The Yield Envelope: Price Ranges for Fixed Income Products

by

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Abstract

There is an extensive literature on the valuation of a fixed income contracts. The present work addresses the problem from a new outlook: we find *upper and lower bounds* for the value of a contract. Constraints are imposed on the evolution of a short-term interest rate and a liability is valued using its present value. A first-order non-linear hyperbolic partial differential equation is formulated for the value, *V*, of the contract. The numerical solution of this equation is then explored. To optimise its value, our contract is hedged with market traded zero-coupon bonds. We can then generate the 'Yield Envelope' (an alternative to the yield curve). At a maturity for which there are no traded instruments, a spread for the yield is obtained. A program implementing our model can be downloaded by clicking here.
 (LINK:http://www.oxfordfinancial.co.uk/oxford/ew_model.zip)

Introduction

The two main classical approaches to pricing and hedging fixed income products may be termed 'stochastic' and 'duration-based.' The former approach assumes that interest rates are driven by a number of random factors. Often a model for just a short-term rate will give as an output the whole yield curve (see Hull, 1996, and Wilmott, Dewynne & Howison(LINK:www.oxfordfinancial.co.uk/), 1993, for example). The latter approach assumes that interest rates are constant for each product, which, of course, is inconsistent across products (see Fabozzi, 1996). In this paper we take a different starting point, by assuming as little as possible about the process underlying the movement of interest rates.

We model a short-term interest rate and price a contract in a *worst (and best) case scenario*, using the short rate as the discount rate. That is, assume that the short-term interest rate evolves in a way that is consistent with our model and such that the present value of the contract is as low (and high) as possible. The resulting problem is non-linear and thus the value of a contract is not always equal to the value of its consituent parts. For the same reason, long and short positions can have different values.

We will construct a stochastic model for the interest rate. But, it will not be a stochastic model in the Brownian motion sense. There will be no probability statements except to assign zero probabilities to certain events. For instance, there is a zero probability that the interest rate will move outside of a certain range and there is a zero probability that the interest rate will grow or decay faster than a certain rate. Our financial world will include all market-traded instruments. These will be used for hedging purposes rather than data fitting, to optimise the value of the contract we are trying to price. The nonlinearity of the problem means that the value of the contract is dependent on what it is hedged with.

The object is to find minimum and maximum levels for the value of our contract. The spread between these values may be large. But this spread will decrease as the number of appropriate hedging instruments increases. The closer the cash flows of the contract are to cash flows of traded instruments, the better the hedge and the narrower the spread will be. If there are no suitable hedging instruments, the contract will be very risky and model dependent. Naturally, the spread will be larger in this case.

This method has two main applications. The model can be used to value a new product *that is not yet traded,* but will be hedged with market-traded instruments to reduce the spread. In this case, the model gives us a spread for the price that is consistent with existing instruments. The model can also be used to value an existing product, with known market price. In this case, the model can spot arbitrage opportunities. The model predicts a spread for the possible value of the product. If the market price lies outside of this range, then there is an obvious arbitrage opportunity. If the market price lies within the range, then we can hedge with this product, as it is market-traded, and find that our minimum and maximum possible values converge to the market price of the product.

We conclude with the *Yield Envelope*. This is a sophisticated version of the yield curve. A yield spread is found at maturities for which there are no traded instruments. At a maturity for which there is a traded instrument, in the absence of arbitrage, the Yield Envelope converges to the observed yield.

Model for the behaviour of the short-term interest rate

Motivated by a desire to model the behaviour of the short-term interest rate, r, over long periods of time with as much freedom as possible, we assume only the following constraints on its movement:

$$r^- \le r \le r^+ \qquad (1)$$

and

$$c^- \le \frac{dr}{dt} \le c^+. \qquad (2)$$

Equation (1) says that the interest rate cannot move outside the range bounded below by the rate r^- and above by the rate r^+. Equation (2) puts similar constraints on the speed of movement of r. The constraints can be time dependent and, in the case of the speed constraints, functions of the spot interest rate, r.

Worst case scenarios and a non-linear equation

In this section the equation governing the worst case price of a fixed income contract is derived. Let $V(r,t)$ be the value of our contract when the short-term interest rate is r and the time is t. The change in the value of this contract during a timestep dt is considered.

Using Taylor's theorem to expand the value of the contract for small changes in its arguments, we find that

$$V(r+dr, t+dt) = V(r,t) + V_r dr + V_t dt + o(dr,dt).$$

This change is to be investigated under our worst case assumption. The change is then given by

$$\min_{dr}(dV) = \min_{dr}(V_r dr + V_t dt).$$

Since the rate of change of r is bounded, we find that

$$\min_{dr}(dV) = \min_{dr}(V_r dr + V_t dt) = (c(V_r)V_r + V_t)dt$$

where

$$c(x) = \begin{cases} c^+ & \text{for} \quad x < 0 \\ c^- & \text{for} \quad x > 0 \end{cases}.$$

In the worst case, it is required that our contract always earns the risk-free rate of interest. This gives us

$$V_t + c(V_r)V_r - rV = 0. \qquad (3)$$

This is the equation to be solved. It is a first-order non-linear hyperbolic partial differential equation, with known final data $V(r,T)$. This is the value of the final cash flow in the contract. In the case of Brownian motion, it is natural to have a convexity term. But in this world, the rate of change of r is bounded and the convexity term disappears in the limit $dr, dt \to 0$.

If there is a cash flow $C(r)$ at time T_C, then over the cash flow date we have
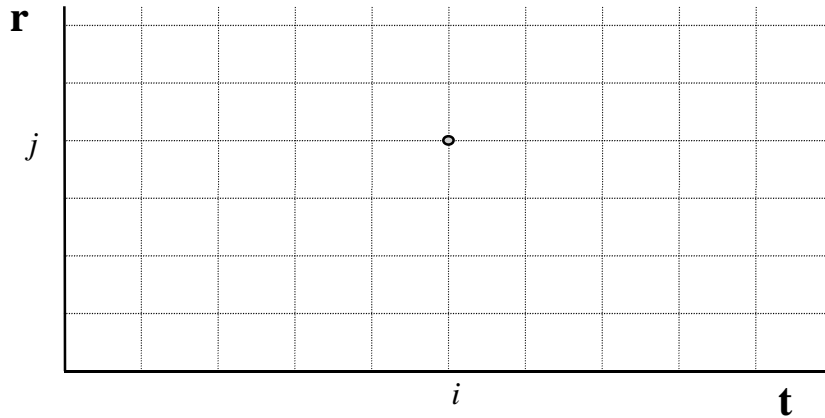
$$V(r, T_C^+) = V(r, T_C^-) - C(r),$$

where T_C^- is just before the cash flow date and T_C^+ is just after.

A simple trinomial model

The problem is discretised, to solve it numerically, using a grid of m space steps and n time steps. A point then has position

$$(r,t) = (r^- + i\Delta r, j\Delta t),$$

where $0 \leq i \leq m, 0 \leq j \leq n$, $\Delta r = \dfrac{(r^+ - r^-)}{m}$ and $\Delta t = T/n$. This grid is shown in Figure 1.
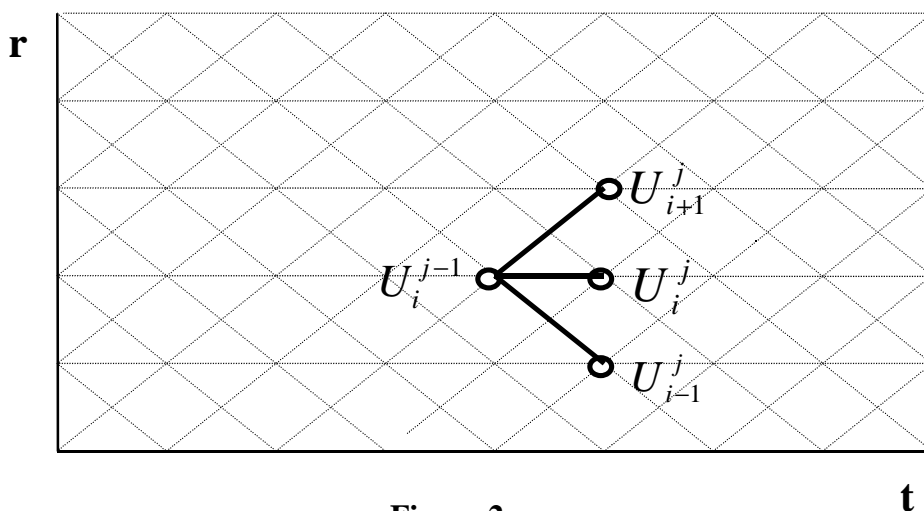

Figure 1

The solution V at a point is approximated by U, where,

$$V(r,t) = V(r^- + i\Delta r, j\Delta t) \approx U_i^j.$$

Equation (3) can then be discretised using an explicit finite-difference scheme. This will solve the problem for general c^- and c^+. In the special case where $c^- = c^+ = c$, a constant, the faster trinomial scheme can be used. This is a lattice method. For both methods, the jump condition at a cash flow date (for a cash flow C) is discretised by including $U_i^j = U_i^j + C$ in our scheme (as we work backwards in time).

To implement the trinomial scheme, a grid for which $c\Delta t = \Delta r$ is formed. In this case, over one time step, the interest rate can jump from r to one of three values: $r - \Delta r$, r, or $r + \Delta r$. There is a lattice structure for the possible interest rate evolution, as shown in Figure 2.

Figure 2

If the solution at time step j is known, then the solution at time step j-1 can be worked out as follows.

The solution at the point U_i^{j-1} must originate from the points U_{i-1}^j, U_i^j and U_{i+1}^j as the interest rate can move at most one space step during a single time step. In a worst case scenario, the value of the contract at time step j-1 is clearly the minimum of the discounted values of the contract at time step j. We discount at the average of the interest rates at time steps j and j-1. This leads to the scheme

$$U_i^{j-1} = \min \begin{cases} U_{i-1}^j(1-(r^- +(i-0.5)\Delta r)\Delta t \\ U_i^j(1-(r^- +i\Delta r)\Delta t \\ U_{i+1}^j(1-(r^- +(i+0.5)\Delta r)\Delta t \end{cases}.$$

The unhedged contract

In addition to the worst case scenario, the value of our contract can be found in a best case scenario. This is equivalent to a worst case scenario where we are short the contract. The above analysis then holds for the best case scenario, where we have

$$-V_{best} = (-V)_{worst}.$$

A contract consisting solely of a single principal payment (i.e. a zero-coupon bond) is shown in Figure 3. Figures 4 and 5 show the worst and best case scenario valuations, respectively, for a zero-coupon bond with principal 1, with varying initial spot rate and time to maturity. Figures 6 and 7 show the yield for these zero-coupon bond prices, in a worst and best case scenario, respectively.
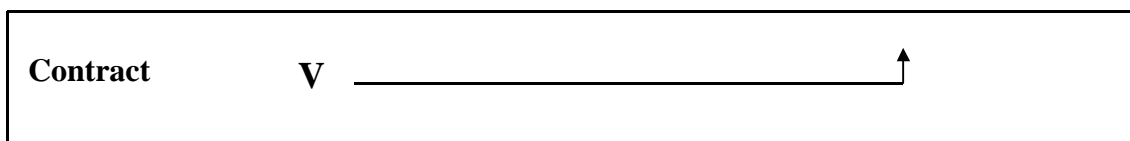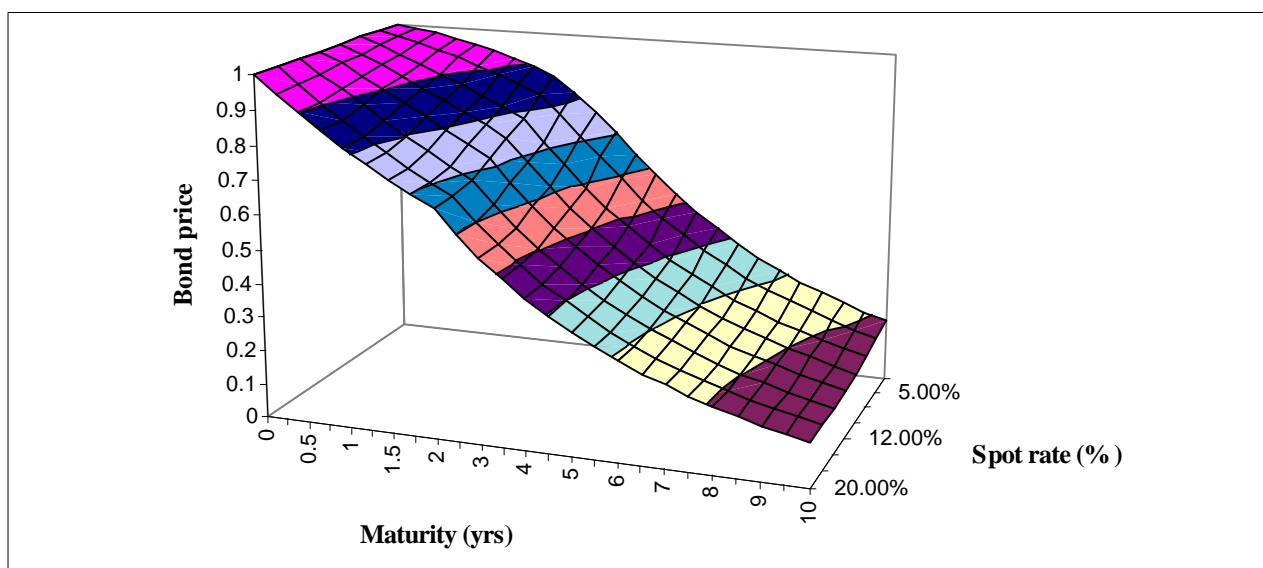

Figure 3

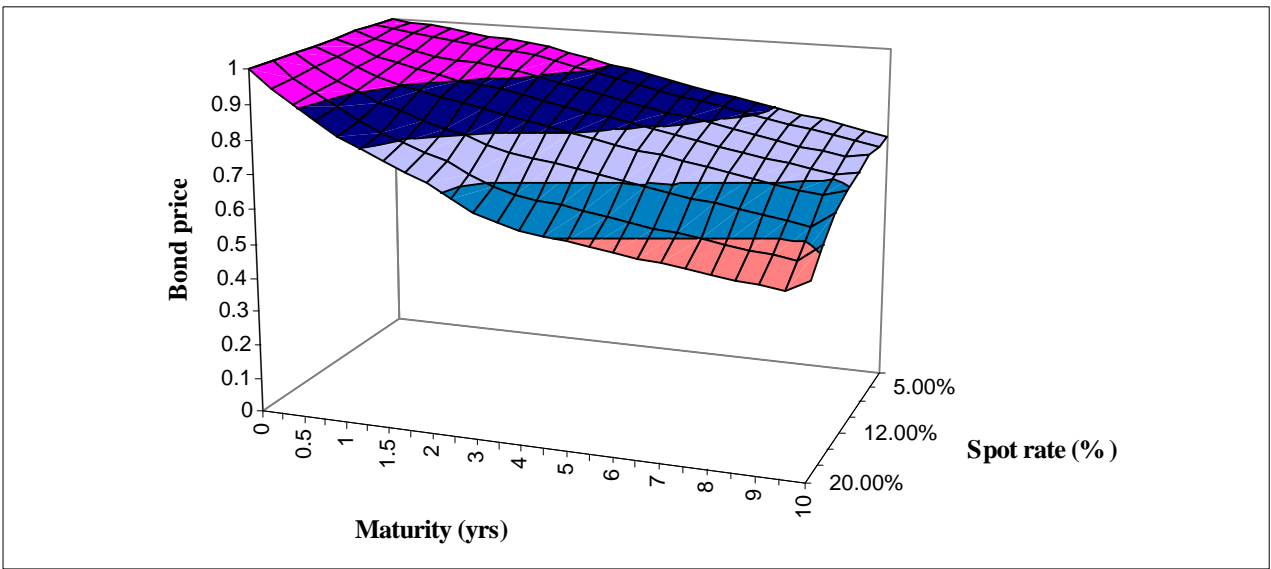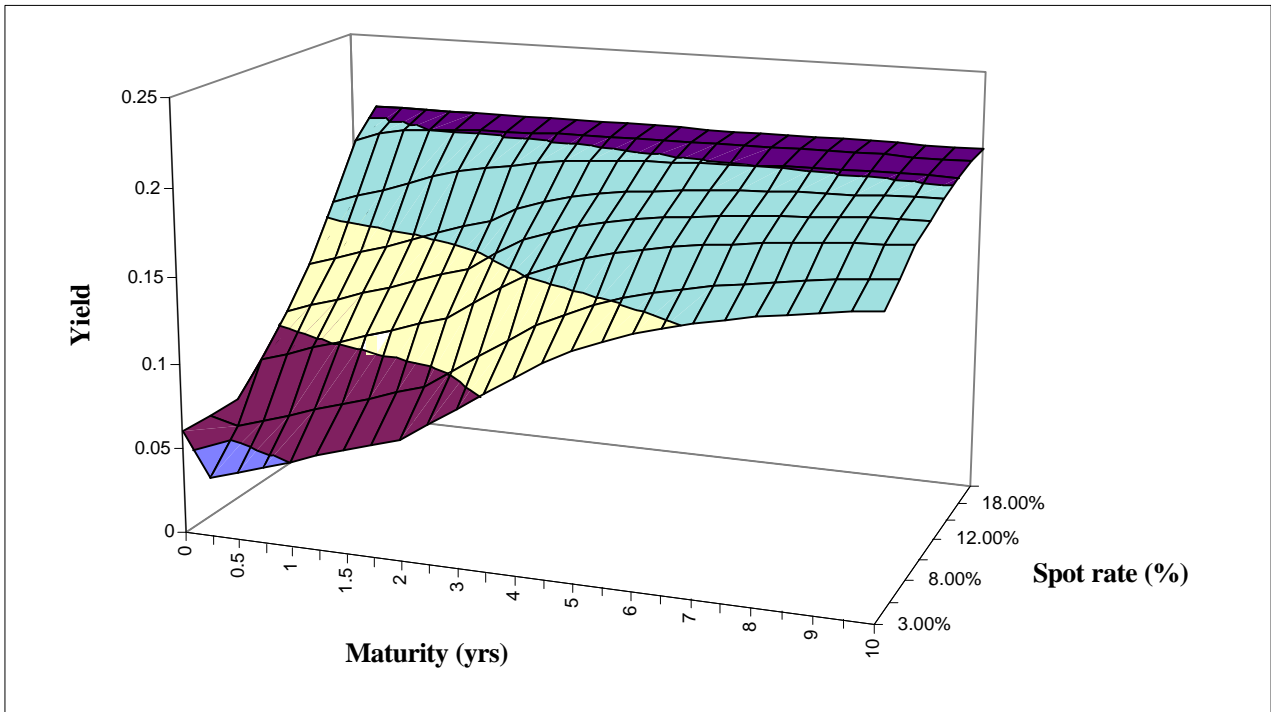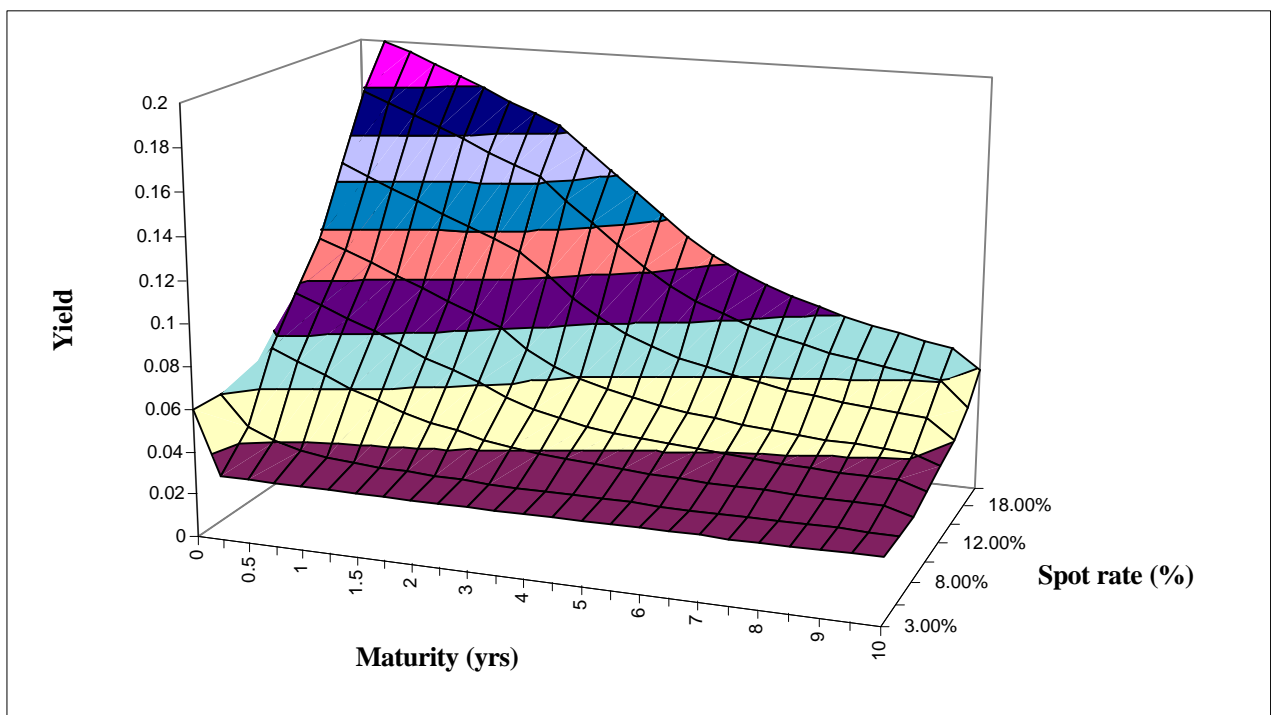

Figure 4

Figure 5

Figure 6

Figure 7

Hedging our contract

We find that the spread between the worst and best case values for both bond price and yield is large. To reduce the spread in the value of a contract, we hedge with market traded instruments. Since the valuation problem is non-linear in nature, the value of our contract in isolation will not necessarily equal the value of the contract when it is hedged with other market-traded instruments. We can write that the value of the contract is the marginal value, after hedging with the traded instruments, i.e.

value(hedged contract) = value(contract + hedging instruments) - cost of hedge.

Here '*value*' means the solution of the non-linear differential equation together with all relevant final and jump conditions. The value of a contract therefore depends on the instruments with which we hedge. We expect a different value if we choose a different static hedge. There is an optimal static hedge for which the worst case value of our contract is as high as possible, and another optimal static hedge for which the best case value of our contract is as low as possible. This optimisation technique is similar in philosophy to that used to hedge volatility risk with derivatives (see Avellaneda, Levy and Paras, 1995). To find this optimal static hedge in the worst case scenario, we maximise the value of our contract with respect to the hedge quantities of the hedging instruments. In the best case scenario, we minimise with respect to the hedge quantities.

This idea is best demonstrated with an example. Suppose that we wish to find the best worst case value for a given contract and that we can hedge it with a particular traded instrument. This traded instrument has a market price of P. The question to ask is 'How many of this instrument should we buy or sell for the optimal hedge?' In our informal notation this problem becomes

maximise over λ [*value(hedged contract)*],

or, equivalently,

maximise over λ [*value(contract + λ hedging instrument) - λ P*].

Here λ is the number of the traded instrument that we buy.

We assume that the market price of a hedging instrument is contained in the spread of values for the instrument generated by our model. If this were not the case, we could make an arbitrage profit by selling (buying) the instrument at a price above (below) its maximum (minimum) possible value, assuming that the interest rate moves within the constraints of our model. If we value a traded contract, in the absence of arbitrage, then the worst and best case values both converge to the market value when we hedge. This is because we can hedge perfectly one for one to leave no residual.

An example of hedging

In practice, we hedge with all the market-traded instruments available i.e. everything contained in our financial world. There will still be an optimal static hedge for which we obtain the maximum value for our contract in a worst case scenario (or the minimum value in a best case scenario). For instance, consider again the contract consisting of a single principal payment, shown in Figure 3. We now hedge our contract with the zero-coupon bonds A, B, C, D, E, F and G, as shown in Figure 8. In a real situation, of course, our original contract, *V*, would contain more than a single cash flow!
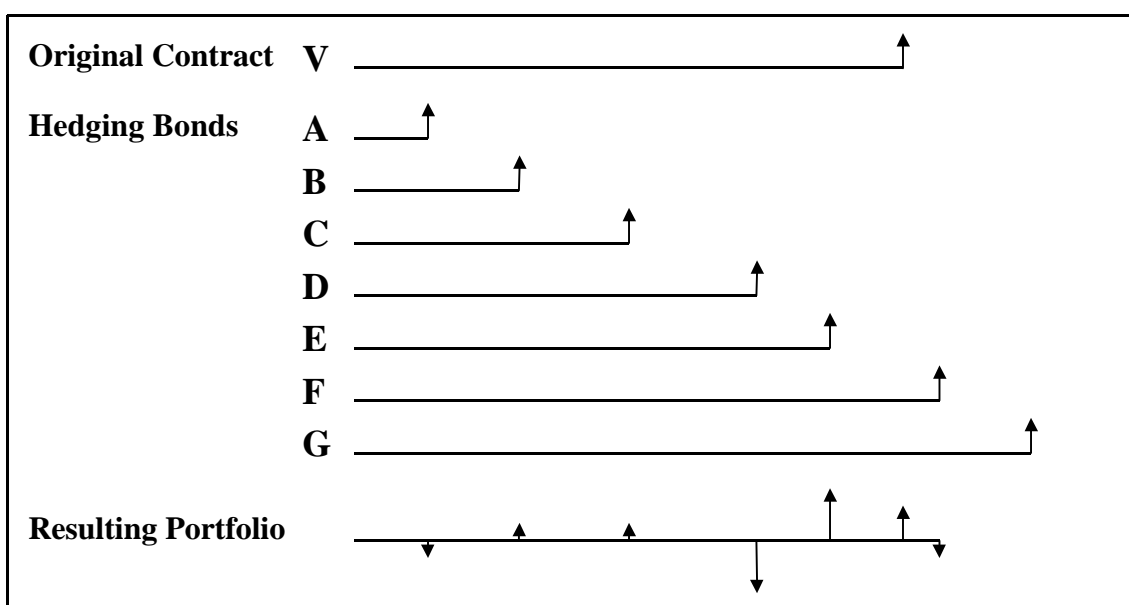

Figure 8

For all the valuations in this paper, we have taken $r^- = 0.03$, $r^+ = 0.2$, $c = 0.04$ and use the trinomial scheme with $\Delta r = 1/500$ and $\Delta t = 1/20$. (To choose the growth rate bounds, we examined market data for the US dollar 1 month rate, from 21/10/86 to 25/4/95, and found that for consistency with our model, we could take $c^+ = -c^- = 0.04$, i.e. a growth rate of between -4% and 4% p.a. Further details may be found in Epstein, 1996).

Example: We hedge a 4 year zero-coupon bond, with principal 1, with market-traded zero-coupon bonds (all with principal 1). These hedging bonds are shown in Table 1. The initial spot rate is 6%.

Hedging bond	Maturity (years)	Market price
A	0.5	0.970
B	1	0.933
C	2	0.868
D	3	0.805
E	5	0.687
F	7	0.579
G	10	0.449

Table 1

The results of the valuation, with and without the optimal static hedge are shown in Table 2. Hedging has reduced the spread in the value of the 4 year bond from 0.299 to 0.028. The optimal static hedges for the worst and best case valuations of the 4 year bond are shown in Table 3.

	Worst case	Best case
No hedge	0.579	0.878
Optimal hedge	0.730	0.758

Table 2

Hedging bond	Worst case hedge quantity	Best case hedge quantity
A	-0.051	0.204
B	0.024	-0.135
C	0.190	0.167
D	-0.734	-0.685
E	-0.466	-0.473
F	0.029	0.000
G	0.000	0.000

Table 3

We observe that the hedge quantities for worst and best case scenario valuations differ and that the optimal static hedge will not necessarily contain all of the hedging instruments.

Generating the 'Yield Envelope'

We continue with our philosophy of finding spreads for prices by generating the Yield Envelope. At a maturity, for which there are no traded instruments, we find a yield spread. But at a maturity where there is a traded instrument, the Yield Envelope converges to the observed yield.

We calculate the worst and best case values of a zero-coupon bond, Z, with principal 1 and maturity T. We then calculate the maximum and minimum yields possible, using

$$Y = -\frac{\log Z}{T}.$$

To reduce the yield spread, we again hedge our zero-coupon bond with market-traded zero-coupon bonds.

Example: We hedge our zero-coupon bond with the bonds in Table 1. The initial spot rate is 6%. The results are shown in Figure 9 and Table 4. Figure 9 also includes plots of the yield for the unhedged bond.

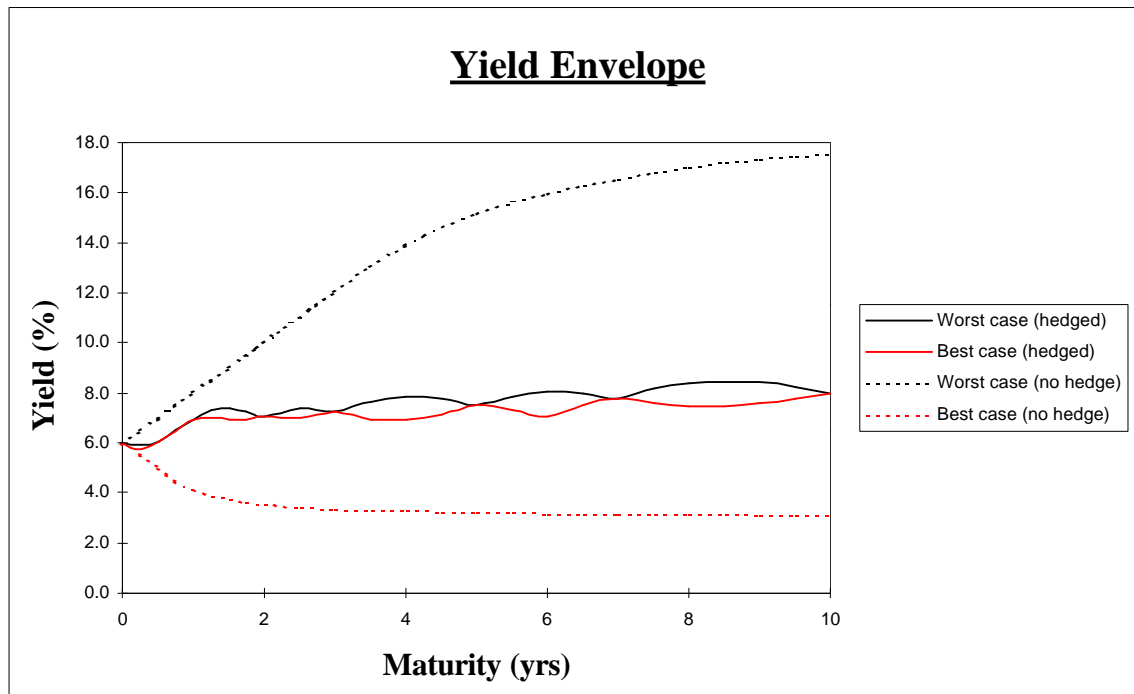| Maturity (yrs)	Yield in worst case (%)	Yield in best case (%)	Maturity (yrs)	Yield in worst case (%)	Yield in best case (%)
0	6.000	6.000	4.5	7.777	7.138
0.25	5.919	5.706	5.0	7.508	7.508
0.5	6.092	6.092	5.5	7.877	7.329
0.75	6.545	6.485	6.0	8.029	7.079
1.0	6.935	6.395	6.5	7.986	7.517
1.25	7.313	7.002	7.0	7.806	7.806
1.5	7.378	6.908	7.5	8.159	7.558
1.75	7.269	6.933	8.0	8.381	7.438
2.0	7.078	7.078	8.5	8.454	7.450
2.5	7.357	6.969	9.0	8.407	7.571
3.0	7.230	7.230	9.5	8.259	7.786
3.5	7.668	6.942	10.0	8.007	8.007
4.0	7.864	6.927			

Table 4

Figure 9

Figures 10 and 11 show the yield in a worst and best case scenario, respectively, with varying spot rate and maturity, when we hedge with the bonds shown in Table 5.
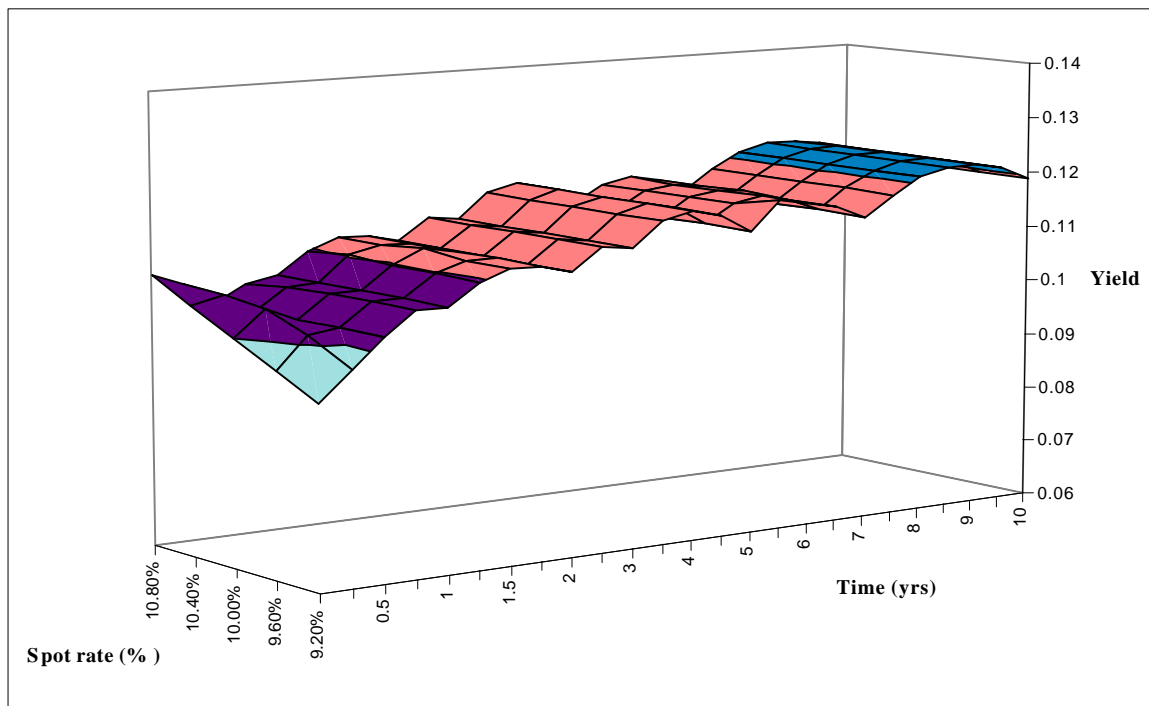
Figure 10
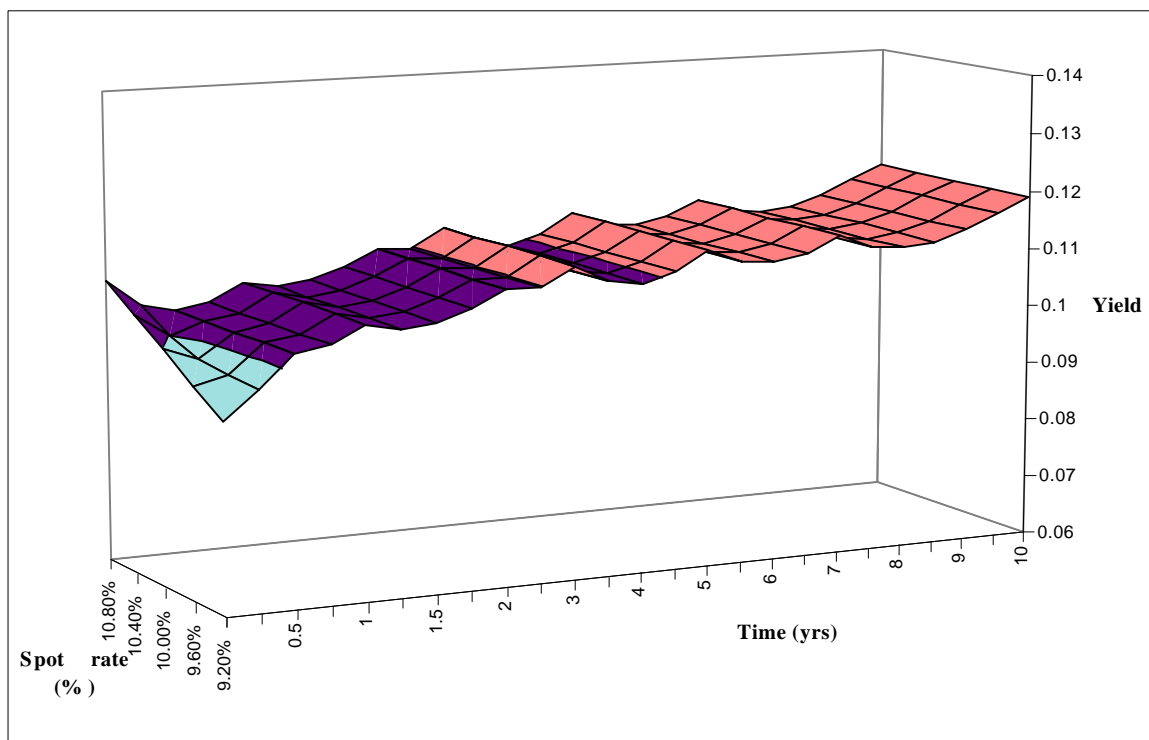
Figure 11

Hedging bond	Maturity (years)	Market price
A	0.5	0.950
B	1	0.899
C	2	0.803
D	3	0.712
E	5	0.566
F	7	0.448
G	10	0.304

Table 5

We can see from Figure 9, or by comparing Figures 6, 7 and 10, 11 that hedging has significantly reduced the spread in the yield. At maturities for which there are traded instruments, the hedged yields in both worst and best case scenarios equal the observed yield. This is because we can fully hedge our zero-coupon bond with the market-traded instrument, leaving a zero residual. This is the optimal static hedge in both scenarios. The value of the zero-coupon bond is then the market price and the yield is therefore the observed value.

Conclusions

In this document we have presented a new model for valuing and hedging interest rate securities. The framework is that of 'worst and best case scenarios' and we have derived a non-linear first-order partial differential equation for the value of a contract. Since the equation is non-linear we find that *the value of a product depends on the contents of the hedging portfolio*. This non-linearity gives the model both advantages and disadvantages compared with other, more traditional approaches. The main disadvantage is that to obtain the full benefit of the model one must solve the equation for the entire portfolio. The principal advantage is that optimal hedges can be found, maximising or minimising the contract's value. A further advantage of the model is that one can be fairly confident in the accuracy of the model parameters. The model can be used to generate the Yield Envelope. This predicts yield spreads at maturities for which there are no market-traded products, but converges to the observed yield at maturities for which there are. In future work we will show how to apply the model to pricing and hedging more complicated contracts (see the working paper by Epstein & Wilmott, 1996).

Acknowledgements

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