Report 3

Pricing Interest Rate Related Instruments

Soad Abuhawas[1], Jeffrey Burnett[2], Oliver Díaz[3], Chuan Hsiang Han[4], Mahendra Panagoda[5], Yi Zhao[6]

Problem Presenter:
Yeol Cheol Seong
Bank of Montreal

Faculty Consultant:
Jean-Pierre Fouque

Abstract

This paper is concerned with implementing a method for pricing interest rate related derivatives. We first show a method for estimating the term structure of interest rates from market data and then show how this term structure is used to calibrate the Black-Derman-Toy model, a binomial model for the evolution of the short rate. An alogrithm for constructing the model is given and prices for several interest rate derivatives are then calculated using the Arrow-Debreu pricing scheme. Lastly, a framework for pricing coumpound options is explored.

3.1 Introduction and Motivation

Our project is concerned with the relationship between market interest rates and the valuation of derivatives on debt instruments. These debt instruments are contracts by which an investor lends money to a borrower in return for the promise of future cashflows. Derivatives on these debt instruments allow an investor to negotiate terms today for a debt instrument contract in the future. Many such derivatives allow the investor the option of entering into the debt agreement. Such derivatives are fittingly called "options." Another class of derivative contracts are "futures," which do not allow the option to exercise, but the terms of the negotiated price may be traded on the open market. Our task is to take public information about the price of traded assets to build

[1]Texas Technical University
[2]North Carolina State Univeristy
[3]University of Texas at Austin
[4]North Carolina State University
[5]Michigan State University
[6]Rutgers University

a framework for understanding the present value of future cashflows and to then determine a fair price for a variety of derivative contracts based on future cashflows.

Constructing an accurate model of expected future interest rates is of tantamount importance to members of the financial community. An accurate model allows both borrowers and lenders of money to protect themselves against unfavorable shifts in the interest rate. Such a model also allows firms the ability to negotiate the terms today for a loan that will begin at some time in the future. Such a model is often called the "term structure of interest rates." The term structure is not one number or one function, but a group of relationships that describe different aspects of future interest rates. There are four main aspects of the term strucutre: the discount curve, a spot interest rate curve, the implied forward rate curve, and the par yield curve. We must also distinguish between a model of today's yield curve and a dynamic model of the term structure that will allow us to price instruments other than bonds. We show a technique for estimating today's yield curve from market data called "bootstrapping" the yield curve. We will also explore one of the many models of the dynamic term structure of interest rates. This model will take the information gleaned from the "bootstrapping" technique as input.

The bootstrapping method estimates a discount curve for present value calculations. Because of a one-to-one relationship between the curves, once one is found the others may be computed. The different curves draw out different aspects of the purchasing power of future cashflows, and which is used depends on the task at hand. In particular, if an option on a financial asset is expected to pay some amout in six months, we must use the discount curve to discount the payoff if we want to know the value in today's prices. This is due to the time value of money, by which the promise of a dollar tomorrow is worth less than a dollar in today's prices. But how can we model interest rates? Interest rates are not traded like stocks, so we cannot observe the price and use it directly in our model. We can, however, observe the price of bonds, and the market determined prices may be our best bet for building a model.

To construct a dynamic model of the term structure we must decide how many dimensions to incorporate into our model. The most accurate model would have a dimension for bonds maturing at every time from now into the infinite future. As such, this would be an infinite dimensional stochastic differential equation and it would be intractable. Instead we could choose to model the short rate of interest, the rate for very short periods of time. We could, in fact, construct a model in which two or more factors would influence the short rate, but the simplest model incorporates only one source of randomness.

Equipped with an understanding of the term structure of interest rates and a model of the short rate, we can begin to price a variety of derivative securities whose payoff is a function of the evolution of the interest rate. There are two basic classes of options: a call option gives the buyer of the option the right to buy something in the future at a price determined today, and a put option gives the buyer of the option the right to sell something at a predetermined price at some future date. Naturally, there are buyers and sellers of options, so depending on the circumstances you might buy a call or sell a put. Combinations of these basic building blocks into so-called structured products allow flexible payoffs to be built that can act as very specialized insurance policies against the unknown. Options based on the interest rate may pay, for example, if the interest rate rises above some predetermined level. This type of option would protect borrowers from high interest rates. Other options allow the buyer to swap a floating interest rate for one that is fixed, if it benefits the buyer. We will use our dynamic model of the term structure to price a family of such interest rate derivatives. We will show techniques for pricing swaps, swaptions, caps and floors, and show prices determined from market data. We will also explore ways to price a compound option, a structured product that gives the option to buy or sell a group of options.

The sticking point with options is determining the fair price for such a right. If the price is too high or too low then someone may make money with probability one; that is, an arbitrage has been created. One way to determine the value of an option today is to calculate the expected payoff of the option and then discount this payoff. Discounting the expected payoff is necessary because of the time value of money. We must be careful, though, to use the correct probabilities when we calculate the payoff. We must use the probabilities as if the game were fair, the so-called risk-neutral probabilities. Luckily, with interest rates we deal solely in the risk-neutral world so we will not have to change our measure.

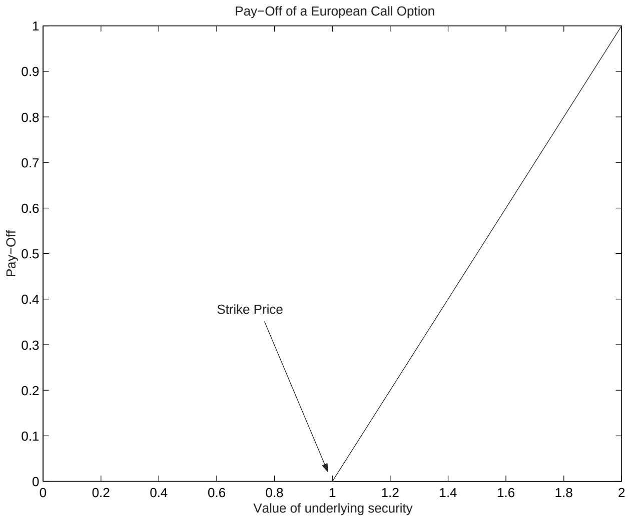

Figure 3.1: Payoff for European Call Option

The call pays $\max(S_T - K, 0)$ *to the buyer of the option, where* S_T *is the asset price at time* T *and* K *is the strike price. The purchaser of the call buys the right to buy an asset below the going market price.*

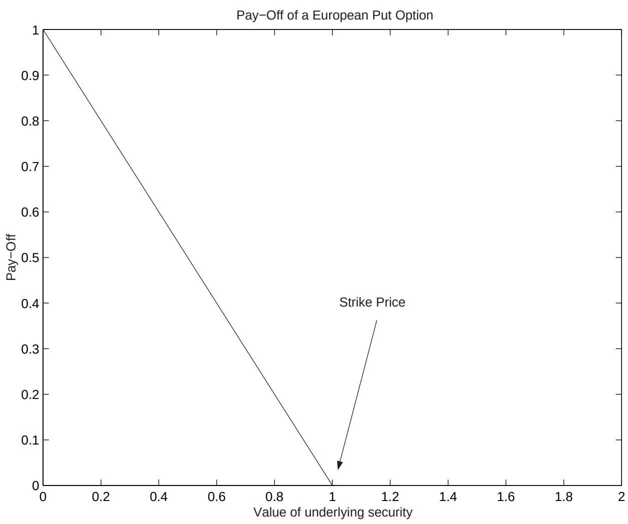

Figure 3.2: Payoff for European Put Option

The put pays $\max(K - S_T, 0)$ *to the buyer of the option, where* S_T *is the asset price at time* T *and* K *is the strike price. Thus, the purchaser of the option buys the right to sell above the market price.*

3.2 The Yield Curve

3.2.1 Estimating the yield curve

We will first describe the “bootstrapping” technique. In risk management, it is important that we value future cash flows of an asset in a consistent and dependable way. The yield on a bond is often described as its internal rate of return and a yield curve is a plot of the yield versus the time to maturity starting today. Since there is no consistent set of yields that we can observe, we must construct a yield curve by stiching together information from several different sources. In a broader sense, the yield curve is important in pricing stocks and futures, managing risk and trading structured derivatives as it encodes information about liquidity, inflation rates and the market risk. In essence, the yield curve reflects the state of the economy. Our goal will be to make one curve for the discount factor for different times to maturity. That is, we will recharacterize everything as a zero-coupon bond. If the cash flow t years from today is C(t) and the discount factor is Z(t), then the present value of the cash flow is:

$$\text{Present Value} = C(t)Z(t).$$

The timing for cash flows are specified in the contracts. Zero-coupon bonds pay nothing until maturity, when all interest and principle is repaid. Coupons bonds usually pay interest twice a year and repay principle and a final interest payment upon maturity. The “yield curve” gives information on $Z(t)$, the discount factor, enabling us to price the asset at present accordingly.

Our goal is to build a yield curve for Eurodollars with the data available from the markets, but which data is best? Economic theory tells us that the prices on actively traded assets will reach an equilibrium so that the price reflects the true demand, so we will use data on actively traded assets. Such assets are described as being liquid, since there is a ready market for them. Since by definition the yield curve describes the term structure of interest rates in the market, we will use money market, futures and swap prices to come up with a single yield curve. Since money market data gives the best information on discount factors for relatively small time scales, we use money market data coupled with the following formula to obtain the first discount factors:

$$Z(t) = \frac{1}{1 + d(t)\alpha(t_0, T)},$$

where $d(t)$ is the cash deposit rate which is given and α is the accrual factor between day t_0 and T. The spot date or the settlement date is t_0.

In calculating the accrual factor we make use of a program often called a calendar generator. This program calculates the number of business days between two given dates. In addition to weekends, the calendar generator takes into account the New York Stock Exchange (NYSE) holidays and United States federal holidays. Different markets use different conventions in evaluating the accrual factor: for example, the Euro data is calculated on the basis known as 30/360. This convention assumes that every year has 360 days and every month is made of 30 days and follows the algorithm

$$\text{number of days between } T_1 \text{ and } T_2 = (D_2 - D_1) + 30(M_2 - M_1) + 360(Y_2 - Y_1),$$

D is day portion of the date, M is month part, and Y is the year portion. So, the number of days between March 5, 2001 and July 15, 2005 is

$$\begin{aligned} \text{number of days} &= (15 - 5) + 30(7 - 3) + 360(2005 - 2001) \\ &= 10 + 120 + 1440 \\ &= 1570 \end{aligned}$$

Other conventions, such as ACT/ACT, ACT/365 and ACT/360 count actual days in each month and the actual number of days in the year (either 365 or 366), and combinations of these. The calendar generator code employes a lookup table containing every holiday for each of the two most common markets in the United States(NYSE and Federal) until 2030. The holidays were calculated algorithmically, so the generator could determine holidays for longer time horizons. Tables are used to economize on computations.

Futures data provide the most reliable information on market conditions for intermediate times to maturity. Therefore, we use the available data on Euro futures to calculate discount factors for times typically from a few months to a few years. With respect to Euro futures, the settlement date is second business day before the third Wednesday of March, June, September and December. And, as before, we calculate the discount factor at time T as

$$Z(T_2) = Z(T_1)\frac{1}{1+(T_1,T_2)\alpha(T_1,T_2)},$$

where $Z(T_i)$ is the discount factor for futures at time T_i. From the given data on $P(T_1,T_2)$ we calculate $f(T_1,T_2)$, the forward rate, using the expression

$$f(T_1,T_2) = \frac{100 - P(T_1,T_2)}{100}.$$

For any i, we have the recurrence formula

$$Z(T_i) = Z(T_{i-1})\frac{1}{1+f(T_{i-1},T_i)\alpha(T_{i-1},T_i)}.$$

An important assumption we make here is that there are no gaps, i.e. settle dates match. Techniques exist to accomodate non-overlapping data, but we will not discuss these here. Also, when we do not have data on a particular discount factor, we interpolate between two known dates. The method of interpolation has a significant effect on the foward rate curve that will be computed from the discount curve. Choice of interplation method is an active area of research and optimal techniques have not yet been established. We used spline interpolation but feel that other methods need to be tested.

As we move 2-3 years into future, the data on swap rates become most reliable. Following our pattern on calculating discount factor and since the par swap rate by definition has zero net present value, we get, after simplifying

$$Z(T_N) = \frac{1 - S(T_N)\sum_{i=1}^{N-1}\alpha(t_{i-1},t_i)Z(T_i)}{1+\alpha(T_{N-1},T_N)S(T_N)},$$

where $S(T_i)$ is the par swap rate in year i. Transforming swap data into a simple discount factor requires that we take into account coupon payments made on swap agreements. For instance, if we use money market and futures data for times to maturity up to two years, and our swap data is on swaps that pay semi-annual payments, we must consider the four payments made on a swap with two years time to maturity.

Finally, putting together data on discount factors for money market rates, futures and swap rates, we obtain a complete discount curve. There is a one-to-one correspondence between the discount curve and the forward rate curve, so we can apply to following formula to obtain the forward rate curve. From period 1 to 2 the forward rate is:

$$F(T_1,T_2) = \left(\frac{Z(T_1)}{Z(T_2)} - 1\right)\frac{1}{\alpha(T_1,T_2)}.$$

3.2.2 Sample data and output from yield curve generator

Sample market data is shown in the following tables. The LIBOR data (London Inter Bank Offered Rate), shown in Table 3.1 is the rate the most creditworthy international banks charge each other for large loans based on Eurodollars. Such loans typically have short maturities. Eurodollars are simply U.S. currency held in banks outside the United States.

Libor	Ask
2 day	5.475
1 month	5.475

Table 3.1: LIBOR data

Contract	Ask
Mar97	94.3900
Jun97	94.2100
Sep97	94.0200
Dec97	93.8100
Mar98	93.7100
Jun98	93.6100
Sep98	93.5300
Dec98	93.4200
Mar99	93.3900
Jun99	93.3300
Sep99	93.2700
Dec99	93.1800

Table 3.2: Futures data*Futures contracts are settled two business days after purchase.*

The data is Table 3.2 is for Euro Futures. The price quoted is for a zero whose par price is 100. Contracts expire two business days before the third Wednesday for the months quoted.

Table 3.3 lists the par swap rate for several swaps with semi-annual coupon payments. Depending on the market, swaps with different tenors may have different coupon payment frequencies. For example, swaps with short tenors may pay quarterly while longer tenors may pay semi-annually. Also note that the day count basis is consistent for these swaps, but it may not be for another market.

The "bootstrapping" method described earlier takes the market data and produces one discount curve. The discount factors for January 29, 1997 is reproduced in Table 3.4 and plotted in Figure 3.3. Implied forward rates calculated from the discount factors are shown in Figure 3.4.

Tenor	Ask
1 SA ACT/365	5.9345
2 SA ACT/365	6.2300
3 SA ACT/365	6.4251
4 SA ACT/365	6.5353
5 SA ACT/365	6.6371
7 SA ACT/365	6.7918
10 SA ACT/365	6.960391
12 SA ACT/365	7.041999
15 SA ACT/365	7.138803

Table 3.3: Swap data *All swaps shown pay semi-annual coupons and use the accounting standard that counts the actual number of days in a month with a 365 day year.*

	Maturity	Discount
LIBOR	29-Jan-97	1.0000
	31-Jan-97	0.9997
	2-Feb-97	0.9994
	28-Feb-97	0.9955
Futures	17-Mar-97	0.9929
	16-Jun-97	0.9790
	15-Sep-97	0.9649
	15-Dec-97	0.9505
	16-Mar-98	0.9359
	15-Jun-98	0.9212
	14-Sep-98	0.9066
	14-Dec-98	0.8920
	15-Mar-99	0.8774
	14-Jun-99	0.8630
	13-Sep-99	0.8487
	13-Dec-99	0.8345
Swaps	13-Mar-00	0.8203
	31-Jul-00	0.7993
	29-Jan-01	0.7723
	30-Jul-01	0.7461
	29-Jan-02	0.7201
	29-Jul-02	0.6953
	29-Jan-03	0.6710
	29-Jul-03	0.6473
	29-Jan-04	0.6240
	29-Jul-04	0.6022
	31-Jan-05	0.5806
	29-Jul-05	0.5595
	30-Jan-06	0.5392
	31-Jul-06	0.5193
	29-Jan-07	0.4998
	30-Jul-07	0.4815
	29-Jan-08	0.4637
	29-Jul-08	0.4466
	29-Jan-09	0.4298
	29-Jul-09	0.4138
	29-Jan-10	0.3984
	29-Jul-10	0.3834
	31-Jan-11	0.3689
	29-Jul-11	0.3547
	30-Jan-12	0.3412

Table 3.4: Discount Factors *The discount curve computed from money market, futures, and swap data. All three data sets are stiched together to form one curve, which is then used to compute the other aspects of the term structure.*

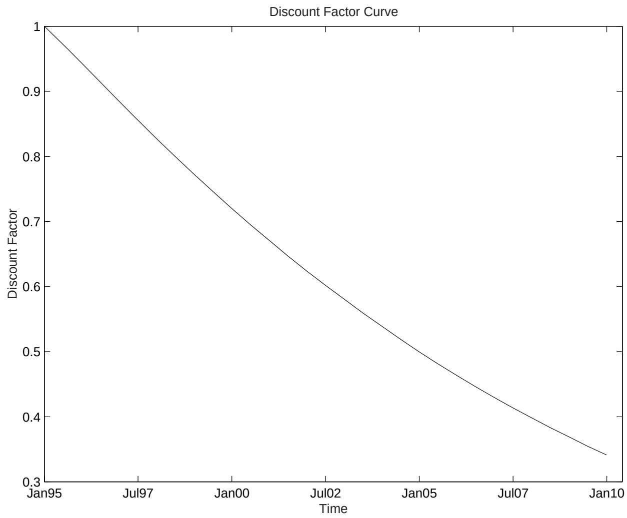

Figure 3.3: Dicount Factor Curve *The discount factor for zero time to maturity is by definition 1.*

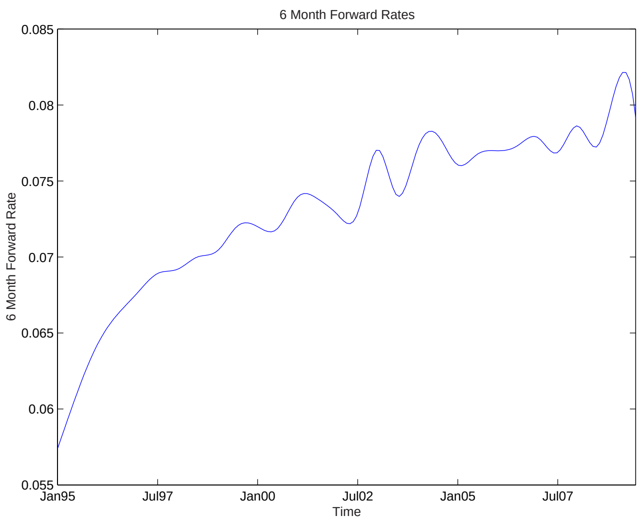

Figure 3.4: Six month forward rates
Forward information calculated from discount factors in Table 3.4

3.3 Interest Rate Models

3.3.1 Basic Binomial Model

The binomial model assumes that the underlying asset price follows a binomial process. Assume that p is a constant between 0 and 1. Some security is worth S_0 today, and a year later it may go up to S_u with probability p or down to S_d with probability $1-p$. If the current one year rate is r, then the value of S at time 1 is

$$S_1 = \frac{pS_u + (1-p)S_d}{1+r} \tag{3.3.1}$$

where $pS_u + (1-p)S_d$ is the expected value and $\frac{1}{1+r}$ is the discount factor. Equation (3.3.1) holds for each node of the price tree, while the short rate r varies from node to node.

3.3.2 Modeling the Interest Rate

A General Framework

Since the 1980s the volume of trading in interest rate related instruments has increased dramatically. Evaluating the interest rate products is more difficult than evaluating equity and foreign exchange derivatives, since interest rate models are concerned with movements of the entire yield cure - not just with changes of a single variable. Also, as mentioned previously, we cannot directly observe interest rates in the same way that we can observe, say, a stock price.

One class of models of the interest rate is based on a process for the short-term risk-free rate, r. The short rate, r, at time t is the rate that applies to infinitesimally short period of time at time t. (Although in practice, the short period can be considered as a day, or even longer.)

Suppose an interest rate derivative provide a payoff $X = \Phi(S_T)$ at time T. If we let $\bar{r}$ be the average value of r in the time interval between t and T, the value of X at time t is given by

$$P(t,X) = E^{\star}\left[\frac{\Phi(S_T)}{1+\bar{r}(T-t)}\right]$$

where $E^{\star}$ denotes expected value in a risk-neutral world. Accordingly, $P(t,T)$, the price at time t of discount bond that pay off \$1 at time T, satisfies:

$$P(t,T) = E^{\star}\left[\frac{1}{1+\bar{r}(T-t)}\right]$$

If $R(t,T)$ is the interest rate at time t for a time to maturity of $T-t$,

$$P(t,T) = E^{\star}\left[\frac{1}{1+R(t,T)(T-t)}\right]$$

Therefore,

$$R(t,T) = \frac{1}{T-t}\left(\frac{1}{E^{\star}\left[\frac{1}{1+\bar{r}(T-t)}\right]} - 1\right)$$

Once we define the process r, we have a model for the evolution of the term structure and a means to price bonds, bond options, and other interest rate products.

One approach, for instance Vasicek (1977), Cox, Ingersoll and Ross (1985), has been to propose a model for the short-term interest rate and deduce a model for the term structure based on an Ornstein-Uhlenbeck process. The Vasicek model is one of the simplest and it admits a solution with a Gaussian distribution. The Cox, Ingersoll, and Ross model, know as the CIR model, makes a slight modification to the volatility term and has a solution with a non-central chi-square distribution. Full information maximum likelihood estimation may

Model	$\mu(r,t)$	$\sigma(r,t)$
Vasicek	$a(b-r)$	σ
Cox, Ingersoll and Ross	$a(b-r)$	$\sigma\sqrt{r}$
Ho and Lee	$\Theta(t)$	σ
Hull and White-Extended Vasicek	$\Theta(t)-a(t)r$	$\sigma(t)$
Hull and White-Extended CIR	$\Theta(t)-a(t)r$	$\sigma(t)\sqrt{r}$

Table 3.5: *Martingale models for the Short Rate*

be used for parameter estimation since distribution of the solutions are known. For a thorough examination of a variety of models, see Rebonato [8].

Ho and Lee(1986)[5], Black, Derman, and Toy(1990)[2], Hull and White(1990)[6] consider how to describe yield curve movement using a "one-factor" model. They take market data, such as the current term structure of interest rates. By specifying the initial yield curve and its volatility structure, they are able to determine a drift structure that makes the model arbitrage free. Heath, Jarrow, and Merton(1992) [4] simulate the evolution of forward rates by allowing the volatility functions, $\sigma_i(\cdot)$, to depend on the entire forward rate curve. The main advantage of the model is th ability to easily specify the initial forward rate (or yield) curve abd their volatilities and correlations.

One Factor Models of The Short Rate

Among the models describing the process of r, those involves one source of uncertainty are called *one-factor* models. In many one-factor models, the short rate r is usually given by a stochastic process:

$$dr = \mu(r,t)dt + \sigma(r,t)dW$$

where W is a Wiener process. Some examples with their specification of μ and σ are given in Table 3.5. For a good discussion of the various models, see Hull[7].

In contrast, the Black, Derman and Toy model (BDT in the following) [2] is a one-factor model that assumes a log-normal process for r:

$$d\log(r) = \left[\Theta(t) + \frac{\sigma'(t)}{\sigma(t)} lnr\right] dt + \sigma(t)dW \qquad (3.3.2)$$

BDT developed a single-factor short-rate model to match the observed term structure of spot interest rate volatilities, as well as the term structure of interest rates, and which has proved popular with practitioners. The BDT model can be used to price any interest-rate-sensitive security (bond options, swaps, etc.) without requiring the explicit specification of investors' risk preferences. The input of model is an array of yield on zero-coupon bonds with various maturities, and an array of yield volatilities of these bonds. In the next section, we will analyze the BDT model algorithmically.

3.3.3 The Black, Derman and Toy Model

As with the original Ho and Lee model, the BDT model is developed algorithmically, describing the evolution of the entire term structure in a discrete-time binomial lattice framwork. A binomial tree is constructed for the short rate in such a way that automatically returns the observed yield function and the volatilities of different yields.

Mathematical Description of the Model

The log-normal process of r prevents the negative rates and make model calibration to some interest rate product much easier. From Equation (3.3.2), one can see that the assumption of decaying short rate volatility is required to prevent the unconditional variance of the short rate, $\sigma(t)^2 t$, from increasing with t without bound, which could be inconsistent with the mean-reverting character of the short rate process.

The solution to equation (3.3.2) is of the form

$$r(t) = u(t)\mathrm{e}^{\sigma}(t)W(t),$$

where $u(t)$ satisfies

$$\frac{d \ln u(t)}{dt} = \Theta(t) + \frac{\sigma'(t)}{\sigma(t)} \ln u(t)$$

is the mean of the short rate distribution at time t and $\sigma(t)$ the short rate volatility. In this model, changes in the short rate are lognomally distributed, with the resulting advantage that interest rates can not become negative. The BDT model incorporates two independent functions of time, $\Theta(t)$ and $\sigma(t)$, chosen so that the model fits the term structure of spot interest rates and the term structure of spot rate volatilities. Once $\Theta(t)$ and $\sigma(t)$ are chosen, the future short rate volatility, by definition, is entirely determined.

Benefits and Problems with the Model

Since there is much evidence that volatility is not constant, the clear benefit of the BDT model is its use of a time-varying volatility. One of the model's strengths is also one of its weaknesses. Since the model is easily calibrated to market data, practitioners favor the model. However, the model must be recalibrated often and will often give inconsistent results. That is, the model is not robust. Secondly, due to its lognormality neither analytic solutions for the price of bonds nor the price of bond options are available and numerical procedures are required to derive the short rate tree that correctly matches market data.

3.3.4 Implementing the BDT Model

We will use a recombining binomial tree to represent the stochastic process of the short rate. If one time step on the tree is Δt, the short rates on the tree are simple Δt-period rates. The usual assumption when a tree is built is that Δt-period rates follow the same stochastic process as the instantaneous rate in the corresponding continuous model.

The Short Rate Tree and Arrow-Debreu Price

The idea of BDT model is to use a multiplicative binomial tree to model the risk-neutral dynamics of the interest rate by calibrating to term structure and volatility data. The general short rate tree looks like the tree shown in Figure 3.5.

If we let $r(i,j)$ be the short interest rate at nodal (i,j), where j means the number of periods and $j-i$ means the number of upward moves. Also define,

(i,j): node at time j and state i.

r_u, r_d: value of r at nodes $(0,1)$ and $(1,1)$, respectively.

S_u, S_d: value of S at nodes $(0,1)$ and $(1,1)$, respectively.

$Y_u(i)$, $Y_d(i)$: yields at nodes $(0,1)$ and $(1,1)$, respectively, on a discount bond maturing at time $i\Delta t$.

Define an Arrow-Debreu security as a contract that pays \$1 at (i,j), and zero at any other nodes. Denote its price at time $t \le j$ and state k by $G(k,t,i,j)$. For example, $G(0,0,2,2)$ would be the price at node $(0,0)$ of a security that pays \$1 at time 2 and in state 2. If we know the value for the short tree up to time j, i.e. $\{r(:,k)\}_{k=0}^{j}$, we could obtain the value of $G(k,t,i,j), 1 \le k \le t, 1 \le t \le j$ simply by repeating the basic pricing formula (3.3.1). Note that BDT assumes $p = \frac{1}{2}$. For instance,

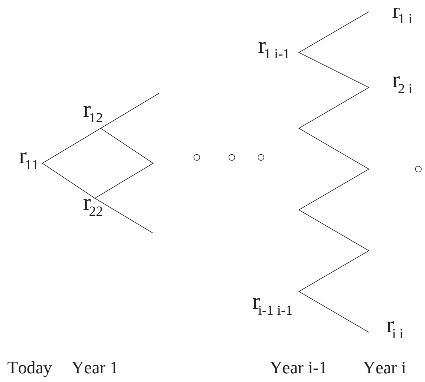

Figure 3.5: *Short Rate Tree*

$$\begin{aligned}
G(0,0,2,2) &= \frac{1}{1+r_{0,0}} + \{\frac{1}{2} \times \frac{1}{1+r_{0,1}} \times [\frac{1}{2} \times 1 + \frac{1}{2} \times 0] + \frac{1}{2} \times \frac{1}{1+r_{1,1}} \times [\frac{1}{2} \times 0 + \frac{1}{2} \times 0]\} \\
&= \frac{1}{4}\frac{1}{1+r_{0,0}}\frac{1}{1+r_{1,1}}
\end{aligned}$$

In general, for any intermediate node

$$G(0,0,i,j) = \frac{1}{2}\frac{G(0,0,i-1,j-1)}{1+r_{i-1,j-1}} + \frac{1}{2}\frac{G(0,0,i+1,j-1)}{1+r_{i+1,j-1}} \tag{3.3.3}$$

and for any extremal node (this means $i = j$ or 0)

$$G(0,0,0,j) = \frac{1}{2}\frac{G(0,0,0,j-1)}{1+r_{0,j-1}} \tag{3.3.4}$$

$$G(0,0,j,j) = \frac{1}{2}\frac{G(0,0,j-1,j-1)}{1+r_{j-1,j-1}}$$

Equations (3.3.3) and (3.3.4) are called the Fokker-Plank equations. Actually, the Arrow-Debreu price G plays a role like Green's function does in PDEs. Once we have the Arrow-Debreu price $G(0,0,i,j)$ for all i and j, we can price any security just by multiplying its payoffs at any node by the Arrow-Debreu price corresponding to that node, which is analogous to a convolution in PDEs.

The key to computing the Arrow-Debreu price is to know the short rate tree in advance. How do we calibrate the short rate tree from given the current zero-coupon yield $\{Y(i)\}_{i=1}^n$ and yield volatility $\{\sigma(i)\}_{i=1}^n$? We use the following algorithm.

1. At time 0, we have of course $r_{0,0}$ given and equal to $Y(1)$.
2. Suppose we knew the values of $\{r(:,k)\}_{k=1}^{j-1}$. Note also that we have all the $G(0,0,l,n)$, $1 \le l \le n$ and $1 \le n \le j-1$, through the Fokker-Plank equations.
3. At time j, we have to find n unknowns $\{r(i,j)\}_{i=1}^{j}$. Because BDT model assumes that the short rate is lognomal with a volatility that depends only on time, which implies

$$\frac{r(1,j)}{r(2,j)} = \frac{r(2,j)}{r(3,j)} = \cdots = \frac{r(j-1,j)}{r(j,j)}.$$

So we need two equations to solve for $r(1,j)$ and $r(2,j)$, say.

Maturity (Years)	Yield (%)	Yield Volatility (%)
1	10	20
2	11	19
3	12	18
4	12.5	17
5	13	16

Table 3.6: Sample Term Structure

Today	*Year1*	*Year2*	*Year3*	*Year4*
0.1	0.1432	0.1942	0.2179	0.2552
	0.0979	0.1377	0.1600	0.1948
		0.0976	0.1183	0.1406
			0.0872	0.1134
				0.0865

Table 3.7: Short Rate Tree

4. Let's initially guess $r(1, j)$ and $r(2, j)$, then we can compute the short rate tree and the corresponding Arrow-Debreu price.
5. Error Correction: It's clear that

$$S(0,0) = \frac{1}{1+r_{0,0}}[\frac{1}{2} \times S_u + \frac{1}{2} \times S_d]. \tag{3.3.5}$$

Also, the volatility of a j-year yield is known to be

$$\sigma(j) = \frac{1}{2} \ln(\frac{Y_u(j)}{Y_d(j)}), \tag{3.3.6}$$

where $Y_u(j) = (\frac{1}{S_u})^{j-1} - 1$ and $Y_d(j) = (\frac{1}{S_d})^{j-1} - 1$.

6. Use Newton-Raphson scheme to solve equations (3.3.5) and (3.3.6) to obtain $r(1, j)$ and $r(2, j)$.

3.3.5 Applications

Given a sample term structure, like the one in Table 3.6, one can get the short rate tree in Table 3.7.
And if we compute Arrow-Debreu price $G(0, 0, 1, 2) = 0.0458$ we get the results in Table 3.8.

Options on Treasuries

Let us value a T-year call option on a N-year($T \leq N$) treasury discounted bond with strike price K. (Put options are calculated similarly.) If the price of N-year bond at time T is S_T, the payoff is $\max(S_T - K, 0)$. Using appropriate combination of the Arrow-Debreu prices for each node of the N-year bond, we obtain possible payoffs $S_T(0), S_T(1), \ldots, S_T(T+1)$. Then apply Arrow-Debreu process again to value the current price

Table 3.8: Arrow-Debreu Price Tree *The first value is the discounted payoff of 1*

0.4058	0.4374	0
	0.4554	1
		0

for call option.

Example: Let $T = 1$, $N = 3$, $K = 0.8$, and face value S=1. The payoffs are

$$S_1(0) = max\{fv * \sum_{j=0}^{2} G(0,1,j,3), 0\} = 0,$$

$$S_1(1) = max\{fv * \sum_{j=1}^{3} G(1,1,j,3), 0\} = 0.0152.$$

Then we repeate the appropriate Arrow-Debreu process again to get the price

$$p = S_1(0) * G(0,0,0,1) + S_1(1) * G(0,0,1,1) = 0.0069.$$

Caps

A popular interest rate option offered by financial institutions is an interest rate cap. Interest rate caps are designed to provide insurance against the rate on a floating-rate loan rising above a certain level (called *cap rate* X). If the principal is L, and interest payments are made at time $t, 2t, \ldots, nt$ from the beginning of the life of the cap, the buyer of the cap will receive a payment at time $(i+1)t$ given by

$$Cap_{(i+1)t} = tL \max(R_i - X, 0)$$

where R_i is the floating rate at time it. Therefore, a cap can be viewed as a portfolio of call options on the floating rate R with different maturities $t, 2t, \ldots, nt$. The individual options comprising a cap are referred to as *caplets*.

We will apply the short rate tree to evaluate the price of cap. Since each caplet is merely a call option, we can apply the method described in the previous section. The today price of cap is the sum of the prices of caplets(call options). Thus, we skip our numerical illustration here.

European Swaptions

Recall that an interest rate swap can be regarded as an agreement to exchange a fixed rate bond for a LIBOR-based floating rate bond. The floating rate is typically of the same maturity as the rate reset frequency. A swaption giving the holder the right to pay fixed and receive floating ("payer" swaption) is equivalent to a put on a fixed rate bond with strike price equal to the pricipal of the swap, and with the coupon payments equal to the quoted swap rate if the reset dates are annual. If the swaption gives the holder the right to pay floating and receive a fixed rate ("receiver" swaption), it is equivalent to a call on a fixed rate coupon bond. Assume that the pricipal is 1. Let $B_{i,j}$ represent the value of the fixed rate bond at node (i, j) in the tree, and C the cash flow at each coupon date.

The first stage to pricing the derivative is to construct the short-rate tree out until the end of the life of the instrument underlying the option. In our example, $T = 1$ with three years left to maturity, i.e. $N = 4$. For convenience, consider the C treasure as a portfolio of three zero coupon bonds - a one-year zero with C face value; a two-year zero with a C face value; and a three-year zero with a $1 + C$ face value.

We initialize the value of the fixed rate bond underlying the swap at each of the states at time N, $S_{i,N} = 1 + C$. We then apply backward induction for the coupon bond price, taking discounted expectation until T

$$S_{i,T} = \sum_{j=1}^{N-T-1} \sum_{k=0}^{T+j-1} C \times G(i,T,k,T+j) + \sum_{k=0}^{N-1} (1+C) \times G(i,T,k,N).$$

Using the state price at all nodes at time step T the swaption price can be evaluated for payer swaptions and receiver swaptions, we have respectively

$$\text{payer swaption} = \sum_{i=0}^{T} G(0,0,i,T)\max\{1 - S_{i,T}, 0\}$$

$$\text{receiver swaption} = \sum_{i=0}^{T} G(0,0,i,T)\max\{S_{i,T} - 1, 0\}$$

In our example, let $C = 0.1$, for payer swaption, $S_{0,1} = 0.08728$ and $S_{1,1} = 0.9731$ such that for the payer swaption

$$p = \max\{1 - S_1(0), 0\} * G(0,0,0,1) + \max\{1 - S_1(1)\} * G(0,0,1,1) = 0.07.$$

3.4 Options on options

3.4.1 Compound options

Since companies often have different kinds of options on the same stock on one hand hand, and they assume that the market is bullish on the other hand, it is useful to come up to strategies to reduce the risk in their investments. One way to handle that situation is by means of compound options, which are options on options.

The compound option gives the holder the right to buy (call) or sell (put) another option. The compound option expires at some date T_1 and the option on which it is contingent, expires at a later time T_2. In some sence, such an option is weakly path dependent. Let us consider the simple Black and Scholes model with a zero coupon bond B and a risky asset S:

$$\begin{aligned} d\,B_t &= r\,B_t\,dt \quad \text{with} \quad B(0) = B_0 \\ d\,S_t &= S_t(\mu dt + \sigma dW_t) \quad \text{with} \quad S(0) = S_0 \end{aligned}$$

where W_t is standard Brownian motion on some probability space $(\Omega, \mathcal{F}_{t\geq 0}, \mathbb{P})$. Then, the pricing of compound options is straightforward and is basically done in two steps: first we price the underlying option and then price the compound option. Suppose that the underlying option has a payoff of $F(S)$ at time T and that the compound option can be exercised at a earlier time $T' < T$ to get a payoff $H(V(S,T'))$, where $V(S,t)$ is the value of the underlying option at time $t \leq T$. Thus the first step means solving the Feynman–Kac equation

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} + rx\frac{\partial V}{\partial x} - rV = 0 \\ V(x,T) = F(x) \end{aligned}$$

to find $V(x = S, T')$, which is the value of the underlying option at time T', where we can exercise the compound option. For the complition of the second step, let us denote by $G(S,t)$ the value of the compound option, which will satisfy the equation

$$\begin{aligned} \frac{\partial G}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 G}{\partial x^2} + rx\frac{\partial G}{\partial x} - rG = 0 \\ G(x,T') = H(V(x,T')) \end{aligned}$$

Using probabilistic methods we can derive *precise* expresions for both V and P as

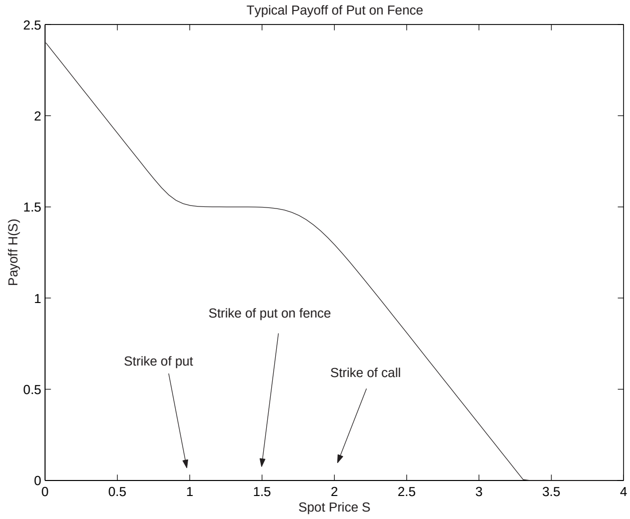

Figure 3.6: Payoff for a put on a fence
The put on a fence pays $(K - (S_T - K_2)^+ - (K_1 - S_T)^+)^+$, *where* $K_1 < K_2$.

$$
\begin{aligned}
V(S,t) &= \mathrm{e}^{-r(T-t)}\mathbb{E}^*\left[F(S_T)|\mathcal{F}_t\right] \\
G(S,t) &= \mathrm{e}^{-r(T'-t)}\mathbb{E}^*\left[H(V(S_{T'}))|\mathcal{F}_t\right]
\end{aligned}
$$

where the conditional expectations are taken with respect to the unique *risk neutral* probability measure $\Pr'$ equivalent to Pr.

3.4.2 Fence option

Let us recall that an european *call* (*put*) option is a contract in which the holder has the right but not the obligation to buy (sell) some finantial instrument, such as stock, at a previously agreed price, strike price, at some time T in the future. Thus the payoffs of the call and put options at a strike price K are $(S_T - K)^+$ and $(K - S_T)^+$ respectively.

A *Fence* is an option in which we long (keep or buy) call option at high strike price while we short (lend or sell) a put option at a low strike price. Thus, the payoff of the fence option is

$$(S_T - K_2)^+ - (K_1 - S_T)^+$$

where $K_1 < K_2$. The payoff diagram for a put on a fence is shown in Figure 3.6.

We will be interested in pricing a put on the fence at strike price K and time of maturity $T' < T$. In practice $K_1 < K < K_2$.

Let us denote by $C(t,T,K,S)$ and $P(t,T,K,S)$ the price at time T of a call and a put respectively, with maturity time T and strike price K. Let $F(t,T,T',K_1,K_2,S)$ be the price of the fence option at time $t < T$. Then, under the Black and Scholes model, we have that

$$
\begin{aligned}
F(t,T,K_1,K_2,S) = C(t,T,K_2,S) - P(t,T,K_1,S)& \\
= S\,N(d_1(S,K_2)) - K_2 e^{-r(T-t)}\,N(d_2(S,K_2))& \\
- K_1 e^{-r(T-t)}\,N(-d_2(S,K_1)) + S\,N(-d_1(S,K_1))& \quad (3.4.7)
\end{aligned}
$$

where

$$\begin{aligned} d_1(S,K) &= \frac{\ln(S/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\ d_2(S,K) &= d_1(S,K)-\sigma\sqrt{T-t} \end{aligned}$$

and $N(z)$ is the standard accumulative normal distribution.

The payoff at maturity time $T' < T$ of a Put option on the Fence with Strike price K is

$$(K - F(T', T, K_1, K_2, S))^+$$

Let be $\tau' = T' - t$. Then, the price of this compound option is given by

$$G_t = \frac{e^{-r\tau'}}{\sqrt{2\pi\tau'}} \int_{-\infty}^{\infty} (K - F(T', T, K_1, K_2, Se^{(r-\frac{\sigma^2}{2})\tau'} e^{\sigma x}))^+ e^{-\frac{x^2}{2\tau'}}\, dx \tag{3.4.8}$$

The expression inside paranthesis is strictly decreasing with respect to x, thus there exists a unique $x^* = x^*(S,\tau')$ such that

$$K - F(T', T, K_1, K_2, Se^{(r-\frac{\sigma^2}{2})\tau'} e^{\sigma x^*}) = 0$$

which also implies that

$$\frac{\partial x^*}{\partial S} = -\frac{1}{\sigma S}.$$

The last expression will be useful for deriving expressions for the hedging protfolio. The derivation of the formula expoits the techniques introduced in Geske [3]. A full derivation will be provided in a future paper.

$$G_t = Ke^{-r(T'-t)}N(\tilde{\gamma_2}) - SM(\tilde{\gamma_1}, a_1;\rho) + K_2e^{-r(T-t)}M(\tilde{\gamma_2}, a_2;\rho) + K_1e^{-r(T-t)}M(\tilde{\gamma_2}, -b_2;-\rho) - SM(\tilde{\gamma_1}, -b_1;-\rho)$$

where

$$\begin{aligned} \tilde{\gamma_1} &= \frac{\log\left(\frac{S}{S^*}\right) + \left(r+\frac{\sigma^2}{2}\right)(T'-t)}{\sigma\sqrt{T'-t}} \\ \tilde{\gamma_2} &= \tilde{\gamma_1} - \sigma\sqrt{T'-t} \\ a_1 &= \frac{\log\left(\frac{S}{K_2}\right) + \left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \\ a_2 &= a_1 - \sigma\sqrt{T-t} \\ b_1 &= \frac{\log\left(\frac{S}{K_1}\right) + \left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \\ b_2 &= b_1 - \sigma\sqrt{T-t} \\ \rho &= \sqrt{\frac{T'-t}{T-t}} \end{aligned}$$

and where

$$S^\star = Se^{x^\star + r(T'-t)}.$$

We define M as a bivariate normal distribution with correlation, given by

$$M(\tilde{\gamma}, a; \rho) = \int_{-\infty}^{\tilde{\gamma}} \int_{-\infty}^{a} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{(\gamma^2 - 2\rho\gamma\xi + \xi^2)}{2(1-\rho^2)}\right) d\gamma d\xi$$

3.5 Conclusions and Future project

Interpolation methods in yield curve estimation needs to be addressed. It is believed that the method currently used is not optimal. The computer code could also be made easier to use by allowing Excel to be used as a front end. This could be accomplished either with a Matlab back end or through a C++ coded .dll automation file.

The BDT model is easily calibrated, but as mentioned before there are some serious shortcomings that leave us in search of a better model. The Hull-White approach may offer a more robust model and should be examined. The appeal of both models is their accomodation of a time-varying volatility, however they both specify volatility as a deterministic function of time. There is ample evidence of a stochastic volatility, so that should also be considered. Derivatives prices based on th BDT model have been implemented, but so far no calculations of the so-called "greeks" has been performed. Calculations of the greeks will show the senstivity of the price to parameter fluctions and will thus show how to build hedging portfolios. Greeks in a binomial model are easily implemented using a finite-difference scheme.

As we have seen, it is possible to find formulae for the price of basic compound options in the simple Black and Scholes model, where volatility remains constant. The expressions obtained, involve the cumulative distribution function of a binormal vector. It is desirable, and will be left as a future project, to find ways to impliment models to price compound options that take into acount the facts that neither the interest nor the volatility are constant. One possible option is to consider models with stochastic volatility combined with stochastic short rate models.

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