Diversified Portfolios in Continuous Time *

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Published in
European Finance Review
Vol. **1**, 361-387. (1998)

Abstract

We study a financial market containing an infinite number of assets, where each asset price is driven by an idiosyncratic random source as well as by a systematic noise term. Introducing "asymptotic assets" which correspond to certain infinitely well diversified portfolios we study absence of (asymptotic) arbitrage, and in this context we obtain continuous time extensions of atemporal APT results. We also study completeness and derivative pricing, showing that the possibility of forming infinitely well diversified portfolios has the property of completing the market. It also turns out that models where the all risk is of diffusion type are qualitatively quite different from models where one risk is of diffusion type and the other is of Poisson type. We also present a simple martingale based theory for absence of asymptotic arbitrage.

*The authors would like to thank David Lando and two anonymous referees for a number of very helpful comments and suggestions.

Key words: Large economies, diversifiable risk, APT, asymptotic arbitrage, completeness, martingales.

JEL Classification: G12, G13,

1 Introduction

The main object of study in this paper is a financial market containing a large number of traded assets. Each asset price is assumed to be driven by a systematic random source as well as by an idiosyncratic source of randomness, and we will study certain well diversified self-financing portfolios in this market. Our model is thus a continuous time extension of the classical arbitrage pricing theory (APT) models studied in Ross (1976), Hubermann (1982) and others. For other continuous time APT models see Chamberlein (1988), Reisman (1992) and Back (1988). More recent studies are those of Kabanov and Kramkov (1994,1996). We will sometimes compare our findings with the classical CAPM results. For the continuous time CAPM, see Merton (1973).

A typical project, in the classical theory as well as in the continuous time models, has been to study the existence (or non-existence) of the asymptotic arbitrage possibilities which may arise because of the existence of diversifiable risk. In the atemporal studies this project has been carried out using geometrical Hilbert space arguments, whereas most of the continuous time authors use the modern theory of martingale measures. The deepest results in this respect are obtained by Kabanov and Kramkov (1994,1996), who consider a very general sequence of increasing markets and show how absence of asymptotic arbitrage is related to contiguity (in the sense of le Cam) properties of certain sequences of probability measures.

It is notable, however, that *pricing of derivatives*, as well as *completeness questions*, has hardly been studied at all within the framework of large markets. One of our main goals is to study precisely these questions.

The structure and aims of the present paper are as follows.

- Every asset price is driven by one idiosyncratic source of randomness as well by one systematic random factor. To make the exposition conceptually as clear as possible we have chosen to work within a very simple model, where every random factor is either a Wiener or a Poisson process.

- We introduce explicitely certain "asymptotic assets" into the model. These assets have a natural interpretation as "infinitely diversified portfolios", and they correspond in reality to large mutual funds. Most of the paper is then devoted to a study of the "asymptotic model", which consists of the original model plus the asymptotic assets.
- By applying standard (finite) no arbitrage techniques to the asymptotic model we easily obtain APT results for the various markets prices of risk, and in particular we show that if the market is free of arbitrage then the market price of diversifiable risk tends to zero.
- In the most central part of the paper we study (asymptotic) completeness, and pricing of derivatives within the asymptotic framework above. The main result is that the inclusion of the asymptotic assets has the effect of completing the market.
- It turns out that, as regards hedging and pricing of derivatives, there is a fundamental difference between, on the one hand models where both the idiosyncratic and the diversifiable risk are of the same type (i.e. Poisson-Poisson or Wiener-Wiener), and on the other hand the cases when the diversifiable risk component is of a different type from from the market risk (i.e. Poisson-Wiener or Wiener-Poisson).
- In particular we show how price and hedge contingent claims in a jump-diffusion model. It is to be noted that here the the asymptotic asset plays a fundamental role in the hedging portfolio.
- We end by presenting a martingale based theory of large markets, and we study our model in the light of this theory.

2 The basic model

2.1 The market

We take as a priori given, a financial market containing a denumerable set of traded assets, and in the sequel this market will be studied on a finite fixed time interval $[0, T]$. The price of asset number i, with $i = 1, 2, \cdots$ is denoted by S_i and we assume that the dynamics of S_i are given as follows, under an objective probability measure P.

$$dS_i(t) = \alpha_i S_i(t)dt + \sigma_i S_i(t-)dZ_i(t) + \beta_i S_i(t)dW(t). \tag{1}$$

Here α_i, σ_i and β_i are assumed to be known deterministic constants (see also the technical assumptions below). The process W, which is common to all assets, represents the systematic risk in the market and W is assumed to be a standard P-Wiener process. The process Z_i, on the other hand, represents the asset specific risk of asset No i. We will consider two cases:

- The case when Z_i is a P-Wiener process ("the diffusion case").
- The case when Z_i is a Poisson process with intensity λ_i under the measure P ("the jump case").

We need some technical assumptions.

Assumption 2.1 *We assume that*

- *The time span under consideration is the a priori fixed interval* $[0, T]$.
- *The parameters* α_i, σ_i, β_i *and* λ_i, $i = 1, 2, \cdots$ *are uniformly bounded deterministic constants.*
- *The processes* $W, \; Z_1, Z_2, \cdots$ *are* P*-independent.*

We also assume the existence of a risk free asset with price process B, i.e.

$$dB = rBdt, \tag{2}$$

where r is the deterministic short rate of interest.

Remark 2.1 The model above is chosen in order to be as simple as possible, while still including the main ideas and leading to nontrivial results. It is of course quite possible, and also natural, to consider more complicated models. One can e.g. construct a model where each asset is driven by a finite number of diversifiable factors, which are common only to a finite number of assets, as well as by a finite number (≥ 2) of systematic factors which are common to all assets. For reasons given above, we do not present such a general case here, but in Section 5 we briefly present an extension to the case where, in addition to the systematic diffusion factor, we also have a systematic jump factor.

Remark 2.2 In the sequel we will often, for the sake of readability, suppress the time index t. In order to discriminate between t and $t-$ we therefore introduce the following notational convention for any process Y:

$$Y^{-}(t) = Y_{-}(t) = Y(t-). \tag{3}$$

2.2 Well diversified portfolios

We assume as above that the dynamics of asset No i are given, under the objective measure P, by equation (1). We are allowed to form finite portfolios in the various assets, and we will now see what happens when we form certain infinitely well diversified portfolios. We need a small technical Lemma, and we recall that λ_i is the P intensity of Z_i in the jump case.

Lemma 2.1 *Define $\hat{\sigma}_i$ by $\hat{\sigma}_i = \sigma_i$ in the diffusion case and $\hat{\sigma}_i = \sigma_i \lambda_i$ in the jump case. Take the sequences $\{\alpha\}_{i=1}^{\infty}$, $\{\sigma\}_{i=1}^{\infty}$, $\{\lambda\}_{i=1}^{\infty}$ and $\{\beta\}_{i=1}^{\infty}$ as given. Assume furthermore that there exists some positive constant A such that*

$$\sup_i |\alpha_i| + \sup_i |\hat{\sigma}_i| + \sup_i |\beta_i| \leq A.$$

Then there exists at least one infinite increasing subsequence $I = \{i_j\}_{j=1}^{\infty}$ of the integers, and real numbers α_I, $\hat{\sigma}_I$ and β_I (depending on the choice of the subsequence), such that

$$\lim_{i \in I} \alpha_i = \alpha_I, \qquad \lim_{i \in I} \hat{\sigma}_i = \hat{\sigma}_I, \qquad \lim_{i \in I} \beta_i = \beta_I.$$

Proof. Choose a subsequence I_0 such that $\lim_{i\in I}\alpha_i = \limsup_i \alpha_i$. Then choose a subsequence $I_1 \subseteq I_0$ such that $\lim_{i\in I_1}\hat{\sigma}_i = \limsup_{i\in I_0}\hat{\sigma}_i$, and at last choose a subsequence $I_2 \subseteq I_1$ such that $\lim_{i\in I_2}\beta_i = \limsup_{i\in I_1}\beta_i$. ∎

Definition 2.1 *An index set (subsequence) I as in the Lemma above will be called a* **converging index set**. *We introduce the natural equivalence relation $\sim$ by writing $\mathcal{I} \sim \mathcal{J}$ if and only if $\alpha_{\mathcal{I}} = \alpha_{\mathcal{J}}$, $\hat{\sigma}_{\mathcal{I}} = \hat{\sigma}_{\mathcal{J}}$ and $\beta_{\mathcal{I}} = \beta_{\mathcal{J}}$. The family of (equivalence classes of) converging index sets is denoted by $\mathcal{C}$.*

Observe that generically there will exist several converging index sets, and that I is unique (i.e. $\mathcal{C}$ is a singleton) if and only if $\lim_i \alpha_i$, $\lim_i \hat{\sigma}_i$ and $\lim_i \beta_i$ all exist, (where the limits are taken with respect to the set of natural numbers).

We now turn to the construction of well diversified self financing portfolio strategies. Let us therefore consider a fixed finite index set $I = \{i_1, ..., i_K\}$ and recall that a self financed portfolio in the assets $\{S_i; i \in I\}$ is specified by

1. The initial capital $V(0)$.
2. A sequence of predictable (and sufficiently integrable) stochastic processses $\{u_i; i \in I\}$ such that $\sum_{i\in I} u_i = 1$.

Remark 2.3 Unless otherwise specified, the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ under consideration will be defined by

$$\mathcal{F}_t = \sigma\left\{W(s), Z_i(s); s \leq t, i = 1, 2, \cdots\right\} \tag{4}$$

Thus the prefix "predictable" above, means $(\mathcal{F}_t)$-predictable. In some cases it will be of interest to consider portfolios which are adapted to a smaller filtration than $\{\mathcal{F}_t\}_{t\geq 0}$, and in such cases we will give separate remarks to clarify the situation.

As usual $u_i(t)$ above is interpreted as the relative share of the total portfolio value invested in asset No i at time t, and we also recall that the evolution of the value process V is given by

$$dV = V^- \cdot \sum_{i\in I} u_i \frac{dS_i}{S_i^-}. \tag{5}$$

Definition 2.2 *Consider a converging index set $I = \{i_j\}_{j=1}^{\infty}$, and define I_n by $I_n = \{i_j\}_{j=1}^{n}$. A* **diversifying strategy along I** *is defined a sequence $\{u^n\}_{n=0}^{\infty}$ of self financing portfolios such that*

1. *For all i, n and t we have $u_i^n(t) \geq 0$.*
2. *The strategy u^n is based on the assets $\{S_i;\ i \in I_n\}$.*
3. *For all t have the following relation P-almost surely.*

$$\lim_{n\to\infty} \sup_{i\in I_n, t\geq 0} u_i^n(t) = 0.$$

This definition formalizes the idea of a well diversified portfolio, in the sense that each asset asymptotically contributes an infinitesimal part to the entire portfolio, and the simplest concrete example is of course given by $u_i^n(t) \equiv 1/n$. Let us now consider a fixed diversifying strategy along I, and let us denote the value process corresponding to u^n by V^n. We then have the following portfolio dynamics.

$$dV^n = V^n \left(\sum_{i\in I_n} u_i^n \alpha_i \right) dt + V_-^n \sum_{i\in I_n} u_i^n \sigma_i dZ_i + V^n \left(\sum_{i\in I_n} u_i^n \beta_i \right) dW, \tag{6}$$

and the immediate project is to study the behaviour of this equation as n tends to infinity.

It turns out that the case of a diversifiable jump risk in some aspects is qualitatively different from the case when the diversifiable risk is of diffusion type. Therefore the two cases are treated separately.

3 The diffusion case

3.1 The asymptotic diffusion model

In the diffusion case the V-dynamics in (6) will be

$$dV^n = V^n \left(\sum_{i\in I_n} u_i^n \alpha_i \right) dt + V^n \sum_{i\in I_n} u_i^n \sigma_i dW_i + V^n \left(\sum_{i\in I_n} u_i^n \beta_i \right) dW, \tag{7}$$

where $W, W_1, \cdots$ are independent P-Wiener processes. We now investigate the asymptotic behaviour of the V^n-sequence.

From the definition of a diversifying strategy it is easy to see that, as n tends to infinity, we have the following results

$$\sum_{i \in I_n} u_i^n \alpha_i \to \alpha_I, \quad \sum_{i \in I_n} u_i^n \beta_i \to \beta_I. \tag{8}$$

This takes care of the first and third term in the right hand side of (7). For the middle term we can look at the infinitesimal variance (recall that the section is heuristic) to obtain

$$Var\left[\sum_{i \in I_n} u_i^n \sigma_i dW_i\right] = \left(\sum_{i \in I_n} (u_i^n)^2 \sigma_i^2\right) dt.$$

Defining B_n by

$$B_n = \sup_{i \in I_n, t \geq 0} |u_i^n(t)|,$$

we have the trivial inequality

$$\sum_{i \in I_n} (u_i^n)^2 \sigma_i^2 \leq B_n \sum_{i \in I_n} u_i^n \sigma_i^2.$$

As n tends to infinity $\sum_{i \in I_n} u_i^n \sigma_i^2 \to \sigma_I^2$, whereas $B_n \to 0$. Thus we see that in the limit the dynamical behaviour of the value process will be given by

$$dV = V\alpha_I dt + V\beta_I dW. \tag{9}$$

This equation gives us the asymptotic behaviour of an "infinitely diversified portfolio", and it is important to notice that the coefficients α_I and β_I depend on the particular choice of converging index set I. We now introduce all processes of the type (9) as formal asset prices in our model.

Definition 3.1 *The* **asymptotic diffusion model** *consists of the following price processes.*

1. *For each i the process S_i with P-dynamics*

$$dS_i = \alpha_i S_i dt + \sigma_i S_i dW_i + \beta_i S_i dW, \quad i = 1, 2, \cdots \tag{10}$$

2. *For each converging index set $I \in \mathcal{C}$, the process S_I with P-dynamics defined by*

$$dS_I = \alpha_I S_I dt + \beta_I S_I dW, \quad I \in \mathcal{C}. \tag{11}$$

3. *The risk free asset price B with*

$$dB = rBdt. \tag{12}$$

The asymptotic model is thus an idelized picture of a financial market where we are allowed to trade infinitely diversified portfolios. It is an idealization in the same way that "continuous trading" and "frictionless markets" are idealizations of real world phenomena. The real world counterparts to the S_I-processes are of course the mutual funds, which can contain hundreds of different assets.

Remark 3.1 If we have only one systematic factor, and if this factor is tradeable, then we get (10)-(11) without needing the asymptotic portfolio. An example of a tradeable systematic factor is the price of oil, or rather the percentage change in the price of oil as tested in Chen *et al.* (1986) with mixed results.

3.2 No arbitrage and the market price of risk

We now go on to investigate the implications of no arbitrage in the asymptotic diffusion model.

Assumption 3.1 *We consider as given the asymptotic model (10)-(12). This model is assumed to be free of arbitrage in the sense that no arbitrage possibilities exist when we are allowed to trade in finite portfolios.*

We have included the requirement of finite portfolios in order to be able to use the "classical" theory of no arbitrage. Note however that since we now are working with the asymptotic model, the assumption of no arbitrage is really an assumption of "no asymptotic arbitrage" in terms of the original model (1)-(2).

We can now apply the standard technique of constructing locally riskfree portfolios in order to see what the implications are of the no arbitrage asumption. We start by fixing two different converging index sets $I,\ J$ and form a portfolio based on S_I and S_J. Then we choose portfolio weights in such a way that the dW-term vanishes from the value process dynamics. Equating the rate of return of this locally riskless portfolio with the short rate of interest (the no arbitrage assumption) gives us the relation

$$\frac{\alpha_I - r}{\beta_I} = \frac{\alpha_J - r}{\beta_J}. \tag{13}$$

Thus we infer the existence of a real number φ with the property that

$$\frac{\alpha_I - r}{\beta_I} = \varphi, \quad \text{for all } I \in \mathcal{C}. \tag{14}$$

We will refer to φ as "the market price of systematic risk".

For each $i = 1, 2, \cdots$ we then define the real number φ_i, referred to as "the market price of type-i-risk", as the solution to the equation

$$\alpha_i = r + \varphi_i \sigma_i + \varphi \beta_i. \tag{15}$$

Summing up we have the following result.

Proposition 3.1 *The market price of systematic risk, φ, and the market price of risk of type i, φ_i, are given by*

$$\varphi = \frac{\alpha_I - r}{\beta_I}, \tag{16}$$

$$\varphi_i = \frac{\alpha_i - r}{\sigma_i} - \frac{\beta_i}{\sigma_i} \cdot \frac{\alpha_I - r}{\beta_I}, \tag{17}$$

where the equations above hold for all $I \in \mathcal{C}$ and all $i = 1, 2, \cdots$.

We note that φ and φ_i above are (apart from a minus sign) precisely the Girsanov kernels one would use naively in order to produce a "martingale measure" Q with the property that the Q-dynamics of the asset prices would have the form

$$dS_i = rS_i dt + \sigma_i S_i d\tilde{W}_i + \beta_i S_i d\tilde{W}, \tag{18}$$

$$dS_I = rS_I dt + \beta_I S_I d\tilde{W}, \tag{19}$$

where $\tilde{W}$ and $\tilde{W}_i$ are independent Q-Wiener processes. The problem with this approach is that if we thus make a Girsanov transformation for each separate Wiener process, we have no guarantee that the measure Q that we obtain is globally equivalent to the objective measure P. This problem will be dealt with in the Section 6.

Remark 3.2 If we write (17) on the form

$$\varphi_i = \frac{1}{\sigma_i}\left\{\alpha_i - r - \frac{\beta_i\beta_I}{\beta_I^2}(\alpha_I - r)\right\}, \tag{20}$$

we see that the market price for idiosyncratic risk can be written as the total risk premium $\alpha_i - r$ minus the risk premium associated with systematic risk,

$$\frac{1}{\beta_I^2} \cdot \frac{Cov\left(\frac{dS_i}{S_i}, \frac{dS_I}{S_I}\right)}{dt}(\alpha_I - r),$$

all divided by σ_i. We also recognize the term $\frac{1}{\beta_I^2} \cdot \frac{Cov\left(\frac{dS_i}{S_i}, \frac{dS_I}{S_I}\right)}{dt}$ as the traditional "beta" of factor i.

Sometimes it is claimed that "the market price of diversifiable risk has to equal zero". Translated into our model a statement of this kind would thus assert that $\varphi_i = 0$ for all i , and we see at once from Proposition 3.1 that this does not have to be the case. On the contrary, and quite in accordance with atemporal APT, we see from (17) that the market price of an individual diversifiable risk source can take any value. There is however an asymptotic result, again extending a previous APT result, which supports the claim made above by saying that, in the limit, the market price of diversifiable risk has to approach zero.

Proposition 3.2

1. *The relation*

$$\varphi_i = 0, \tag{21}$$

holds for a particular i if and only if

$$\frac{\alpha_i - r}{\beta_i} = \frac{\alpha_I - r}{\beta_I}, \quad \text{for all } I \in \mathcal{C}. \tag{22}$$

2. We have the following limit result.

$$\lim_{i\to\infty} \varphi_i = 0. \tag{23}$$

Proof. The first part of the proposition follows immediately from (17). To prove the second part we first note that, for any converging index set I, we have

$$\lim_{i\in I} \varphi_i = \frac{\alpha_I - r}{\sigma_I} - \frac{\beta_I}{\sigma_I} \cdot \frac{\alpha_I - r}{\beta_I} = 0.$$

Now take any infinite subsequence J of the natural numbers. It is easily seen that J will contain some converging set I. Thus, using the result above, we see that any subsequence of the full sequence $\{\varphi_i\}_1^\infty$ will contain a further subsequence converging to zero. Hence the full sequence has to converge to zero. ∎

Remark 3.3 From Proposition 3.2 and (20) we also see that asymptotically we are on the CAPM line

$$\alpha_i - r = \frac{\beta_i \beta_I}{\beta_I^2} (\alpha_I - r)$$

3.3 Hedging and completeness

In this section we will study how to price and how to hedge against a contingent claim in the original model as well as in the asymptotic one. To keep things simple we will assume that we have only one converging index set I namely the full sequence $I = \{1, 2, \cdots\}$. This means that all coefficient sequences converge, i.e. we have $\lim_i \alpha_i = \alpha$, $\lim_i \sigma_i = \sigma$ and $\lim_i \beta_i = \beta$. Furthermore we have only one asymptotic asset, which we will denote by S, with price dynamics given by

$$dS = \alpha S dt + \beta S dW. \tag{24}$$

We keep the assumption of only allowing our hedging portfolios containing a finite number of assets (though for the asymptotic model we are again allowed to trade in the asymptotic asset S).

We start by looking at the original model and the question is whether it is complete or not. The answer to this question will of course depend on what we allow as a contingent claim, and here there are several choices. We list some of them.

Definition 3.2 *Fix some point in time T. A (sufficiently integrable) stochastic variable X is said to be a*

1. **contingent claim** *if*

$$X \in \sigma\{W(t),\ W_i(t);\ t \leq T,\ i = 1, \cdots\}$$

2. **market observable contingent claim** *if*

$$X \in \sigma\{S_i(t);\ t \leq T,\ i = 1, \cdots\}$$

3. **finite contingent claim** *if there exist a number n such that*

$$X \in \sigma\{W(t),\ W_i(t);\ t \leq T,\ i = 1, \cdots, n\}$$

4. **finite market observable contingent claim** *if there exists a number n such that*

$$X \in \sigma\{S_i(t);\ t \leq T,\ i = 1, \cdots, n\}$$

In this section we will only consider claims of finite type, and we have the following results.

Proposition 3.3 *The original model is incomplete with respect to the set of finite claims. It is complete with respect to finite market observable contingent claims.*

Proof. Fix T and define X by $X = W_1(T)$. Then X is finite and it is easily seen that X can not be replicated by a portfolio containing a finite number of assets. Thus the original model is incomplete with respect to the set of finite claims.

Consider on the other hand a market observable claim

$$X \in \sigma \left\{ S_i(t);\ t \leq T,\ i = 1, \cdots, n \right\}.$$

Defining the Wiener processes $\left\{ \hat{W}_i;\ i = 1, \cdots n \right\}$ by

$$\hat{W}_i(t) = \frac{1}{d_i} \left[\sigma_i W_i(t) + \beta_i W(t) \right],$$

with

$$d_i = \sqrt{\sigma_i^2 + \beta_i^2},$$

we can write the price dynamics for $S_1, \cdots, S_n$ as

$$dS_i = \alpha_i S_i dt + d_i S_i d\hat{W}_i, \quad i = 1, \cdots, n.$$

Using e.g. Gram-Schmidt we can now orthogonalize the $\hat{W}_i$-processes to obtain

$$dS_i = \alpha_i S_i dt + S_i \sum_{j=1}^{n} D_{ij} d\bar{W}_j, \tag{25}$$

where $\bar{W}_1, \cdots, \bar{W}_n$ are independent Wiener processes and the diffusion matrix $D = [D_{ij}]$ is nonsingular. We are now back in a standard situation and completeness follows. ∎

Remark 3.4 In the previous proposition we note that, by the convention of Remark 2.3, the portfolios are allowed to be adapted to the "big" filtration $\sigma \left\{ W(s),\ W_i(s);\ s \leq t,\ i = 1, \cdots, n \right\}$. For the completeness part of the propsition, it is easily seen from the proof, that the replicating portfolio is in fact adpted to the smaller filtration $\sigma \left\{ S_i(s);\ s \leq t,\ i = 1, \cdots, n \right\}$.

Proposition 3.4 *The asymptotic model is complete with respect to the set of finite claims.*

Proof. Suppose that $X \in \sigma \left\{ W(t), W_i(t);\ t \leq T,\ i = 1, \cdots, n \right\}$. We now adjoin the asymptotic asset to the finite set of assets $S_1, \cdots, S_n$. Thus we obtain the $(n+1)$-dimensional asset price vector $S, S_1, \cdots, S_n$ for which the diffusion matrix clearly is nonsingular. Completeness now follows from standard results. ∎

Remark 3.5 A priori, the replicating portfolio above is only $\mathcal{F}_t$-adapted, but a closer look reveals that it is in fact adapted to the filtration $\sigma\{W(s),\ W_i(s);\ s \leq t,\ i = 1, \cdots, n\}$ or (equivalently) to the filtration $\sigma\{S(s),\ S_i(s);\ s \leq t,\ i = 1, \cdots, n\}$.

The moral of the results above is that the possibility of using infinitely diversifiable portfolios has the effect of completing the market. For the present case, where we have only Wiener processes as driving noise, this result should however not be overemphazised and the reason is as follows.

The distinction between a (finite) general contingent claim and a market observable claim is indicated by the name: a market observable claim is actually defined in terms of the various asset prices, which in our interpretation of the model, can be observed on the market. A claim like $W_i(T)$ on the contrary, is a claim that can never be paid out in a market where the available information is generated by the prices only. The reason that it can not be paid out is that, based upon price information only, it is impossible to determine the value of X at time T.

Thus we may say that while the original model is incomplete (in a certain sense), the incompleteness stems from the fact that we have overspecified the number of driving Wiener processes. Except for certain "pathological" claims, which will never occur in a concrete market situation, all (finite) claims can in fact be replicated. We emphasize, however, that this phenomenon is highly dependent upon the fact that, in the present model, both the asset specific risk and the systematic risk are of diffusion type. As we shall see below the picture is radically changed when the asset specific risk is of jump type.

3.4 Derivative pricing

Let us fix a finite market observable contingent claim X. From the preceeding section we know that X can be replicated, so it has a unique arbitrage free price process $\Pi(t; X)$ given by $\Pi(t; X) = V(t)$ where V is the value process for the replicating portfolio. Using standard martingale arguments we may of course also express the price as an expected value.

Proposition 3.5 *For any square integrable finite market observable contingent claim $X \in \sigma\{S_1, \cdots, S_n\}$ the price process $\Pi(t; X)$ is given by*

$$\Pi(t; X) = e^{-r(T-t)} E^Q [X | \sigma\{S_1(s), \cdots, S_n(s);\ s \le t\}] \tag{26}$$

Here the Q-dynamics of $(S_1, \cdots, S_n)$ are given by

$$dS_i = rS_i dt + \sigma_i S_i d\tilde{W}_i + \beta_i S_i d\tilde{W}, \quad i = 1, \cdots, n, \tag{27}$$

where $\tilde{W}, \tilde{W}_1, \cdots, \tilde{W}_n$ are independent Q-Wiener processes.

As we already have noted, the price dynamics in (27) contains one redundant Wiener process. For computational reasons it may therefore be advantageous to reduce the model for $(S_1, \cdots, S_n)$ to a model containing exactly n Wiener processes, as we did in (25). This will give us Q-dynamics of the form

$$dS_i = rS_i dt + S_i \sum_{j=1}^{n} D_{ij} dW_j^\star, \quad i = 1, \cdots, n,$$

where $W_n^\star, \cdots, W_n^\star$ are independent Wiener processes under Q. To give a simple concrete example let X be a standard European call on $S_1(T)$, with strike price K. Then we write can the Q-dynamics for S_1 as

$$dS_1 = rS_1 dt + d_i S_i d\tilde{W}_1, \tag{28}$$

where $d_1 = \sqrt{\sigma_1^2 + \beta_1^2}$, and the value of the call is given by Black-Scholes formula using the volatility d_1.

We thus see that for the pure diffusion case we have the following moral.

Moral: "When it comes to pricing and hedging of derivatives defined in terms of a finite number of underlying assets, the possibility of forming well diversified portfolios plays no role whatsoever. In particular, the market prices of diversifiable and asset specific risk do not have to be known in order to price the derivative."

As we shall see below, the jump case presents a totally different picture.

4 The jump case

4.1 The jump model

In the jump case, the asset dynamics are given by

$$dS_i(t) = \alpha_i S_i(t)dt + \sigma_i S_i(t-)dN_i(t) + \beta_i S_i(t)dW(t). \tag{29}$$

where $N_1, N_2, \cdots$ are independent P-Poisson processes. We recall that the constant P-intensity of N_i is denoted by λ_i.

In this model, the Poisson process N_i represents the occurence of sudden asset specific shocks. If $\sigma_i > 0$ then the shocks will increase the value of the stock, and this could be interpreted as a technological breakthrough or suddenly improved market conditions. When $\sigma_i < 0$, on the other hand, the stock price will decrease at every jump time of the Poisson process, representing negative shocks. The most striking example is perhaps the case when $\sigma_i = -1$, in which case the stock price will drop to zero at the fist jump time of N_i. This has a very natural interpretation as a model of (complete) *default* for asset No i, and in this context it would be natural to study various credit risk derivatives. See Sections 4.4-4.5 below for a concrete worked out example.

The first paper treating jump risk is the classic stock price model in Merton (1976). There exists a large literature on jump risk in interest rate models, where the focus often has been on credit risk. See Artzner and Delbaen (1995), Duffie and Singleton (1995), Jarrow and Turnbull (1995), and Madan and Unal (1997). For a fairly general theory of point process driven interest rate models, see Björk *et al.* (1997a,b).

4.2 The asymptotic jump model

Turning to well diversified portfolios, the V-dynamics in (6) will now have the form

$$dV^n = V^n \left(\sum_{i \in I_n} u_i^n \alpha_i \right) dt + V_-^n \sum_{i \in I_n} u_i^n \sigma_i dN_i + V^n \left(\sum_{i \in I_n} u_i^n \beta_i \right) dW, \tag{30}$$

In order to facilitate the analysis it is convenient to write (30) on P-semimartingale form as

$$
\begin{aligned}
dV^n &= V^n \left(\sum_{i \in I_n} u_i^n \left[\alpha_i + \hat{\sigma}_i \right] \right) dt + V_-^n \sum_{i \in I_n} u_i^n \sigma_i \left[dN_i - \lambda_i dt \right] \\
&+ V^n \left(\sum_{i \in I_n} u_i^n \beta_i \right) dW, \qquad (31)
\end{aligned}
$$

where as before $\hat{\sigma}_i$ is defined as

$$\hat{\sigma}_i = \sigma_i \lambda_i. \qquad (32)$$

We now let n tend to infinity, and as in Section 3.1 we have

$$
\begin{aligned}
\sum_{i \in I_n} u_i^n \left[\alpha_i + \hat{\sigma}_i \right] &\rightarrow \alpha_I + \hat{\sigma}_I, \qquad (33) \\
\sum_{i \in I_n} u_i^n \beta_i &\rightarrow \beta_I. \qquad (34)
\end{aligned}
$$

For the middle term in (31) we see that the differential $dN_i - \lambda_i dt$ is a martingale increment, and thus it has expected value zero. The infinitesimal variance is given by

$$Var_P[\sum_{i \in I_n} u_i^n \sigma_i \left[dN_i - \lambda_i dt \right]] = \sum_{i \in I_n} (u_i^n)^2 \sigma_i^2 \lambda_i dt.$$

As in Section 3.1 it is easily seen that $\sum_{i \in I_n} (u_i^n)^2 \sigma_i^2 \lambda_i \rightarrow 0$ as $n \rightarrow \infty$. Thus the asymptotic behaviour of (31) is given by the equation

$$dV = (\alpha_I + \hat{\sigma}_I) V dt + V \beta_I dW, \qquad (35)$$

and we now define the asymptotic jump model.

Definition 4.1 *The* **asymptotic jump model** *consists of the following price processes.*

1. *For each i the process S_i with P-dynamics*

$$dS_i = \alpha_i S_i dt + \sigma_i S_i^- dN_i + \beta_i S_i dW, \quad i = 1, 2, \cdots \qquad (36)$$

2. *For each converging index set $I \in \mathcal{C}$, the process S_I with P-dynamics defined by*
$$dS_I = (\alpha_I + \hat{\sigma}_I)S_I dt + \beta_I S_I dW, \quad I \in \mathcal{C}. \tag{37}$$

3. *The risk free asset price B with*
$$dB = rBdt. \tag{38}$$

Remark 4.1 Note that in equation (36), the parameter α_i does **not** represent the systematic drift (under P) of the S_i process. More precisely: α_i is the P-drift of S_i *between jumps*. The overall drift of S_i (under P) is given by $\alpha_i + \hat{\sigma}_i$, which can be seen from the P-semimartingale representation

$$dS_i = S_i(\alpha_i + \lambda_i \sigma_i)dt + \sigma_i S_i [dN_i - \lambda_i dt] + \beta_i S_i dW, \quad i = 1, 2, \cdots \tag{39}$$

Since we will work under the martingale measure Q, as well as under P, we will keep to the measure invariant formulation (36).

4.3 The market price of risk

As before we assume that the asymptotic model is free of arbitrage and we now go on to find the market prices of risk for the diffusion part as well as for the various jump parts. Again we perform the analysis the following naive way:

- Carry out a Girsanov transformation for the Wiener process using a likelihood process of the form
$$\begin{aligned} dL_0 &= L_0 \cdot \varphi^{\star} dW, \\ L_0(0) &= 1. \end{aligned}$$

- For each i we carry out a Girsanov transformation using a likelihood process of the form
$$\begin{aligned} dL_i &= L_i^{-} \mu_i^{\star} [dN_i - \lambda_i dt] \\ L_i(0) &= 1. \end{aligned}$$

In this way we obtain a measure Q with the property that

- Under Q the process W has the decomposition

$$dW = \varphi^\star dt + d\tilde{W},$$

where $\tilde{W}$ is Q-Wiener.

- The process N_i is a Q-Poisson process with Q-intensity λ_i^Q given by

$$\lambda_i^Q = (1 + \mu_i^\star)\lambda_i.$$

The asset price dynamics under Q are thus given by

$$dS_i = [\alpha_i + \beta_i\varphi^\star + \hat{\sigma}_i(1 + \mu_i^\star)]\, S_i dt + \sigma_i S_i^- [dN_i - (1 + \mu_i^\star)\lambda_i] + \beta_i S_i d\tilde{W}, \quad (40)$$

$$dS_I = \{\alpha_I + \hat{\sigma}_I + \beta_I\varphi^\star\}\, S_I dt + \beta_I S_I d\tilde{W}, \quad I \in \mathcal{C}. \quad (41)$$

Now we want to choose the Girsanov kernels in such a way that all asset prices have a local rate of return equal to the short rate of interest, i.e we want to solve the following set of equations

$$\alpha_i + \beta_i\varphi^\star + \hat{\sigma}_i(1 + \mu_i^\star) = r, \quad (42)$$

$$\alpha_I + \hat{\sigma}_I + \beta_I\varphi^\star = r. \quad (43)$$

This system is easily solved as

$$\varphi^\star = \frac{r - \alpha_I - \hat{\sigma}_I}{\beta_I}, \quad (44)$$

$$\mu_i^\star = \frac{r - \alpha_i - \hat{\sigma}_i - \frac{\beta_i}{\beta_I}(r - \alpha_I - \hat{\sigma}_I)}{\hat{\sigma}_i}. \quad (45)$$

At last we may define "the market price of diffusion risk", φ, by

$$\varphi = -\varphi^\star \quad (46)$$

and "the market price of jump risk of type i", μ_i, by

$$\mu_i = -\mu_i^\star. \quad (47)$$

Again we see that absence of arbitrage generically does *not* imply that the market price of diversifiable risk equals zero. Instead we have the following results, the proof of which are identical to the proof for the Wiener case.

Proposition 4.1

1. *The market price of jump risk of type i equals zero if and only if the following relation hold for all $I \in \mathcal{C}$.*

$$\frac{\alpha_i + \hat{\sigma}_i - r}{\beta_i} = \frac{\alpha_I + \hat{\sigma}_I - r}{\beta_I}. \tag{48}$$

2. *We have the following asymptotic result*

$$\lim_{i \to \infty} \mu_i = 0.$$

We thus see that the market price of jump risk of type i equals zero if and only if the risk premium per unit of systematic volatility of asset i equals the risk premium per unit of systematic volatility for any asymptotic asset. Furthermore we see that if all assets are identical under P, i.e. the coefficients for asset i does not depend on the index i, then in fact all market prices of diversifiable risks equal zero.

Remark 4.2 We may rewrite (45) as

$$-\mu_i^\star = \frac{1}{\hat{\sigma}_i} \left\{ \alpha_i + \hat{\sigma}_i - r - \frac{1}{\beta_I^2} \cdot \frac{Cov\left(\frac{dS_i}{S_i}, \frac{dS_I}{S_I}\right)}{dt} (\alpha_I + \hat{\sigma}_I - r) \right\}$$

and then argue as in Remark 3.2. Using Proposition 4.1 we see that we again are asymptotically on the CAPM line

$$\alpha_i + \hat{\sigma}_i - r = \frac{1}{\beta_I^2} \cdot \frac{Cov\left(\frac{dS_i}{S_i}, \frac{dS_I}{S_I}\right)}{dt} (\alpha_I + \hat{\sigma}_I - r)$$

As in the diffusion case we stress the fact that the arguments above are partly heuristic, since there is no guarantee that the prospective "martingale measure" Q obtained above is globally equivalent to the objective measure P. This question will be handled in Section 6 below.

Remark 4.3 Contemplating the possible signs of the various market prices of risk, we have from (46)-(47)

$$\varphi = \frac{\alpha_I + \hat{\sigma}_I - r}{\beta_I},$$

$$\mu_i = \frac{\alpha_i + \hat{\sigma}_i - r - \frac{\beta_i}{\beta_I}(\alpha_I + \hat{\sigma}_I - r)}{\hat{\sigma}_i}$$

Assuming that $\beta_i > 0$ for all i, and thus implying $\beta_I > 0$, we then see that the market price of diffusion risk, φ, is positive if and only if we have the relation $\alpha_I + \hat{\sigma}_I - r > 0$, i.e. if and only if the asymptotic asset has a positive risk premium. This relation will of course hold in every risk averse market.

In the same way we see that the market price of jump risk of type i, μ_i, is positive if and only if we have the relation

$$\frac{\alpha_i + \hat{\sigma}_i - r}{\beta_i} > \frac{\alpha_I + \hat{\sigma}_I - r}{\beta_I}$$

i.e. if and only if the risk premium per unit of systematic volatility is greater for asset No i than for the asymptotic asset.

4.4 Completeness for the jump model

In order to study completeness we again have to specify the relevant classes of contingent claims to be considered. To make things simple we assume that the class of convergent index sets is a singleton, i.e. that all model coefficients converge. Thus we only have one asymptotic asset, which will be denoted by S, and its P-dynamics will be written as

$$dS = (\alpha + \hat{\sigma})Sdt + \beta SdW.$$

We now come to the first major difference between the diffusion model and the jump model. For the pure diffusion model the "large" sigma algebra $\sigma\{W(t),\ W_i(t);\ t \leq T,\ i = 1, \cdots, n\}$ was strictly included in the "market observable" sigma algebra $\sigma\{S_i(t);\ t \leq T,\ i = 1, \cdots, n\}$, which meant that we had to introduce the concept of "market observable" claims. In the present mixed case it is easy to see that the sigma algebras

$$\sigma\{W(t),\ N_i(t);\ t \leq T,\ i = 1, \cdots, n\}$$

and

$$\sigma \{S_i(t);\ t \leq T,\ i = 1, \cdots, n\}$$

are equal. This motivates the following definitions.

Definition 4.2 *Fix some point in time T. A (sufficiently integrable) stochastic variable X is said to be a*

1. **contingent claim** *if $X \in \sigma \{W(t),\ N_i(t);\ t \leq T,\ i = 1, \cdots\}$.*
2. **finite contingent claim** *if there exist a number n such that $X \in \sigma \{W(t),\ N_i(t);\ t \leq T,\ i = 1, \cdots, n\}$.*

The main result concerning the original model is not surprising. Remember that, both in the original, and in the asymptotic model, we are by assumption only allowed to trade in finite portfolios.

Proposition 4.2 *The original model is not even complete with repect to finite (market observable) claims.*

Proof. It is immediately clear that no claim claim of the form $\Phi(N_1(T))$, where Φ is any nonconstant function, can be replicated using a finite set of assets. ∎

Note that, in contrast with the pure diffusion case earlier, these claims are in no way "pathological". As an example consider the case when $\sigma_1 = -1$. Then we have $S_1(t) \equiv 0$ for all $t \geq \tau$, where τ is the first jump time of N_1, and we can interpet τ as the time of default for asset 1. In this example it is extremely natural to consider a claim of the form $X = I\{N_1(T) = 0\}$, where I is the indicator function, since such a claim acts as an insurance against default. As we have seen there is no way of replicating such a claim in the original model, but for the asymptotic model the situation is brighter.

Proposition 4.3 *In the asymptotic model every (Q-square integrable) finite claim can be replicated.*

Proof. Suppose that the claim X is of the form

$$X \in \sigma\left\{W(t),\ N_i(t);\ t \le T,\ i = 1, \cdots, n\right\}.$$

Then we adjoin the asymptotic asset S to $S_1, \cdots, S_n$, and it is a standard exercise to see that X can be replicated using a portfolio based upon $S, S_1, \cdots, S_n$. ∎

Once again we thus see that the introduction of the asymptotic asset – i.e. the possibility of forming well diversified portfolios – has the function of completing the model (see the end of the next section for a simple concrete example). We stress the fact that, in contrast to the pure diffusion case, the completness of the asymptotic model really amounts to a substantial improvement over the original model.

4.5 Pricing in the jump model

We now turn to the task of computing arbitrage free prices for (finite) derivatives in the asymptotic jump model. The results are obvious given those of the preceeding section.

Proposition 4.4 *Assume that $\mathcal{C}$ is a singleton and consider any finite claim*

$$X \in \sigma\left\{W(t),\ N_i(t);\ t \le T,\ i = 1, \cdots, n\right\}.$$

Then the arbitrage free price process, $\Pi(t; X)$, of the claim is given by

$$\Pi(t; X) = e^{-r(T-t)} E^Q\left[X \middle|\, \sigma\left\{S(u),\ S_i(u);\ u \le t,\ i = 1, \cdots, n\right\}\right]. \tag{49}$$

Here the Q-dynamics are given by

$$dS = rSdt + \beta S d\tilde{W}, \tag{50}$$

$$dS_i = \left(r - \sigma_i^Q\right) S_i dt + \sigma_i S_i^- dN_i + \beta_i S_i d\tilde{W},\ i = 1, \cdots, n. \tag{51}$$

where

$$\sigma_i^Q = \sigma_i \lambda_i (1 + \mu_i^\star), \tag{52}$$

and where $\mu_i^\star$ is given by (45). Furthermore, the processes $N_1, \cdots, N_n$ are independent Q-Poisson processes, and the Q-intensity of N_i is given by $\lambda_i^Q = \lambda_i(1 + \mu_i^\star)$ $i = 1, \cdots n$.

For the case of a "simple" claim we may compute the price by solving a difference-differential equation.

Proposition 4.5 *Assume that the claim is of the form*

$$X = \Phi\left(S_1(T), \cdots, S_n(T)\right), \tag{53}$$

then the price process is given by $\Pi\left(t; X\right) = F\left(t, S(t), S_1(t), \cdots, S_n(t)\right)$ *where* F *is the solution of the boundary value problem*

$$\begin{aligned}\frac{\partial F}{\partial t} &+ rs\frac{\partial F}{\partial s} + \sum_1^n (r - \sigma_i^Q) s_i \frac{\partial F}{\partial s_i} + \frac{1}{2}\beta^2 s^2 \frac{\partial^2 F}{\partial s^2} + \frac{1}{2}\sum_1^n \beta_i^2 s_i^2 \frac{\partial^2 F}{\partial {s_i}^2} \\ &+ \sum_{i,j=1}^n \beta_i \beta_j s_i s_j \frac{\partial^2 F}{\partial s_i \partial s_j} + \sum_1^n \beta_i \beta s_i s \frac{\partial^2 F}{\partial s_i \partial s} \\ &+ \sum_1^n F_i^\star - rF = 0, \end{aligned} \tag{54}$$

with the boundary condition

$$F\left(T, s, s_1, \cdots, s_n\right) = \Phi\left(s_1, \cdots, s_n\right). \tag{55}$$

Here we have used the notation

$$F_i^\star(s, s_1, \cdots, s_n) = F(s, s_1, \cdots, s_i(1 + \sigma_i), \cdots s_n) - F(s, s_1, \cdots, s_i, \cdots s_n). \tag{56}$$

Proof. The Kolmogorov backward equation. ∎

Example:
As an example to illustrate ideas, let us replicate the "binary" claim discussed above. i.e. we consider a claim X of the form

$$X = I\left\{N_1(T) = 0\right\}. \tag{57}$$

We start by writing down the Q-dynamics of the discounted asset price processes $Z(t) = \frac{S(t)}{B(t)}$ and $Z_1(t) = \frac{S_1(t)}{B(t)}$. These are easily obtained from (50)-(51) as

$$dZ = \beta Z d\tilde{W}, \tag{58}$$
$$dZ_1 = \sigma_1 Z_1^- d\tilde{N}_1 + \beta_1 Z_1 d\tilde{W}, \tag{59}$$

where $\tilde{N}_1$ is the Q-compensated N_1-process, defined as

$$\tilde{N}_1(t) = N_1(t) - \lambda_1^Q t.$$

Next we define the Q-martingale M by $M(t) = E^Q\left[e^{-rT} I\left\{N_1(T) = 0\right\}\middle|\mathcal{F}_t\right]$, where $\mathcal{F}_t$ as usual is defined by (4). It is easily seen that we have

$$M(t) = e^{-rT} e^{-\lambda_1^Q (T-t)} I\left\{N_1(t) = 0\right\},$$

and that the stochastic differential of M is given by

$$dM = -M(t-) d\tilde{N}_1. \tag{60}$$

It is well known, and easy to see, that if we can find $\mathcal{F}_t$-predictable processes h and h_1 such that

$$h(t) dZ(t) + h_1(t) dZ_1(t) = dM(t), \tag{61}$$

Then the claim can be replicated by a portfolio consisting, at each t, of $h(t)$ units of the asymptotic asset, h_1 units of asset S_1 and h_B units of the risk free asset, where h_B is defined by

$$h_B(t) = M(t) - h(t)Z(t) - h_1(t)Z_1(t) \tag{62}$$

Plugging (58) and (59) into (61) and equating coefficients gives us

$$h(t) = \frac{M(t-)\beta_1}{\sigma_1 \beta Z(t-)} \tag{63}$$
$$h_1(t) = \frac{M(t-)}{\sigma_1 Z_1(t-)}, \tag{64}$$

Thus the portfolio defined by (62), (63) and (64) will indeed replicate the claim. Notice how the introduction of the well diversified portfolio S in this case helps us to hedge a claim which, without the possibility of diversifying, would have been unhedgeable. Also notice that the portfolio weights, which a priori only are adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, actually are adapted to the much smaller filtration generated by S_1 and S.

The arbitrage free price process, $\Pi(t; X)$ for the claim above is of course given by the value process,$V(t)$ of the replicating portfolio. It is again easily seen that the value process is given by $V(t) = e^{rt}M(t)$, so we have the pricing formula

$$\Pi(t; X) = e^{-r(T-t)}e^{-\lambda_1^Q(T-t)}I\{N_1(t) = 0\}. \tag{65}$$

5 A systematic jump component

In this section we will see how the framework above can be extended to more general situations. Rather than doing this in complete generality, we have chosen to study a simple extension of the jump model above, where we, apart from the systematic diffusion factor, also include a systematic jump component. The arguments are very similar to those of the previous sections, so our discussion will be rather brief.

5.1 The model

The asset dynamics are given, under the objective measure P, by

$$dS_i(t) = \alpha_i S_i(t)dt + \sigma_i S_i(t-)dN_i(t) + \beta_i S_i(t)dW(t) + S_i(t-)\gamma_i dN. \tag{66}$$

where $N, N_1, N_2, \cdots$ are independent P-Poisson processes. The sequence $\{\gamma_i\}_i$ is assumed to be uniformly bounded, and the concept of a converging index set is extended in the obvious way. The constant P-intensity of N is denoted by λ. The novelty is the systematic component N, which represents market shocks, as opposed to N_i which represents an idiosyncratic shock.

Arguing exactly as in Section 4.2, we obtain the following P-dynamics for a well diversified portfolio along a converging index set.

$$dV = (\alpha_I + \hat{\sigma}_I)Vdt + V\beta_I dW + V^- \gamma_I dN. \tag{67}$$

We can now present the corresponding asymptotic model.

Definition 5.1 *The* **asymptotic model** *consists of the following price processes.*

1. *For each i the process S_i with P-dynamics*

$$dS_i = \alpha_i S_i dt + \sigma_i S_i^- dN_i + \beta_i S_i dW + \gamma_i S_i^- dN, \quad i = 1, 2, \cdots \tag{68}$$

2. *For each converging index set $I \in \mathcal{C}$, the process S_I with P-dynamics defined by*

$$dS_I = (\alpha_I + \hat{\sigma}_I)S_I dt + \beta_I S_I dW + \gamma_I S_I^- dN, \quad I \in \mathcal{C}. \tag{69}$$

5.2 The market price of risk

We follow the methodology and notation of Section 4.3, and change measure from P to Q, while adding a Girsanov transformation of the form

$$\begin{aligned} dL_N &= L_N^- \cdot \mu^\star \left[dN - \lambda dt \right], \\ L_N(0) &= 1. \end{aligned}$$

As before obtain a measure Q with the the properties of Section 4.3, and with the added property that the process N is a Q-Poisson process with Q-intensity λ^Q given by

$$\lambda^Q = (1 + \mu^\star)\lambda.$$

The connection between the "Girsanov kernels" $\varphi^\star$ $\mu_i^\star$, $\mu^\star$, and the various "market prices of risk" φ, μ_i, μ is as before given by

$$\varphi = -\varphi^\star, \quad \mu = -\mu^\star, \quad \mu_i = -\mu_i^\star.$$

Arguing as in Section 4.3, and introducing the notation

$$\hat{\gamma}_i = \gamma_i \lambda,$$

we are led to the following set of equations for the determination of the Girsanov kernels.

$$\alpha_i + \beta_i \varphi^\star + \hat{\sigma}_i(1 + \mu_i^\star) + \hat{\gamma}_i(1 + \mu^\star) = r, \quad i = 1, 2, \cdots \tag{70}$$

$$\alpha_I + \hat{\sigma}_I + \beta_I \varphi^\star + \hat{\gamma}_I(1 + \mu^\star) = r, \quad I \in \mathcal{C} \tag{71}$$

We now come to the main difference between the present setup and the one encountered earlier. In Section 4.3 the corresponding system of equations (42)-(43), was shown to have a unique solution, but in our present setting this is no longer necessarily the case. If e.g. the family $\mathcal{C}$ of converging index sets, is a singleton, then there are infinitely many solutions to (70)-(71). The economic reason is that in this case, the asymptotic model is *incomplete*. In order to have completeness, we intuitively need one traded asset for every source of randomness. We obviously have one asset for each source of idiosyncratic randomness, and by adding the asymptotic asset we also have one asset for one of the systematic jump factors. Since we now have two systematic factors, we will however need two linearly independent asymptotic assets in order to have completeness, and with a singleton $\mathcal{C}$ we only have one. The formal expression of this intuitive reasoning is given as follows.

Proposition 5.1 *Assume that $\mathcal{C}$ contains two converging index sets I and J with the property that the volatility matrix*

$$\begin{pmatrix} \beta_I & \hat{\gamma}_I \\ \beta_J & \hat{\gamma}_J \end{pmatrix} \tag{72}$$

is nonsingular. Then there is a unique solution to the system (70)-(71). In other words, the market prices of risk are uniquely determined within the asymptotic model.

Proof. Under the assumptions above, $\mu^\star$ and $\varphi^\star$ are uniquely determined by the system

$$\begin{aligned} \alpha_I + \hat{\sigma}_I + \beta_I \varphi^\star + \hat{\gamma}_I(1 + \mu^\star) &= r, \\ \alpha_J + \hat{\sigma}_J + \beta_J \varphi^\star + \hat{\gamma}_J(1 + \mu^\star) &= r. \end{aligned}$$

The coefficient $\mu_i^\star$ is then determined by the equation

$$\alpha_i + \beta_i \varphi^\star + \hat{\sigma}_i(1 + \mu_i^\star) + \hat{\gamma}_i(1 + \mu^\star) = r. \quad \blacksquare$$

Regardless of whether the market is complete or not, the market prices of diversifiable risk will still asymptotically tend to zero.

Proposition 5.2 *Assume that $\mu_i^\star$, $\mu^\star$ and $\varphi^\star$ is a solution of the system (70)-(71). Then*

$$\lim_{i \to \infty} \mu_i^\star = 0.$$

Proof. Take any subsequence of the natural numbers. Then this sequence will contain a convergent index set I. Letting $i \to \infty$ along I in equation (70) shows that in fact $\mu_i^\star$ converges to a limit $\mu_I^\star$, and we obtain

$$\alpha_I + \beta_I \varphi^\star + \hat{\sigma}_I(1 + \mu_I^\star) + \hat{\gamma}_I(1 + \mu^\star) = r.$$

From this equation, and from (71) (with the same I) we obtain $\mu_I^\star = 0$. $\blacksquare$

5.3 Hedging and pricing

Given the arguments in the previous section, the completeness question is already more or less resolved: In order to have a complete market we need two linearly independent asymptotic assets.

Proposition 5.3 *Assume that $\mathcal{C}$ contains two converging index sets I and J with the property that the volatility matrix*

$$\begin{pmatrix} \beta_I & \hat{\gamma}_I \\ \beta_J & \hat{\gamma}_J \end{pmatrix} \tag{73}$$

is nonsingular. Then the asymptotic market is complete in the sense that every (Q-square integrable) claim can be hedged.

Proof. Suppose that the claim X is of the form

$$X \in \sigma \left\{ W(t),\ N(t),\ N_i(t);\ t \leq T,\ i = 1, \cdots, n \right\}.$$

Then we adjoin the asymptotic assets S_I and S_J to $S_1, \cdots, S_n$, and it is a standard exercise to see that X can be replicated using a portfolio based upon $S_I, S_J, S_1, \cdots, S_n$. ∎

Proposition 5.4 *Consider any finite claim*

$$X \in \sigma \left\{ W(t),\ N_i(t);\ t \leq T,\ i = 1, \cdots, n \right\}.$$

Then, under the assumptions of Proposition 5.3 the arbitrage free price process, $\Pi(t; X)$, *of the claim is given by*

$$\Pi(t; X) = e^{-r(T-t)} E^Q \left[X \middle| \sigma \left\{ S_I(u),\ S_J(u)\ S_i(u);\ u \leq t,\ i = 1, \cdots, n \right\} \right]. \quad (74)$$

Here the Q-dynamics are given by

$$dS_I = \left(r - \gamma_I^Q \right) S dt + \beta_I S d\tilde{W} + \gamma_I S^- dN, \quad (75)$$

$$dS_J = \left(r - \gamma_J^Q \right) S dt + \beta_J S d\tilde{W} + \gamma_J S^- dN, \quad (76)$$

$$dS_i = \left(r - \sigma_i^Q - \gamma_i^Q \right) S_i dt + \sigma_i S_i^- dN_i + \beta_i S_i d\tilde{W} + \gamma_i S^- dN. \quad (77)$$

where

$$\begin{aligned} \gamma_I^Q &= \gamma_I \lambda (1 + \mu^\star), \\ \gamma_J^Q &= \gamma_J \lambda (1 + \mu^\star), \\ \sigma_i^Q &= \sigma_i \lambda_i (1 + \mu_i^\star), \end{aligned}$$

The processes $N, N_1, \cdots, N_n$ are independent Q-Poisson processes, and the Q-intensity of N and N_i is given by $\lambda(1+\mu^\star)$ and $\lambda_i^Q = \lambda_i(1+\mu_i^\star)$ respectively.

For the case of a simple claim of the form $X = \Phi(S_1(T), \cdots, S_n(T))$ we may as usual compute the arbitrage free price as the solution of a boundary value problem. Since the arguments are exactly the same as in Section 4.5, and since the expressions become quite messy, this is however left to the interested reader.

6 The martingale approach

The strategy in the previous sections has roughly been the the following

- We extended the original model to the asymptotic model by including the set of well diversified portfolios as asymptotic assets.
- We assumed that the asymptotic model was free of arbitrage in the sense that no arbitrage opportunities existed for any finite submodel.
- In this way we claimed that absence of (finite) arbitrage in the asymptotic model could be interpreted as absence of "asymptotic arbitrage" in the original model.

A different approach would be to give a precise definition of "asymptotic arbitrage" for the original model and then to investigate the implications of absence of asymptotic arbitrage. A natural conjecture in this context is of course that absence of asymptotic arbitrage is connected to the existence of a "martingale measure" $Q \sim P$, such that all discounted asset prices are Q-martingales.

In this section we present a simple version of such a theory. We show that the existence of a global martingale measure implies absence of arbitrage and we also investigate the models above in the light of this theory.

A very deep study of asymptotic arbitrage has been carried out by Kabanov and Kramkov (1994),(1996). Instead of studying the existence of globally defined martingale measures, as we do below, they relate absence of asymptotic arbitrage to contiguity (in the sense of Le Cam) properties of certain sequences of local martingale measures. Their theory is considerably more general than the one presented here, but it also requires much harder mathematical tools. Our theory, which is a simple and straightforward generalization of the corresponding finite theory, is presented in the belief that it is more easily understood by the general reader.

Let us denote the finite market consisting only of the assets $S_1, \cdots, S_n$ by $\mathcal{M}_n$. We consider a fixed time horizon T. By h^n we denote a self financing trading strategy on $\mathcal{M}_n$ and the corresponding value process is denoted by

V^n. There are several resonable definitions of asymptotic arbitrage, and the simplest to use is perhaps the following.

Definition 6.1 *An* **asymptotic arbitrage** *is a sequence of strategies* $\{h^n\}_1^\infty$ *such that, for some real number* $c > 0$, *we have*

1.
$$V^n(t) \geq -c, \quad \forall t \leq T, \ \forall n$$

2.
$$V^n(0) = 0, \quad \forall n.$$

3.
$$\liminf_{n\to\infty} V^n(T) \geq 0, \quad P - a.s.$$

4.
$$P\left(\liminf_{n\to\infty} V^n(T) > 0\right) > 0.$$

We use a straightforward definition of martingale measures.

Definition 6.2 *A measure* Q *is called a* **martingale measure** *if the following conditions are satisfied.*

1. $P \sim Q \quad on\ \sigma\{W(s),\ N_i(s); s \leq T\}$
2. *All discounted asset prices* $B^{-1}(t)S_i(t)$ *are* Q*-martingales.*

Exactly as in the finite case we now have the following central (and extremely easy) result.

Proposition 6.1 *Assume that there exists a martingale measure* Q. *Then there is no asymptotic aritrage.*

Proof. Without loss of generality we may assume that the short rate of interest equals zero. Suppose now that $\{h^n\}_1^\infty$ actually realises an asymptotic arbitrage. Then every V^n is a local martingale under Q, and because of the requirement $V^n \geq -c$ we see that every V^n is in fact a Q-supermartingale. Thus we have $0 = V^n(0) \geq E^Q[V^n(T)]$, and Fatou's lemma gives us $E^Q[\liminf V^n(T)] \leq 0$. However,the definition of an asymptotic arbitrage, plus the equivalence between P and Q implies the inequality $E^P[\liminf V^n(T)] > 0$ which leads to a contradiction. ∎

We will now study our earlier models in the light of Proposition 6.1, and since the treatment of the diffusion model and the point process model are completely parallell we confine ourselves to a discussion of the diffusion model.

Viewing each W_n as the coordinate process on canonical space $C[0,T]$ we may write $\Omega = \bigotimes_{n=0}^\infty \Omega_n$, where $\Omega_n = C[0,T]$, and P as $P = \prod_{n=0}^\infty P_n$, where P_n is Wiener measure.

Following Section3.2 we denote the (deterministic and constant) Girsanov kernel for P_n by $-\varphi_n$ for $n > 0$ and $-\varphi$ for $n = 0$. This will transform P_n into the measure Q_n On Ω_n, and the new measure Q on Ω is defined by $Q = \prod_0^\infty Q_n$. In order to make all discounted asset prices into Q-martingales, we furthermore require that the kernels satisfy the relations

$$\alpha_i = r + \varphi_i\sigma_i + \varphi\beta_i, \quad i = 1, 2, \cdots \tag{78}$$

To see if Q is a martingale measure according to the definition above it now only remains to check if $\prod_{n=0}^\infty P_n$ is equivalent to $\prod_{n=0}^\infty Q_n$. To this end we use the Kakutani Dichotomy Theorem (see e.g. Durrett (1996), p.244) which says that $\prod_{n=0}^\infty P_n \sim \prod_{n=0}^\infty Q_n$ if and only if the following condition holds:

$$\prod_{n=0}^{\infty} E^P\left[\sqrt{L_n}\right] > 0 \tag{79}$$

where $L_n = dQ_n/dP_n$ on $\sigma\{W_n(s); s \leq T\}$.

We immediately have

$$L_n = e^{-\varphi_n W_n(T) - \frac{1}{2}\varphi_n^2 \cdot T},$$

and the Kakutani condition (79) is easily seen to be equivalent to the condition

$$\varphi^2 + \sum_1^\infty \varphi_n^2 < \infty. \tag{80}$$

Using (78), the condition (80) reads

$$\varphi^2 + \sum_{n=0}^{\infty} \frac{1}{\sigma_n^2} \{\alpha_n - r - \beta_n \varphi\}^2 < \infty, \tag{81}$$

which implies that

$$\alpha_n - r - \beta_n \varphi(t) \to 0, \tag{82}$$

and if we take the limit along any converging index set I we obtain

$$\varphi(t) = \frac{\alpha_I - r}{\beta_I}, \quad \forall I \in \mathcal{C} \tag{83}$$

which, together with (78) gives us

$$\varphi_n(t) = \frac{\alpha_n - r}{\sigma_n} - \frac{\beta_n}{\sigma_i} \frac{\alpha_I - r}{\beta_I}, \quad \forall I \in \mathcal{C}. \tag{84}$$

Thus we obtain exactly the same formulas as in Proposition 3.1. The difference between our different approaches is that the martingale approach is more general than the "asymptotic model appraoch". Concretely this is seen by the fact that while we only obtain the result that $\varphi_n \to 0$ in the analysis using asymptotic assets, we have the stronger result $\sum_0^\infty \varphi_n^2 < \infty$ using the martingale approach. The point of the asymptotic model approach is of course that it gives an intuitive interpretation of the abstract martingale results in terms of the well diversified portfolios

References

[1] Artzner, P. & Delbaen, F. (1995): Default risk insurance and incomplete markets. *Mathematical Finance* **5**, 187-195.

[2] Back, K. (1988): Factorization of returns in multiperiod securities markets. Working paper. *School of Business, Indiana University.*

[3] Björk, T. & Kabanov, Y. & Runggaldier, W. (1997a): Bond market structure in the presence of marked point processes. *Mathematical Finance,* **7**, 211-239.

[4] Björk, T. & Di Masi, G. & Kabanov, Y. & Runggaldier, W. (1997b): Towards a general theory of bond markets.*Finance and Stochastics,* **1**, 141-174.

[5] Brémaud, P. (1981) *Point Processes and Queues: Martingale Dynamics.* Springer-Verlag, Berlin.

[6] Chamberlain, G.(1988) Asset pricing in multiperiod securities markets. *Econometrica.* **56**, 1283-1300.

[7] Chen, N. & Roll, R & Ross, S.A. (1986) Economic forces and the stock market. *journal of Business* **59**, 673-700.

[8] Duffie, D. & Singleton, K. (1995) Modeling of term structures of defaultable bonds. Working paper, *graduate School of Business, Stanford University.*

[9] Durrett, R (1996) Probability: *Theory and Examples* (2:nd ed.). Duxbury Press.

[10] Hubermann, G. (1982) A simple approach to Arbitrage Pricing Theory. *Journal of Economic Theory.* **28**, 183-191.

[11] Jacod, J. & Shiryaev A.N. (1987) *Limit Theorems for Stochastic Processes.* Springer-Verlag, Berlin.

[12] Jarrow, R. & Turnbull. (1995) Pricing derivatives on financial securities subject to credit risk. *Journal of Finance* **50**, 53-86.

[13] Kabanov, Y.M. & Kramkov D.O. (1994) Large financial markets: asymptotic arbitrage and contiguity. *Probability Theory and its Applications.* **39**, 222-229.

[14] Kabanov, Y.M. & Kramkov D.O. (1996) Asymptotic arbitrage in large financial markets. Preprint.

[15] Madan, D. & Unal, H. (1997)Pricing the risks of default. Working paper *College of Business and Managemant, University of Maryland.*

[16] Merton, R. (1973) An intertemporal capital asset pricing model. *Econometrica.* **41**, 867-888.

[17] Merton, R. (1976) Option pricing when the underlying stock returns are discontinuous. *Journal of Financial Economics.* **5**, 125-144.

[18] Ross, S. (1976) The arbitrage theory of asset pricing. *Journal of Economic Theory.* **13**, 341-360.

[19] Reisman, H. (1992) Intertemporal Arbitrage Pricing Theory. *The Review of Financial Studies* **5** No.1, 105-122.