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Volatility Structures of Forward Rates and the Dynamics of the Term Structure*

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For general volatility structures for forward rates, the evolution of interest rates may not be Markovian and the entire path may be necessary to capture the dynamics of the term structure. This chapter identifies conditions on the volatility structure of forward rates that permit the dynamics of the term structure to be represented by a two-dimensional state variable Markov process. The permissible set of volatility structures that accomplishes this goal is shown to be quite large and includes many stochastic structures. In general, analytical characterisation of the terminal distributions of the two state variables is unlikely, and numerical procedures are required to value claims. Efficient simulation algorithms using control variates are developed to price claims against the term structure.

This chapter deals with the pricing of contingent claims when interest rates are stochastic. The methodology used incorporates all current information in the yield curve. This approach, pioneered by Ho and Lee (1986) and significantly generalised by Heath, Jarrow and Morton (1992), relies upon markets being dynamically complete with continuous trading opportunities. Analogous to the Black–Scholes model, where preferences are embedded into the stock price and the volatility is exogenously provided, in this approach preferences are embedded into the observable term structure, and the volatility function for forward rates is exogenously specified.

Heath, Jarrow and Morton (1992) show that to preclude arbitrage opportunities among bonds of different maturities, the drift and volatility terms in the evolution of all forward rates must be related to a common market price of interest rate risk. Further, they identify a unique martingale measure that can be used to price all interest rate claims. Unfortunately, the computation of prices is complex because the evolution of the term structure under the martingale measure is usually not Markovian with respect to a finite-dimensioned state space.[1] The path dependence is also made apparent in a

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related paper by Heath, Jarrow and Morton (1990) where they use a binomial approximation to provide an alternative derivation of their main results. Due to path dependency the lattice usually grows exponentially with the number of time periods. The path dependence causes complications even if simulation models are used to compute the prices of European claims. In particular, since the *entire* term structure must be manipulated along each simulated path, simulations are computationally intensive, and the build-up of errors due to discreteness may be severe. Many single-factor models of the term structure resolve the complications introduced by path dependence by imposing sufficient structure on the problem so that the evolution of the term structure is *path-independent*. In contrast, this chapter identifies conditions which permit the term structure's dynamics to be captured by two variables rather than the entire term structure. In particular, we identify conditions on the volatility structure of forward rates that permits the term structure's dynamics to be represented by a two-dimensional sufficient statistic. Unlike the general Heath, Jarrow and Morton (HJM) models, our models thus lead to a *two-state-dimensional* Markov representation of the term structure. The permissible set of volatility structures that accomplishes this goal is shown to be quite large and includes many stochastic and deterministic structures. For example, the volatility structure can be specified in such a way that the resulting spot rate dynamics is generated by a square root process. In general, an analytical characterisation of the terminal distributions of the two state variables is unlikely, and hence numerical procedures are required. We illustrate how simulation techniques, combined with control variables, can be used to establish efficient pricing algorithms for all types of claims against the term structure.

The chapter proceeds as follows. In the next section we identify the path dependence that is implicit in models of the term structure in which the volatility structure of forward rates is arbitrary. We then show that if the volatility structure is carefully curtailed, bond prices can be expressed in terms of the spot rate and an additional statistic which captures all the relevant history of the term structure dynamics. The resulting spot rate process induced by the volatility constraint is developed. This process is a two-state-variable Markovian process and has the desirable property of incorporating into its dynamics all the information provided from the term structure. When the volatility structures are deterministic then simple analytical solutions for pricing interest rate claims are available. For all other cases numerical procedures are required. Fortunately, since the martingale measure which is relevant for pricing is Markovian, models can be established which do not require the entire term structure to be manipulated along each path. The third section discusses simulation mechanisms and provides examples of pricing interest-sensitive options, when the volatility structure of forward rates is non-deterministic. The convergence of simulated results is substantially accelerated by using appropriate control variables. The valuation of claims that depend on prices of risky assets as well as on yields drawn from the term structure is also considered. The fourth section summarises the chapter.

Path dependence and volatility structures

Forward rates are assumed to follow a diffusion process of the form

$$df(t,T) = \mu_f(t,T)dt + \sigma_f(t,T)dw(t); \quad \text{given } f(0,T)\,\forall T \tag{1}$$

Here $\mu_f(t, T)$ and $\sigma_f(t, T)$ are the drift and volatility parameters which could depend on the level of the term structure itself, and $\delta\omega(\tau)$ is the Wiener increment.

Integrating (1) yields the relation between current forward rates and their values at time t. In particular,

$$f(t,T) = f(0,T) + \int_0^t \mu_f(s,t)ds + \int_0^t \sigma_f(s,T)dw(s) \quad \text{for } t \le T \tag{2}$$

Since the spot rate at time t, r(t), is given by f(t, t), we obtain that

$$r(t) = f(t,t) = f(0,t) + \int_0^t \mu_f(s,t)\,ds + \int_0^t \sigma_f(s,t)\,dw(s) \tag{3}$$

The dynamics of the spot rate are then obtained as

$$dr(t) = df(t,t) + \frac{d}{du} f(t,u)\Big|_{u \to t} dt \tag{4}$$

Let P(t, T) be the price at date t of a pure discount bond that matures at time T. By definition, the bond price is given by

$$P(t,T) = \exp\left(-\int_t^T f(t,s)\,ds \right) \tag{5}$$

Since bond prices depend on forward rates, as evidenced by (5), the drift and volatility structure of bond returns must be related to the drift and volatility structure for forward rates. In particular, let

$$\frac{dP(t,T)}{P(t,T)} = \mu_P(t,T)\,dt + \sigma_P(t,T)\,dw(t) \tag{6}$$

Usual arbitrage arguments then lead to

$$\lambda(t) = \frac{\mu_P(t,T) - r(t)}{\sigma_P(t,T)} \tag{7}$$

and

$$\mu_f(t,T) = \sigma_f(t,T)\left[\lambda(t) - \sigma_P(t,T)\right] \tag{8}$$

where

$$\sigma_P(t,T) = -\int_t^T \sigma_f(t,s)\,ds \tag{9}$$

Here λ(t) is the market price of interest rate risk at time t, and is perhaps stochastic, but is the same for all bonds regardless of maturity. Equation (1) can now be written as

$$df(t,T) = \sigma_f(t,T)\left[\lambda(t) - \sigma_P(t,T)\right]dt + \sigma_f(t,T)\,dw(t) \tag{10}$$

Equation (10) makes explicit the relation between the drift component of the evolution of forward rates, the volatility structure of forward rates, and the market price of risk, which must be satisfied to preclude dynamic arbitrage opportunities. Equivalently, integrating (10) we obtain

$$f(t,T) = f(0,T) + \int_0^t \sigma_f(u,T)\left[\lambda(u) - \sigma_P(u,T)\right]du + \int_0^t \sigma_f(u,T)\,dw(u) \tag{11}$$

Using (6) and (11) leads to the relation between forward prices and future bond prices:

$$P(t,T) = \left[\frac{P(0,T)}{P(0,t)}\right] \exp\left\{ \int_t^T \int_0^t \sigma_f(u,s)\,\sigma_P(u,s)\,du\,ds - \int_t^T \left[\int_0^t \sigma_f(u,s)\left[\lambda(u)\,du + dw(u)\right] \right] ds \right\} \tag{12}$$

This form of the bond pricing equation is given in Heath, Jarrow and Morton (1992). As they emphasise, this general form is quite distinct from the traditional literature. In the traditional literature, bond prices are *assumed* to be functions of some state variables and explicit formulae linking bond prices to these state variables are then established. For example, in many single-factor models, the state variable is the spot rate and bond prices are developed as a function of the spot rate. In the above term-structure-constrained approach, bond prices may not be Markovian with respect to a finite set of state variables. Indeed, the above single-factor representation of the term structure highlights the fact that bond prices at date t depend on prices at an earlier date, on the entire path taken from date 0 to date t, and of course on the evolution of the market price of risk.

From a valuation perspective, (11) and (12) are of little use. It would be desirable to obtain a bond pricing equation that depends on a few state variables, common across all maturities. It would also be desirable for the equation to be preference free in the sense that the market prices of risk do not appear. To accomplish this goal, we impose further restrictions on the process by which forward rates are generated. This point marks our departure from Heath, Jarrow and Morton (1992).

With the goal of obtaining a proxy for the path of interest rates that is relevant for pricing all bonds, rewrite (11) as

$$f(t,T)-f(0,T)-\int_0^t \sigma_f(u,T)\left[\lambda(u)-\sigma_P(u,T)\right]du=\int_0^t \sigma_f(u,T)\,dw(u)=\Lambda(t,T)$$

$\Lambda(t, T)$ reflects the path dependence as the "weighted sum" of all Brownian disturbances realised from time 0 to time t. This path dependence can be captured by a single statistic common across all maturities, T, without imposing any additional restrictions on the initial term structure, $f(0, T)$, or on the structure for the market price of risk, $\lambda(t)$, providing a common "weighting scheme" exists for forward rates of all maturities. However, if a unique weighting scheme is to exists for all forward rates, it must be the case that the weighting function is independent of T. This in turn implies that

$$\frac{\sigma_f(u,T)}{\int_0^t \sigma_f(x,T)\,dx}=\frac{\sigma_f(u,t)}{\int_0^t \sigma_f(x,t)\,dx}\qquad \forall u\in[0,t]$$

Equivalently,

$$\sigma_f(u,T)=\sigma_f(u,t)\left[\frac{\int_0^t \sigma_f(x,T)\,dx}{\int_0^t \sigma_f(x,t)\,dx}\right]=\sigma_f(u,t)k(t,T)$$

where the new quantity $k(t, T)$ is well defined for all $t > 0$.

Of course $\sigma_f(u, T)$ is fully determined at date u. This implies that the right-hand side is also fully determined at date u. In general, both the numerator and the denominator of $k(t, T)$ are only known at a later date t, but from the above, their ratio must be independent of the path taken from the earlier date u. Hence, $k(t, T)$ must be deterministic. Substituting $t = u$ leads to

$$\sigma_f(u,T)=\sigma_r(u)k(u,T)$$

where $\sigma_r(u) = \sigma_f(u, u)$. Similarly, $\sigma_f(u, t) = \sigma_r(u)k(u, t)$. Hence

$$k(u,T)=k(u,t)k(t,T)\qquad \forall t\in[u,T] \tag{13}$$

with

$$k(t,T)=1\qquad \forall t\in[u,T]$$

If $\sigma_f(t, T)$ is differentiable with respect to T, the $k(t, T)$ is differentiable with respect to T as well. Let $g(t, T) = \ln k(t, T)$. Then

$$g(u,T)=g(u,t)+g(t,T)\qquad \forall t\in[u,T]$$

Differentiating with respect to T yields

$$\frac{\partial}{\partial T} g(u,T) = \frac{\partial}{\partial T} g(t,T)$$

This implies that the above derivative is independent of u. Let $\kappa(T) = -\partial g(u,T)/\partial T$. Then, by definition,

$$g(u,T) - g(u,u) = \int_u^T \frac{\partial}{\partial x}\left[g(u,x)\right]dx = -\int_u^T \kappa(x)\,dx$$

Further, $g(u, u) = 0$. Hence the above equation reduces to

$$k(u,T) = \exp\left(-\int_u^T \kappa(x)\,dx\right) \tag{14}$$

Equation (14) is a necessary condition that must be satisfied by the volatility structure of forward rates if the path dependence is to be captured by a single statistic. It turns out that this is sufficient as well, and in the resulting models all bond prices can be expressed in terms of two state variables. These results are summarised below.

PROPOSITION 1 *(a) If the volatilities of forward rates are differentiable with respect to their maturity dates, then for any initial term structure, a necessary and sufficient condition for the prices of all interest contingent claims at time t to be completely determined by a two-state-variable Markov process is that the volatility structure of forward rates satisfy*

$$\sigma_f(t,T) = \sigma_r(t)\exp\left(-\int_t^T \kappa(x)\,dx\right) \tag{15}$$

(b) Under the restriction imposed by (15), the price of a bond, at any future date t, can be represented in terms of its forward price at date 0, the short interest rate at date t, and the path of interest rates as

$$P(t,T) = \left(\frac{P(0,T)}{P(0,t)}\right)\exp\left\{-\tfrac{1}{2}\beta^2(t,T)\phi(t) + \beta(t,T)\left[f(0,T) - r(t)\right]\right\} \tag{16}$$

where

$$\beta(t,T) = -\frac{\sigma_P(t,T)}{\sigma_f(t,t)} = \int_t^T e^{-\int_t^T \kappa(x)\,dx}\,du$$

$$\phi(t) = \int_0^t \sigma_f^2(s,t)\,ds$$

PROOF See the Appendix.

Equation (16) identifies the two state variables as the spot interest rate, $r(t)$, and the "integrated variance" factor, $\phi(t)$. Given these two values at date t, the entire term structure can be reconstructed. Further, under the forward rate volatility restriction, we can express the evolution of the two state variables, $r(t)$ and $\phi(t)$, in terms of their current values and the level of the forward rate curve at an earlier date 0. Specifically, substituting in (4) from (15) and (16) leads to

$$dr(t) = \mu_r(t)\,dt + \sigma_r(t)\,dw(t) \tag{17}$$

and

$$d\phi(t) = \left(\sigma_r^2(t) - 2\kappa(t)\phi(t)\right)dt \qquad (18)$$

where

$$\mu_r(t) = \kappa(t)\left[f(0,t) - r(t)\right] + \phi(t) + \sigma_r(t)\lambda(t) + \frac{d}{dt}f(0,t)$$

and

$$\sigma_r(t) = \sigma_f(t,t)$$

Equation (17) describes the evolution of the spot interest rate by a two-state Markovian process. This is in sharp contrast to the general HJM models in which the interest rate process cannot be described by a finite-state Markov process.

By constraining the volatility structure of forward rates we are able to capture the dynamics of the entire term structure by a two-state Markovian representation. If the volatility structure is not of the form in (15), then such a representation is not possible, and either full path dependence (as in the general HJM models) will be present, or additional assumptions on volatilities and investment behaviour will be necessary to obtain a Markovian representation.

The class of volatility structures admitted by (15) is quite large. In particular, no restrictions are placed on the volatility of the spot interest rate. Indeed, the spot rate volatility at date t could depend on the full set of information available at date t. As an example of a feasible representation, consider the special case of (15) where

$$\sigma_f(t,T) = \sigma_r(t)e^{\kappa(T-t)} \qquad (19)$$

and

$$\sigma_r(t) = \sigma\left[r(t)\right]^{\gamma} \qquad (20)$$

Notice that for $\gamma = 0$ the volatility structure becomes deterministic. In this case the spot interest rate becomes the only state variable, and the above proposition reduces to a statement on necessary and sufficient conditions that permit the term structure to be Markovian with respect to the spot interest rate. The conditions have previously been derived by Carverhill (1994) and by Hull and White (1993). The above proposition generalises their results by considering a larger class of volatility structures that permit a two-state Markovian representation.

For $\gamma = 0.5$ the spot rate volatility is square root and forward rates are linked to spot rates via an exponentially dampened function. In general, the specification given in (19) and (20) leads to models of the term structure that are completely described by three parameters, namely σ, κ and γ.

Valuation of claims under the restricted volatility structure

Heath, Jarrow and Morton (1992) develop the martingale measures under which claims on the term structure may be priced as expectations of their terminal payoffs relative to a money market fund. In practice, computing these expectations is extremely difficult, except for the simplest of cases where the volatility structure is deterministic. For the general case, computing the terminal term structures under the risk-neutralised process requires constructing the *entire* term structure at every point in time. This burden rapidly becomes computationally expensive, but is necessary for valuing all types of interest rate claims.

In contrast, models under the restricted volatility structure identified in this chapter are two-state Markovian. As a result, all the information contained in the term structure is captured by two variables. Hence, to construct the terminal term structure, any algorithm need only keep track of these two variables. In this section we develop a discrete approximation algorithm to value European claims on the term structure. Further, we show how the performance of the algorithm can be significantly enhanced by using control variate techniques, in conjunction with pricing relationships developed when interest rates have deterministic volatilities.

To make matters specific, we shall restrict attention to the case where the structure for the spot rate volatility depends on the level of the spot rate and $\kappa(\cdot)$ is a constant. Specifically, we assume that the volatility structure is given by (19) and (20). With these restrictions the interest rate process given in (17) reduces to

$$dr(t) = \left(\kappa\left[f(0,t) - r(t)\right] + \phi(t) + \lambda(t)\sigma\left[r(t)\right]^{\gamma} + \frac{d}{dt}f(0,t)\right)dt$$

$$+\sigma\left[r(t)\right]^{\gamma}dw(t) \qquad (21)$$

The volatility of the resulting spot rate dynamics is similar to that of models nested in the structure

$$dr(t) = \kappa\left[\mu - r(t)\right]dt + \sigma\left[r(t)\right]^{\gamma}dw$$

The main difference, however, is that the drift term depends on information from a past term structure as well as on the path statistic, $\phi(t)$, which is described as

$$\phi(t) = \sigma^2\int_0^t\left[r(u)\right]^{2\gamma}e^{-2\kappa(t-u)}du \qquad (22)$$

Now, let $g(0)$ represent the date 0 value of a claim having a terminal payout at date s that is fully determined by yields drawn from the term structure. Using standard arbitrage arguments the martingale measure under which all claims are priced is obtained by setting $\lambda(t) = 0$ in (21). In particular, following Heath, Jarrow and Morton we obtain

$$g(0,s) = \tilde{E}_0\left[\frac{g(s)}{M(s)}\right] \qquad (23)$$

where $M(s)$ is the value of a money fund that is initialised with US$1.0 at date 0 and rolls over at the current riskless rate of return. The expectation in the above equation is taken under the joint *risk-neutralised process*

$$dr(t) = \left(\kappa(t)\left[f(0,t) - r(t)\right] + \phi(t) + \frac{d}{dt}f(0,t)\right)dt + \sigma_r(t)dw(t) \qquad (24)$$

and

$$d\phi(t) = \left(\sigma_r^2(t) - 2\kappa(t)\phi(t)\right)dt \qquad (25)$$

In general, because of path dependency, it is unlikely that analytical expressions for the terminal joint distribution of the state variables $r(s)$ and $\phi(s)$ can be obtained.[2] To establish numerical approximations of the term structure, we begin by partitioning the interval $[0, s]$ into n equal time increments of width Δt. Over the discrete intervals let

$$\sigma_f^a(i,j) = \sigma\left[r(i\Delta t)\right]^{\gamma}e^{-\kappa(j-t)\Delta t}, \qquad j \geq i$$

The approximating path statistic and change in interest rate are $a(\cdot)$ and $\Delta[r^a(\cdot)]$, respectively, where

$$\phi^a(j) = \sigma^2\sum_{i=0}^{j-1}\left[r^a(i)\right]^{2\sigma}e^{-2\kappa(j-i)\Delta t} \qquad (26)$$

and

$$\Delta\left[r^a(i+1)\right] = r^a(i+1) - r^a(i)$$

where $r^a(0) = r(0)$ and $r^a(i)$ approximates the interest rate $r(i\Delta t)$. Then

$$\Delta\left[r^a(i+1)\right] =$$
$$\left(\kappa\left[f(0,i\Delta t) - r^a(i)\right] + \phi^a(i) + \frac{f\left(0,[i+1]\Delta t\right) - f(0,i\Delta t)}{\Delta t}\right)\Delta t + \sigma_f^a(i,i)Z(i) \quad (27)$$

where $Z(i)$ is a normal random variable with mean zero and variance Δt. Using (26) and (22) the path statistic, $\phi^a(\cdot)$ can be updated as follows:

$$\phi^a(i+1) = e^{-2\kappa\Delta t}\phi^a(i) + \frac{\sigma^2\left[r^a(i)\right]^{2\gamma}}{2\kappa}\left[1 - e^{-2\kappa\Delta t}\right] \quad (28)$$

To investigate the convergence of (27) extensive simulations were performed over a wide range of parameter settings. Specifically, given particular values of the parameters, κ, σ, and γ, we simulated a path of 800 partitions, updating every time increment. Using the same sequence of random numbers we then considered updating less frequently. The percentage error in the resulting terminal value of $r(t)$ and $\phi(t)$ was computed for each path and solely reflects the bias from less frequent updates. Ten thousand paths were simulated, and the first four central sample moments of the marginal distribution of these percentage errors were computed. Table 1 summarises the mean percentage error when the number of updates over the three-month period ranged from 25 to 200. The table only reports the average percentage errors for the marginal distributions. The magnitudes of the second, third and fourth moments of the percentage errors were negligibly small and are not reported.

The table illustrates the fact that the percentage errors decline with the number of partitions with no systematic bias. For the parameters chosen in Table 1 it appears that the moments of the "true" marginal distributions of $r(t)$ and $\phi(t)$ are well approximated with 100 updates. For a three-month period this corresponds to updating the path once a day.[3] The analysis was repeated for different time periods, and similar results were obtained. Indeed, in all our simulations the rule of allowing one revision period per day appeared to provide results that were not economically different from finer partitions. In all our future analyses, our simulated joint distributions were based on approximately two revisions per day.

Table 1. Convergence of the mean percentage error of terminal values of the state variables

		$\kappa = 0.0$		$\kappa = 0.5$		$\kappa = 1.0$	
N	VAR	Vol = 1%	Vol = 3%	Vol = 1%	Vol = 3%	Vol = 1%	Vol = 3%
25	r	–0.010	–0.009	–0.010	–0.009	–0.010	–0.008
	ϕ	–0.007	–0.017	0.145	–0.138	0.302	–0.292
50	r	–0.010	–0.009	–0.009	–0.009	–0.009	–0.008
	ϕ	–0.007	–0.012	0.064	0.059	0.137	0.131
100	r	–0.008	–0.008	–0.008	–0.008	–0.008	–0.007
	ϕ	–0.006	–0.008	0.024	0.020	0.055	0.052
200	r	–0.005	–0.005	–0.005	0.005	–0.005	0.005
	ϕ	–0.004	–0.005	0.005	0.004	0.016	0.014

The volatility (Vol) of the short rate process was chosen to be $\sigma\sqrt{(r(0))}$. The yield curve was flat at 10%. The time is three months. Numbers displayed are the average of the percentage errors computed as $100(r_N - r_{800})/r_{800}$ and $100(\phi_N - \phi_{800})/\phi_{800}$ where r_N (ϕ_N) is the interest rate (path variable) generated with N updates. For example, when the mean reversion parameter, κ, equals 0.5, and the volatility of the short rate is set at 3%, then the mean percentage error in computing the average terminal interest rate with 50 updates is –0.009% of its value using 800 updates.

The simulation procedure was used to generate $\phi(s)$, $r(s)$, and the money market account, $M(s)$, under the risk-neutralised process. For each path, the entire term structure could then be obtained using (16), and the terminal payout of any interest rate claim computed. We simulated 10,000 paths and averaged the resulting terminal payouts of the interest rate claim relative to the money fund. This leads to an approximation to the fair value in (23).

The convergence of the numerical prices to their true values can be significantly accelerated by using control variables. A candidate for the control variable is obtained by setting γ to zero in (19) and (20). For this case, however, $\phi(s)$ is deterministic and the two-state model reduces to a path-independent, single-state-variable representation. Specifically, from (22) we obtain

$$\phi(s) = \frac{\sigma^2}{2\kappa}\left[1 - e^{-2\kappa s}\right]$$

For this specification, interest rates are normally distributed and analytical solutions are available for a wide variety of interest rate claims. For example, the price of an s-period European call option with strike X on a discount bond with maturity date T is given by

$$C(0,s) = P(0,T)N\left(d_1\right) - XP(0,s)N\left(d_2\right) \tag{29}$$

where

$$d_1 = \frac{\ln\left[P(0,T)/XP(0,s)\right] + \varphi^2}{\varphi}$$

$$d_2 = d_1 - \varphi$$

$$\varphi^2 = \frac{\sigma^2}{2\kappa^3}\left[1 - e^{-\kappa(T-s)}\right]^2\left[1 - e^{-2\kappa s}\right]$$

This model was first developed by Jamshidian (1989). Also, the price of an s-period European yield option, that provides the owner with the option to buy the yield $y(s, T)$ for X dollars is given by

$$C(0,s) = P(0,s)\,\xi\left[M(q) - q\right] \tag{30}$$

where

$$\xi^2 = \frac{1}{2\kappa^3(T-s)^2}\left[1 - e^{-\kappa(T-s)}\right]^2\left[1 - e^{-2\kappa s}\right]$$

and

$$q = \frac{X - y_0(s,T) - \frac{1}{2}\xi^2(T-s)}{\xi}$$

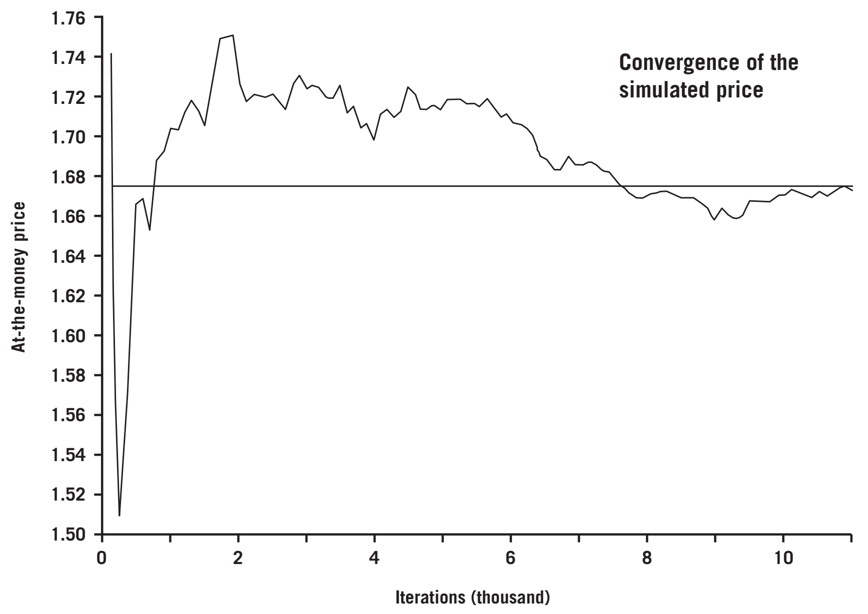
and $y_0(s, T)$ is the forward yield to maturity viewed from time 0, and

$$M(q) = \int_{-\infty}^{q} N(x)\,dx$$

where $N(x)$ in the cumulative standard normal distribution.

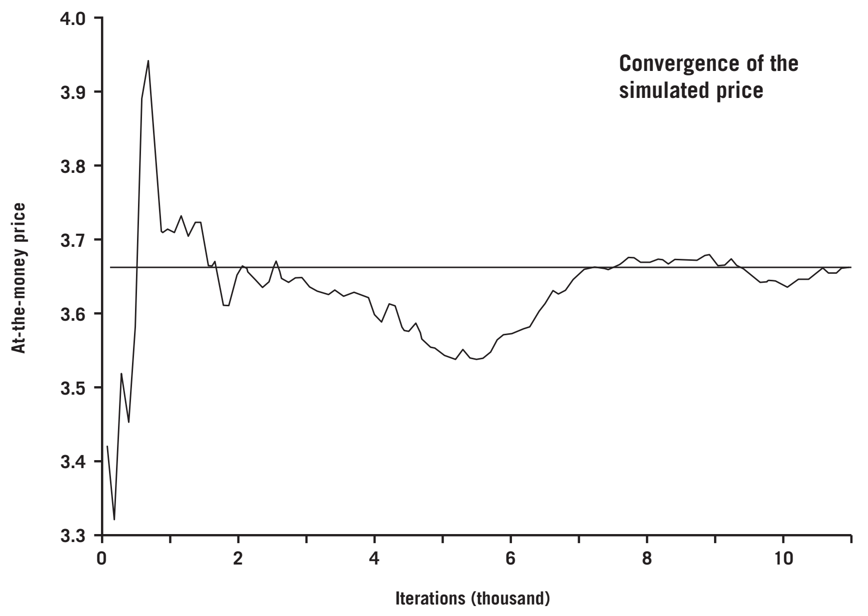
Figures 1 and 2 illustrate the rate of convergence of the simulated prices to their true values given in (29) and (30). The contract in Figure 1 is a three-month at-the-money call option on a 15-year discount bond, while in Figure 2 the claim is an at-the-money three-month interest rate call option on the spot rate. The mean reversion parameter, κ, was set at 0.5 and the annual volatility parameter, σ, was set at 2%. Notice that as the itera-

1. Rate of convergence of the simulated price of an at-the-money, three-month call option on a 15-year discount bond



The initial term structure is flat at 10%, and the volatility structure of forward rates is given by (19) and (20) with $\kappa = 0.5$, $\sigma = 0.02$ and $\gamma = 0$. The flat horizontal line is the theoretical value given by (29)

2. Rate of convergence of the simulated price of an at-the-money, three-month call option on the spot rate

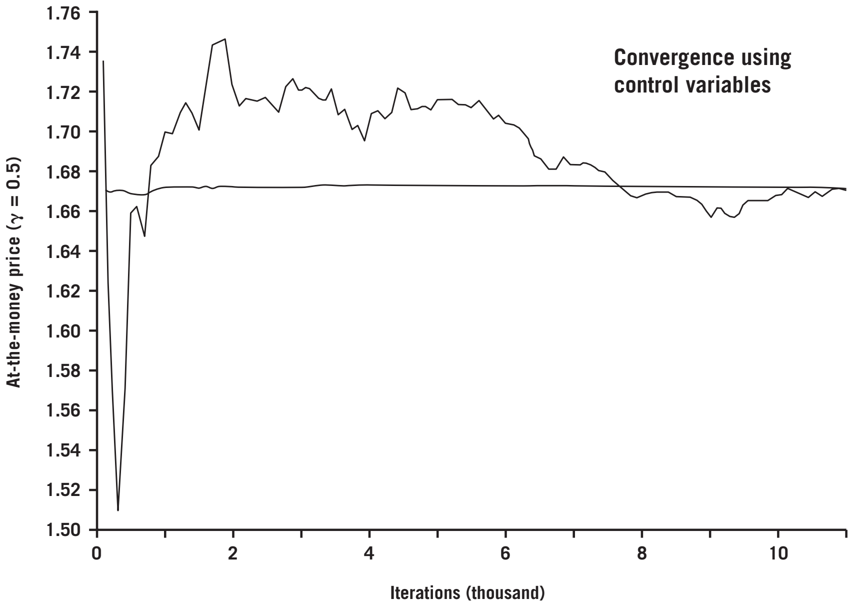


The initial term structure is flat at 10%, and the volatility structure of forward rates is given by (19) and (20) with $\kappa = 0.5$, $\sigma = 0.02$ and $\gamma = 0$. The flat horizontal line is the theoretical value given by (30)

tions increase, the simulated results converge to the analytical, which are indicated by horizontal lines. As can be seen, with 10,000 replications the algorithm produced results indistinguishable from their analytical counterparts.

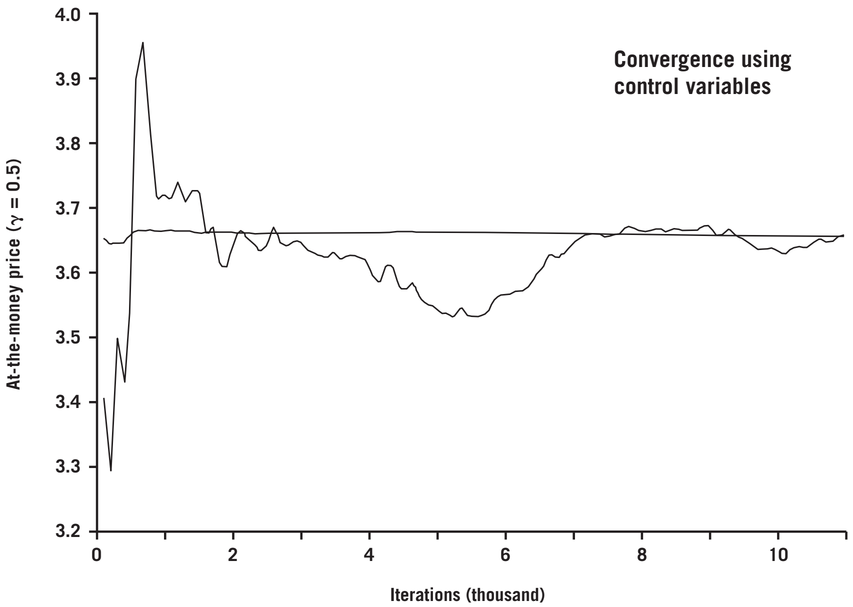
Figures 3 and 4 illustrate the rate of convergence of the same option contracts for $\gamma = 0.5$. Since analytical solutions are available for $\gamma = 0$, control variate procedures are

3. Comparison of the rate of convergence of a three-month at-the-money call option on a 15-year discount bond with and without control variables



The initial term structure is flat at 10%, and the volatility structure of forward rates is given by (19) and (20) with $\kappa = 0.5$, $\sigma^* = 0.02$ and $\gamma = 0$. The nearly flat line is the value computed using the control variate

4. Comparison of the rate of convergence of a three-month at-the-money call option on the spot interest rate with and without control variables



The initial term structure is flat at 10%, and the volatility structure of forward rates is given by (19) and (20) with $\kappa = 0.5$, $\sigma^* = 0.02$ and $\gamma = 0$. The nearly flat line is the value computed using the control variate

used to enhance the speed of convergence.[4] The figures compare the rate of convergence with and without the control variable. In each figure, the nearly horizontal line represents the price of the option using the $\gamma = 0$ model price as a control variable, while the more volatile line is the simulated price obtained without using the control variable. The figures suggest that the prices generated using the control variable stabilise by 2,000 iterations.

If the economy is expanded to include risky assets, claims having terminal values determined by both risky assets and yields drawn from the term structure can be readily valued. To illustrate this, assume

$$\frac{dS(t)}{S(t)} = \mu_s(t)dt + \sigma_s dv(t), \qquad S(0) \text{ given} \tag{31}$$

Here $\mu_s(t)$ is the instantaneous drift, $\sigma_s(t)$ is the instantaneous volatility, and $dv(t)$ is the standard Wiener increment with $\mathbf{E}[dw(t)dv(t)] = \rho_{vw}dt$. Using standard arbitrage arguments, the equivalent martingale measure under which all securities are priced is given by

$$\frac{dS(t)}{S(t)} = r(t)dt + \sigma_s(t)dv(t)$$

$$dr(t) = \left(\kappa(t)\left[f(0,t) - r(t)\right] + \varphi(t) + \frac{d}{dt}f(0,t)\right)dt + \sigma_r(t)dw(t)$$

$$d\phi(t) = \left(\sigma_r^2(t) - 2\kappa(t)\phi(t)\right)dt$$

As an example, consider the price of a call option on a stock when the forward rates are given by (19) and (20) and the volatility of the instantaneous returns on the stock are constant. For the special case where $\gamma = 0$, the price of a call option on the stock with strike X and maturity s can be computed as

$$C^E(0,s) = S(0)N\left(d_1\right) - XP(0,\tau)N\left(d_2\right)$$

$$d_1 = \frac{\ln\left[S(0)/XP(0,s)\right] + \frac{1}{2}\sigma_y^2}{\sigma_y}$$

$$d_2 = \frac{\ln\left[S(0)/XP(0,s)\right] - \frac{1}{2}\sigma_y^2}{\sigma_y}$$

$$\sigma_y^2 = \int_0^s \left[\sigma_s^2 + \sigma_p^2(t,s) - 2\rho_{vw}\sigma_s(t)\sigma_p(t,s)\right]dt$$

$$= \sigma_s^2 s + \frac{\sigma^2}{2\kappa^3}\left[2\kappa s - 3 + 4e^{-\kappa s} - e^{-2\kappa s}\right] - \frac{2\rho_{vw}\sigma_s\sigma}{\kappa^2}\left[\kappa s - 1 + e^{-\kappa s}\right] \tag{32}$$

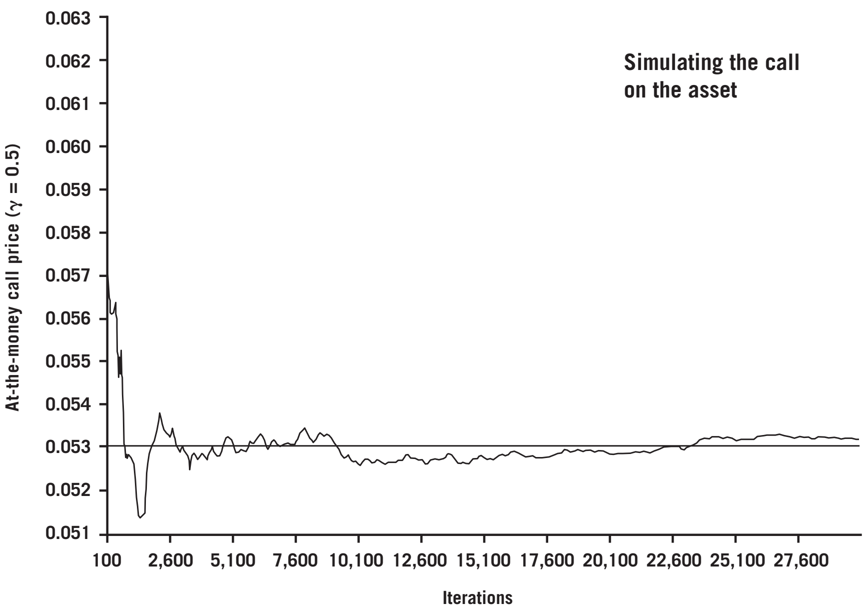
The convergence of the at-the-money call option for $\gamma = ½$, using the above model as a control variable, is demonstrated in Figure 5. For the example presented, we chose $\kappa = 0.5$, $\sigma^* = 0.02$, $\sigma_s = 0.2$, and $\rho = -0.5$.

The nearly horizontal line plots the price generated using the $\gamma = 0$ price as a control variable, while the variable line is the simulated value for $\gamma = 0.5$. The above examples illustrate how the Markovian property of the term structure can be fully exploited to yield simple algorithms for pricing claims against the term structure.

Conclusion

For general volatility structures for forward rates, the evolution of interest rates is not Markovian, and the entire path is necessary to capture the dynamics of the process. Numerical models such as simulation and lattice procedures for valuing claims have to manipulate the entire term structure along each path. In this chapter we identified necessary and sufficient conditions on the volatility structure of forward rates that eliminate the market price of risk from the bond-pricing equation while allowing prices to be rep-

5. Comparison of the rate of convergence of a three-month at-the-money call option on the risky stock with and without control variables



The initial term structure is flat at 10%, and the volatility structure of forward rates is given by (19) and (20) with $\kappa = 0.5$, $\sigma^* = 0.02$ and $\gamma = 0$. The volatility of the stock $\sigma_s = 0.2$ and the correlation $\rho = -0.5$. The nearly flat line is the value computed using the control variate generated using (32)

resented by a two-state Markov process, where the spot interest rate is one of two state variables, the second representing a measure of the history of its evolution. As a consequence, the burden of carrying full information on the term structure at each point is dramatically reduced.

The class of volatility structures that permit this reduction is quite large and imposes no restrictions on the volatility of spot interest rates. We also showed that if the volatility structure of forward rates does not belong to this class then path dependence cannot be captured by a two-dimensional state variable Markovian model. More precisely, to make the process Markovian additional assumptions on volatilities and/or on investor behaviour will be required. Given that the volatility structure belongs to the restricted class, simple simulation models were established for pricing interest rate claims. The efficiency of such algorithms were improved using control variates.

It remains for future research to identify non-deterministic volatility structures belonging to the restricted class that lead to analytical solutions for the terminal risk-neutralised process. It also remains for future research to establish the path reconnecting lattice approximations for the risk-neutralised process. Such lattices would permit American claims to be valued. Finally, the analysis presented here focused on a single-factor model of the term structure. These results can readily be extended to multifactor economies. In a two-factor economy, for example, path dependence will be captured by two sufficient statistics and the term structure could be made Markovian with respect to four state variables. Such extensions, together with empirical tests are also the subject of future research.

Appendix

PROOF OF PROPOSITION 1

1(a). The necessity of the restriction has already been outlined in the motivation of the proposition and will not be repeated here. The sufficiency obtains from the development of the bond pricing equation below.

1(b). From equation (11), let

$$R(t;T) \equiv f(t,T) - f(0,T)$$

$$= \int_0^t \sigma_f(s,T)\left[\lambda(s) - \sigma_P(s,T)\right] ds \int_0^t \sigma_f(s,T) dw(s) \tag{33}$$

Here, $R(t;T)$ is defined as the difference between the forward rate at date t and that at the original date. Using the restriction imposed by (16) on (33), we obtain

$$\frac{R(t;T)}{k(t,T)} + \int_0^t \sigma_f(s,t)\sigma_p(s,T) ds = \int_0^t \sigma_f(s,t)\lambda(s) ds + \int_0^t \sigma_f(s,t) dw(s) \tag{34}$$

Since the right-hand side of (34) is independent of T, take $T = t$ to obtain

$$\frac{R(t;T)}{k(t,T)} + \int_0^t \sigma_f(s,t)\sigma_p(s,T) ds = \frac{R(t;t)}{k(t,t)} + \int_0^t \sigma_f(s,t)\sigma_p(s,t) ds$$

which simplifies to

$$R(t;T) = k(t,T)\left\{R(t;t) + \int_0^t \sigma_f(s,t)\left[\sigma_p(s,t) - \sigma_p(s,T)\right] ds\right\}$$

$$= k(t,T)\left\{R(t;t) + \int_0^t \sigma_f(s,t) \int_t^T \sigma_f(s,x) dx\, ds\right\}$$

$$= \frac{\sigma_f(t,T)}{\sigma_f(t,t)}\left\{R(t;t) + \int_0^t \sigma_f^2(s,t) ds \int_t^T k(t,x) dx\right\}$$

$$= \frac{\sigma_f(t,T)}{\sigma_f(t,t)}\left\{R(t;t) + \int_0^t \sigma_f^2(s,t) ds \int_t^T \frac{\sigma_f(t,x)}{\sigma_f(t,t)} dx\right\}$$

$$= \sigma_f(t,T)\frac{R(t;t)}{\sigma_f(t,t)} - \frac{\sigma_f(t,T)}{\sigma_f^2(t,t)}\sigma_p(t,T)\phi(t)$$

where

$$\phi(t) = \int_0^t \sigma_f^2(s,t) ds$$

Equivalently,

$$f(t,T) = f(0,T) + \frac{\sigma_f(t,T)\left[f(t,t) - f(0,t)\right]}{\sigma_f(t,t)} - \frac{\sigma_f(t,T)}{\sigma_f^2(t,t)}\sigma_p(t,T)\phi(t)$$

This leads to

$$f(t,T) - f(0,T) = -\frac{d}{dT}\left[\sigma_p(t,T)\frac{R(t;t)}{\sigma_f(t,t)} - \frac{\phi(t)}{2\sigma_f^2(t,t)}\sigma_p^2(t,T)\right]$$

Further, from (5) we have

$$P(t,T) = \exp\left\{ -\int_t^T f(0,s)\,ds + \int_t^T \frac{d}{dx}\left[\sigma_p(t,x)\frac{R(t;t)}{\sigma_f(t,t)} - \frac{\phi(t)}{2\sigma_f^2(t,t)}\,\sigma_p^2(t,x) \right] dx \right\}$$

which upon simplification yields the result.

Observe that the state variable, (t) which captures the information relating to the path of interest rates is independent of bond maturity. Further, since the two state variables, r(t) and (t), permit us to compute the prices of all discount bonds, they also permit us to construct the entire term structure at time t. This then allows one to compute the prices of all other interest contingent claims.

1 *An exception to this case is when volatility of all forward rates are deterministic. In this case simplifications result and analytical solutions for certain claims are available. For example Jamshidian (1989) and Heath, Jarrow and Morton (1992) have developed term-structure-constrained models for debt options in a dynamically complete market. Turnbull and Milne (1991) establish an equilibrium model where prices follow diffusion processes but trading dates are simple pricing mechanisms for a large variety of interest rate claims.*

2 *Exceptions to this of course are the cases where the volatility structure of all forward rates are deterministic.*

3 *Similar results held true over all parameter settings. Specifically, we tested the models with* κ *ranging from 0 to 1,* σ *ranging from 0.1 to 0.4 and* γ *from 0 to 1.*

4 *For a discussion on control variate techniques and their use in pricing options see Boyle (1977) and Hull and White (1988). In our control variate scheme, we chose the parameter of the volatility of the spot rate process: ie,* $\sigma = \sigma^*$*, say, so that it equalled the initial volatility of the interest rate under the stochastic volatility structure. That is,* $\sigma^* = \sigma[r(0)]^\gamma$*. For the results presented in Figures 3 and 4, we chose* σ^* *at 2% and* κ *at 0.5.*

BIBLIOGRAPHY

Boyle, P., 1977, "Options: A Monte Carlo Approach", *Journal of Financial Economics* 4, pp. 323–38.

Carverhill, A., 1994, "When Is the Short Rate Markovian?", *Mathematical Finance* 4, pp. 305–12; reprinted as Chapter 00 of the present volume.

Heath, D., R. Jarrow and A. Morton, 1990, "Bond Pricing and the Term Structure of Interest Rates: A Discrete Time Approximation", *Journal of Financial and Quantitative Analysis* 25, pp. 419–40.

Heath, D., R. Jarrow and A. Morton, 1992, "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation", *Econometrica* 60, pp. 77–105.

Ho, T., and S. B. Lee, 1986, "Term Structure Movements and the Pricing of Interest Rate Contingent Claims", *Journal of Finance* 41, pp. 1011–30.

Hull, J., and A. White, 1988, "The Use of the Control Variate Technique in Option Pricing", *Journal of Financial and Quantitative Analysis* 23, pp. 237–52.

Hull, J., and A. White, 1993, "Bond Option Pricing Based on a Model for the Evolution of Bond Prices", *Advances in Futures and Options Research* 6, pp. 1–13.

Jamshidian, F., 1989, "An Exact Bond Option Formula", *Journal of Finance* 44, pp. 205–09.

Turnbull, S., and F. Milne, 1991, "A Simple Approach to Interest-Rate Option Pricing", *The Review of Financial Studies* 4, pp. 87–120.

Vasicek, O., 1977, "An Equilibrium Characterization of the Term Structure", *Journal of Financial Economics* 1, pp. 177–88.