Minimizing Basis Risk from Nonparallel Shifts in the Yield Curve

Eric Falkenstein
KeyCorp
efalken@ix.netcom.com

Jerry Hanweck
J.P. Morgan
hanweck_j@JPMorgan.com

The Journal of Fixed Income
Volume 6 Number 1
June 1996

Introduction

A duration hedge immunizes a portfolio against parallel shifts in yield curves, but does not immunize a portfolio against the nonparallel shifts. Duration hedges can thus leave a portfolio with a considerable amount risk from less structured yield curve fluctuations (for discussion of this point see Fong and Vasicek (1984), Ho (1992), or Barrett, Gosnell and Heuson (1995)).

We describe a new hedging algorithm that minimizes yield curve risk due to all movements in the yield curve. The building blocks of our approach consist of the relevant securities' basis point values (also called the present value of a basis point) for different key rates on the yield curve and the covariance matrix of these key rates.[1] We compare our approach to the standard duration and yield beta approaches and show our approach significantly reduces the risk of a hedged portfolio of Treasury bonds.

Our aim is to construct a hedge that minimizes the variance of the hedged portfolio within a class of hedges while we incorporate the covariances among and between the hedging securities and the portfolio itself. The technique we propose extends the standard immunization or duration hedge to use multiple hedging securities. Using multiple hedging securities with different maturities and durations provides certain advantages over using a single hedging security. First, hedging with two or more securities of differing durations ("barbelling") can protect against changes in the slope of the yield curve. Second, using multiple hedging securities can reduce the variance or risk in the portfolio through diversification.

The trick is to choose the combination of hedging instruments in a reasonable, consistent way. Our hedging technique constructs a "covariance-consistent hedge" that is consistent with minimizing the portfolio's risk, while taking into account the covariances of key interest rates that influence the yields of the portfolio and the hedging securities.

In general, there is no way to completely immunize a bond portfolio against non-parallel fluctuations in the yield curve, so the portfolio is subject to yield curve risk. This problem arises because the hedge securities and the bond portfolio almost always differ in maturity or coupon rates, creating a mismatch in the value of a basis point movement in forward rates between a portfolio and its hedge.

For example, hedging a six-year bond with a seven-year bond exposes the hedged portfolio to risk from independent fluctuations in the one-year rate six-years forward. The imperfect correlation between a portfolio and its hedge is due to the imperfect correlation in forward rates.

Hilliard (1984) demonstrates that if hedge securities and bond portfolios are not perfectly correlated because of mismatched maturities, the minimum-variance hedge uses the full covariance matrix of interest rates and uses all relevant hedge securities. Operating on this insight, he solves, for the optimal minimum-variance hedge.

The hedge we describe in this paper works for reasons illustrated by Hilliard, but our algorithm is considerably easier to implement. While Hilliard uses the present value of cash flows in his analysis, we use basis point values; where Hilliard used the covariance matrix of zero rates to each of these cash flows, we use a more standard covariance matrix. Basis point values are calculated in a method similar to that used by those who hedge swaps with Eurodollar futures, and the optimal hedging weights arise from a weighted least squares regression.[2]

There is a subtle similarity between our hedging method and the yield beta approach, which is an extension of the duration hedge. Ederington (1979) demonstrates that to minimize basis risk (the variance of the portfolio consisting of a security and a hedge instrument) the optimal hedge security is the one that has the highest correlation with the security to be hedged. Given this hedge security, Fabozzi (1991) outlines the method of finding the optimal hedge ratio by using the ratio of the basis point value of the bond to be hedged over the basis point value of the hedge security.

The yield beta approach multiplies this hedge ratio by a yield beta, where the beta is found through a regression of the changes in the yield of the security to be hedged on the yield of the hedge security (see Fabozzi (1993)). In this way, the yield beta approach incorporates specific information on the correlation of the different interest rates. Our approach uses the estimated covariance structure of key rates, and thus is comparable to the yield beta approach since both use information on the covariance of different interest rates.

Using a set of key rates and their covariance matrix has certain advantages and disadvantages over the yield-beta approach. Employing a set of key rates (e.g., zeros or forwards) as the factors of the model allows one to model conveniently a wide variety of instruments conveniently, from bonds to futures contracts. The key rates are usually derived from the most liquid points on the yield curve (typically the benchmark issues) and thus provide high-quality, readily available data for estimating the covariance of rates. Moreover, many commercial systems use key rates to compute interest-rate sensitivities (e.g., partial durations or key-rate durations). However, certain securities can exhibit idiosyncratic behavior relative to the key-rate curve, which the yield-beta method may more accurately model.

We construct two separate tests to demonstrate the efficiency of our approach. In the first we use a structured Monte Carlo method to generate random shocks to the yield curve consistent with its covariance structure. Using this method, we can demonstrate the relative efficiency of our hedge against a simple duration hedge and also highlight how mispecification of the covariance matrix affects the results. For hypothetically constructed bond portfolios with and without coupons with maturities ranging from two to ten years, Monte Carlo tests show that using the covariance matrix reduces the basis risk by 22% to 74%, depending on the nature of the portfolio.

The second tests use Treasury and futures prices during 1995 over fifty-one week-long periods. The covariance matrix and yield betas are constructed using the same methodology so that a direct comparison between the approaches can made. These empirical tests show our method reduces basis risk an average of 13% compared to the yield beta approach and an average of 25% compared to the duration approach.

1. A Simple Example

We motivate our approach using an example of a simple hedging problem. Let $BPV_i(k)$ represent the value of a basis point change in key rate i for security k. As with any hedge using basis point values, these sensitivities are relevant only for local changes in rates, which implies the hedged portfolio involves continual rebalancing.

The key rates are forward rates, although we use zero rates in subsequent sections. The security we wish to hedge is a two year zero-coupon bond, B_2, and the two available hedge securities are the one year and three year zero-coupon bonds, H_1 and H_3. The forward curve can be divided up to arbitrarily small lengths, but in this example we will restrict our attention to changes in the one, two and three year forward rates.

First we set up a system of equations where each equation refers to the change in value of the hedged portfolio during a shock to a different forward rate. Let the hedge error e_i represent the change in the value of the portfolio given changes in the one year forward rate r_i. w_1 and w_3 are the as yet undetermined number of hedge securities H_1 and H_3 used in the portfolio.

$$\begin{aligned} BPV_1(B_2) &= w_1 BPV_1(H_1) + w_3 BPV_1(H_3) + e_1 \\ BPV_2(B_2) &= w_1 BPV_2(H_1) + w_3 BPV_2(H_3) + e_2 \\ BPV_3(B_2) &= w_1 BPV_3(H_1) + w_3 BPV_3(H_3) + e_3 \end{aligned} \tag{1}$$

Because we have more equations than unknowns, we cannot find weights *w* that ensure all of the elements of *e* are zero (which would imply complete immunization to all curve movements). If the vector of hedge errors *e* were independent and had equal variances, the method of solving (1) is simply an ordinary least squares regression that minimizes the sum of squared residuals, $\sum_{i=1}^{3} e_i^2$.

This approach would give equal weight to each equation, although we know that 1) the variance of e_1 will be different than the variance of e_2 because in general the "volatility curve" is not flat and 2) as forward rates are positively correlated, positive shocks to e_1 usually imply positive shocks to e_2. The latter point is most important, since non-zero elements of the non-diagonal terms in the error matrix (i.e., correlation between errors) imply that OLS would give not only inefficient parameter estimates but biased ones as well.

Thus our criterion of minimization Var(e), should be amended conditional upon the covariance matrix of forward rates, so that $Var(e) = \sum_{j=1}^{T} \sum_{i=1}^{j} e_i e_j s_{i,j}$, where $\sigma_{i,j}$ is the covariance of the i and j period forward rates. This is a classic weighted least squares approach. Letting Ω represent the covariance matrix of forward rates, the hedge weights *w* that minimize the basis risk in Equation (1)) are found by solving

$$\underset{w}{Min}\, E[e'\Omega e] \tag{2}$$

In our weighted least squares problem

$$y = \begin{pmatrix} BPV_1(B) \\ BPV_2(B) \\ BPV_3(B) \end{pmatrix} x = \begin{pmatrix} BPV_1(H_1) & BPV_1(H_3) \\ BPV_2(H_1) & BPV_2(H_3) \\ BPV_3(H_1) & BPV_3(H_3) \end{pmatrix} b = \begin{pmatrix} w_1 & w_3 \end{pmatrix} \tag{3}$$

and the weighting matrix is the forward rate covariance matrix Ω. Since e=y-xb, substituting for *e* using the definitions in (1), (2), and (3), and solving for the optimal hedge weights gives us the formula

$$b = (x'\Omega x)^{-1} x'\Omega y \tag{4}$$

To use a concrete example, assume we are hedging a $1000 cash flow in two years using $1000 cash flows in years 1 and 3. The assumed yield curve is flat at 10%. The basis point values give us

$$y = \begin{pmatrix} BPV_1(B) \\ BPV_2(B) \\ BPV_3(B) \end{pmatrix} = \begin{pmatrix} .075 \\ .075 \\ 0 \end{pmatrix},\ x = \begin{pmatrix} BPV_1(H_1) & BPV_1(H_2) \\ BPV_2(H_1) & BPV_2(H_2) \\ BPV_3(H_1) & BPV_3(H_2) \end{pmatrix} = \begin{pmatrix} .083 & .068 \\ 0 & .068 \\ 0 & .068 \end{pmatrix} \tag{5}$$

Assume the volatility and correlation matrix of forward rates is

$$v = \begin{pmatrix} vol(F_1) \\ vol(F_2) \\ vol(F_3) \end{pmatrix} = \begin{pmatrix} .20 \\ .18 \\ .16 \end{pmatrix}, r = \begin{pmatrix} 1 & .9 & .8 \\ .9 & 1 & .9 \\ .8 & .9 & 1 \end{pmatrix} \tag{6}$$

The covariance matrix of forward rates is therefore

$$vv' \bullet r = \Omega = \begin{pmatrix} .0400 & .0324 & .0256 \\ .0324 & .0324 & .0259 \\ .0256 & .0259 & .0256 \end{pmatrix} \tag{6}$$

Multiplication of the correlation matrix by the matrix vv' is element-by-element. The solution to the optimally weighted hedge weights are 0.660 and 0.486 for the first and second hedge securities (w_1 and w_3), respectively.

2. A General Method of Implementing the Optimal Hedge

The example is highly restrictive, and we need to clarify the way to apply the approach to a more realistic case. A general derivation of the covariance-consistent hedge as the minimum variance hedge is given in the appendix. We are hedging a portfolio of bonds with three Treasury notes futures traded on the Chicago Board of Trade (CBOT). The first step is to find the basis point values for the cash portfolio and the relevant hedge securities for movements in the key rates, which in this example are the ten annual zero rates.

Let *BPV(B)* represent the 10x1 vector of the basis point value of the bond portfolio for changes for the ten zero coupon rates, and *BPV(H)* represent the 10x3 vector representing the basis point values for the three different futures contracts to the ten different zero rates. We solve for the optimal hedge weights *w** using the equation

$$w^* = \left(BPV(H)'\,\Omega BPV(H)\right)^{-1}\left(BPV(H)'\,\Omega BPV(B)\right) \qquad (8)$$

Here *w** is a 3x1 vector whose elements correspond to the number of CBOT Treasury note futures in the optimal hedge. This approach extends to other cases using similar methodology. For example, using the first four Treasury bill futures, our approach would use the covariance matrix of the monthly forward rates and calculate the basis point values for the four different Treasury bill futures to perturbations in each monthly zero rate.

3. Monte Carlo Results

We construct three hedges for the Monte Carlo tests. In the first method, the hedge weights are found by allocating each cash bond to the hedge security with the closest duration, creating a classic duration hedge. A second duration approach allocates a bond to its "barbells", the hedge securities with durations just higher and lower than the cash bond's duration, such that the total duration of the two barbell positions is equal to that of the bond. This method is an important comparison because it may be that much of the benefit of our approach is simply due to the splitting of a bond to several futures contracts and not the covariance matrix of key rates.[3]

A bond with a duration at an extremum of the hedge securities is allocated to only the nearest duration hedge security. Present value is also maintained. The covariance-consistent hedge is computed using the algorithm supplied in Equation (8).

We use six different portfolios in order to see if different parts of the curve generate different magnitudes of relative efficiency for the covariance-consistent hedge. All portfolios contain fifty different bonds with equally spaced maturities (all with identical par values). Portfolio 1 has maturities from two to ten years, portfolio 2 has maturities from two to five years and portfolio 3 has maturities from five to ten years. Portfolios 4 through 6 are identical to portfolios 1 through 3, except they have no coupons. The hedge securities considered are constructed to resemble the 2, 5 and 10 year CBOT US Treasury contracts. We used 2 year, 4.5 year, and 7.5 year 8% coupon bonds as hedge securities.

Using Citibase data of monthly rate changes from October 1982 to June 1995 we estimate a covariance matrix of zero rates, Ω, which is used to generate 1,000 random shocks taking into account the covariance structure of the zero curve.[4] This same covariance matrix is used to solve for the "covariance-consistent" hedge weights, while the "covariance-consistent w/ error" hedge weights are formed using Ω+E and Ω-E, where E is a noise matrix.

E is a symmetric matrix in which the terms increase in absolute magnitude as they diverge from the diagonal. Ω+E overestimates the cross covariance terms up to 40% (at the covariance of the one and ten year rates), while Ω-E does the opposite. This is done to gauge the robustness of the covariance-consistent hedge in the face of measurement error of the covariance matrix.

The initial rate structure is assumed to be flat at 6%. Using the 1,000 rate change scenarios, we have 1,000 different changes in yield curves, and thus 1,000 observed changes in the value of the hedged portfolio (i.e., basis changes). Table 1 shows the standard deviation of basis changes for the various portfolios using the two duration approaches and three variants of the covariance hedge (these three variants include using the true covariance matrix, and two using the a covariance matrix that over- and under-estimates the cross correlation terms).

The first thing to note from Exhibit 1 is that the covariance-consistent hedge reduces basis risk from between 21% and 76% compared to the duration approach. Further, the duration approach and the barbell approach produce similar results; in fact, in these tests the bucket hedge often outperformed the barbell hedge. For this reason and also for simplicity, our subsequent tests that use actual portfolios consider only the bucket hedge.

The superiority of the covariance-consistent hedge relative to the barbell approach demonstrates that it is not more efficient simply because it splits the bond between two hedge securities with different maturities; the splitting of the bond to different futures has to be done in a way that incorporates the covariance of the zero rates in order to elicit the beneficial results.

The covariance hedge outperforms the duration hedge because the hedged portfolios in this environment are subject to risk from non-parallel shifts in the yield curve. As opposed to immunizing the portfolio perfectly against an arbitrary movement in rates (a parallel shift), the covariance hedge minimizes basis risk over all yield curve movements in proportion to their estimated probabilities.

The advantage of our approach increases the greater the curve risk in the portfolio. For example, the portfolio of bonds with maturities from five to ten years demonstrates the greatest relative efficiency of the covariance approach. For the coupon bond portfolios, using the covariance-consistent hedge ratios reduces basis risk is reduced 52%, 22%, and 62% for the two- to ten- year, one- to five- year, and five- to ten- year maturity portfolios, respectively by.

Zero-coupon bonds are subject to more basis risk than the coupon bonds because the hedge securities entail coupon payments and are subject to curve risk not contained in the zero coupon bonds. The covariance-consistent hedge takes this into account while the duration approach does not, and thus the covariance approach delivers its greatest relative efficiency in this context. Compared to the duration hedge, the covariance-consistent hedge produces on average 48% reduction in basis risk for the coupon portfolios and an average 65% reduction in basis risk for the zero-coupon portfolios.

Exhibit 1 also shows the performance of the covariance-consistent hedge when the hedge is constructed with a misestimated covariance matrix of interest rates. To demonstrate this we add and subtract “noise” from Ω when determining our hedging weights, yet kept the simulation of rate changes based on Ω. This increases the basis risk relative to using the true covariance when generating the covariance-consistent hedge, but it remains significantly less risky than using the duration approach.

Note that the basis risk is affected asymmetrically affected by error. On average over all six portfolios, overestimation of the cross-correlation terms increases basis risk 25% compared to using the true covariance matrix, while underestimation of the cross-correlations increases basis risk only 5%.

In comparison to the duration hedge (on average over all six portfolios), basis risk is reduced 58% using the true covariance matrix, while the covariance-consistent hedge calculated with error reduces risk an average of 50%. This suggests the covariance-consistent hedge provides substantial benefits even when accounting for the practicalities of estimating the covariance matrix of zero rates.

IV Results from Actual Bond and Futures Data, 1995

Our tests using actual Treasury and CBOT futures data through 1995 involves five different US Treasury bonds. We used the thirty, ten, and three year on-the-run Treasury notes, which are the most recently auctioned thiry, ten and three and ten year Tnotes. These represent very liquid securities, and involved 5 different issues over the testing period (January 3, 1995 to December 26, 1995). We also used the 8% 5/15/01 Treasury note and the zero-coupon Treasury note of 5/15/05, which have approximate maturities during the test period of six- and ten-years, respectively.

The futures we use are the two, five, and ten year tnote and the Treasury bond futures. We used the front month futures contract and switches out of that contract when there were 3 weeks remaining to delivery. This avoids complications caused by liquidity and option issues at the end of a futures contract.[5]

Data are from Bloomberg. We use the closing price for the futures and the 3pm New York price for the Treasuries, so that Treasuries and the futures are synchronous prices. Holding periods in the portfolio are weekly, from Monday close to Monday close (if there was a Monday holiday we used Tuesday close for both Treasuries and futures).

We compare our approach to the duration approach and the yield beta approach. The duration hedge calculates a hedge ratio by finding the value of a basis point (i.e., the change in the security's value when the zero curve changes by a basis point) for the security to be hedged and the Treasury underlying the futures contract. That is:

$$\frac{\text{Price value of a basis point for bond to be hedged}}{\text{Price value of a basis point for hedge security}} = \text{Duration hedge ratio}$$

The yield beta approach multiplies the prior hedge ratio by a yield beta. Specifically:

$$\frac{\text{Price value of a basis point for bond to be hedged}}{\text{Price value of a basis point for hedge security}} \times \text{Yield beta} = \text{Yield beta duration hedge ratio}$$

Yield betas are estimated by regressing the change in yield of the security to be hedged on the security underlying the futures contract. For construction of yield beta only, we simply use two, three, and five year constant maturity Treasury rates to proxy for the two year futures, the two year on-the-run, and the five year futures. Yield betas and covariance matrices can be specified in various ways, and to keep them consistent we use a similar methodology, using the past twenty-six weekly changes in yields as inputs. The issue of finding the optimal forecast of future covariances (and also yield betas) is not addressed here, and improving on the forecasting methodology would most likely help the yield beta and the covariance-consistent approach similarly.[6]

Each week we generate a new set of hedge weights, depending on the current interest rates and the most recent estimate of the yield beta and covariance of key interest rates. At the end of each week, the profit or loss of the hedged portfolio is documented and used to calculate the standard deviation of basis changes for each approach over the sample period. The lower the standard error, the greater the hedge efficiency.

Exhibit 2 displays the average portfolio hedge weights for the five different bonds using the three different hedge methods. Exhibit 3 shows the sample standard deviations of basis changes over the fifty-one weeks from the sample period. The yield beta hedge dominates the duration method on average, but does worse than the duration method for both the three year on-the-run and the six year bond. It appears that as one moves down the yield curve towards zero the yield beta method does worse, although the yield beta approach is significantly better than a duration hedge when using the longer maturity 10 year on-the-run and zero bonds.

The covariance-consistent hedge bests both approaches for four of the five bonds. The covariance-consistent method shows its greatest advantage vis-à-vis the duration method for the 10 year zero-coupon bond, while vis-à-vis the yield beta method it does best using the three and six year bonds. For all but the 30 year on-the-run Treasury, the covariance-consistent hedge reduces basis risk an average of 30% and 19% relative to the duration and yield beta methods.

The yield beta method works best for the 30 year on-the-run Treasury primarily because all potential hedges have lower durations, and therefore we constrain the covariance-consistent hedge to use only one hedge security (for reasons below). Unsurprisingly in this context, the specific information implicit in the yield beta approach dominates using standardized zero rate correlations.

5. Adding all Futures in the Covariance-Consistent Hedge

Exhibit 3 also shows the results when using all four futures contracts for each bond portfolio (listed under the column Full Cov-Cons.). As illustrated in the appendix, the optimal estimator uses all available hedges, as the ability to diversify the random errors of these forecast variables makes for the most efficient hedge. In practice, there is little difference between all using the futures and using the futures with maturities just above and below the maturity of the bond to be hedged. The reason is collinearity.

When two hedge securities have durations strictly above or below the duration of the bond to be hedged, the high correlation between the two hedges increases the absolute values and standard errors of the coefficient estimates generated by the weighted least squares regression (Equation 8). Even with this consideration, however, using all futures contracts does not significantly worsen the performance of the covariance-consistent hedge. Thus a black box that uses all four futures contracts appears to dominate the yield beta approach for all but the thirty- year on-the-run Treasury.

6. Conclusion

Complete immunization against arbitrary shifts in the yield curve is trivial if one can choose an arbitrary array of hedge securities. For most situations, however, one simply does not have enough liquid hedges available to make this remotely practical. In these environments, the covariance-consistent hedge provides a significant increase in hedge efficiency over duration and yield beta approaches. Further, unlike other more efficient hedging techniques such as factor analysis, this hedge does not require highly sophisticated statistics. The necessary data are the basis point values of the relevant securities, the covariance matrix of interest rates, and the ability to do matrix algebra.

Our method is based on the fact that most hedges do not have underlying cash flows identical to those in the portfolio they are hedging, and thus will produce basis risk. By taking into account the covariance of the portfolio basis point values due to the covariance of key points on the zero curve, our approach provides hedging weights aimed at the minimizing the basis risk to the portfolio over all changes in these key rates. Since long and short rate movements are positively correlated, this method is still largely insulated against parallel shifts in the yield curve. Unlike the duration method, however, this approach also takes into account the performance of the hedge under non-parallel shifts in proportion to their estimated probability of occurring.

It is also shown that misestimations in the covariance matrix still allow significant outperformance of traditional hedges. This is reinforced by the empirical tests using actual data (which of course include measurement error of the covariance matrix).

Table 1
Mean Standard Deviation in Basis Changes
(as a % of par value of the bonds)

The basis is the difference in value between the portfolio of bonds and the portfolio of hedge securities, thus the change in basis is the change in the value of the hedged portfolio. The initial rate structure is assumed to be flat at 6%. The numbers represent the mean absolute value of the change in the value of the basis. Each portfolio contained 50 securities with evenly spaced maturities between the top and bottom maturity bond. All bonds were assumed to have principal amounts. The various columns represent the performance of the different hedge construction techniques. The Classic hedge allocates each bond in the portfolio to a single hedge security while the Barbell hedge uses the two nearest hedge securities. For the three hedges using the covariance-consistent method, Ω uses the true covariance matrix, Ω+noise adds a noise matrix which overestimates the cross correlation terms, and Ω-noise subtracts the noise matrix and thus underestimates the cross correlation terms. The 1,000 rate change scenarios representing monthly changes in yield curves were generated consistent with the Ω.

	Duration Hedges		Covariance-Consistent Hedges Using		
8% coupon	Classic	Barbell	Ω	Ω+noise	Ω-noise
2-10 year mat.	0.0192	0.0203	0.0093	0.0140	0.0101
2-5 year mat.	0.0186	0.0183	0.0145	0.0150	0.0147
5-10 year mat.	0.0296	0.0353	0.0113	0.0176	0.0121
Average	0.0225	0.0246	0.0117	0.0155	0.0123
zero coupon					
2-10 year mat.	0.0200	0.0203	0.0086	0.0132	0.0094
2-5 year mat.	0.0310	0.0329	0.0133	0.0138	0.0135
5-10 year mat.	0.0428	0.0462	0.0111	0.0170	0.0118
Average	0.0313	0.0331	0.0110	0.0147	0.0116
Total Average	0.0269	0.0289	0.0113	0.0151	0.0119

Table 2
Average Portfolio Hedge Weights, Weekly Holding periods 1/3/95-12/26/95

Each hedge used a different number of hedge contracts. The Duration and Duration with Yield Beta approach used the futures contract with the nearest duration to the cash bond, while the covariance-consistent hedge used only those futures with maturities just above and just below the cash bond. The hedged portfolio was assumed to contain one bond with a par amount of 100, while the futures contracts were also assumed to have a par amount of 100.

	Duration	Duration w/ Yield Beta	Covariance-Consistent
3 year Hot Run	0.67 5yr	0.70 5yr	0.74 2yr & 0.36 5yr
8% 5/15/01	0.88 10yr	0.90 10yr	0.67 5yr & 0.42 10yr
10 year Hot Run	1.28 10yr	1.15 10yr	0.86 5yr & 0.20 10yr
5/15/05 Strip	0.86 10yr	0.73 10yr	0.43 10yr & 0.22 Tbond
30 year Hot Run	1.20 Tbond	1.08 Tbond	1.05 Tbond

Table 3
Portfolio Basis Risk, Weekly Holding periods 1/3/95-12/26/95
(as a % of par value of the bonds)

The basis risk is the standard deviation in the change in value of the hedged portfolio over one week holding periods. Each portfolio used one bond with a par amount of 100, and hedged with the appropriate futures according the hedge methodology described in the paper. Futures contracts included the 2, 5, 10 year Tnote and the Tbond futures traded at the CBOT. All prices were calculated as of 3PM Monday (Tuesday if a holiday). Yield betas and the covariance matrix used the prior 26 weekly interest rate changes as of the Friday close prior to implementing the hedge at the close of Monday. Cov-Cons. is the covariance-consistent hedge using two futures (with just the Tbond futures for the 30 year bond), while Full Cov-Cons. is the covariance-consistent hedge using all four futures contract. The final two columns represent the basis risk of the covariance-consistent hedge relative to the Duration and Duration w/ Yield Beta hedges, respectively.

	1	2	3	4	3/1	3/2
	Duration	Duration w/ Yield Beta	Cov-Cons.	Full Cov.-Cons.	Relative Efficiency	
3 year on the run	0.096	0.106	0.082	0.084	.85	.78
8% 5/15/01	0.119	0.123	0.090	0.098	.76	.73
10 year on the run	0.238	0.161	0.149	0.147	.63	.92
5/15/05 Strip	0.201	0.137	0.113	0.105	.56	.83
30 year on the run	0.225	0.192	0.211	0.228	.96	1.1
Average	0.176	0.144	0.130	0.132	0.75	0.87

Appendix A General Exposition of the Minimum Variance Hedge

Let P be an nx1 vector of n bond prices with maturities between 0 and T periods, and let F be a kx1 vector of k futures contract prices. Let α be an nx1 vector of weights representing the number of bonds in the cash portfolio, and let γ be a kx1 vector of weights representing the number of futures contracts. Define the basis, *b*, as the difference in price between the portfolio of cash bonds and the futures.

$$b = aP - gF \tag{A1}$$

Although in general E(*b*)≠0, over short intervals this has negligible practical impact, and so we shall assume that E(*b*)=0. Futures and bond prices are assumed both determined by the same set of forward rates, r_t, t=1 to T. The variance of the basis is due to shocks to r_t, so that the variance of the basis can be stated in terms of the random vector r as Var(*b*(*r*)). Since E(*b*)=0, var(*b*(r)) is simply $\{db/dr\}^2$var(r). Since r is a vector we must sum a collection of terms

$$Var(b) = \sum_{j=1}^{T} \sum_{i=1}^{j} \frac{db}{dr_i}\frac{db}{dr_j} E\left(r_i r_j\right) = \sum_{j=1}^{T} \sum_{i=1}^{j} \frac{db}{dr_i}\frac{db}{dr_j} s_{i,j} \tag{A2}$$

Here db/dr_j is the sensitivity of the basis to a basis point change in the forward rate r_j, and $\sigma_{i,j}$ is the covariance of forward rates r_i and r_j. Letting Ω be the TxT covariance matrix of the forward curve this can be written more simply as

$$Var(b) = E\left\{\frac{db}{dr}' \, \Omega \frac{db}{dr}\right\} \tag{A3}$$

Substituting for $\frac{db}{dr}$ by taking the derivative of (A1) we get

$$Var(b) = E\left(a\frac{dP}{dr} - g\frac{dF}{dr}\right)' \Omega \left(a\frac{dP}{dr} - g\frac{dF}{dr}\right) \tag{A4}$$

The gradients $\frac{dF}{dr}$ and $\frac{dP}{dr}$ can be constructed as the basis point sensitivities of the cash portfolio and the various potential hedge securities. Thus splitting the curve up into 10 annual forwards, $\frac{dP}{dr_5}$ would be the change in the value of the cash portfolio to a basis point change in the 5 year forward. The objective of minimizing a hedged portfolio's interest rate risk can thus be formally stated as

$$\underset{g}{Min}\, E\left(a\,\frac{dP}{dr} - g\,\frac{dF}{dr} \right)' \Omega \left(a\,\frac{dP}{dr} - g\,\frac{dF}{dr} \right) \qquad \text{(A5)}$$

Where α is an nx1 vector, $\frac{dP}{dr}$ is an nxT matrix, γ is a kx1 vector, $\frac{dF}{dr}$ is a kxT matrix, and Ω is a TxT matrix. The first order condition which solves (A5) is

$$-2\,\frac{dF}{dr}'\,\Omega a\,\frac{dP}{dr} - 2\,\frac{dF}{dr}'\,\Omega g\,\frac{dF}{dr} = 0 \qquad \text{(A6)}$$

which solving for the optimal hedge ratio γ* gives

$$g^* = \left(\frac{dF}{dr}'\,\Omega a\,\frac{dP}{dr} \right) \left(\frac{dF}{dr}'\,\Omega\,\frac{dF}{dr} \right)^{-1} \qquad \text{(A7)}$$

The resulting formula for γ* is thus a weighted least squares equation where the weighting matrix is equal to the covariance matrix of forward rates.

References

Barrett, W. Brian, Thomas F. Gosnell, Jr., and Andrea J. Heuson. “Yield Curve Shifts and the Selection of Immunization Strategies.” *Journal of Fixed Income*. September, (1995).

Burghardt, Galen, Terry Belton, Morton Lane, Geoffrey Luce, and Rick McVey. *Eurodollar Futures and Options.* Probus: Chicago. 1991.

Ederington, L. “The Hedging Performance of the New Futures Markets.” *Journal of Finance*, 34 (1979), pp. 157-170.

Fabozzi, Frank J. *Handbook of Fixed Income Securities, Third Edition.* 1991. Business One Irwin. Homewood Illinois. pp. 1036-57.

Fabozzi, Frank J. *Fixed Income Mathematics, Revised Edition.* 1993. Probus, Chicago Illinois. pp. 152-3

Fong, H.G., and O. Vasicek. “The Tradeoff Between Return and Risk in Immunized Portfolios.” *Financial Analysts Journal* (September-October 1983), pp. 73-78.

Guldimann, Till. *J.P. Morgan’s RiskMetrics^{TM}-Technical Document*, Third Edition. 1995.

Hilliard,, Jimmy. “Hedging Interest Rate Risk with Futures Portfolios under Term Structure Effects.” *Journal of Finance*, (1984), pp. 1547-1569.

Ho, T.S.Y. “Key Rate Durations: Measures of Interest Rate Risks.” *Journal of fixed Income*, September (1992), pp. 29-44.

Reitano, Robert R., “Non-Parallel Yield Curve Shifts and Stochastic Immunization.” *Journal of Portfolio Management*, Winter (1996), pp. 71-78.

---

[1] A key rate is defined by Ho (1992) as maturity points on the interest rate curve to be used in generating security sensitivities, such as the 1, 2, 3, 5, 10 and 30 year points. Reitano (1996) refers to these points as yield curve drivers.

[2] See Burghardt et al (1991) for a description of the use of basis point values when hedging swaps with Eurodollar contracts.

[3] For example, assume only the forward rates 1, 2, and 3 can move. A 2 year cash bond allocated to hedges with maturities of 1 and 3 would perforce be susceptible to the 3 year forward rate. Further, it would not be perfectly immunized against the 1 and 2 year forwards since the duration added by the 3 year hedge would have to be offset by reductions in duration added by the 1 year hedge.

[4] This is done by taking the Cholesky decomposition of Ω and multiplying it by the 10x1 vector of numbers generated independently from a standard normal distribution, resulting in a 10x1 vector of zero rate changes consistent with the covariance matrix Ω.

---

[5] For example, on May 29 we used the June futures contracts, and switched to the September contracts on June 1.

[6] Alternatively one could use the covariance matrix provided by J.P. Morgan, which now make available daily and monthly covariance estimates of swap and Treasury curves for most major currencies updated daily on the internet (RiskMetrics[TM]).