine pricing function – was provided by Dai and Singleton (2000) and extended by Duffee (2002).

The multifactor affine term structure model consists of the following three components: There is linear relation between the short rate and the factors:

$$r_t = \delta_0 + \delta' Y_t, \tag{14}$$

where Y_t denotes the N-vector of time t factor realizations. The factor dynamics conform to an affine diffusion

$$dY_t = \mathcal{K}\left(\theta - Y_t\right) dt + \Sigma \sqrt{S_t} dW_t^{\mathbb{Q}}, \tag{15}$$

where K and Σ are $N \times N$ matrices (with no general restrictions) and S_t is a diagonal matrix with the i-th diagonal element equal to

$$\left[S_t^{ii}\right] = \alpha_i + \beta_i' Y_t. \tag{16}$$

The S_t matrix allows for the instantaneous conditional variance of the factors to be linear functions of factor levels. If every element of Y_t can affect the conditional volatility of every other factor, then (15) is a multifactor generalization of the CIR model from the last section. Of course, the fact that volatility is linear in the level of Y requires strong restrictions on the parameters of the model in order to ensure that variances are non-negative.

If no elements of Y affect the conditional volatility, then (15) is a multifactor generalization of the Vasicek model. If $m < N$ factors affect the conditional volatility, then the multifactor affine model is a mixture of the CIR and Vasicek forms. Dai and Singleton (2000) define different classes of affine models by the number of factors that affect the conditional factor volatilities, with $\mathbb{A}_m(N)$ being the general notation for an N-factor model with m-factors driving conditional volatilities.

Under these assumptions, bond prices satisfy a multivariate generalization of (6) given by

$$P_t(\tau) = \exp\left[A(\tau) + B(\tau)' Y_t\right]. \tag{17}$$

The functions $A(\tau)$ and $B(\tau)$ are the solutions to the ordinary differential equations

$$\frac{dA(\tau)}{d\tau} = -\theta\mathcal{K}' B(\tau) + \frac{1}{2}\sum_{i=1}^{N}\left[\Sigma' B(\tau)\right]_i^2 \alpha_i - \delta_0 \tag{18}$$

and

$$\frac{dB(\tau)}{d\tau} = -\mathcal{K}' B(\tau) + \frac{1}{2}\sum_{i=1}^{N}\left[\Sigma' B(\tau)\right]_i^2 \beta_i - \delta. \tag{19}$$

The third – and final – component of the general multifactor affine model is the specification of the market prices of risk, which connects pricing under the risk-neutral measure to pricing under the physical measure:

$$\Lambda_t = \sqrt{S_t}\lambda_0 + \sqrt{S_t^-}\lambda Y_t, \tag{20}$$

where λ_0 is an N-vector of constants, λ is an $N \times N$ matrix of constants, and S_t^- is an N-dimensional diagonal matrix with diagonal elements equal to

$$S_t^-(ii) = \begin{cases} \left(\alpha_i + \beta_i' Y_t\right)^{-1/2}, & \text{if } \inf\left(\alpha_i + \beta_i' Y_t\right) > 0; \\ 0, & \text{otherwise.} \end{cases} \tag{21}$$

The first term in (20) is a straightforward generalization of the single-factor risk premium specifications: risk premiums are proportional to factor volatilities. The second component is an important source of additional flexibility in multifactor affine models. It allows these models to provide a better fit to the distribution of bond excess returns, and it is also useful in rationalizing the observed violations of the expectations hypothesis discussed earlier.

As noted earlier, the general multifactor affine model can be viewed as a blending of the Vasicek and CIR forms. These extreme specifications also reveal a critical trade-off in multifactor term structure modelling. The CIR form offers the greatest flexibility in specifying the volatility dynamics of bond prices. However, this flexibility comes at a cost. The parameter restrictions that are ¡required to ensure that (16) provides a valid description of factor variances, impose substantial restrictions on the permissible correlations between the factors. In the extreme case of the pure multifactor CIR model, the factors must be uncorrelated to ensure an admissible volatility specification.

Dai and Singleton (2002), Duffee (2002), and Brandt and Chapman (2005) fit multifactor affine term structure models to more than 25 years of monthly U.S. bond data. Each paper considers the ability of different versions of $\mathbb{A}_m(3)$ models (for $m \in \{0, 1, 2, 3\}$) to explain both the rejections of the expectations hypothesis generated by regressions of the form of (1) and the ability of these models to provide meaningful forecasts of future yields. Both Dai and Singleton (2002) and Brandt and Chapman (2005) find that a Gaussian version (an $\mathbb{A}_0(3)$ model) can rationalize the risk premiums revealed by yield change regressions. Duffee (2002) demonstrates that an $\mathbb{A}_0(3)$ model with the expanded risk premium specification of (20) can produce meaningful multistep forecasts of Treasury yields at different maturities.

Although the ability to explain risk premiums and yield movements is an important success for multifactor affine models, their biggest failing is that these explanations require that conditional yield volatilities are constant. Essentially, the flexibility in factor correlations that are required to explain these features of the data require a stochastic structure that precludes the volatility dynamics that are an equally important feature of interest rate data. A final important issue in evaluating these models is whether or not their latent factors can be connected in any meaningful way to structural (macroeconomic) explanations of term structure dynamics.

Models with Macroeconomic Factors

Ang and Piazzesi (2003) and Ang, Dong, and Piazzesi (2005) are recent papers that begin to answer the question about how to connect no-arbitrage term structure models with elements of the macroeconomy. In Ang and Piazzesi (2003), a term structure model is developed where some of the factors are macroeconomic variables and some factors remain unobservable (or latent). In particular, they retain the structure of equation (14) in a multifactor affine model, and they consider a Gaussian model with five factors. The first two factors are observable macroeconomic factors that are extracted as principal components from two groups of observable variables, one that captures measures of inflation and one that captures measures of real activity. The three (latent) factors are allowed to be correlated with each other, but they are assumed to be orthogonal to the macroeconomic factors. The results of estimating this model suggests that macroeconomic variables explain a significant portion of return variances. This is particularly true for short and intermediate maturity bonds.

Ang, Dong, and Piazzesi (2005) extends the analysis in Ang and Piazzesi (2003) by allowing macroeconomic variables to have a feedback effect on the dynamics of the latent factors and vice versa. This structure is then interpreted as a Taylor rule for monetary policy, where the monetary authority is interested in controlling both deviations from a target real growth rate and deviations from a target inflation rate. This multifactor model includes three factors – two macroeconomic variables and one latent factor – that follow a Gaussian structure. This structure allows for a decomposition of yield changes into responses to monetary policy (as specified in the Taylor rule) and responses to monetary policy shocks. They find that 30 to 40 percent of the variability of short and intermediate maturity yields can be explained by responses to policy. The approach followed in this paper can be extended and combined with different forms of latent-factor affine term structure models by making additional assumptions about either factor dynamics or risk premiums.

Conclusions

Affine term structure models have come a long way in the nearly thirty years since their initial formulation. Flexible multifactor models have been specified that are both amenable to econometric estimation and capable of rationalizing many important features of U.S. Treasury bond prices. In particular, these models have been capable of producing term premiums that are both consistent with no-arbitrage and with observed rejections of the expectations hypothesis of the term structure.

There are two directions for future work in this area that seem particularly important. One direction consists of extending and strengthening the connections between no-arbitrage models and models of the actions of monetary authorities. This research holds out the promise of greater intuition behind the factors in no-arbitrage models as well as a greater understanding of how capital markets perceive the actions of monetary authorities. A second direction for multifactor affine models is to explain simultaneously rejections of the expectations hypothesis and yield volatility dynamics.

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