Minimal Realizations of Forward Rates *

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Abstract

We consider interest rate models where the forward rates are allowed to be driven by a multidimensional Wiener process as well as by a marked point process. Assuming a deterministic volatility structure, and using ideas from systems and control theory, we investigate when the input-output map generated by such a model can be realized by a finite dimensional stochastic differential equation. We give necessary and sufficient conditions, in terms of the given volatility structure, for the existence of a finite dimensional realization and we provide a formula for the determination of the dimension of a minimal realization. The abstract state space for a minimal realization is shown to have an immediate economic interpretation in terms of a minimal set of benchmark forward rates, and we give explicit formulas for bond prices in terms of the benchmark rates as well as for the computation of derivative prices.

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1 Introduction

The purpose of the present paper is to investigate when and how the input output behavior of a given, a priori infinite dimensional, forward rate model can be realized by a finite dimensional linear stochastic system. More precisely we take a forward rate volatility structure $\sigma(t, x)$ in a Heath-Jarrow-Morton type model as given, and we then try to answer the following questions.

- When does there exist a finite dimesnional state space realization of the forward rate model defined by σ above?
- If there exist a finite dimensional realization, what is the minimal dimension of the state space?
- How do we construct a realization, given only a specification of the forward rate volatility structure σ?
- Is there an economic interpretation of the abstract state space?

In this paper we only study the linear case, and the main reasons for the project are as follows.

Firstly we want to obtain a deeper understanding of the geometric structure of the forward rate dynamics. Put in other words we want to lay bare the "minimal essential dynamics" of the interest rate model under study. The linear case is then a natural starting point, and in a subsequent paper we hope to carry out parts of the same program for the nonlinear case.

Secondly we hope that this theoretical paper can be used as a foundation for future empirical research. In particular we view our results as a first step towards an application of stochastic realization theory (see e.g. [17]) to the area of interest rates. In statistical terms this means that we hope to be able to carry out "dynamic factor analysis" based on observed bond price information.

Using ideas and concepts from systems and control theory we are able to provide a fairly complete analysis of the given problems above. The structure and main results of the paper are as follows.

In Section 3 we connect our problem to systems theory by computing the transfer function for the given forward rate system. Using the structure of the transfer function, the main result is Theorem 3.1 which gives necessary and sufficient conditions for the existence of a finite dimensional linear realization. In Section 4 we study realizations having minimal dimension, and the main result is Proposition 4.1, which gives an easy test for the existence of a finite dimensional realisation, as well as a formula for the minimal dimension. Section 5 is devoted to the economic interpetation of the abstract state space, and in Proposition 5.1 it is shown that the states of a *minimal* realization can be interpeted as a set of benchmark forward rates. In Section 6 we give some examples to illustrate the theory, and in Section 8 we extend the theory by allowing also a driving multivariate point process.

As far as we know, the idea of looking at interest rates from a realization point of view is new, but there are a number of related results in the literature.

In a recent paper [1], it is investigated when the forward rate curve for a given (nonlinear) interest rate model actually evolves on a given finite dimensional submanifold in the forward curve space.

Because of the linear structure, the present paper is also connected to the well known theory of affine term structures developed in [7] and [10]. Note however that our approach is, in a sense, inverse to that of e.g. Duffie-Kan [10]. Duffie-Kan take as given a state space model, and then investigate when there is an afine term structure, whereas we take as given the forward rate volatilities and ask when a state space realization exists at all.

The preprint [11] contains interesting results related to ours. As in the affine term structure theory, the authors take as given a deterministic volatility structure and they also take as given a factor model with invertible diffusion matrix. They then (among other things) show that the state dynamics are indeed linear and that it is possible to identify the state space with a set of forward rates. In the present paper, in contrast, we do not take a state space model as given but instead we give conditions which ensures the existence of a finite state space. Furthermore we are not confined to assume an invertible diffusion matrix for the state dynamics in order to obtain the forward rate interpretation of the state space. Instead we show that for a minimal realisation (see further comments after proposition 5.1) we automatically have this interpretation.

A connection between control theory and infinite dimensional linear stochastic equations is made, within a Hilbert space framework, in [9]. In [19], which has been an inspiration of the present paper, the results of [9] are applied to forward rate dynamics.

Many of the ideas and techniques below come from finite dimensional linear systems theory, and for ease of reference we have included an appendix, where the standard results and definitions can be found. For further information about linear systems the reader is referred to [6] or to any textbook on the subject.

2 The model

We consider a bond market model living on a stochastic basis (filtered probability space) $(\Omega, \mathcal{F}, \mathbf{F}, Q)$ where $\mathbf{F} = \{\mathcal{F}_t\}_{t \geq 0}$. The basis is assumed to carry a standard d-dimensional Wiener process W, and we also assume that the filtration $\mathbf{F}$ is the internal one generated by W. The restriction to a pure Wiener process framework is made for readability reasons, and in Section 8 we will also consider a driving marked point process.

By $p(t, x)$ we denote the price, at t, of a zero coupon bond maturing at $t + x$, and the forward rates $r(t, x)$ are defined by

$$r(t, x) = -\frac{\partial \log p(t, x)}{\partial x}. \tag{1}$$

Note that we use the Musiela parameterization, where x denotes the time **to** maturity. The short rate R is defined as $R(t) = r(t, 0)$, and the money account B is given by $B(t) = \exp\left\{\int_0^t R(s)ds\right\}$. The model is assumed to be free of arbitrage in the sense that the measure Q above is a martingale measure for the model. In other words, for every $x \geq 0$, the process

$$Z(t, x) = \frac{p(t, x)}{B(t)},$$

is a Q-martingale.

Let us now consider a given forward rate model of the form

$$dr(t, x) = \beta(t, x)dt + \sigma(t, x)dW, \tag{2}$$

$$r(0, x) = r^\star(0, x). \tag{3}$$

where, for each x, β and σ are given optional processes. The initial curve $\{r^\star(0, x);\ x \geq 0\}$ is taken as given. It is interpreted as the *observed forward rate curve.*

The standard Heath-Jarrow-Morton drift condition ([13]) can easily be transferred to the Musiela parameterization. The result (see [4], [19]) is as follows.

Proposition 2.1 (The Musiela Equation) *Under the martingale measure Q the r-dynamics are given by*

$$\begin{aligned} dr(t, x) &= \left\{\frac{\partial}{\partial x} r(t, x) + \sigma(t, x) \int_0^x \sigma(t, u)du\right\} dt + \sigma(t, x)dW(t), \\ r(0, x) &= r^\star(0, x). \end{aligned}$$

We recall that we have the relation

$$p(t,x) = e^{-y(t,x)} \tag{4}$$

where the **yield** $y(t,x)$ is defined by

$$y(t,x) = \int_0^x r(t,s)ds. \tag{5}$$

In the present paper we will only consider the case when the volatility $\sigma(t,x)$ is a deterministic time-independent function $\sigma(x)$ of x only. Denoting the function $x \longmapsto r(t,x)$ by $r(t)$, and correspondingly for y and p we then have

$$dr(t) = \{\mathbf{F}r(t) + D\}\, dt + \sigma dW(t), \quad r(0) = r^\star(0). \tag{6}$$

$$y(t) = \mathbf{H}r(t), \tag{7}$$

$$p(t) = e^{y(t)}. \tag{8}$$

Here the linear operators $\mathbf{F}$ and $\mathbf{H}$ are defined by

$$\mathbf{F} = \frac{\partial}{\partial x}, \tag{9}$$

$$\mathbf{H}g(x) = \int_0^x g(s)ds, \tag{10}$$

whereas the function D is defined by

$$D(x) = \sigma(x) \int_0^x \sigma(s)ds, \tag{11}$$

i.e.

$$D = \frac{1}{2}\mathbf{F}\|\mathbf{H}\sigma\|^2. \tag{12}$$

We now note that the system (6)-(7) constitutes a **linear** (or rather *affine*) **infinite dimensional** input-output system, which takes a Wiener trajectory $W(\cdot,\omega)$ into the corresponding (infinite dimensional) yield trajectory $y(\cdot,\omega)$.

The purpose of the present paper is now to investigate when and how the input-output map defined by (6)-(7) can be realized by means of a linear (or affine) **finite dimensional** system driven by the same Wiener process W. In order to make our problem precise we now transform our model from the case of affine dynamics and linear output to the case of linear dynamics and affine output.

The operator $\mathbf{F}$ is the infinitesimal generator of the group of translations, i.e. for any $f \in C[0,\infty)$ we have

$$\left[e^{\mathbf{F}t} f\right](x) = f(t+x), \tag{13}$$

and we can write the solution of (6) as

$$r(t,x) = e^{\mathbf{F}t} r^\star(0,x) + \int_0^t e^{\mathbf{F}(t-s)} D(x)ds + \int_0^t e^{\mathbf{F}(t-s)} \sigma(x) dW(s). \qquad (14)$$

Thus we have

$$r(t,x) = r^\star(0,x+t) + \int_0^t D(x+t-s)ds + \int_0^t e^{\mathbf{F}(t-s)} \sigma(x) dW(s), \qquad (15)$$

and, defining r_0 by

$$dr_0(t,x) = \mathbf{F} r_0(t,x)dt + \sigma(x)dW(t), \quad r_0(0,x) = 0 \qquad (16)$$

it is clear by inspection that the forward rate process $r(t,x)$, and the yield process $y(t,x)$ have the representations

$$\begin{aligned} r(t,x) &= r_0(t,x) + \delta(t,x), & (17)\\ y(t,x) &= \mathbf{H} r_0(t,x) + \Delta(t,x),, & (18) \end{aligned}$$

where δ and Δ are given by

$$\begin{aligned} \delta(t,x) &= r^\star(0,x+t) + \int_0^t D(x+t-s)ds \\ &= r^\star(0,x+t) + \frac{1}{2}\left(\|\mathbf{H}\sigma(x+t)\|^2 - \|\mathbf{H}\sigma(x)\|^2 \right), & (19)\\ \Delta(t,x) &= \int_0^x \delta(t,u)du, \\ &= y^\star(0,t+x) - y^\star(0,t) \\ &+ \frac{1}{2}\int_0^x \left\{ \|\mathbf{H}\sigma(t+u)\|^2 - \|\mathbf{H}\sigma(u)\|^2 \right\} du & (20) \end{aligned}$$

To connect to the standard formulation of systems theory we may finally write the equations (6)-(7) as

$$\begin{aligned} dr_0(t,x) &= \mathbf{F} r_0(t,x)dt + \sigma(x)dW(t), \quad r_0(0,x) = 0 & (21)\\ y_0(t,x) &= \mathbf{H} r_0(t,x) & (22)\\ r(t,x) &= r_0(t,x) + \delta(t,x) & (23)\\ y(t,x) &= y_0(t,x) + \Delta(t,x) & (24) \end{aligned}$$

Since $\Delta(t,x)$ is not affected by the input W, we see that the input output behavior of the term structure system (6)-(7) is completely determined by the linear system (21)-(22). We are thus led to the following definition.

Definition 2.1 *A triple $[A, B, C(x)]$, where A is an $n \times n$-matrix , B is $n \times d$-matrix and C is an n-dimensional row-vector function, is called an n-dimensional* **realization** *of the system (21)-(22) if y_0 has the representation*

$$\begin{aligned} dz(t) &= Az(t)dt + BdW(t), \quad z(0) = 0. & (25) \\ y_0(t, x) &= C(x)z(t), & (26) \end{aligned}$$

The triple $[A, B, C(x)]$ is called an n-dimensional realization of the system (21) if r_0 has the representation

$$\begin{aligned} dz(t) &= Az(t)dt + BdW(t), \quad z(0) = 0. & (27) \\ r_0(t, x) &= C(x)z(t), & (28) \end{aligned}$$

It is obvious that $[A, B, C(x)]$ is a finite dimensional realization for r_0 if and only if $[A, B, \mathbf{H}C(x)]$ is a realization for y_0. Thus it is enough to consider realizations of the r_0 process.

Our main problems are now as follows.

- Take as a priori given a volatility structure $\sigma(x)$.
- When does there exists a finite dimensional realization?
- If there exists a finite dimensional realization, what is the minimal dimension?
- How do we construct a minimal realization from knowledge of σ?
- Is there an economic interpretation of the "state process" z in the realization?

3 Existence of finite dimensional realizations

We will study the existence of a finite dimensional realization of the stochastic system

$$dr_0(t, x) = \mathbf{F}r_0(t, x)dt + \sigma(x)dW(t), \quad r_0(0, x) = 0 \tag{29}$$

by studying the transfer function of an associated deterministic system.

To this end we solve (29) explicitly. Integrating by parts (the integrand is deterministic) then gives us r_0 as

$$
\begin{aligned}
r_0(t,x) &= \int_0^t \sigma(x+t-s)dW(s) && (30)\\
&= W(t)\sigma(x) + \int_0^t W(s)\sigma'(x+t-s)ds && (31)
\end{aligned}
$$

We now see that, for each x, we have

$$
r_0(t,x,\omega) = \Phi[x, W(\cdot,\omega)](t), \tag{32}
$$

where that mapping

$$
\Phi[x,\cdot] : C[0,\infty) \to C[0,\infty), \tag{33}
$$

is defined by

$$
\Phi[x,v](t) = v(t)\sigma(x) + \int_0^t v(s)\sigma'(x+t-s)ds \tag{34}
$$

We now observe that, for each x, the mapping $\Phi[x,\cdot]$ is continuous in the topology of uniform convergence on compacts. Thus it is completely determined by its behavior on any dense subset, $S \subset C[0,\infty)$, and in particular we may choose $S = C^1[0,\infty)$. For any $v \in C^1$ it is however easily seen that $\Phi[x,v](t) = r_0(t,x)$, where r_0 is defined as the solution of the deterministic system

$$
\frac{dr_0}{dt}(t,x) = \mathbf{F}r_0(t,x) + \sigma(x)u(t), \tag{35}
$$

and where $u(t) = \frac{dv}{dt}(t)$.

The same argument applied to any finite dimensional realization gives us the following result.

Lemma 3.1 *The system*

$$
\begin{aligned}
dz(t) &= Az(t)dt + BdW(t), \quad z(0) = 0. && (36)\\
r_0(t,x) &= C(x)z(t), && (37)
\end{aligned}
$$

is a realization of

$$
dr_0(t,x) = \mathbf{F}r_0(t,x)dt + \sigma(x)dW(t), \quad r_0(0,x) = 0 \tag{38}
$$

if and only if the deterministic system

$$
\frac{dr_0}{dt}(t,x) = \mathbf{F}r_0(t,x) + \sigma(x)u(t), \tag{39}
$$

has the same input-output map as the system

$$\frac{dz}{dt}(t) = Az(t) + Bu(t), \tag{40}$$

$$r_0(t,x) = C(x)z(t), \tag{41}$$

The point of all this is that we may now restate the problem in terms of transfer functions. Denoting the Laplace transform by ˜ it is easily shown (see below) that for the linear system (39) as well as for the system (40)-(40) we have a relation between $\tilde{u}(s)$ and $\tilde{r}_0(s,x)$ of the form

$$\tilde{r}_0(s,x) = G(s,x)\tilde{u}(s).$$

The function G is called the *transfer function* for the system in question, and the uniqueness of the Laplace transform immediately gives us the following result.

Corollary 3.1 *The system (36)-(37) is a realization of (38) if and only if the system (40)-(41) has the same transfer function as (39).*

We now go on to determine the transfer function for the system (39). In the formulas below subindex denotes translation, i.e. $g_x(y) = g(x+y)$. We denote Laplace transforms by ˜ or by $\mathcal{L}$. We have the following easy, but for our purposes fundamental, result.

Lemma 3.2 *The transfer function $G(s,x)$ of (39) is given by*

$$G(s,x) = \mathcal{L}\left[\sigma_x\right](s), \tag{42}$$

Proof. From (39) we have

$$r_0(t,x) = \int_0^t \sigma(x+t-s)u(s)ds = \left[\sigma_x \star u\right](t), \tag{43}$$

and thus

$$\tilde{r}_0(s,x) = \mathcal{L}\left[\sigma_x\right](s)\tilde{u}(s). \blacksquare$$

The main result now follows immediately.

Proposition 3.1

1. *The system*

$$dr_0(t,x) = \mathbf{F}r_0(t,x)dt + \sigma(x)dW(t), \quad r_0(0,x) = 0 \tag{44}$$

has a finite dimensional realization if and only if the volatility function σ can be written on the form

$$\sigma(x) = C_0 e^{Ax} B \tag{45}$$

2. *If σ has the form (45) then a concrete realization is given by*

$$dz(t) = Az(t)dt + BdW(t), \quad z(0) = 0. \tag{46}$$
$$r_0(t,x) = C(x)z(t), \tag{47}$$

with A, B as in (45), and with $C(x) = C_0 e^{Ax}$.

Proof. Suppose that (40)-(41) is a realization. It is easy to see that the transfer function $H(s,x)$ of (40)-(41) is given by

$$H(s,x) = C(x)\left[sI - A\right]^{-1} B. \tag{48}$$

so we must also have $G(s,x) = C(x)\left[sI - A\right]^{-1} B$. Thus for each x, $G(s,x)$ is a rational matrix function with a zero at infinity, and by Lemma 3.2 this also holds for $\mathcal{L}\left\{\sigma_x\right\}(s)$. By a standard result from systems theory we thus infer the existence of functions $A(x), B(x), C_0(x)$ such that

$$\sigma_x(t) = C_0(x)e^{A(x)t}B(x). \tag{49}$$

We thus obtain

$$\sigma(t) = \sigma_0(t) = C_0(0)e^{A(0)t}B(0), \tag{50}$$

which gives us (45) with $C_0 = C_0(0)$, $A = A(0)$, and $B = B(0)$.

Suppose on the other hand that σ has the form $\sigma(x) = C_0 e^{Ax} B$. Then we have $\sigma_x(t) = C_0 e^{Ax} e^{At} B$ and an easy calculation shows that

$$G(s,X) = \mathcal{L}\left\{\sigma_x\right\}(s) = C_0 e^{Ax}\left[sI - A\right]^{-1} B.$$

This, however, is precisely the transfer function of the system (40)-(41) with $C(x) = C_0 e^{Ax}$. ∎

There are a number of standard algorithms in the systems theoretic literature which constructs a realization, given knowledge of the transfer functions. When you try to find a realization of the r_0 system, it is thus generally easier to find a realization of the corresponding transfer function $G(s,x)$, rather than to factor $\sigma(x)$ directly as in (45). For that purpose the following observation, which is an immediate consequence of the preceding result, can be useful. It shows that we do not have to find a realization for the general expression $G(s,x)$. It is enough to apply the standard realization algorithms from linear systems theory to the transfer function $G(s,0)$.

Lemma 3.3 *Assume that the r_0 system has a finite realization. Assume furthermore that we have fund A, B and C such that $[A,B,C]$ is a realization of the transfer function $G(s,0)$, i.e.*

$$G(s,0) = C\,[sI - A]^{-1}\,B,$$

then a realization of r_0 is given by $\left[A, B, Ce^{Ax}\right]$. The forward rates $r(t,x)$ have then the representation

$$r(t,x) = \delta(t,x) + \int_0^t e^{A(t-s)}\sigma(x)dW(s), \tag{51}$$

where δ is given by (19).

Proof. The first assertion follows at once from Proposition 3.1. it remains to show is (51). But this follows immediately from (15) and Proposition 3.1.

Remark 3.1 Note that in order to use the lemma above, we must have a priori knowledge of the existence of a finite dimensional realisation. An easy test is given in the next section.

4 Minimal realizations

The purpose of this section is to determine the minimal dimension of a finite dimensional realization.

Definition 4.1 *The* **dimension** *of a realization $[A,B,C(x)]$ is defined as the dimension of the corresponding state space. A realization $[A,B,C(x)]$ is said to be* **minimal** *if there is no other realization with smaller dimension. The* **McMillan-degree**, *$\mathcal{M}$, of the forward rate system is defined as the dimension of a minimal realization.*

Since the forward rate system (21) in systems theoretic terms is observable (we observe the state directly), one conjectures immediately that the McMillan degree equals the dimension of the "reachable subspace" $\mathcal{R}$ of the system (39). In analogy with finite dimensional theory we furthermore expect $\mathcal{R}$ to be given by $\mathcal{R} = \text{span}\left[\sigma, \mathbf{F}\sigma, \mathbf{F}^2\sigma, \cdots\right]$, where as usual $\mathbf{F} = \frac{d}{dx}$.

Remark 4.1 The ideas above are of course quite standard in linear systems theory, and there also exist a fairly well developed theory for linear systems in a Hilbert or Banach space (see e.g. [8]). It is therefore tempting to try to attack our realization problem by abstract methods. The problem with such an approach is that the natural space of forward rate curves seems to be the topological vector space $C[0,\infty)$ with the topology of uniform convergence on compacts, and on this space the translation semigroup is not equicontinuous, thus making it hard to use the standard abstract theory.

For linear stochastic systems, Da Prato and Zabczyk have developed a deep theory for linear stochastic differential equations in a Hilbert space, and they characterize the support space of the solution of a linear Hilbert space valued SDE as the space $\mathcal{R}$ above. In the context of forward rates these results of [9] are referred to by Musiela in [19] who identifies the support of the forward rate process with the space $\mathcal{R}$. It is, however, not clear that the results from [9] can be transferred to our case, and the reason is that Da Prato & Zabczyk work within a Hilbert space framework whereas again the natural space of forward rate curves seems to be $C[0,\infty)$.

Rather than developing a full theory for systems in topological vector spaces, we have thus chosen to give a direct proof or the conjecture above.

Proposition 4.1 *Consider the volatility function*

$$\sigma = [\sigma_1, \cdots, \sigma_m]$$

as given. Then the McMillan degree, $\mathcal{M}$, is given by

$$\mathcal{M} = \dim \mathcal{R}, \tag{52}$$

with

$$\begin{aligned} \mathcal{R} &= \mathit{span}\left[\sigma, \mathbf{F}\sigma,\ \mathbf{F}^2\sigma,\ \cdots\right] & (53) \\ &= \mathit{span}\left[\mathbf{F}^k\sigma_i\ ;\ i = 1, \cdots, m.\ k = 0, 1, \cdots\right] & (54) \end{aligned}$$

Proof. It is obvious that $\dim \mathcal{R} < \infty$ if and only if σ satisfies a linear ODE with constant coefficients, so from ODE theory it follows that $\dim \mathcal{R} < \infty$ if and only if σ can be written on the form $\sigma(x) = Ce^{Ax}B$. Thus $\dim \mathcal{R} < \infty$ if and only if there exists a finite dimensional realization, so the proposition is proved for the case $\dim \mathcal{R} = \infty$.

Assume now that $\mathcal{M} = n$, and that $\left[A, B, Ce^{Ax}\right]$ is a finite dimensional minimal realization. In particular A is then an $n \times n$ matrix, $[A, C]$ is observable, and $[A, B]$ is reachable. Using Proposition 3.1 we can write

$$\sigma(x) = Ce^{Ax}B, \tag{55}$$

and we have

$$\mathbf{F}^k\sigma(x) = Ce^{Ax}A^kB, \quad k = 0, 1, \cdots \tag{56}$$

Now consider the linear mapping $\Lambda : R^n \to C[0, \infty)$ defined by

$$[\Lambda b](x) = Ce^{Ax}b.$$

This map is injective. To see this assume that $\Lambda b = 0$, i.e. $Ce^{Ax}b = 0$ for all x. Taking derivatives and setting $x = 0$ gives us the equation

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} b = 0,$$

so $b = 0$ since (A, C) is observable.

Furthermore (A, B) is reachable, so we know that

$$\text{span}[B, AB, A^2B, \cdots] = \text{span}[B, AB, A^2B, \cdots, A^{n-1}] = R^n,$$

where the span operation is interpreted as the linear hull of the column vectors of the partitioned matrices. From (55) and (56) we see that

$$\mathcal{R} = \text{span}\left[\sigma,\ \mathbf{F}\sigma,\ \mathbf{F}^2\sigma,\ \cdots\right] = \Lambda\Big(\ \text{span}[B, AB, A^2B, \cdots]\Big) = \text{Im}\Lambda,$$

and the injectivity of Λ gives us $\dim \mathcal{R} = n$. ∎

5 Economic interpretation of the state space

In many practical applications of linear systems theory, the state space for the minimal realization has no concrete (e.g. physical) interpretation. In our case, however, it turns out that the states of the *minimal* realization have a very simple economic interpretation. The states are, modulo an affine transformation, a minimal set of "benchmark" forward rates.

We will need the following technical result. For generality the result below is stated and proved for a general triple of matrices (A, B, C), whereas in our applications the matrix C always is a row vector.

Lemma 5.1 *Assume that (A, C) is observable and that (A, B) is reachable, where A is $n \times n$. Then, on any compact interval I, apart from a finite set of forbidden values, the matrices*

$$\begin{bmatrix} Ce^{Ax_1} \\ \vdots \\ Ce^{Ax_n} \end{bmatrix}$$

and

$$\left[e^{Ax_1}B, e^{Ax_2}B, \cdots, e^{Ax_n}B\right]$$

have rank n, for all distinct choices of $x_1, \cdots, x_n$ in I.

Proof. The two cases have exactly the same proofs, so we consider only the first matrix above. For each k with $1 \leq k \leq n$ we consider matrices of the form

$$H_k = \begin{bmatrix} Ce^{Ax_1} \\ \vdots \\ Ce^{Ax_k} \end{bmatrix}$$

and proceed by induction to show that we can choose $x_1, \cdots, x_k$ such that $\text{rank}[H_k] \geq k$.

The case $k = 1$ is clear since $C \neq 0$ and e^{Ax} is invertible for all choices of x. For any k with $1 \leq k \leq n-1$ we now assume that we have chosen $x_1, \cdots, x_k$ so that $\text{rank}[H_k] \geq k$. If $\text{rank}[H_k] > k$ the induction step is already finished, so we need only to consider the case when $\text{rank}[H_k] = k$. We now consider the time dependent matrix $H(t)$ defined by

$$H(t) = \begin{bmatrix} Ce^{Ax_1} \\ \vdots \\ Ce^{Ax_k} \\ Ce^{At} \end{bmatrix}$$

and we want to prove that we can choose a t such that $\text{rank}[H(t)] \geq k+1$. To this end we define the real valued function $f(t)$ by

$$f(t) = \sum_j [D_j(t)]^2$$

where the D_j is the determinant of a $(k+1) \times (k+1)$ submatrix of $H(t)$. We thus sum the squares of the determinants of all $(k+1) \times (k+1)$ minors. We now want to prove that, for almost all t, $\text{rank}[H(t)] \geq k+1$, so it is enough to prove that for almost all t we have $f(t) \neq 0$. Now, all D_j are real analytic functions, so $f(t)$ is real analytic. Thus: either it has only a finite number of zeroes on any compact, or it is identically zero. In the first case we are finished, so it only remains to show that we can not have the second case.

Assume that in fact $f(t) = 0$, for all $t \in R$. Using the induction hypothesis, this means that, for all t, all rows of the matrix $y(t) = Ce^{At}$ belongs to the k-dimensional linear hull M of the rows of

$$\begin{bmatrix} Ce^{Ax_1} \\ \vdots \\ Ce^{Ax_k} \end{bmatrix}$$

This also means that, for all t, the rows of all derivatives of $y(t)$ belongs to M, and in particular all the rows of the matrices

$$CA^j = \frac{d^j y}{dt^j}(0)$$

belong to M for $0 \leq j \leq n-1$. This however is impossible, since the dimension of M is k, whereas, by observability, we have

$$\text{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n. \blacksquare$$

We now proceed to the interpretation of the state space, and therefore assume that $[A, B, C]$ is a minimal realization of the forward rates as in (46)-(47). Let us choose a set of "benchmark " maturities $x_1, \cdots, x_n$. We use the notation $\bar{x} = (x_1, \cdots, x_n)$. Assume furthermore that the maturity vector $\bar{x}$ is chosen so that the matrix

$$T(\bar{x}) = \begin{bmatrix} Ce^{Ax_1} \\ \vdots \\ Ce^{Ax_n} \end{bmatrix} \tag{57}$$

is invertible. According to Lemma 5.1 this can "almost always" be done as long as the maturities are distinct. We use the notation

$$r_0(t,\bar{x}) = \begin{bmatrix} r_0(t,x_1) \\ \vdots \\ r_0(t,x_n) \end{bmatrix}$$

and corresponding interpretations for column vectors like $r(t,\bar{x})$, $\delta(t,\bar{x})$ etc. Since $T = T(\bar{x})$ is invertible we now have, from (47)

$$\begin{aligned} r_0(t,\bar{x}) &= Tz(t), && (58)\\ z(t) &= T^{-1}r_0(t,\bar{x}), && (59) \end{aligned}$$

and from (46) and (58) we obtain

$$\begin{aligned} dr_0(t,\bar{x}) &= Tdz = TAzdt + TBdW, \\ &= TAT^{-1}r_0(t,\bar{x})dt + TBdW. && (60) \end{aligned}$$

From (23) we have

$$r(t,x) = r_0(t,x) + \delta(t,x), \tag{61}$$

which gives us

$$dr(t,x) = dr_0(t,x) + \frac{\partial}{\partial t}\delta(t,x)dt. \tag{62}$$

Plugging (60) into (62), and using (61) as well as (19) we finally obtain the following result.

Proposition 5.1 *Assume that (46)-(47) is a minimal realization of the forward rates, and assume furthermore that a maturity vector $\bar{x} = (x_1, \cdots, x_n)$ is chosen as in Lemma 5.1. Then the following hold.*

- *With notation as above, the vector $r(t,\bar{x})$ of benchmark forward rates has the dynamics*

$$\begin{aligned} dr(t,\bar{x}) &= \left[T(\bar{x})AT^{-1}(\bar{x})r(t,\bar{x}) + \Psi(t,\bar{x})\right]dt + T(\bar{x})BdW(t), && (63)\\ r(0,\bar{x}) &= r^\star(0,\bar{x}), && (64) \end{aligned}$$

where the deterministic function Ψ is given by

$$\Psi(t,\bar{x}) = \frac{\partial r^\star}{\partial x}(0, t\bar{e}+\bar{x}) + D(t\bar{e}+\bar{x}) - T(\bar{x})AT^{-1}(\bar{x})\delta(t,\bar{x}) \tag{65}$$

Here $\bar{e} \in R^n$ denotes the vector with unit components, i.e.

$$\bar{e} = \begin{bmatrix} 1, \\ 1, \\ \vdots \\ 1 \end{bmatrix}$$

- *The system of benchmark forward rates determine the entire forward rate process according to the formula*

$$r(t,x) = Ce^{Ax}T^{-1}(\bar{x})r(t,\bar{x}) - Ce^{Ax}T^{-1}(\bar{x})\delta(t,\bar{x}) + \delta(t,x). \tag{66}$$

Proof. The only thing left to prove is (66). We have

$$\begin{aligned} r(t,x) &= r_0(t,x) + \delta(t,x) = Ce^{Ax}z(t) + \delta(t,x) \\ &= Ce^{Ax}T^{-1}(\bar{x})r_0(t,\bar{x}) + \delta(t,x) \\ &= Ce^{Ax}T^{-1}(\bar{x})\left[r(t,\bar{x}) - \delta(t,\bar{x})\right] + \delta(t,x), \end{aligned}$$

which gives us (66). ∎

The conclusion is thus that the state variables of a *minimal* realization can be interpreted as an affine transformation of a vector of benchmark forward rates. Note that we can easily handle multiple roots of the matrix A, and that the input noise can actually have dimension smaller than the dimension of A. Neither of these cases are usually treated by the standard theory of multifactor models.

According to Lemma 5.1 we see that the set of benchmark maturities can be chosen almost arbitrarily, and we also see that the SDE (63) for the benchmark forward rates resembles the Hull-White model (the extension of the Vasiček model) in the sense that we have an affine system where the function $\Psi(t,\bar{x})$ is chosen to obtain a perfect fit between the theoretical initial forward rates and the observed initial forward curve.

For completeness sake we end by giving the formula which connects bond prices to the set of benchmark forward rates. Since we are in a linear enviroment we of course have an affine term structure, so the formula below could as well be obtained from [10]. We again stress the fact that the point of the present paper is not the affine term structure, but the existence and constructability of a finite dimensional state space representation given only a specification of the volatility structure.

Proposition 5.2 *With the assumptions above, bond prices $p(t,x)$ are given by*

$$p(t,x) = \frac{p^\star(0,t+x)}{p^\star(0,t)} \exp\left\{\Gamma(t,x,\bar{x}) - \Phi(x,\bar{x})r(t,\bar{x})\right\},$$

where

$$\Phi(x,\bar{x}) = C\left(\int_0^x e^{As}ds\right)T^{-1}(\bar{x}), \tag{67}$$

$$\Gamma(t,x,\bar{x}) = C\left(\int_0^x e^{As}ds\right)T^{-1}(\bar{x})\delta(t,\bar{x}) \tag{68}$$

$$- \frac{1}{2}\int_0^x \left\{\|\mathbf{H}\sigma(t+u)\|^2 - \|\mathbf{H}\sigma(u)\|^2\right\} du \tag{69}$$

Proof. Use the formula $p(t,x) = \exp\left\{-\int_0^x r(t,s)ds\right\}$, (66) and (20) . ∎

6 Examples

In this section we will give some simple illustrations of the theory. Note again the handling of multiple roots of the matrix A, and the fact that the input noise can have dimension smaller than the dimension of A.

Example 6.1 $\sigma(\mathbf{x}) = \sigma\mathbf{e}^{-\mathbf{ax}}$
We consider a model driven by a one dimensional Wiener process, having the forward rate volatility structure

$$\sigma(x) = \sigma e^{-ax},$$

where σ in the right hand side denotes a constant. (The reader will probably recognize this example as the Hull-White model.) We start by determining the McMillan degree $\mathcal{M}$, and by Proposition 4.1 we have

$$\mathcal{M} = dim\mathcal{R},$$

where the reachable subspace $\mathcal{R}$ is given by

$$\mathcal{R} = span\left[\frac{d^k}{dx^k}\sigma e^{-ax}\ ;\ k \geq 0\right].$$

It is obvious that $\mathcal{R}$ is one dimensional, and that it is spanned by the single function e^{-ax}. Thus the McMillan degree is given by $\mathcal{M} = 1$. We now want to apply Proposition 3.1 to find a realization, so we must factor the volatility function. In this case this is easy, since we have the trivial factorization $\sigma(x) = 1 \cdot e^{-ax} \cdot \sigma$. In the notation of Proposition 3.1 we thus have

$$\begin{aligned} C_0 &= 1, \\ A &= -a, \\ B &= \sigma. \end{aligned}$$

A realization of the forward rates is thus given by

$$\begin{aligned} dz(t) &= -az(t)dt + \sigma dW(t), \\ r_0(t,x) &= e^{-ax}z(t), \\ r(t,x) &= r_0(t,x) + \delta(t,x), \end{aligned}$$

and since the state space in this realization is of dimension one, the realization is minimal. We see that if $a > 0$ then the system is asymptotically stable.

We now go on to the interpretation of the state space, and since $\mathcal{M} = 1$ we can choose a single benchmark maturity. The canonical choice is of course $x_1 = 0$, i.e. we choose the instantaneous short rate $R(t)$ as the state variable. In the notation of Proposition 5.1 we then have

$$\begin{aligned} T(\bar{x}) &= 1, \\ r(t,\bar{x}) &= R(t), \end{aligned}$$

and we get rate dynamics

$$dR(t) = \{\Psi(t,0) - aR(t)\}\, dt + \sigma dW(t).$$

Thus we see that we have indeed the Hull-White extension of the Vasiček model. Note however that we do not have to choose the benchmark maturity as $x_1 = 0$. We can in fact choose *any* fixed maturity, x_1 and then use the corresponding forward rate as benchmark. This will give us the dynamics

$$dr(t,x_1) = \{\Psi(t,x_1) - ar(t,x_1)\}\, dt + e^{-ax_1}dW(t),$$

and now the entire forward rate curve will be determined by the x_1-rate according to formula (66).

Example 6.2 $\sigma(\mathbf{x}) = \mathbf{x}\mathbf{e}^{-\mathbf{a}\mathbf{x}}$
In this example we still have a single driving Wiener process, but the volatility function is now "hump-shaped".

By taking derivatives of $\sigma(x)$ we immediately see, from Proposition 4.1 that $\mathcal{R}$ is given by

$$\mathcal{R} = span\left[xe^{-ax},\ e^{-ax}\right],$$

so in this case $\mathcal{M} = 2$, and we have a two dimensional minimal state space. In order to obtain a realization we compute the transfer function $G(s, x)$, which is given by Lemma 3.2 as

$$G(s, x) = \mathcal{L}\left[(x + \cdot)e^{-a(x+\cdot)}\right](s).$$

An easy calculation gives us

$$G(s, x) = \frac{e^{-ax}}{(a+s)^2} + \frac{xe^{-ax}}{(a+s)} = \frac{sxe^{-ax} + (1+ax)e^{-ax}}{(a+s)^2},$$

and we now look for a realization of this transfer function (for a fixed x). The obvious thing to do is to use the standard controllable realization (see [6]), and we obtain

$$\begin{aligned} C(x) &= \left[xe^{-ax}, (1+ax)e^{-ax}\right] \\ A &= \begin{bmatrix} -2a & -a^2 \\ 1 & 0 \end{bmatrix}, \\ B &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{aligned}$$

Since $\mathcal{M} = 2$ and this realization is two-dimensional we have a minimal realization, given by

$$\begin{aligned} dz_1(t) &= -2az_1(t)dt - a^2 z_2(t)dt + dW(t), \\ dz_2(t) &= z_1(t)dt \\ r_0(t, x) &= xe^{-ax}z_1(t) + (1+ax)e^{-ax}z_2(t) \\ r(t, x) &= r_0(t, x) + \delta(t, x). \end{aligned}$$

We have a double eigenvalue of the system matrix A at $\lambda_1 = -a$, so if $a > 0$ the system is asymptotically stable.

Example 6.3 $\sigma(\mathbf{x}) = [\sigma_1 \mathbf{x}, \sigma_2 \mathbf{e}^{-\mathbf{ax}}]$
We have a two-dimensional driving Wiener process but, as we shall see, the McMillan degree is not equal to two. Taking derivatives it is easy to see that in fact

$$\mathcal{R} = span\left[1, x, e^{-ax}\right].$$

Thus we have $\mathcal{M} = 3$. We may now use Lemma 3.3, and conclude that in order to find a realization for the transfer function $G(s, x)$ it is enough to find one for $G(s, 0)$. We have

$$G(s,0) = \left[\frac{1}{s^2}, \frac{1}{s+a}\right].$$

For this transfer function it is natural to use the standard observable realization. The result is as follows

$$\begin{aligned} C &= [1, 0, 0] \\ A &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -a \end{bmatrix}, \\ B &= \begin{bmatrix} 0 & 1 \\ 1 & -a \\ 0 & a^2 \end{bmatrix}. \end{aligned}$$

We see that we have two stable modes which are not asymptotically stable, both corresponding to the pole at $s = 0$, and one asymptotically stable mode corresponding to the pole at $s = -a$.

7 Calibration and actual computations

In this section we discuss how to actually compute the prices $p(t, x)$, and how to construct the matrices A, B, C from market data. There is no loss of generality in assuming that A, B, C are already the matrices relative to the representation in terms of the benchmarks, so that $T(\bar{x}) \equiv I$ and $r(t, x)$ is given by (51). Then, to compute the prices, we need to compute δ and Δ in (19) and (20).

If the matrix A is invertible, these correction terms can be easily computed by solving a system of linear equations. To see this we need a simple lemma.

Lemma 7.1 *Let $P(t) = \int_0^t e^{As} Q e^{A's} ds$. Then $P(t)$ is the solution to the Lyapunov equation:*

$$AP(t) + P(t)A' + Q - e^{At} Q e^{A't} = 0 \tag{70}$$

Proof. We can write $e^{At} Q e^{A't} - Q$ as the integral of its derivative, i.e.:

$$e^{At} Q e^{A't} - Q = \int_0^t (A e^{As} Q e^{A's} + e^{As} Q e^{A's} A') ds = AP(t) + P(t)A'$$

from which the conclusion follows. ∎

Equation (70) is, for each fixed time t, a standard Lyapunov equation, and since algorithms to solve these equations are very fast (they are linear), $\delta(t,x)$ can in fact be computed online.

The computation of $\delta(t,x)$ then goes as follows:

$$\delta(t,x) = r^*(0,t+x) + \int_0^t e^{\mathbf{F}(t-\tau)} D(x) d\tau$$

where

$$\begin{aligned} D(x) &= \sigma(x) \int_0^t \sigma'(s) ds = Ce^{Ax} BB' \int_0^x e^{A's} ds C' \\ &= Ce^{Ax} BB'(A')^{-1} [e^{A'x} - I] C' \end{aligned}$$

and so

$$(e^{\mathbf{F}(t-\tau)} D)(x) = Ce^{A(t-\tau+x)} BB'(A')^{-1} [e^{A'(t-\tau+x)} - I] C'$$

Integrating between 0 and t and adding $r^*(0,x)$ we get:

$$\begin{aligned} \delta(t,x) &= r^*(0,x) + CA^{-1} e^{Ax} [I - e^{At}] BB'(A')^{-1} C' \qquad (71) \\ &+ Ce^{A(x+t)} \left[\int_0^t e^{-A\tau} BB' e^{-A'\tau} d\tau \right] e^{A'(x+t)} (A')^{-1} C' \end{aligned}$$

Setting $P(t) := \int_0^t e^{-A\tau} BB' e^{-A'\tau} d\tau$ and integrating once more with respect to x, we obtain

$$\begin{aligned} \Delta(t,x) &:= y^*(0,t+x) - y^*(0,t) + \qquad (72) \\ &+ CA^{-2} [e^{Ax} - I][I - e^{At}] BB'(A')^{-1} C' \qquad (73) \\ &+ Ce^{At} \left[\int_0^x e^{As} P(t) e^{A's} ds \right] e^{A't} (A')^{-1} C' \end{aligned}$$

Notice that the integral

$$\Pi(t) := \int_0^x e^{As} P(t) e^{A's} ds$$

can also be easily computed as a solution to a Lyapunov equation of the form (70).

How do we calibrate the model on real data, i.e. how do we actually determine A, B, C? Two approaches are possible.

1. Statistical analysis of past data to estimate the forward rates volatility: this is a standard procedure. Nevertheless, due to the particular structure of the volatility, very effective system identification methods can be used: the literature on this subject is quite extensive and there is a vast library of computer algorithms to solve this kind of problems (see [18] or [21]).The computational load is very low.

2. Implied volatility based on some derivatives market price. In other words, given the market prices of n derivatives $D_1^*, ..., D_n^*$, find matrices A, B, C such that, for each i, the theoretical derivative price D_i coincides with the market price D_i^*. This is, as usual, computationally more demanding. Nevertheless, since all quantities can be computed, and there exist good atlases for equivalence classes of matrices A, B, C (we refer to the appendix for the concept of equivalence), we can formulate the problem as a minimization problem where the value function is, for example

$$V(A,B,C) := \sum_{i=1}^{n} |D_i(A,B,C) - D_i^*|^2$$

then the equations for the critical points of this value function (again referred to the equivalence classes) can be derived explicitly and gradient algorithms can be used to solve them numerically. Or, more simply, a direct minimization algorithm can be implemented directly on the value function (without computing the gradient).

The details of these approaches will be discussed elsewhere. But, to illustrate the relative ease of computation, we calculate the value $C(t, x_1, x_2, K)$ at time t of a european call option with strike price K and time to maturity x_1 on a bond with time to maturity x_2 (obviously $x_1 < x_2$).

It is well known (see [12]) that

$$C(t, x_1, x_2, K) = q(t, x_2)N(d_2) - Kq(t, x_1)N(d_1) \tag{74}$$

where

$$d_1 := \frac{\ln \frac{q(t,x_2)}{Kq(t,x_1)} + \frac{1}{2}\Sigma_{x_1,x_2}(x_1)}{\sqrt{\Sigma_{x_1,x_2}(x_1)}}$$

$$d_2 := d_1 - \sqrt{\Sigma_{x_1,x_2}(x_1)}$$

$$Z(s) := \frac{q(t+s, x_2 - s)}{q(t+s, x_1 - s)}$$

and $\Sigma_{x_1,x_2}(s)$ is the variance of $\ln Z(s)$, with respect to the forward measure Q^{x_1} with numeraire $q(t+s, x_1-s)$ for which the process $Z(s)$ is a martingale. In view this fact, of (51), and the fact that a change of measure preserves the volatility, the process $Z(t)$ satisfies

$$dZ(s) = Z(s) \int_{x_1-s}^{x_2-s} \sigma(u) du dW^{x_1}(s)$$

where W^{x_1} denotes the Brownian Motion with respect to the new forward measure Q^{x_1}. Therefore, we can conclude that $\ln Z(s)$ has variance

$$\Sigma_{x_1,x_2}(s) = \int_0^s \left[\int_{x_1-\tau}^{x_2-\tau} \sigma(u)du\right]\left[\int_{x_1-\tau}^{x_2-\tau} \sigma(u)'du\right] d\tau$$
$$= \int_0^s \left[\int_{x_1-\tau}^{x_2-\tau} Ce^{Au}Bdu\right]\left[\int_{x_1-\tau}^{x_2-\tau} B'e^{A'u}C'du\right] d\tau$$
$$= \int_0^s CA^{-1}[e^{A(x_2-\tau)} - e^{A(x_1-\tau)}]BB'[e^{A'(x_2-\tau)} - e^{A'(x_1-\tau)}](A')^{-1}C'd\tau$$
$$= CA^{-1}[e^{Ax_2} - e^{Ax_1}]\int_0^s e^{-A\tau}BB'e^{A'\tau}d\tau[e^{A'(x_2-\tau)} - e^{A'(x_1-\tau)}](A')^{-1}C'$$
$$= CA^{-1}[e^{Ax_2} - e^{Ax_1}]P(-s)[e^{A'(x_2-\tau)} - e^{A'(x_1-\tau)}](A')^{-1}C'$$

where $P(-s)$ is computed as in Lemma 7.1. In conclusion, the option value (74) can be computed exactly. Observe that, as usual, when $t = 0$, the only term to compute is $\Sigma_{x_1,x_2}(x_1)$ and this computation is, as we have just shown, fairly straightforward.

8 Point Process Inputs

In this section we will extend our results by allowing also for a marked point process as driving noise in the forward rate model. The ideas are, modulo some minor technicalities, exactly the same a for the purely Wiener driven case, so we will give the results and only sketch the arguments.

The stochastic basis is thus assumed to carry, beside the usual d-dimensional Wiener process W, a marked point process $\mu(dt, dx)$ on a measurable Lusin mark space $(E, \mathcal{E})$ with compensator $\nu(dt, dy)$. We assume that $\nu([0, t] \times E) < \infty$ Q-a.s. for all finite t, i.e. μ is a multivariate point process in the terminology of [16].

Assumption 8.1 *We assume that, under a martingale measure Q, the forward rate dynamics are of the form*

$$dr(t, x) = \alpha(t, x)dt + \sigma(x)dW(t) + \int_E \eta(x, y)\mu(dt, dy), \tag{75}$$

where $\sigma(x)$ and $\eta(x, y)$ are deterministic functions. We also assume that the compensator ν has a predictable deterministic time invariant intensity measure, i.e. $\nu(dt, dy) = \lambda(dy)dt$.

We now have the following result from [2], [3], which generalizes the HJM drift condition to the point process case.

Proposition 8.1 *Under the martingale measure Q the following relation hold*

$$\alpha(t,x) = \frac{\partial r}{\partial x}(t,x) + \sigma(x)\int_0^x \sigma(s)ds - \int_E \eta(x,y)e^{N(x,y)}\lambda(dy), \tag{76}$$

where

$$N(x,y) = -\int_0^x \eta(s,y)ds. \tag{77}$$

Again we view all functions of x as vectors in $C[0,\infty)$, and, suppressing the x-dependence, we then have our basic system

$$\begin{aligned} dr(t) &= \{\mathbf{F}r(t) + D\}\,dt + \sigma dW(t) + \int_E \eta(y)\mu(dt,dy), && (78)\\ r(0) &= r^\star(0). && (79)\\ y(t) &= \mathbf{H}r(t), && (80)\\ p(t) &= e^{y(t)}. && (81) \end{aligned}$$

Here the linear operators $\mathbf{F}$ and $\mathbf{H}$ are defined as before whereas the function D is defined by

$$D(x) = \sigma(x)\int_0^x \sigma(s)ds - \int_E \eta(x,y)e^{N(x,y)}\lambda(dy) \tag{82}$$

i.e.

$$D = \mathbf{F}\left\{\frac{1}{2}\|\mathbf{H}\sigma\|^2 + \int_E e^{N(\cdot,y)}\lambda(dy)\right\}. \tag{83}$$

As before we introduce the process r_0, and write

$$\begin{aligned} dr_0(t,x) &= \mathbf{F}r_0(t,x)dt + \sigma(x)dW(t) + \int_E \eta(x,y)\mu(dt,dy), && (84)\\ r_0(0,x) &= 0 && (85)\\ y_0(t,x) &= \mathbf{H}r_0(t,x) && (86)\\ r(t,x) &= r_0(t,x) + \delta(t,x) && (87)\\ y(t,x) &= y_0(t,x) + \Delta(t,x) && (88) \end{aligned}$$

where

$$\begin{aligned} \delta(t,x) &= r^\star(x+t) + \int_0^t D(x+t-s)ds \\ &= r^\star(x+t) + \frac{1}{2}\left(\|\mathbf{H}\sigma(x+t)\|^2 - \|\mathbf{H}\sigma(x)\|^2\right) \\ &+ \int_E \left\{e^{N(x+t,y)} - e^{N(x,y)}\right\}\lambda(dy), && (89)\\ \Delta(t,x) &= \int_0^x \delta(t,u)du, \end{aligned}$$

$$
\begin{aligned}
&= q^\star(0,t) - q^\star(0,t+x) \\
&+ \frac{1}{2}\int_0^x \left\{\|\mathbf{H}\sigma(t+u)\|^2 - \|\mathbf{H}\sigma(u)\|^2\right\} du \\
&+ \int_0^x \int_E \left\{e^{N(t+u,y)} - e^{N(u,y)}\right\} \lambda(dy)
\end{aligned} \tag{90}
$$

The main problem is again to find out when and how the input-output map generated by the r_0-process can be realized by a finite dimensional system.

Definition 8.1 *A list $[A, B, K, C]$ where where A is an $n \times n$-matrix , B is $n \times d$-matrix $K : E \to R^n$ is a column-vector function, and $C : R \to R^n$ is a row-vector function, is called an n-dimensional* **realization** *of r_0, if r_0 has the representation*

$$
dz(t) = Az(t)dt + BdW(t) + \int_E K(y)\mu(dt, dy), \tag{91}
$$

$$
r_0(t, x) = C(x)z(t). \tag{92}
$$

As before we study the existence of a finite dimensional realization for r_0 by comparing the transfer function for the deterministic system

$$
dr_0(t, x) = \mathbf{F}r_0(t, x)dt + Bu(t)dt + \int_E \eta(x, y)m(dt, dy). \tag{93}
$$

with the transfer function for the system

$$
dz(t) = Az(t)dt + Bu(t) + \int_E K(y)m(dt, dy), \tag{94}
$$

$$
r_0(t, x) = C(x)z(t). \tag{95}
$$

For these control systems we have deterministic inputs of two kinds: u, which is an arbitrary continuous d-dimensional column vector function, and m, which is an arbitrary point process trajectory (an "E-valued impulse control").

For both these systems the transfer function will be of the form

$$
G(s, x) = [G_1(s, x), G_2(s, x, \cdot)],
$$

i.e.

$$
\tilde{r}_0(s, x) = G_1(s, x)\tilde{u}(s) + \int_E G_2(s, x, y)\tilde{m}(s, dy)
$$

where ˜ as before denotes the Laplace transform in the t-variable. A straightforward calculation gives us the following results, where $\mathcal{L}$ denotes the Laplace transform, and x as subindex denotes translation.

Proposition 8.2

1. *The transfer function for (93) is given by*

$$G(s,x) = \left[\mathcal{L}\left[\sigma_x\right](s), \mathcal{L}\left[\eta_x\right](s,\cdot)\right], \tag{96}$$

i.e.

$$\begin{aligned} G_1(s,x) &= \int_0^\infty \sigma(t+x)e^{-st}dt, \\ G_2(s,x,y) &= \int_0^\infty \eta(t+x,y)e^{-st}dt. \end{aligned}$$

2. *The transfer function for (94)-(95) is given by*

$$G_1(s,x) = C(x)\left[sI - A\right]^{-1} B, \tag{97}$$

$$G_2(s,x,y) = C(x)\left[sI - A\right]^{-1} K(y). \tag{98}$$

We thus see that (94)-(95) is a realization of the r_0-process if and only if we have

$$\begin{aligned} \mathcal{L}\left[\sigma_x\right](s) &= C(x)\left[sI - A\right]^{-1} B, \\ \mathcal{L}\left[\eta_x\right](s,y) &= C(x)\left[sI - A\right]^{-1} K(y), \end{aligned}$$

and an argument along the lines of the pure Wiener case gives us the following basic result.

Proposition 8.3

1. *There exists a finite dimensional realization of the r_0-process if and only if the volatility functions can be factored as*

$$\sigma(x) = Ce^{Ax}B, \tag{99}$$

$$\eta(x,y) = Ce^{Ax}K(y). \tag{100}$$

2. *If σ and η have the forms above, then a concrete realization is given by $[A, B, K(y), C(x)]$, with A, B, and K as in (99)-(100), and with $C(x) = Ce^{Ax}$.*

Using this result it is easy to show the following result for the McMillan degree.

Proposition 8.4 *The McMillan degree $\mathcal{M}$ for the forward rate system is given by*

$$\mathcal{M} = span\left[\mathbf{F}^k\sigma(\cdot), \mathbf{F}^k\eta(\cdot,y); k \geq 0, y \in E\right]. \tag{101}$$

As in Section 5 we can interpret the states in a minimal realization as a set of benchmark forward rates. Using Lemma 5.1 the generalization of Propositions 5.1 and 5.2 is as follows.

Proposition 8.5 *Assume that* $\left[A, B, K(y), Ce^{Ax}\right]$ *is a minimal realization of the forward rates. Assume furthermore that a maturity vector* $\bar{x} = (x_1, \cdots, x_n)$ *is chosen as in Lemma 5.1. Then, with* $T = T(\bar{x})$ *as in (57), the following hold.*

- *The vector* $r(t, \bar{x})$ *of benchmark forward rates has the dynamics*

$$\begin{aligned} dr(t, \bar{x}) &= \left[TAT^{-1}r(t, \bar{x}) + \Psi(t, \bar{x})\right] dt + TBdW(t) & (102) \\ &+ \int_E TK(y)m(dt, dy), & (103) \\ r(0, \bar{x}) &= r^{\star}(0, \bar{x}), & (104) \end{aligned}$$

  where the deterministic function Ψ *is given by*

$$\Psi(t, \bar{x}) = \frac{\partial r^{\star}}{\partial x}(0, t\bar{e} + \bar{x}) - D(t\bar{e} + \bar{x}) - T(\bar{x})AT^{-1}(\bar{x})\delta(t, \bar{x}) \qquad (105)$$

- *The system of benchmark forward rates determine the entire forward rate process according to the formula*

$$r(t, x) = Ce^{Ax}T^{-1}r(t, \bar{x}) - Ce^{Ax}T^{-1}\delta(t, \bar{x}) + \delta(t, x). \qquad (106)$$

- *Bond prices* $p(t, x)$ *are given by*

$$p(t, x) = \frac{p^{\star}(0, t + x)}{p^{\star}(0, t)} \exp\left\{\Gamma(t, x, \bar{x}) - \Phi(x, \bar{x})r(t, \bar{x})\right\}, \qquad (107)$$

  where

$$\begin{aligned} \Phi(x, \bar{x}) &= C\left(\int_0^x e^{As}ds\right)T^{-1}, & (108) \\ \Gamma(t, x, \bar{x}) &= C\left(\int_0^x e^{As}ds\right)T^{-1}\delta(t, \bar{x}) \\ &- \frac{1}{2}\int_0^x \left\{\|\mathbf{H}\sigma(t+u)\|^2 - \|\mathbf{H}\sigma(u)\|^2\right\} du \\ &- \int_0^x \int_E \left\{e^{N(t+u,y)} - e^{N(u,y)}\right\} \lambda(dy)du & (109) \end{aligned}$$

9 Appendix on Linear Systems

In this appendix we give a brief review of some basic concepts and results in linear systems theory. For details the reader is referred to [6] or almost any textbook on the subject.

We consider an input-output system of the form

$$\frac{dz}{dt}(t) = Az(t) + Bu(t), \quad (110)$$

$$y(t) = Cz(t). \quad (111)$$

Here A, B and C are matrices of dimension $n \times n$, $n \times d$, and $k \times n$ respectively. The interpretation is that we feed the d-dimensional *input signal* u into the system. We can not observe the *state* z directly, but we can observe the k-dimensional *output signal* y. We will denote such a system by $[A, B, C]$. Sometimes we will have no input, and then the system will be denoted by by $[A, C]$. At other times we will disregard the output, and the system will be denoted by $[A, B]$. Given a starting point $z(0) = z_0$ and an input signal u, $z(t; x_0, u)$ will denote the value of z at time t, with the corresponding interpretation for $y(t; x_0, u)$ We note that we have

$$z(t; z_0, u) = e^{At} z_0 + \int_0^t e^{A(t-s)} Bu(s) ds, \quad (112)$$

$$y(t; z_0, u) = Ce^{At} z_0 + \int_0^t Ce^{A(t-s)} Bu(s) ds. \quad (113)$$

9.1 Reachability and Observability

If $\xi = z(t; x_0, u)$ we say that we have *reached* the point ξ by starting at z_0 and using the control u.

Definition 9.1

- *The* **reachable subspace**, *$\mathcal{R}$ is the set of points in R^n which, for some choice of input signal u and some choice of time t, can be reached starting from the origin $z(0) = 0$.*
- *If all points can be reached, i.e. if $\mathcal{R} = R^n$, then the system $[A, B]$ is said to be* **reachable**.

The basic result concerning reachability follows fairly easy from (112).

Proposition 9.1

- *The reachable subspace is given by*

$$\mathcal{R} = span\left[B, AB, A^2B, \cdots, A^{n-1}B\right], \tag{114}$$

where span *denotes the linear hull of the column vectors.*

- *The system* $[A, B]$ *is reachable if and only if*

$$span\left[B, AB, A^2B, \cdots, A^{n-1}B\right] = R^n. \tag{115}$$

It should be noted that the "basic" result is that

$$\mathcal{R} = span\left[B, AB, A^2B, \cdots\right].$$

It is then the finite dimensionality of the system, together with the Cayley-Hamilton Theorem, which gives us the relation (114).

The dual concept of reachability is observability. Consider the system $[A, C]$, and consider the mapping $\Gamma : R^n \to C[0, \infty)$ taking a starting point z_0 into the corresponding output trajectory y, i.e.

$$[\Gamma z](t) = y(t; z).$$

Definition 9.2

- *The* **silent subspace** $\mathcal{S}$ *of the system* $[A, C]$ *is defined as*

$$\mathcal{S} = ker\Gamma \tag{116}$$

- *The system* $[A, C]$ *is said to be* **observable** *if* $\mathcal{S} = \{0\}$.

The silent subspace is thus the set of points which will give rise to the zero output signal. If a system is observable it is possible to determine the starting point z_0 from observation of the output signal. The basic result is the following.

Proposition 9.2 *The system* $[A, C]$ *is observable if and only if*

$$ker\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \{0\}.$$

i.e. if and only if

$$rank\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n.$$

9.2 Realization Theory

Consider the systems

$$(\Sigma)\begin{cases} \frac{dz}{dt}(t) = Az(t) + Bu(t), \\ y(t) = Cz(t). \end{cases}$$

It is easily seen that, if we set, for example, $z(0) = 0$, for each input $u(t)$ the function

$$y(t) = \int_0^t Ce^{A(t-s)}Bu(s)ds$$

is uniquely determined. Typically $u(\cdot)$ is taken in $\mathcal{C}[0,\infty)$ (or $L^\infty[0,\infty)$). Then the output $y(t)$ is also a continuous function (or an essentially bounded function if the system is stable). Thus we can define the *impulse response* $\Phi(u) := e^{Au}B$ and a map $\mathcal{I}_\Phi$

$$\begin{aligned} \mathcal{I}_\Phi : \qquad & \mathcal{C}[0,\infty) \mapsto \mathcal{C}[0,\infty) \\ & u(t) \mapsto \int_0^t \Phi(t-s)u(s)ds \end{aligned}$$

The map $\mathcal{I}_\Phi$ is called *input-output map* of the system (Σ) (The same concept can be defined for bounded functions).

Conversely, we have the following:

Definition 9.3 *Given an input-output map $\mathcal{I}_\Phi$, we say that a system (Σ) is a* realization *of $\mathcal{I}_\Phi$ if the the input output map of (Σ) is $\mathcal{I}_\Phi$. Two systems (Σ) and (Σ') are said to be* equivalent *if they have the same input-output map.*

Although the input-output map is a very powerful concept, it is not immediate to represent it or to construct a realization. It turns out that, with our assumptions, it is equivalent and much simpler to work in the frequency domain with the Laplace transform of the impulse response. With ˜ denoting the Laplace transform, we can set

$$G(s) := \tilde{\Phi}(s)$$

Definition 9.4 *The matrix function $G(s)$ defined above is called the* **transfer function** *for the input-output map $\mathcal{I}_\Phi$.*

In view of the invertibility of the Laplace transform, an input-output map is uniquely determined by its transfer function. Thus, to describe or realize an input-output map we can use its transfer function. Now, it is immediately seen that if $\Phi = Ce^{Au}B$, then

$$G(s) = C[sI - A]^{-1}B$$

Thus it is quite obvious that different systems may have the same transfer function and that also this relation is also an equivalence. Clearly these two equivalences must coincide:

Lemma 9.1 *Two systems are equivalent if they have the same transfer function.*

From the Cramer rule it follows that if G is a transfer function of a system (Σ), then it is a rational matrix function, which is strictly proper, i.e. the degree of each component is ≤ -1. Rational matrix functions are extremely simple to describe in terms of their coefficients. The interesting point is that we can always go back from a transfer function to a system:

Proposition 9.3 *Let G be any $k \times m$-dimensional strictly proper rational matrix function. Then there exist matrices A, B and C such that $G(s) = C\,[sI - A]^{-1}\,B$.*

We say that the system $[A, B, C]$ is a realization of the transfer function G. Although, as we said, the realization of a transfer function is far from unique, there exist some properties which can be imposed on the realization to turn it into a useful object.

Definition 9.5

- *The* **dimension** *n of a realization $[A, B, C]$ is the dimension of the matrix A.*
- *A realization is* **minimal** *if there exist no realization with smaller dimension.*

Clearly two minimal realizations must have the same degree. There is a simple way to characterize a minimal realization and to relate two different minimal realizations.

Proposition 9.4 *Let transfer function G be given, and assume that $[A, B, C]$ is a realization of G. Then*

- *$[A, B, C]$ is minimal if and only if $[A, B]$ is reachable and $[A, C]$ is observable.*
- *Let $[A, B, C]$ and $[A', B', C']$ be minimal realizations of the same transfer function G. Then there exists an invertible transformation T such that*

$$A' = TAT^{-1} \qquad B' = TB \qquad C' = CT^{-1}$$

The other immediate result which we use in this paper is the following corollary.

Corollary 9.1 *Let G be any $k \times m$-dimensional strictly proper rational matrix function. Then there exist matrices A, B and C such that $\mathcal{L}^{-1}[G](s) = Ce^{At}B$.*

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