On the construction of finite dimensional realizations for nonlinear forward rate models

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Abstract

We consider interest rate models of Heath-Jarrow-Morton type where the forward rates are driven by a multidimensional Wiener process, and where the volatility structure is allowed to be a smooth functional of the present forward rate curve. In a recent paper (to appear in *Mathematical Finance*) Björk and Svensson give necessary and sufficient conditions for the existence of a finite dimensional Markovian state space realization (FDR) for such a forward rate model, and in the present paper we provide a general method for the actual construction of an FDR.

The method works as follows: From the results of Björk and Svensson we know that there exists an FDR if and only if a certain Lie algebra is finite dimensional. Given a set of generators for this Lie algebra we show how to construct an FDR by solving a finite number of ordinary differential equations in Hilbert space.

We illustrate the method by constructing FDR:s for a number of concrete models. These FDR:s generalize previous results by allowing for a more general volatility structure. Furthermore the dimension of the realizations obtained by using our method is typically smaller than that of the corresponding previously known realizations.

We also show how to obtain realizations in terms of benchmark forward rates from the realizations obtained using our method, and finally we present a bond pricing formula for the realizations we have obtained.

Keywords HJM models, factor models, state space models, Markovian realizations

JEL Classification: E43, G13

1 Introduction

The main objective of this paper is to present a systematic method to construct finite dimensional realizations of nonlinear forward rate models, provided that such realizations exist. More specifically we consider a Heath-Jarrow-Morton model of the forward rates driven by an m-dimensional Wiener process. Furthermore we assume that the forward rate volatilities are allowed to be smooth functionals of the present forward rate curve. Given that this model can be realized in terms of a finite dimensional Markovian state space model, we supply a systematic procedure to find such a realization.

Several volatility structures which give rise to finite dimensional realizations have already been presented in the literature. Short rate realizations, that is realizations where the only state variable of the realization is the short rate, has been studied in [5], [14], [12], and [8]. In [5] only deterministic volatilities are considered, whereas in [14] and in [12] the volatilities are allowed to be functions of the short rate. In [8] the case with a deterministic volatility structure is studied, but then with a driving Levy process.

There is a substantial literature providing sufficient conditions for the existence of multidimensional (but finite) realizations. The case of a deterministic volatility structure was completely solved in [2]. In [16], [1] and [13] the authors consider volatilities which are assumed to be functions of the short rate. More precisely they consider a multiplicative volatility structure which will be referred to as a "deterministic direction volatility" below. Whereas the case with only one driving Wiener process is considered in [16] and [1], the case with a multi-dimensional driving Wiener process is treated in [13]. Finally, a model where the volatility is allowed to depend on a finite number of benchmark forward rates has been considered in [6]. Also this case, however, turns out to be a special case of the class of "deterministic direction volatilities" to be treated below. In all these references the authors present various sufficient conditions for the existence of an FDR, and the existence proofs are carried out by actually constructing finite realizations.

In [3] the general FDR problem is completely solved by presenting necessary and sufficient conditions for the existence of an FDR for the case of a general volatility structure. However, as opposed to the references above, [3] only provide pure existence results whereas no concrete realizations are derived.

The purpose of our paper is to present a systematic procedure for the construction of finite dimensional realizations for *any* model possessing a finite dimensional realization. This adds to the results of [3] by providing concrete realizations where [3] only give existence results. It is also a substantial improvment of the other references above by being applicable to *any* model for which there exists an FDR.

The method basically works as follows: From [3] we know hat there exists an FDR if and only if a certain Lie algebra is finite dimensional. Given a set of generators for this Lie algebra we show how to construct an FDR by essentially solving a finite number of ordinary differential equations in Hilbert space.

The main advantage of our procedure is thus that it is systematic and generally applicable, and we illustrate the method by constructing FDR:s for a number of concrete models. These FDR:s generalize previous results by allowing for a more general volatility structure, and the the dimension of the realizations obained by using our method is typically smaller than that of the corresponding previously known realizations (when these exist).

The rest of the paper is organized as follows. In Section 2 we briefly go through the HJM-framework we will be working within and the Musiela parameterization. The method and the results it relies on are presented in Section 3

The method is then applied, first to deterministic volatilities in Section 4 and then to deterministic direction volatilities in Section 5, and concrete realizations are constructed. The sections 4 and 5 only treat time homogeneous systems, but the method can be applied to time-varying systems as well, and this is done in Section 6.

The state variables in the realizations we find using our method may have no economic interpretation. However, we show in Section 7 that a realization in terms of benchmark forward rates can readily be obtained from the realizations we found using our method.

Finally we show in Section 8 that it is easy to price zero-coupon bonds using the realizations we have obtained. The formula is an extension of the formula given in [6].

2 Basics

As in Heath, Jarrow and Morton [9], we consider a default free bond market living on a filtered probability space $\{\Omega, \mathcal{F}, Q, \{\mathcal{F}_t\}_{t\geq 0}\}$ carrying an m-dimensional Wiener process W. Let $p(t,T)$ denote the price at time $t \geq 0$ of a zero-coupon bond with maturity $T \geq 0$. Then the instantaneous forward rate $f(t,T)$ is defined by

$$f(t,T) = -\frac{\partial \ln p(t,T)}{\partial T},$$

and the short rate R is defined by

$$R(t) = f(t,t).$$

We assume that the market is frictionless and that the bonds are perfectly divisible. We also assume that the model is arbitrage-free in the sense that the probability measure Q is a martingale measure for the model, i.e. for each $T \geq 0$ we have that $p(t,T)/B(t)$ is a martingale for $t \leq T$. Here B denotes the money account defined by $B(t) = \exp\left\{\int_0^t R(s)ds\right\}$. The forward rates are assumed to have dynamics of the following form under the martingale measure Q

$$df(t,T) = \alpha(t,T)dt + \sigma_0(t,T)dW_t.$$

A more suitable parameterization of the forward rates for our purposes is the Musiela parameterization ([4] and [15]) given by

$$r(t,x) = f(t,t+x).$$

Here x denotes time **to** maturity in contrast to T, which denotes time **of** maturity. The process r will have the following induced dynamics

$$dr(t,x) = \beta(t,x)dt + \sigma(t,x)dW_t,$$

where

$$\begin{cases} \beta(t,x) & = & \frac{\partial}{\partial x} r(t,x) + \alpha(t, t+x), \\ \sigma(t,x) & = & \sigma_0(t, t+x). \end{cases}$$

Let us now define the Hilbert space we will be working with, i.e. the space of forward rate curves.

Definition 2.1 *Consider fixed real numbers $\beta > 1$, and $\gamma > 0$. The space $\mathcal{H}_{\beta,\gamma}$ is defined as the space of all infinitely differentiable functions*

$$r : R_+ \to R$$

satisfying the norm condition $\|r\|_{\beta,\gamma} < \infty$. Here the norm is defined as

$$\|r\|^2_{\beta,\gamma} = \sum_{n=0}^{\infty} \beta^{-n} \int_0^\infty \left(\frac{d^n r}{dx^n}(x) \right)^2 e^{-\gamma x} dx.$$

In the sequel we will suppress the subindices (β, γ) and only write $\mathcal{H}$, since the results are uniform w.r.t. (β, γ). Proposition 4.2 in [3] states that $\mathcal{H}$ is a Hilbert space, when equipped with the obvious inner product.

To start with we will consider homogeneous models and we take as given a deterministic mapping

$$\sigma : \mathcal{H} \times R_+ \to R^m.$$

This means that each component of $\sigma(r,x) = [\sigma_1(r,x), \ldots, \sigma_m(r,x)]$ is a functional of the infinite dimensional r-variable, and a function of x.

Now pick a volatility of the form described above. If we translate the HJM no arbitrage condition to the Musiela parameterization (see [15]), and express the dynamics in terms of a Stratonovich SDE, rather than an Itô SDE, we find that the dynamics of r under the martingale measure Q are given by

$$dr_t = \mu(r_t)dt + \sigma(r_t) \circ dW_t, \tag{1}$$

where $\circ$ denotes the Stratonovich integral, and

$$\mu(r) = \frac{\partial}{\partial x} r + \sigma(r)\mathbf{H}\sigma(r)^* - \frac{1}{2}\sigma'_r(r)[\sigma(r)]. \tag{2}$$

Here $*$ denotes transpose, $\sigma'_r(r)[\sigma(r)]$ denotes the Frechet derivative $\sigma'_r(r)$ operating on $\sigma(r)$, and $\mathbf{H}\sigma$ is defined by

$$\mathbf{H}\sigma(r,x) = \int_0^x \sigma(r,s)ds.$$

3 A Lie algebraic approach to constructing finite dimensional realization

Let us begin this section by specifying exactly what we mean with a finite dimensional realization of the forward rates generated by a volatility. To this end chose a volatility $\sigma : \mathcal{H} \times R_+ \rightarrow R^m$ and recall from (1) that the system which describes how the forward rates evolve is given by

$$\begin{cases} dr_t &= \mu(r_t)dt + \sigma(r_t) \circ dW_t, \\ r_0 &= r^0, \end{cases} \tag{3}$$

where μ was defined in (2).

Definition 3.1 *We say that the SDE (3) has a (local) d-dimensional* **realization** *at r^0, if there exists a point $z_0 \in R^d$, smooth vector fields $a, b_1, \ldots, b_m$ on some open subset $\mathcal{Z}$ of R^d and a smooth (submanifold) map $G : \mathcal{Z} \rightarrow \mathcal{H}$, such that r has the local representation*

$$r_t = G(Z_t), \quad a.s.$$

where Z is the strong solution of the d-dimensional Stratonovich SDE

$$\begin{cases} dZ_t &= a(Z_t)dt + b(Z_t) \circ dW_t, \\ Z_0 &= z_0. \end{cases} \tag{4}$$

The driving Wiener process W in (4) should be the same as in (3). The term "local" above means that the representation is assumed to hold for all t with $0 \leq t \leq \tau(r^0)$, a.s. where, for each $r^0 \in \mathcal{H}$, $\tau(r^0)$ is a strictly positive stopping time.

The method for constructing a finite dimensional realization of a HJM model which we will present, requires some basic concepts from infinite dimensional differential geometry. These are introduced in the next section.

3.1 Basic concepts in differential geometry

Let us recall the following concepts from infinite dimensional differential geometry, since they are needed to understand the results on which we base our construction of finite dimensional realizations. The presentation follows that in [3].

Consider a real Hilbert space X. By an n-dimensional **distribution** we mean a mapping F, which to each x in an open subset V of X associates an n-dimensional subspace $F(x) \subseteq X$. A mapping (vector field) $f : U \rightarrow X$, where U is an open subset of X, is said to **lie in** F (on U) if $U \subseteq V$ and $f(x) \in F(x)$ for every $x \in U$. A collection $f_1, \ldots, f_n$ of vector fields lying in F on U **generates** (or spans) F on U if $span\{f_1, \ldots, f_n\} = F(x)$ for every x in U, where *span* denotes the linear hull over the real field. The distribution is **smooth** if, to every x in V, there exists an open set U such that $x \in U \cap V$, and smooth vector fields $f_1, \ldots, f_n$ spanning F on U. A vector field is smooth if it belongs to C^∞. If F and G are distributions and $G(x) \subseteq F(x)$ for all x we say that F **contains** G, and we write $G \subseteq F$. The **dimension** of a distribution F is defined pointwise as $dim F(x)$.

Let f and g be smooth vector fields on U. Their **Lie bracket** is the vector field

$$[f, g](x) = f'(x)g(x) - g'(x)f(x),$$

where $f'(x)$ denotes the Frechet derivative of f at x, and $g'(x)$ is the analogue for g. We will sometimes write $f'(x)[g(x)]$ instead of $f'(x)g(x)$ to emphasize that the Frechet derivative is operating on g. A distribution F is called **involutive** if for all smooth vector fields f and g lying in F on U, their lie bracket also lies in F, i.e.

$$[f, g](x) \in F(x) \qquad \forall x \in U.$$

We are now ready to define the concept of a Lie algebra which will play a central role in what follows.

Definition 3.2 *Let F be a smooth distribution on X. The* **Lie algebra** *generated by F, denoted by $\{F\}_{LA}$, is defined as the minimal (under inclusion) involutive distribution containing F.*

When trying to determine a concrete Lie algebra the following observations often come in handy.

Lemma 3.1 *Take the vector fields $f_1, \ldots, f_k$ as given. It then holds that the Lie algebra $\{f_1, \ldots, f_k\}_{LA}$ remains unchanged under the following operations.*

- *The vector field f_i may be replaced by αf_i, where α is any smooth nonzero scalar field.*
- *The vector field f_i may be replaced by*

  $$f_i + \sum_{j \neq i} \alpha_j f_j,$$

  where $\alpha_1, \ldots, \alpha_k$ are any smooth scalar fields.

Let F be a distribution and let $\varphi : V \to W$ be a diffeomorphism between the open subsets V and W of X. Then we can define a new distribution $\varphi_\star F$ on W by

$$(\varphi_\star F)(\varphi(x)) = \varphi'(x)F(x).$$

For any smooth vector field $f \in C^\infty(U, X)$ the field $\varphi_\star f$ is defined analogously. It is straightforward to verify that

$$\varphi_\star[f, g] = [\varphi_\star f, \varphi_\star g].$$

This implies that if F is generated by $f_1, \ldots, f_n$, then $\varphi_\star F$ is generated by $\varphi_\star f_1, \ldots, \varphi_\star f_n$, and that F is involutive if and only if $\varphi_\star F$ is involutive.

The final concept we will need is that of a tangential manifold.

Definition 3.3 *Let F be a smooth distribution, and let x_0 be a fixed point in X. A submanifold $\mathcal{G} \subseteq X$ with $x_0 \in \mathcal{G}$ is called a* **tangential manifold** *through x_0 for F, if $F(x) \subseteq T_{\mathcal{G}}(x)$ for each x in a neighbourhood of x_0 in $\mathcal{G}$. Here $T_{\mathcal{G}}(x)$ denotes the tangent space to $\mathcal{G}$ at x.*

3.2 The general method

In this section we describe our method for constructing a finite dimensional realization of a forward rate system, given that we know that such a realization exists. The approach we use is Lie algebraic and the main advantage is that it provides a systematic way of finding a finite dimensional realization.

Assumption 3.1 *We assume that the dimension of the Lie algebra $\{\mu, \sigma_1, \dots, \sigma_m\}_{LA}$ is constant near the initial forward rate curve r^0.*

The method relies on the following two theorems from [3]. The first tells us when the forward rate system possesses a finite dimensional realization.

Theorem 3.1 (Björk and Svensson) *Take as given the volatility mapping $\sigma = (\sigma_1, \dots, \sigma_m)$ as well as an initial forward rate curve $r^0 \in \mathcal{H}$. Then the forward rate model generated by σ generically admits a finite dimensional realization at r^0, if and only if*

$$dim\{\mu, \sigma_1, \dots, \sigma_m\}_{LA} < \infty$$

in a neighbourhood of r^0, where μ is given by

$$\mu(r) = \frac{\partial}{\partial x} r + \sigma(r)\mathbf{H}\sigma(r)^* - \frac{1}{2}\sigma_r'(r)[\sigma(r)],$$

and $\mathbf{H}\sigma$ is defined by

$$\mathbf{H}\sigma(r, x) = \int_0^x \sigma(r, s) ds.$$

The second theorem gives us a parameterization of the forward rate curves produced by the model. To state this theorem we need the following definition.

Definition 3.4 *Let f be a smooth vector field on $\mathcal{H}$, and let y be a fixed point in $\mathcal{H}$. Consider the ODE*

$$\begin{cases} \dfrac{dy_t}{dt} &= f(y_t), \\ y_0 &= y. \end{cases}$$

We denote the solution y_t as $y_t = e^{ft}y$.

The second theorem now reads as follows.

Theorem 3.2 (Björk and Svensson) *Assume that the Lie algebra* $\{\mu, \sigma\}_{LA}$ *is spanned by the smooth vector fields* $f_1, \dots, f_d$. *Then, for the initial point* r^0, *all forward rate curves produced by the model will belong to the induced tangential manifold* $\mathcal{G}$, *which can be parameterized as* $\mathcal{G} = Im[G]$, *where*

$$G(z_1, \dots, z_d) = e^{f_d z_d} \dots e^{f_1 z_1} r^0,$$

and where the operator $e^{f_i z_i}$ *is given in Definition 3.4*

The tangential manifold $\mathcal{G}$ in the above theorem is invariant under the forward rate dynamics. It will be therefore be referred to as the invariant manifold in the sequel.

We are now ready to describe our method for finding a finite dimensional realization of a forward rate system. Take as given a volatility $\sigma : \mathcal{H} \times R_+ \to R^m$ for which $\{\mu, \sigma\}_{LA} < \infty$, that is take as given a volatility such that the forward rates generated by this volatility can generically be realized by means of a finite dimensional SDE. Then a finite dimensional realization can be constructed in the following way:

- Choose a finite number of vector fields $f_1, \dots, f_d$ which span $\{\mu, \sigma\}_{LA}$. Lemma 3.1 is often useful for "simplifying" the vector fields.
- Compute the invariant manifold $G(z_1, \dots, z_d)$ using Theorem 3.2.
- We now have that $r = G(Z)$. Make the following *Ansatz* for the dynamics of the state space variables Z

  $$dZ = a(Z)dt + b(Z) \circ dW_t.$$

  It must then hold that

  $$G_\star a = \mu, \qquad G_\star b = \sigma. \tag{5}$$

  Use the equations in (5) to obtain the vector fields a and b.

Remark 3.1

- It may be that the equations in (5) do not have unique solutions, but for us it is enough to find one solution, and any solution will do.
- Although we have to solve for the Stratonovich dynamics of the state variables, it turns out that the Itô-dynamics are much nicer looking (see the realizations in the following sections). This is perhaps not surprising since this is also true for the forward rate dynamics themselves.

Again we emphasize that this method can be applied quite mechanically, the only choice to be made is that of vector fields which span the Lie algebra $\{\mu, \sigma\}_{LA}$. Generally you will want to choose these vector fields as "simple" as possible and to do this you use Lemma 3.1. The reason you want simple vector fields is that this simplifies the computation of the parameterization of the forward rate curves in the next step (recall that this requires solving $\mathcal{H}$-valued ODEs with right hand sides equal to the generating vector fields).

In the next few sections we will apply this scheme repeatedly to various volatilities σ and derive finite dimensional realizations.

4 Deterministic volatility

Assume that

$$\sigma(r, x) = \sigma(x), \tag{6}$$

where each component of the vector σ is of the following form

$$\sigma_i(x) = \sigma_i \lambda_i(x), \qquad i = 1, \ldots, m \tag{7}$$

Here, with a slight abuse of notation, σ_i on the right hand side denotes a constant, and λ_i is a constant vector field. According to Proposition 5.1 in [3] the forward rates generated by this volatility structure has a finite dimensional realization if and only if

$$dim(span\{\sigma, \mathbf{F}\sigma, \mathbf{F}^2\sigma, \ldots\}) < \infty,$$

where $\mathbf{F}$ denotes the operator $\frac{\partial}{\partial x}$. We therefore assume that λ_i solves the ODE

$$\mathbf{F}^{n_i+1}\lambda_i(x) = \sum_{k=0}^{n_i} c_k^i \mathbf{F}^k \lambda_i(x), \tag{8}$$

where the c_k^i:s are constants. Since the Lie algebra spanned by μ and σ for this case is given by

$$\{\mu, \sigma\}_{LA} = span\{\mu, \sigma, \mathbf{F}\sigma, \mathbf{F}^2\sigma, \ldots\},$$

we can choose the following generator system for the Lie algebra

$$\{\mu, \sigma\}_{LA} = span\{\mu, \mathbf{F}^k\lambda_i;\ i = 1, \ldots, m;\ k = 0, 1, \ldots, n_i\}.$$

The next step in constructing a finite dimensional realization is to compute the invariant manifold $G(z_0, z_k^i;\ i = 1, \ldots, m;\ k = 0, 1, \ldots, n_i)$. This means computing the operators $\exp\{\mu t\}$ and $\exp\{\mathbf{F}^k\lambda_i\}$, $i = 1, \ldots, m$, $k = 0, \ldots, n_i$. This has been done in Proposition 5.2 in [3] and the invariant manifold generated by the initial forward rate curve r_0 is parameterized as

$$\begin{aligned} & G(z_0, z_k^i;\ i = 1, \ldots, m;\ k = 0, 1, \ldots, n_i)(x) \\ = \quad & r(x + z_0) + \frac{1}{2}(\|S(x + z_0)\|^2 - \|S(x)\|^2) + \sum_{i=1}^{m}\sum_{k=0}^{n_i} \mathbf{F}^k\lambda_i(x) z_k^i, \end{aligned} \tag{9}$$

where

$$S(x) = \int_0^x \sigma(u)du.$$

We now proceed to the last step of the procedure, which is finding the dynamics of the state space variables. This means solving the equations (5). We therefore need the Frechet derivative G' of G. Simple calculations give

$$G'(z_0, z_k^i;\ i=1,\ldots,m;\ k=0,1,\ldots,n_i)\begin{pmatrix} h_0 \\ h_0^1 \\ h_1^1 \\ \vdots \\ h_{n_m}^m \end{pmatrix}(x)$$

$$= \frac{\partial}{\partial x} r_0(x+z_0)h_0 + D(x+z_0)h_0 + \sum_{i=1}^{m}\sum_{k=0}^{n_i} \mathbf{F}^k \lambda_i(x) h_k^i,$$

where D is the constant field given by

$$D(x) = \sum_{i=1}^{m} \sigma_i^2 \lambda_i(x) \int_0^x \lambda_i(u)du.$$

Since for this model the Frechet derivative with respect to r of each component of the volatility is zero, i.e. $\sigma_i'(r,x) = 0$, we obtain the following expression for μ from (2)

$$\mu(r) = \mathbf{F}r + D.$$

If we use that $r = G(z)$ we can obtain an expression for $\mathbf{F}r$, and the equation $G_\star a = \mu$ then reads

$$\frac{\partial}{\partial x} r_0(x+z_0)a_0 + D(x+z_0)a_0 + \sum_{j=1}^{m}\sum_{k=0}^{n_j} \mathbf{F}^k \lambda_j(x) a_{jk}$$

$$= \frac{\partial}{\partial x} r_0(x+z_0) + D(x+z_0) + \sum_{j=1}^{m}\sum_{k=0}^{n_j} \mathbf{F}^{k+1} \lambda_j(x) z_k^j.$$

Since this equality is to hold for all x, and a is not allowed to depend on x it is possible to identify what a must look like. If we recall that λ_i solves the ODE defined in (8) we obtain

$$\begin{aligned} a_0 &= 1, \\ a_{j0} &= c_0^j z_{n_j}^j, && j=1,\ldots,m, \\ a_{jk} &= c_k^j z_{n_j}^j + z_{k-1}^j, && j=1,\ldots,m;\ k=1,\ldots,n_j. \end{aligned}$$

From $G_\star b^i(z)(x) = \sigma_i(x)$ we obtain the equation

$$\frac{\partial}{\partial x} r_0(x+z_0)b_0^i + D(x+z_0)b_0^i + \sum_{j=1}^{m}\sum_{k=0}^{n_j} \mathbf{F}^k \lambda_j(x) b_{jk}^i$$

$$= \sigma_i \lambda_i(x),$$

where σ_i denotes a constant. Therefore we have that

$$\begin{aligned} b^i_{jk} &= \sigma_i, & j = i,\ k = 0, \\ b^i_{jk} &= 0, & \text{all other } j \text{ and } k. \end{aligned}$$

From this we see that to each Wiener process there corresponds one state variable which is driven by this, and only this, Wiener process. The dynamics for these state variables are given by

$$dZ^j_0 = c^j_0 Z^j_{n_j} dt + \sigma_j \circ dW^j_t, \quad j = 1, \ldots, m.$$

Since σ_j is a constant, the Itô-dynamics will look the same, and we have thus proved the following proposition.

Proposition 4.1 *Given the initial forward rate curve r_0 the forward rate system generated by the volatilities described in equations (6) through (8) has a finite dimensional realization given by*

$$r_t = G(Z_t),$$

where G was defined in (9) and the dynamics of the state space variables Z are given by

$$\begin{cases} dZ_0 = dt, \\ dZ^j_0 = c^j_0 Z^j_{n_j} dt + \sigma_j dW^j_t, \quad j = 1, \ldots, m, \\ dZ^j_k = (c^j_k Z^j_{n_j} + Z^j_{k-1}) dt, \quad j = 1, \ldots, m;\ k = 1, \ldots, n_j. \end{cases}$$

Remark 4.1 Note that the first state space variable represents running time. This will be the case for all realizations derived in this paper.

4.1 Ho-Lee

As a special case of the deterministic volatilities studied in the previous section consider a volatility given by

$$\sigma(x) = \sigma, \tag{10}$$

where σ is a scalar constant, that is we have only one driving Wiener process. In the formalism of the previous paragraph we have $\lambda(x) \equiv 1$, which satisfies the trivial ODE $\mathbf{F}\lambda(x) = 0$. A direct application of Proposition 4.1 gives the following result.

Proposition 4.2 *Given the initial forward rate curve r_0 the forward rate system generated by the volatility of equation (10) has a finite dimensional realization given by*

$$r_t = G(Z_t),$$

where G is given by

$$G(z_0, z_1)(x) = r(x + z_0) + \sigma^2 \left(x z_0 + \frac{1}{2} z_0^2 \right) + z_1,$$

and the dynamics of the state space variables Z are given by

$$\begin{cases} dZ_0(t) &= dt, \\ dZ_1(t) &= \sigma dW_t. \end{cases}$$

4.2 Hull-White

Another special case of deterministic volatilities is

$$\sigma(x) = \sigma e^{-cx}, \tag{11}$$

where σ and c are scalar constants, so again there is only one driving Wiener process. This time we have $\lambda(x) = e^{-cx}$, which satisfies the ordinary differential equation $\mathbf{F}\lambda(x) = -c\lambda(x)$. Applying Proposition 4.1 once more we obtain the following.

Proposition 4.3 *Given the initial forward rate curve r_0 the forward rate system generated by the volatility of equation (11) has a finite dimensional realization given by*

$$r_t = G(Z_t),$$

where G is given by

$$G(z_0, z_1)(x) = r(x + z_0) + \frac{\sigma^2}{c^2} \left(e^{-cx}(1 - e^{-cz_0}) + \frac{e^{-2cx}}{2}(e^{-2cz_0} - 1) \right) + z_1,$$

and the dynamics of the state space variables Z are given by

$$\begin{cases} dZ_0(t) &= dt, \\ dZ_1(t) &= -cZ_1(t)dt + \sigma dW_t. \end{cases}$$

5 Deterministic direction volatility

Consider a volatility structure of the form

$$\sigma(r, x) = \varphi(r)\lambda(x). \tag{12}$$

Here φ is a smooth functional of r, and λ is a constant vector field. Note that we are now dealing with the case with only one driving Wiener process. Depending on whether φ satisfies a certain non-degeneracy condition or not we get two cases. We next study these two cases separately.

5.1 The generic case

In the generic case φ satisfies the following assumption.

Assumption 5.1 *We assume that*

- $\varphi(r) \neq 0$ *for all* $r \in \mathcal{H}$ *and for all* $i = 1, \ldots, m$.
- $\Phi''(r)[\lambda; \lambda] \neq 0$ *for all* $r \in \mathcal{H}$*, where* $\Phi(r) = \varphi^2(r)$ *and* $\Phi''(r)[\lambda; \lambda]$ *denotes the second order Frechet derivative of* Φ *operating on* $[\lambda; \lambda]$.

Given these assumptions, Proposition 6.1 in [3] states that the system of forward rates generated by the volatility (12) possesses a finite dimensional realization if and only if λ is a quasi-exponential function, i.e. of the form $\lambda(x) = ce^{Ax}b$, where c is a row vector, A is a square matrix and b is a column vector. We will therefore assume that λ is of the form

$$\lambda(x) = p(x)e^{\alpha x}, \tag{13}$$

where p is a polynomial of degree n and α is a scalar constant.

It is also shown in [3] that, given Assumption 5.1, the Lie algebra generated by μ and σ is given by

$$\{\mu, \sigma\}_{LA} = span\{\mathbf{F}r, \mathbf{F}^i\lambda, \mathbf{F}^i D;\ i = 0, 1, \ldots\},$$

where

$$D(x) = \lambda(x) \int_0^x \lambda(u)du. \tag{14}$$

We may now note that λ, regardless of what p looks like, satisfies the following ODE of order $n+1$

$$(\mathbf{F} - \alpha)^{n+1}\lambda(x) = 0.$$

This can also be written in the following way

$$\mathbf{F}^{n+1}\lambda(x) = -\sum_{i=0}^{n} \binom{n+1}{i} (-\alpha)^{n+1-i}\mathbf{F}^i\lambda(x). \tag{15}$$

Partial integration reveals that D can be written as $D(x) = u(x)e^{2\alpha x} + \gamma\lambda(x)$, where u is a polynomial of degree $q = 2n$ and γ is a constant. Using Lemma 3.1 we see that we can use $\widetilde{D}$ instead of D to generate the Lie algebra, where $\widetilde{D}$ is given by

$$\widetilde{D}(x) = D(x) - \left[\sum_{i=0}^{n} \left(\frac{-1}{\alpha}\right)^{i+1} \mathbf{F}^i p(0)\right] \cdot \lambda(x).$$

Here the sum on the right hand side equals γ. Therefore $\widetilde{D}(x) = u(x)e^{2\alpha x}$ and thus $\widetilde{D}$ satisfies the following ODE of order $q+1$

$$(\mathbf{F} - 2\alpha)^{q+1}\widetilde{D}(x) = 0,$$

which we can also write as

$$\mathbf{F}^{q+1}\widetilde{D}(x) = -\sum_{j=0}^{q} \begin{pmatrix} q+1 \\ j \end{pmatrix} (-2\alpha)^{q+1-j}\mathbf{F}^{j}\widetilde{D}(x). \tag{16}$$

After these considerations we choose the following generator system for the Lie algebra

$$\{\mu, \sigma\}_{LA} = span\{\mathbf{F}r, \mathbf{F}^{i}\lambda, \mathbf{F}^{j}\widetilde{D};\ i = 0, 1, \ldots, n;\ j = 0, 1, \ldots, q\},$$

We now turn to the task of finding a parameterization of the invariant manifold $G(z_0, z_i^1, z_j^2; i = 0, 1, \ldots, n; j = 0, 1, \ldots, q)$, which amounts to computing the operators $\exp\{\mathbf{F}rt\}\ \exp\{\mathbf{F}^{i}\lambda t\}$, $i = 0, 1, \ldots, n$ and $\exp\{\mathbf{F}^{j}\widetilde{D}t\}$, $j = 0, 1, \ldots, q$. The operator $\exp\{\mathbf{F}rt\}$ is obtained as the solution to

$$\frac{dy_t}{dt} = \mathbf{F}r.$$

This is a linear equation and the solution is

$$y_t = e^{\mathbf{F}t}y_0,$$

which means that

$$(e^{\mathbf{F}t}r_0)(x) = r_0(x+t).$$

Since the rest of the generating fields are constant, the corresponding ODEs are trivial, and we have

$$(e^{\mathbf{F}^{i}\lambda t}r_0)(x) = r_0(x) + \mathbf{F}^{i}\lambda t,$$

and

$$(e^{\mathbf{F}^{j}\widetilde{D}t}r_0)(x) = r_0(x) + \mathbf{F}^{j}\tilde{D}t,$$

respectively. The invariant manifold generated by the initial forward rate curve r_0 is thus parameterized as

$$\begin{aligned} & G(z_0, z_i^1, z_j^2;\ i = 0, 1, \ldots, n;\ j = 0, 1, \ldots, q)(x) \\ = \quad & r(x+z_0) + \sum_{i=0}^{n} \mathbf{F}^{i}\lambda(x)z_i^1 + \sum_{j=0}^{q} \mathbf{F}^{j}\widetilde{D}(x)z_j^2. \end{aligned} \tag{17}$$

To obtain the state space dynamics we solve the equations (5). The Frechet derivative G' of G is given by

$$\begin{aligned} & G'(z_0, z_i^1, z_j^2;\ i = 0, 1, \ldots, n;\ j = 0, 1, \ldots, q) \begin{pmatrix} h_0 \\ h_0^1 \\ h_1^1 \\ \vdots \\ h_q^2 \end{pmatrix} (x) \\ = \quad & \frac{\partial}{\partial x} r_0(x+z_0)h_0 + \sum_{i=0}^{n} \mathbf{F}^{i}\lambda(x)h_i^1 + \sum_{j=0}^{q} \mathbf{F}^{j}\widetilde{D}(x)h_j^2. \end{aligned}$$

From (2) we obtain the following expression for μ

$$\mu(r) = \mathbf{F}r + \varphi^2(r)D - \frac{1}{2}\varphi'(r)[\lambda]\varphi(r)\lambda,$$

where D was defined in (14). Using that $r = G(z)$, the equation $G_\star a = \mu$ reads

$$\begin{aligned} & \frac{\partial}{\partial x} r_0(x+z_0)a_0 + \sum_{i=0}^{n} \mathbf{F}^i \lambda(x) a_i^1 + \sum_{j=0}^{q} \mathbf{F}^j \widetilde{D}(x) a_j^2 \\ = \quad & \frac{\partial}{\partial x} r_0(x+z_0) + \sum_{i=0}^{n} \mathbf{F}^{i+1} \lambda(x) z_i^1 + \sum_{j=0}^{q} \mathbf{F}^{j+1} \widetilde{D}(x) z_j^2 \\ & + \varphi^2(G(z))D(x) - \frac{1}{2}\varphi'(G(z))[\lambda]\varphi(G(z))\lambda(x). \end{aligned}$$

This equality has to hold for all x, and a is not allowed to depend on x. This allows us to identify what a must look like. Recall that λ solves the ODE defined in (15), and that $\widetilde{D}$ solves the ODE in (16). Furthermore, recall that $D(x) = \widetilde{D}(x) + \gamma\lambda(x)$, and let

$$c_i = -\binom{n+1}{i}(-\alpha)^{n+1-i} \quad \text{and} \quad d_j = -\binom{q+1}{j}(-2\alpha)^{q+1-j}. \tag{18}$$

We then obtain

$$\begin{aligned} a_0 &= 1, \\ a_0^1 &= c_0 z_n^1 + \gamma\varphi^2(G(z)) - \tfrac{1}{2}\varphi'(G(z))[\lambda]\varphi(G(z)), \\ a_i^1 &= c_i z_n^1 + z_{i-1}^1, && i = 1, \dots, n, \\ a_0^2 &= d_0 z_q^2 + \varphi^2(G(z)), \\ a_j^2 &= d_j z_q^2 + z_{j-1}^2, && j = 1, \dots, q. \end{aligned}$$

From $G_\star b = \sigma$ we obtain the equation

$$\begin{aligned} & \frac{\partial}{\partial x} r_0(x+z_0)b_0 + \sum_{i=0}^{n} \mathbf{F}^i \lambda(x) b_i^1 + \sum_{j=0}^{q} \mathbf{F}^j \widetilde{D}(x) b_j^2 \\ = \quad & \varphi(G(z))\lambda(x), \end{aligned}$$

where we have used that $r = G(z)$. This gives us

$$\begin{aligned} b_0 &= 0, \\ b_j^i &= \varphi(G(z)), && i = 1,\ j = 0, \\ b_j^i &= 0, && \text{all other } i, j. \end{aligned}$$

Just as for the case with deterministic volatilities we see that the Wiener process only drives one of the state variables. On Stratonovich form the dynamics of Z_0^1 are

$$dZ_0^1 = \left(c_0 Z_n^1 + \gamma\varphi^2(G(Z)) - \frac{1}{2}\varphi'(G(Z))[\lambda]\varphi(G(Z))\right) dt + \varphi(G(Z)) \circ dW_t.$$

Changing to Itô-dynamics for Z_0^1 we have the following proposition.

Proposition 5.1 *Given the initial forward rate curve r_0 the forward rate system generated by the volatility defined by the equations (12) and (13) has a finite dimensional realization given by*

$$r_t = G(Z_t),$$

where G was defined in (17) and the dynamics of the state space variables Z are given by

$$\begin{cases} dZ_0 &= dt, \\ dZ_0^1 &= [c_0 Z_n^1 + \gamma\varphi^2(G(Z))]dt + \varphi(G(Z))dW_t, \\ dZ_i^1 &= (c_i Z_n^1 + Z_{i-1}^1)dt, \qquad i = 1, \dots, n, \\ dZ_0^2 &= [d_0 Z_q^2 + \varphi^2(G(Z))]dt, \\ dZ_j^2 &= (d_j Z_q^2 + Z_{j-1}^2)dt, \qquad j = 1, \dots, q. \end{cases}$$

Here c_i and d_j are given by (18)

5.2 A degenerate case: CIR

A deterministic direction volatility for which Assumption 5.1 does not hold is

$$\sigma(r, x) = \sqrt{\alpha R + \beta} \cdot \lambda(x), \tag{19}$$

where α and β are constants. Note that only a point evaluation of the forward rate curve is used. It has been shown in Proposition 7.3 in [3] that in order for the forward rate system generated by this volatility to have a short rate realization, λ has to satisfy the following integral differential equation

$$\frac{\partial\lambda}{\partial x}(x) + \alpha\lambda(x)\int_0^x \lambda(u)du + \gamma\lambda(x) = 0. \tag{20}$$

Here γ is a constant. We choose to normalize λ in such a way that $\lambda(0) = 1$. Since there is a short rate realization the Lie algebra has to be two dimensional, and thus it is generated by μ and σ. For this model μ is given by

$$\mu(r) = \mathbf{F}r + (\alpha R + \beta)D - \frac{\alpha}{4}\lambda, \tag{21}$$

where

$$D(x) = \lambda(x) \int_0^x \lambda(u) du. \tag{22}$$

Using Lemma 3.1 we see that the following simpler fields also generate the Lie algebra

$$\begin{aligned} f_0 &= \mathbf{F}r + (\alpha r(0) + \beta)D, \\ f_1 &= \lambda. \end{aligned}$$

Again, the operator associated with the constant field f_1 is trivial to obtain. We have

$$(e^{f_1 t} r_0)(x) = r_0(x) + \lambda(x)t.$$

The operator associated with f_0 is given by the solution of

$$\frac{dy}{dt} = \mathbf{F}y + (\alpha y(0) + \beta)D.$$

This looks very similar to the kind of equations we have solved earlier, except for the appearance of $y(0)$ on the right hand side. The solution is given by

$$y_t = e^{\mathbf{F}t} y_0 + \int_0^t e^{\mathbf{F}(t-s)} (\alpha y_s(0) + \beta) D ds.$$

This means that $y(0)$ satisfies

$$y_t(0) = y_0(t) + \int_0^t (\alpha y_s(0) + \beta) D(t-s) ds.$$

To simplify keeping track of things, we define u as the solution of the following integral equation

$$u_t = y_0(t) + \int_0^t (\alpha u_s + \beta) D(t-s) ds.$$

We then have that

$$y_t(x) = y_0(x+t) + \int_0^t (\alpha u_s + \beta) D(x+t-s) ds.$$

The operator $\exp\{f_0 t\}$ is thus given by

$$(e^{f_0 t} r_0)(x) = r_0(x+t) + \int_0^t (\alpha u_s + \beta) D(x+t-s) ds,$$

where

$$u_t = r_0(t) + \int_0^t (\alpha u_s + \beta) D(t-s) ds. \tag{23}$$

The invariant manifold is therefore parameterized as

$$G(z_0, z_1)(x) = r(x+z_0) + \int_0^{z_0} (\alpha u_s + \beta) D(x+z_0-s) ds + \lambda(x) z_1. \tag{24}$$

Turning to the dynamics of the state space variables, we need G' in order to solve (5). G' is given by

$$G'(z_0, z_1)\begin{pmatrix} h_0 \\ h_1 \end{pmatrix}(x) = \frac{\partial}{\partial x} r_0(x+z_0)h_0 + (\alpha u_{z_0} + \beta)D(x)h_0 + \int_0^{z_0} (\alpha u_s + \beta)D'(x+z_0-s)ds \cdot h_0 + \lambda(x)h_1.$$

If we use that $r(0) = G(z_0, z_1)(0)$ the equation $G_\star a = \mu$ reads

$$\begin{aligned} &\frac{\partial}{\partial x} r_0(x+z_0)a_0 + (\alpha u_{z_0} + \beta)D(x)a_0 \\ &+ \int_0^{z_0} (\alpha u_s + \beta)D'(x+z_0-s)ds \cdot a_0 + \lambda(x)a_1 \\ = &\frac{\partial}{\partial x} r_0(x+z_0) + \int_0^{z_0} (\alpha u_s + \beta)D'(x+z_0-s)ds \\ &+ \mathbf{F}\lambda(x)z_1 + [\alpha G(z_0, z_1)(0) + \beta]D(x) - \frac{\alpha}{4}\lambda(x) \end{aligned} \tag{25}$$

From (24) and (23) we have that

$$\begin{aligned} G(z_0, z_1)(0) &= r(z_0) + \int_0^{z_0} (\alpha u_s + \beta)D(z_0 - s)ds + z_1 \\ &= u_{z_0} + z_1 \end{aligned}$$

Here we have used the normalization $\lambda(0) = 1$. Inserting this expression into (25) we have

$$\begin{aligned} &\frac{\partial}{\partial x} r_0(x+z_0)a_0 + (\alpha u_{z_0} + \beta)D(x)a_0 \\ &+ \int_0^{z_0} (\alpha u_s + \beta)D'(x+z_0-s)ds \cdot a_0 + \lambda(x)a_1 \\ = &\frac{\partial}{\partial x} r_0(x+z_0) + \int_0^{z_0} (\alpha u_s + \beta)D'(x+z_0-s)ds + \mathbf{F}\lambda(x)z_1 \\ &+ (\alpha u_{z_0} + \beta)D(x) + \alpha D(x)z_1 - \frac{\alpha}{4}\lambda(x) \end{aligned}$$

Recall that λ satisfies equation (20). We can then identify terms to obtain

$$\begin{aligned} a_0 &= 1, \\ a_1 &= -\left(\gamma z_1(t) + \frac{\alpha}{4}\right). \end{aligned}$$

From the equation $G_\star b = \sigma$, which reads as

$$\begin{aligned} &\frac{\partial}{\partial x} r_0(x+z_0)b_0 + (\alpha u_{z_0} + \beta)D(x)b_0 \\ &+ \int_0^{z_0} (\alpha u_s + \beta)D'(x+z_0-s)ds \cdot b_0 + \lambda(x)b_1 \\ = &\sqrt{\alpha G(z_0, z_1)(0) + \beta} \cdot \lambda(x), \end{aligned}$$

we obtain

$$\begin{aligned} b_0 &= 0, \\ b_1 &= \sqrt{\alpha G(z_0, z_1)(0) + \beta}. \end{aligned}$$

Only Z_1 will be stochastic, and its Stratonovich-dynamics are given by

$$dZ_1(t) = -\left(\gamma Z_1(t) + \frac{\alpha}{4}\right) dt + \sqrt{\alpha G(Z_0, Z_1)(0) + \beta} \circ dW_t.$$

Translating these dynamics into Itô-form we arrive at the following proposition.

Proposition 5.2 *Given the initial forward rate curve r_0 the forward rate system generated by the volatility defined by the equations (19) and (20) has a finite dimensional realization given by*

$$r_t = G(Z_t),$$

where G is given by (24), and the dynamics of the state space variables Z are given by

$$\begin{cases} dZ_0(t) &= dt, \\ dZ_1(t) &= \left(\frac{\alpha}{4} - \gamma Z_1(t)\right) dt + \sqrt{\alpha G(Z_0, Z_1)(0) + \beta} \cdot dW_t. \end{cases}$$

6 Time varying systems

So far we have only considered homogeneous systems. In this section we introduce the slight modifications needed in order for the method to be applicable to time varying systems. Consider the following system of forward rate equations

$$\begin{cases} dr_t &= \mu(r_t, t)dt + \sigma(r_t, t) \circ dW_t, \\ r_s &= r^0. \end{cases} \tag{26}$$

The volatility is now of the form $\sigma : \mathcal{H} \times R \times R_+ \to R^m$. The drift μ is still given by the expression in (2), except that there is now an explicit time dependence. The definition of a realization is given below.

Definition 6.1 *We say that the SDE (26) has a (local) d-dimensional* **realization** *at (s, r^0), if there exists a point $z_s \in R^d$, smooth vector fields $a, b_1, \dots, b_m$ on some open subset $\mathcal{Z}$ of R^d and a smooth (submanifold) map $G : \mathcal{Z} \to \mathcal{H}$, such that r has the local representation*

$$r_t = G(Z_t), \qquad t \geq s,$$

where Z is the solution of the d-dimensional Stratonovich SDE

$$\begin{cases} dZ_t &= a(Z_t)dt + b(Z_t) \circ dW_t, \\ Z_s &= z_s. \end{cases}$$

The way to handle the explicit time dependence is to enlarge the state space to include running time as a state variable.

Definition 6.2 *Define the following extended objects.*

$$
\begin{aligned}
\widehat{\mathcal{H}} &= \mathcal{H} \times R, \\
\hat{r} &= \begin{bmatrix} r \\ t \end{bmatrix}, \\
\hat{\mu}(\hat{r}) &= \begin{bmatrix} \mu(r,t) \\ 1 \end{bmatrix}, \\
\hat{\sigma}(\hat{r}) &= \begin{bmatrix} \sigma(r,t) \\ 0 \end{bmatrix}.
\end{aligned}
$$

We have the following theorem.

Theorem 6.1 (Björk and Svensson) *The time varying system (26) has a finite dimensional realization if and only if*

$$dim\{\hat{\mu}, \hat{\sigma}\}_{LA} < \infty.$$

6.1 Deterministic direction volatility

For the general time-dependent deterministic direction volatility each volatility component is of the form

$$\sigma_i(r,t,x) = \sum_{j=0}^{n_i} \varphi_j^i(r,t)\lambda_j^i(t,x). \tag{27}$$

We make the following assumption.

Assumption 6.1 *We assume that $\varphi_j^i(r,t) \neq 0$ for all $r \in \mathcal{H}$, for all $i = 1, \ldots, m$; $j = 0, 1, \ldots, n_i$ and for all t.*

From a slight extension of the results in Section 6 in [3], we have that a sufficient condition for the forward rate system generated by this volatility structure to have a finite dimensional realization is that for each t the function space

$$\left\{ \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right)^l \lambda_j^i(t), \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right)^l D_{j,k}^i(t) \right\}, \tag{28}$$

where $i = 1, \ldots, m$; $j, k = 0, 1, \ldots, n_i$ and $l = 0, 1, \ldots$ has finite dimension. Here $D_{j,k}^i$ is given by

$$D_{j,k}^i(t,x) = \lambda_j^i(t,x) \int_0^x \lambda_k^i(t,u)du.$$

In order for this condition to be satisfied, it is clear that some assumption has to be made on λ_j^i. For instance we could assume that λ_j^i satisfies an ordinary differential equation. The difficulty is then to sort out what this implies for $D^i_{j,k}$, i.e. which ordinary differential equation $D^i_{j,k}$ satisfies. In any given case this is of course possible, but we will not pursue this here.

Instead we will exemplify our method in the time varying case by studying an extension of the volatility structure considered in [6]. We assume that each component i of σ, that is σ_i, solves an ODE of the form

$$\frac{\partial^{(n_i+1)}}{\partial T^{(n_i+1)}}\sigma_i(r,t,T) = \sum_{j=0}^{n_i} \kappa^i_j(T)\frac{\partial^j}{\partial T^j}\sigma_i(r,t,T).$$

Note that we have temporarily switched back to time of maturity T, instead of time to maturity x. Volatility structures satisfying ordinary differential equations of this type have been studied in both [13] and [6]. In [13] only first order ODEs are considered, and furthermore an initial condition of the form $\sigma_i(r,t,t) = \sigma_i(R,t,t)$ is used, that is the short rate volatility is assumed to be a function of the time t and the present short rate R. In [6] general n-order ODEs are considered and an initial condition of the form $\sigma_i(r,t,t) = \sigma_i(r(t,x_1),\dots,r(t,x_q),t,t)$ is used, i.e. the short rate volatility is assumed to be a function of the time t and a finite number q of the present forward rates $r(t,x_1),\dots,r(t,x_q)$. We generalize this further by allowing the short rate volatility to be a functional of the entire forward rate curve.

To see that these volatilities are of the form (27) we write the ODE of order n_i+1 as a system of n_i+1 first order ODEs in the standard way. Let $\Phi^i(T,t)$ solve

$$\begin{cases} \dfrac{\partial \Phi^i(T,t)}{\partial T} &=& A^i(T)\Phi^i(T,t), \\ \Phi^i(t,t) &=& I^i, \end{cases}$$

where I^i denotes the $(n_i+1)\times(n_i+1)$ identity matrix, and A^i is given by

$$A^i(T) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & & & 1 \\ \kappa^i_0(T) & \kappa^i_1(T) & \kappa^i_2(T) & \dots & \kappa^i_{n_i}(T) \end{pmatrix}.$$

Furthermore let $y^i = [y^i_0, y^i_1, \dots, y^i_{n_i}]^*$ where $y^i_j(T) = \frac{\partial^j}{\partial T^j}\sigma_i(r,t,T)$. Then we have that

$$y^i(T) = \Phi^i(T,t)y^i(t),$$

which means that

$$\sigma_i(r,t,T) = \sum_{j=0}^{n_i} \phi^i_j(T,t)\frac{\partial^j}{\partial T^j}\sigma_i(r,t,t).$$

Here $\phi^i(T,t)$ denotes the first row in $\Phi^i(T,t)$. Thus σ_i is of the form (27) with

$$\begin{aligned}\varphi_j^i(r,t) &= \frac{\partial^j}{\partial T^j}\sigma_i(r,t,t),\\ \lambda_j^i(t,x) &= \phi_j^i(t+x,t).\end{aligned} \tag{29}$$

To see that the function space (28) is finite dimensional for this case, we use the relations

$$\frac{\partial}{\partial T}\Phi^i(T,t) = A^i(T)\Phi^i(T,t), \quad \text{and} \quad \frac{\partial}{\partial t}\Phi^i(T,t) = -\Phi^i(T,t)A^i(t),$$

to obtain that

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\lambda_j^i(t) = -\left[\lambda_{j-1}^i(t) + \kappa_j^i(t)\lambda_{n_i}^i(t)\right], \tag{30}$$

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)D_{j,k}^i(t) &= -\left[D_{j-1,k}^i(t) + \kappa_j^i(t)D_{n_i,k}^i(t)\right.\\ &\quad \left.+D_{j,k-1}^i(t) + \kappa_k^i(t)D_{j,n_i}^i(t) + \lambda_j^i(t)\delta_{0k}\right],\end{aligned} \tag{31}$$

where the convention $\lambda_{-1}^i = 0$ is used, and δ_{0k} denotes the Kronecker delta, which equals one if $k = 0$ and zero otherwise. From this it is clear that the function space (28) is finite dimensional.

For this model we have from (2) that

$$\begin{aligned}\mu(r,t) &= \mathbf{F}r + \sum_{i=1}^{m}\sum_{j=0}^{n_i}\sum_{k=0}^{n_i}\varphi_j^i(r,t)\varphi_k^i(r,t)D_{j,k}^i(t)\\ &\quad -\frac{1}{2}\sum_{i=1}^{m}\sum_{j=0}^{n_i}\sum_{k=0}^{n_i}(\varphi_j^i)_r'(r,t)[\lambda_k^i(t)]\varphi_k^i(r,t)\lambda_j^i(t).\end{aligned} \tag{32}$$

Now, using Lemma 3.1 and Assumption 6.1 it is straight forward to see that the Lie algebra $\{\hat{\mu},\hat{\sigma}\}_{LA}$ is included in

$$span\left\{\begin{bmatrix}\mathbf{F}r\\ 1\end{bmatrix}, \begin{bmatrix}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)^l \lambda_j^i(t)\\ 0\end{bmatrix}, \begin{bmatrix}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)^l D_{j,k}^i(t)\\ 0\end{bmatrix}\right\}.$$

Here $i = 1,\dots,m$; $j,k = 0,1,\dots,n_i$ and $l = 0,1,\dots$ From the equations (30) and (31) follows that the Lie algebra is actually included in the following span

$$span\left\{f_0, f_j^i, g_{j,k}^i\right\},$$

where

$$\begin{aligned}f_0 &= \begin{bmatrix}\mathbf{F}r\\ 1\end{bmatrix},\\ f_j^i &= \begin{bmatrix}\lambda_j^i(t)\\ 0\end{bmatrix},\\ g_{j,k}^i &= \begin{bmatrix}D_{j,k}^i(t)\\ 0\end{bmatrix}.\end{aligned}$$

where $i = 1, \ldots, m$ and $j, k = 0, 1, \ldots, n_i$. This is the functions we choose as generating functions.

Having decided which generating functions to use, we proceed by computing the invariant manifold $\widehat{G}(z_0, z^i_j, z^i_{j,k};\ i = 1, \ldots, m;\ j, k = 0, 1, \ldots, n_i)$. This amounts to computing the operators $\exp\{f_0\tau\}$, $\exp\{f^i_j\tau\}$ and $\exp\{g^i_{j,k}\tau\}$. The operator $\exp\{f_0\tau\}$ is obtained as the solution of the equation

$$\frac{d}{d\tau}\begin{bmatrix} r \\ t \end{bmatrix} = \begin{bmatrix} \mathbf{F}r \\ 1 \end{bmatrix}.$$

From this we obtain that

$$(e^{f_0\tau}\hat{r}_0) = \begin{bmatrix} \tilde{r}_0 \\ s + \tau \end{bmatrix},$$

where $\hat{r}_0 = [r_0, s]^*$ and $\tilde{r}_0(x) = r_0(x + \tau)$. The ODEs corresponding to generating fields which do not depend on r are trivial to solve, and we have that

$$(e^{f^i_j\tau}\hat{r}_0) = \begin{bmatrix} r_0 + \lambda^i_j(s)\tau \\ s \end{bmatrix},$$

and that

$$(e^{g^i_{j,k}\tau}\hat{r}_0) = \begin{bmatrix} r_0 + D^i_{j,k}(s)\tau \\ s \end{bmatrix}.$$

The invariant manifold generated by the initial forward rate curve r_0 is thus parameterized as

$$\begin{aligned} &\widehat{G}(z_0, z^i_j, z^i_{j,k};\ i = 1, \ldots, m;\ j = 0, 1, \ldots, n_i;\ k = 0, 1, \ldots, n_i) \\ = &\begin{bmatrix} \tilde{r}_0 + \sum_{i=1}^{m}\sum_{j=0}^{n_i}\lambda^i_j(s + z_0)z^i_j + \sum_{i=1}^{m}\sum_{j=0}^{n_i}\sum_{k=0}^{n_i} D^i_{j,k}(s + z_0)z^i_{j,k} \\ s + z_0 \end{bmatrix} \end{aligned} \tag{33}$$

To obtain the state space dynamics we solve the equations (5), where we should use $\widehat{G}_\star$, $\hat{\mu}$ and $\hat{\sigma}$. The Frechet derivative $\widehat{G}'$ of $\widehat{G}$ is given by

$$\begin{aligned} &\widehat{G}'(z_0, z^i_j, z^i_{j,k};\ i = 1, \ldots, m;\ j, k = 0, 1, \ldots, n_i)\begin{pmatrix} h_0 \\ h^1_0 \\ \vdots \\ h^m_{n_m} \\ h^1_{0,0} \\ \vdots \\ h^m_{n_m,n_m} \end{pmatrix} \\ &= \begin{bmatrix} \eta \\ h_0 \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned}\eta(x) &= \frac{\partial}{\partial x} r_0(x+z_0)h_0 + \sum_{i=1}^{m}\sum_{j=0}^{n_i} \frac{\partial}{\partial t}\lambda_j^i(s+z_0,x) z_j^i h_0 \\ &\quad + \sum_{i=1}^{m}\sum_{j=0}^{n_i} \lambda_j^i(s+z_0,x) h_j^i + \sum_{i=1}^{m}\sum_{j=0}^{n_i}\sum_{k=0}^{n_i} \frac{\partial}{\partial t} D_{j,k}^i(s+z_0,x) z_{j,k}^i h_0 \\ &\quad + \sum_{i=1}^{m}\sum_{j=0}^{n_i}\sum_{k=0}^{n_i} D_{j,k}^i(s+z_0,x) h_{j,k}^i.\end{aligned}$$

To simplify notation we introduce

$$F_{j,k}^i(r,t) = \varphi_j^i(r,t)\varphi_k^i(r,t),$$

and

$$H_{j,k}^i(r,t) = (\varphi_j^i)'_r(r,t)[\lambda_k^i(t)]\varphi_k^i(r,t).$$

We also introduce

$$\begin{aligned} & G(z_0, z_j^i, z_{j,k}^i;\ i=1,\dots,m;\ j=0,1,\dots,n_i;\ k=0,1,\dots,n_i) \\ = \ & \tilde{r}_0 + \sum_{i=1}^{m}\sum_{j=0}^{n_i} \lambda_j^i(s+z_0) z_j^i + \sum_{i=1}^{m}\sum_{j=0}^{n_i}\sum_{k=0}^{n_i} D_{j,k}^i(s+z_0) z_{j,k}^i,\end{aligned}$$

so that we can write

$$\widehat{G}(z) = \begin{bmatrix} G(z) \\ s+z_0 \end{bmatrix}.$$

Using that $\hat{r} = \widehat{G}(z)$, the equation $\widehat{G}_\star a = \hat{\mu}$ gives the following two equations

$$\begin{aligned} & \frac{\partial}{\partial x} r_0(x+z_0)a_0 + \sum_{i=1}^{m}\sum_{j=0}^{n_i} \frac{\partial}{\partial t}\lambda_j^i(s+z_0,x) z_j^i a_0 \\ & + \sum_{i=1}^{m}\sum_{j=0}^{n_i} \lambda_j^i(s+z_0,x) a_{i,j} + \sum_{i=1}^{m}\sum_{j=0}^{n_i}\sum_{k=0}^{n_i} \frac{\partial}{\partial t} D_{j,k}^i(s+z_0,x) z_{j,k}^i a_0 \\ & + \sum_{i=1}^{m}\sum_{j=0}^{n_i}\sum_{k=0}^{n_i} D_{j,k}^i(s+z_0,x) a_{i,j,k} \\ = \ & \frac{\partial}{\partial x} r_0(x+z_0) + \sum_{i=1}^{m}\sum_{j=0}^{n_i} \frac{\partial}{\partial x}\lambda_j^i(s+z_0,x) z_j^i \\ & + \sum_{i=1}^{m}\sum_{j=0}^{n_i}\sum_{k=0}^{n_i} \frac{\partial}{\partial x} D_{j,k}^i(s+z_0,x) z_{j,k}^i \\ & + \sum_{i=1}^{m}\sum_{j=0}^{n_i}\sum_{k=0}^{n_i} F_{j,k}^i(G(z), s+z_0) D_{j,k}^i(s+z_0,x) \\ & - \frac{1}{2}\sum_{i=1}^{m}\sum_{j=0}^{n_i}\sum_{k=0}^{n_i} H_{j,k}^i(G(z), s+z_0)\lambda_j^i(s+z_0,x),\end{aligned}$$

and

$$a_0 = 1.$$

Recall that λ_j^i and $D_{j,k}^i$ satisfies (30) and (31), respectively. This together with the fact that the equations have to hold for all x, and that a is not allowed to depend on x, makes it possible to identify what a must look like. We have that

$$\begin{aligned}
a_0 &= 1, \\
a_{i,j} &= z_{j+1}^i + z_{j,0}^i - \frac{1}{2}\sum_{k=0}^{n_i} H_{j,k}^i(G(z), s+z_0), \\
a_{i,n_i} &= \sum_{j=0}^{n_i} \kappa_j^i(s+z_0) z_j^i + z_{n_i,0}^i - \frac{1}{2}\sum_{k=0}^{n_i} H_{n_i,k}^i(G(z), s+z_0),
\end{aligned}$$

where the second formula holds for all $i = 1.\dots,m$ and $j = 0,1,\dots,n_i - 1$, and the third formula holds for all $i = 1.\dots,m$. Furthermore we obtain that

$$\begin{aligned}
a_{i,j,k} &= z_{j+1,k}^i + z_{j,k+1}^i + F_{j,k}^i(G(z), s+z_0), \\
a_{i,n_i,k} &= \sum_{j=0}^{n_i} \kappa_j^i(s+z_0) z_{j,k}^i + z_{n_i,k+1}^i + F_{n_i,k}^i(G(z), s+z_0), \\
a_{i,j,n_i} &= z_{j+1,n_i}^i + \sum_{k=0}^{n_i} \kappa_k^i(s+z_0) z_{j,k}^i, + F_{j,n_i}^i(G(z), s+z_0), \\
a_{i,n_i,n_i} &= \sum_{j=0}^{n_i} \kappa_j^i(s+z_0) z_{j,n_i}^i + \sum_{k=0}^{n_i} \kappa_k^i(s+z_0) z_{n_i,k}^i + F_{n_i,n_i}^i(G(z), s+z_0),
\end{aligned}$$

where the first formula holds for all $i = 1.\dots,m$ and $j,k = 0,1,\dots,n_i - 1$, the second formula holds for $i = 1.\dots,m$ and $k \neq n_i$, and the third formula holds for $i = 1.\dots,m$ and $j \neq n_i$. From $\widehat{G}_\star b^l = \hat{\sigma}_l$ we obtain the equations

$$\begin{aligned}
&\frac{\partial}{\partial x} r_0(x+z_0) b_0^l + \sum_{i=1}^{m}\sum_{j=0}^{n_i} \frac{\partial}{\partial t}\lambda_j^i(s+z_0,x) z_j^i b_0^l \\
&+ \sum_{i=1}^{m}\sum_{j=0}^{n_i} \lambda_j^i(s+z_0,x) b_{i,j}^l + \sum_{i=1}^{m}\sum_{j=0}^{n_i}\sum_{k=0}^{n_i} \frac{\partial}{\partial t} D_{j,k}^i(s+z_0,x) z_{j,k}^i b_0^l \\
&+ \sum_{i=1}^{m}\sum_{j=0}^{n_i}\sum_{k=0}^{n_i} D_{j,k}^i(s+z_0,x) b_{i,j,k}^l \\
= &\sum_{j=0}^{n_l} \varphi_j^l(G(z), s+z_0) \lambda_j^l(s+z_0,x),
\end{aligned}$$

where we have used that $r = G(z)$ and $t = s + z_0$, and

$$b_0^l = 0.$$

This gives us

$$
\begin{array}{lcll}
b_0^l & = & 0, & l = 0, 1, \ldots, m, \\
b_{l,j}^l & = & \varphi_j^l(G(z), s + z_0), & l = 0, 1, \ldots, m,\ j = 0, 1, \ldots, n_l, \\
b_{i,j}^l & = & 0, & l = 0, 1, \ldots, m,\ i \neq l, \\
b_{i,j,k}^l & = & 0, & \text{all } i, j, k, l.
\end{array}
$$

Again only a few state space variables are driven by the Wiener processes. The dynamics of these variables on Stratonovich form are

$$
\begin{cases}
dZ_j^i & = \left(Z_{j+1}^i + Z_{j,0}^i - \dfrac{1}{2} \displaystyle\sum_{k=0}^{n_i} H_{j,k}^i(G(Z), s + Z_0) \right) dt \\
& \quad + \varphi_j^i(G(Z), s + Z_0) \circ dW_t^i, \\
dZ_{n_i}^i & = \left(\displaystyle\sum_{j=0}^{n_i} \kappa_j^i(s + Z_0) Z_j^i + Z_{n_i,0}^i - \dfrac{1}{2} \displaystyle\sum_{k=0}^{n_i} H_{n_i,k}^i(G(Z), s + Z_0) \right) dt \\
& \quad + \varphi_{n_i}^i(G(Z), s + Z_0) \circ dW_t^i,
\end{cases}
$$

where the first formula holds for all $i = 1 \ldots, m$ and $j = 0, 1, \ldots, n_i - 1$, while the second formula holds for all $i = 1 \ldots, m$. Rewriting the dynamics on Itô-differential form we have proved the following proposition.

Proposition 6.1 *Given the initial forward rate curve r_0 the forward rate system generated by the volatility defined by the equations (27) and (29) has a finite dimensional realization given by*

$$
r_t = G(Z_t),
$$

where G was defined in (33) and the dynamics of the state space variables Z are *given by*

$$
\begin{cases}
dZ_0 & = & dt, \\
dZ^i_j & = & \left(Z^i_{j+1} + Z^i_{j,0}\right) dt + \varphi^i_j(G(Z), s+Z_0) dW^i_t, \\
dZ^i_{n_i} & = & \left(\sum_{j=0}^{n_i} \kappa^i_j (s+Z_0) Z^i_j + Z^i_{n_i,0} \right) dt \\
 & & + \varphi^i_{n_i}(G(Z), s+Z_0) dW^i_t, \\
dZ^i_{j,k} & = & \left(Z^i_{j+1,k} + Z^i_{j,k+1} + F^i_{j,k}(G(Z), s+Z_0)\right) dt, \\
dZ^i_{n_i,k} & = & \left(\sum_{j=0}^{n_i} \kappa^i_j (s+Z_0) Z^i_{j,k} + Z^i_{n_i,k+1} + F^i_{n_i,k}(G(Z), s+Z_0) \right) dt, \\
dZ^i_{j,n_i} & = & \left(Z^i_{j+1,n_i} + \sum_{k=0}^{n_i} \kappa^i_k (s+Z_0) Z^i_{j,k} + F^i_{j,n_i}(G(Z), s+Z_0) \right) dt, \\
dZ^i_{n_i,n_i} & = & \left(\sum_{j=0}^{n_i} \kappa^i_j (s+Z_0) Z^i_{j,n_i} + \sum_{k=0}^{n_i} \kappa^i_k (s+Z_0) Z^i_{n_i,k} \right. \\
 & & \left. + F^i_{n_i,n_i}(G(Z), s+Z_0) \right) dt.
\end{cases}
$$

where in all formulas $i = 1, \ldots, m,$ *and* $j, k = 0, 1, \ldots, n_i - 1.$

Remark 6.1 The above proposition improves on the results obtained in [6] in two ways. Firstly, as has already been pointed out, the short rate volatility is allowed to be a general functional of the entire forward rate curve, instead of a point evaluation in finitely many benchmark forward rates. Since the realization in [6] has one state variable for each benchmark forward rate, their method can not handle this case. Secondly the realization obtained here is of lower dimension than the realization obtained in [6]. This is true even for the case when the short rate volatility is a point evaluation in finitely many benchmark forward rates. For this case the realization given in [6] has at most dimension $q \sum_{i=1}^m (n_i+1)^2(n_i+4)/2$, where q is the number of benchmark forward rates used, whereas our realization will have dimension $\sum_{i=1}^m (n_i+1)(n_i+2)$, excluding running time.

It should be noted however, that unless the coefficients κ^i_j are constants, the transfer matrices $\Phi^i(T,t)$ can not be computed analytically. If we are in the situation considered in [6], the benchmark forward rates used in the point evaluation can be adjoined as state variables. This will give us a system of state variables, the dynamics of which do not involve the components of the transfer matrices. It will of course also increase the dimension of the realization. The components of the transfer matrices will still appear in the parameterization of the invariant manifold, and there numerical techniques will have to be used to solve for these components.

The above proposition also contains [13] as a special case (the realizations obtained are not the same, but the dimensions of them are). For this case the transfer matrices can be computed explicitly even if the coefficients κ_j^i are time dependent, since the ODEs are of order one.

7 Benchmark realizations

The state space variables in the realizations found using our method do not readily lend themselves to an economic interpretation. However, from [3] we have the following theorem.

Theorem 7.1 (Björk and Svensson) *Suppose that* $dim\{\mu, \sigma_1, \ldots, \sigma_m\}_{LA} = d$*. Then, for almost every choice of distinct benchmark maturities* $x_1, \ldots, x_d$*, the realization can be chosen such that the state process* Z_t *is given by* $Z_t = (r_t(x_1), \ldots, r_t(x_d))$*. The expression "almost every choice" above means that, apart from a discrete set of forbidden values,* $x_1, \ldots, x_d$ *can be chosen freely.*

A nice feature of the volatility structures considered in this paper is that they all lead to parameterizations of the invariant manifold which are affine in the state variables. Actually they are affine in all the state variables except for the first, but as we know this variable really represents running time. The parameterization of the invariant manifold is thus of the following form

$$G(z_0, z_1, \ldots, z_{d-1})(x) = g(z_0, x) + \sum_{i=1}^{d-1} h_i(z_0, x) z_i. \tag{34}$$

For a model with a parameterization of the invariant manifold which is affine in the state variables, a realization in terms of benchmark forward rates can be obtained in a straight forward manner. Choose benchmark maturities $x_1, \ldots, x_q$, where $q = d - 1$ (we choose $d - 1$ benchmark forward rates since we will keep running time as one variable). Since the parameterization of the invariant manifold is of the form (34), we can solve the following system of linear equations to obtain the old state variables as functions of the benchmark forward rates

$$\begin{bmatrix} h_1(z_0, x_1) & h_2(z_0, x_1) & \cdots & h_q(z_0, x_1) \\ h_1(z_0, x_2) & h_2(z_0, x_2) & \cdots & h_q(z_0, x_2) \\ \vdots & \ddots & & \vdots \\ h_1(z_0, x_q) & h_2(z_0, x_q) & \cdots & h_q(z_0, x_q) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_q \end{bmatrix} = \begin{bmatrix} f(z_0, x_1) \\ f(z_0, x_2) \\ \vdots \\ f(z_0, x_q) \end{bmatrix}.$$

Here $f(z_0, x_i) = r(x_i) - g(z_0, x_i)$. If we let $r_i = r(x_i)$ the parameterization of the invariant manifold in terms of the benchmark forward rates, $\widetilde{G}$, is given by

$$\widetilde{G}(z_0, r_1, \ldots, r_q) = G(z_0, z_1(r_1, \ldots, r_q), \ldots, z_q(r_1, \ldots, r_q)).$$

The dynamics of the benchmark forward rate $r(x_i)$ are of course given by the specified volatilities and the HJM-drift condition. Inserting $r = \widetilde{G}(z_0, r_1, \ldots, r_q)$ into the volatilities they are given as a system of SDEs.

Remark 7.1 Note that Theorem 7.1 holds for the minimal realization. However the above technique can be used to substitute benchmark forward rates for some of the state space variables in the original realization, even if the realization is not of minimal degree. The requirement for this to work is that the functions $h_1(z_0), h_2(z_0), \ldots, h_k(z_0)$ are linearly independent, where k is the number of state space variables we want to substitute.

A natural question is now for which volatility structures can we find a finite dimensional realization such that the parameterization of the invariant manifold is affine in all state space variables, except for the variable representing running time. We will not answer the general question, instead we give a sufficient condition for deterministic direction volatilities in the following lemma.

Lemma 7.1 *Consider a deterministic direction volatility for which each component is of the form*

$$\sigma_i(r,t,x) = \sum_{j=1}^{n_i} \varphi_j^i(r,t)\lambda_j^i(t,x).$$

Suppose that the following function space is finite dimensional

$$\left\{ \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right)^l \lambda_j^i(t), \ \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right)^l D_{j,k}^i(t) \right\}, \tag{35}$$

where $i = 1, \ldots, m$*;* $j, k = 0, 1, \ldots, n_i$ *and* $l = 0, 1, \ldots$ *and* $D_{j,k}^i$ *is given by*

$$D_{j,k}^i(t,x) = \lambda_j^i(t,x) \int_0^x \lambda_k^i(t,u)du.$$

Then there is a finite dimensional realization of the forward rate system generated by this volatility, and furthermore there is a realization for which the parameterization of the invariant manifold is affine in all the state space variables, except for the one which represents running time.

Proof. In the beginning of Section 6.1 it was established that there is a finite dimensional realization for these volatilities. It remains to show that there is a realization for which the parameterization of the invariant manifold is affine in all the state space variables, except for the one representing running time.

It was also established in Section 6.1 that the Lie algebra $\{\hat{\mu}, \hat{\sigma}\}_{LA}$ for this model is included in

$$span \left\{ \begin{bmatrix} \mathbf{F}r \\ 1 \end{bmatrix}, \begin{bmatrix} \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right)^l \lambda_j^i(t) \\ 0 \end{bmatrix}, \begin{bmatrix} \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right)^l D_{j,k}^i(t) \\ 0 \end{bmatrix} \right\}.$$

Here $i = 1, \ldots, m$; $j, k = 0, 1, \ldots, n_i$ and $l = 0, 1, \ldots$ Since the function space in (35) is assumed to be finite dimensional, it is generated by the functions

$f_1(t), \ldots, f_q(t)$. Since neither $\lambda_j^i(t)$ nor $D_{j,k}^i(t)$ depend on r, neither will the functions $f_1(t), \ldots, f_q(t)$. The Lie algebra is then generated by the following functions

$$span \left\{ \begin{bmatrix} \mathbf{F}r \\ 1 \end{bmatrix}, \begin{bmatrix} f_1(t) \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} f_q(t) \\ 0 \end{bmatrix} \right\}.$$

Calculations very similar to those in Section 6.1 show that the state space variables corresponding to $f_i(t)$, appear linearly in the parameterization of the invariant manifold. The only r-dependent field is $[\mathbf{F}r, 1]^*$, and just as in Section 6.1 the state space variable corresponding to this field represents running time. ∎

7.1 Ho-Lee, Hull-White and CIR

From Section 4.2 we have that forward rate system generated by the volatility

$$\sigma(x) = \sigma,$$

is realized by $r_t = G(Z_t)$ where

$$G(z_0, z_1)(x) = r(x + z_0) + \sigma^2 \left(x z_0 + \frac{1}{2} z_0^2 \right) + z_1,$$

and the state variable dynamics are given by

$$\begin{cases} dZ_0(t) &= dt, \\ dZ_1(t) &= \sigma dW_t. \end{cases}$$

Suppose that we want the forward rate with maturity $t + \xi$, that is $r(t, \xi)$ to be the state variable instead of Z_1. We have that

$$r(\xi) = G(Z_0, Z_1)(\xi) = r_0(\xi + Z_0) + \sigma^2 \left(\xi z_0 + \frac{1}{2} z_0^2 \right) + Z_1.$$

Thus,

$$Z_1 = r(\xi) - r_0(\xi + Z_0) - \sigma^2 \left(\xi z_0 + \frac{1}{2} z_0^2 \right).$$

Let $r_\xi = r(\xi)$. Then the new parameterization of the invariant manifold is given by

$$r(x) = \widetilde{G}(Z_0, r_\xi)(x) = r_0(x + Z_0) - r_0(\xi + Z_0) + \sigma^2 Z_0 (x - \xi) + r_\xi.$$

The dynamics of $r(\xi)$ are found to be

$$dr(\xi) = [\mathbf{F}r_0(\xi + Z_0) + \sigma^2(\xi + Z_0)]dt + \sigma dW_t.$$

Note that if $\xi = 0$ we have chosen the short rate to be the state space variable and the formulas then reveal that we have retrieved the ordinary short rate model of Ho and Lee [10].

The same procedure can be applied to the volatilities in the sections named Hull-White and CIR, and it should come as no surprise that you can then retrieve the short rate models of Hull and White [11] and Cox *et al.* [7], respectively.

8 Bond prices

Another nice consequence of the fact that the parameterization of the invariant manifold is affine in the state space variables, is that it makes bond prices easy to compute. Suppose that the parameterization of the invariant manifold is given by

$$G(z_0, z_1, \ldots, z_d)(x) = g(z_0, x) + \sum_{i=1}^{d} h_i(z_0, x) z_i. \tag{36}$$

From the definition of forward rates we have that

$$\begin{aligned} p(t,T) &= \exp\left\{-\int_t^T f(t,u)du\right\} \\ &= \exp\left\{-\int_0^{T-t} f(t,t+u)du\right\} \\ &= \exp\left\{-\int_0^x r(t,u)du\right\}. \end{aligned}$$

Recall that Lemma 7.1 gives sufficient conditions for a deterministic direction volatility to have a finite dimensional realization, for which the parameterization of the invariant manifold is affine in the state space variables. Now insert $r = G(Z)$ into the above formula to obtain the following proposition.

Proposition 8.1 *For all models with a deterministic direction volatility satisfying the condition in Lemma 7.1, there is a finite dimensional realization for which the parameterization of the invariant manifold is of the form (36). Given this realization, bond prices are given by*

$$p(t,T) = p(t,t+x) = \exp\left\{-\int_0^x g(z_0,u)du - \sum_{i=1}^{d}\left[\int_0^x h_i(z_0,u)du \cdot z_i\right]\right\}.$$

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