Pricing and Hedging of Forwards, Futures and Swaps by Change of Numéraire

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Abstract

We derive prices and hedging strategies for some contingent claims which were treated by Jamshidian [12]. For this we discuss price functionals and the technique of "change of numéraire" in a general semimartingale framework. These tools allow us to develop a unified method based on the explicit computation of the price processes via the multiplicative Doob-Meyer decomposition and the assumption that certain (co-)variation processes have a deterministic terminal value.

Keywords: price functionals, change of numéraire, hedging strategies

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1 Introduction

In his 1995 article "Hedging Quantos, Differential Swaps and Ratios" F. Jamshidian [12] prices a variety of contingent claims by means of explicitly given hedging strategies. However, these strategies are presented somehow ad hoc without being derived in a systematic manner from the structure of the contingent claim in question. Also, there are no admissibility considerations for the given strategies. Therefore, recalling the possibility of "suicide strategies" (cf. Harrison-Pliska [8]) the employed duplication argument might lead to contradictions.

In this paper we will proceed in a different way. Instead of pricing by duplication we will develop a price functional yielding the whole value process of the given contingent claim. After that we determine the hedging strategy by use of Itô's formula. This method excludes "suicide effects" already from the start and provides a systematic derivation of prices and portfolio strategies.

The outline of the paper is as follows: In Section 2 we present the basic model. We recall some general results on pricing contingent claims in general semimartingale models and in examine in detail the technique of "change of numéraire" introduced in El Karoui-Geman-Rochet [6]. This technique will be combined with the multiplicative Doob-Meyer decomposition to formulate our "general" approach to pricing and hedging contingent claims.

Section 3 then provides some applications dealing with forwards, futures and swaps. All of these examples are also treated in Jamshidian [12]. Under the assumption of [12] that certain (co-)variation processes have deterministic final value, the method of Section 2 allows us to derive the pricing formulas of [12] in a systematic manner. As explicit examples we will treat forwards on ratios of equity indexes, or options on forwards for which we recover a Black-Scholes-type valuation formula. Furthermore we will treat quanto futures and simulate a dollar-denominated future by its pound-denominated analogue. Finally, we will price and hedge continuous differential swaps which allow an American investor to participate in the British interest rate without bearing an exchange rate risk. In contrast to [12] our approach will involve the unbiased expectation hypothesis and a Heath-Jarrow-Morton type dynamics of bond prices.

2 General Theory

Let us start with an informal description of the financial market. We assume that there are the following primary assets:

- *stocks* whose price processes will be denoted by $P = (P_t)_{0 \le t \le T}$ where $T < \infty$ is the *time horizon* for our financial market.

- a continuum of *zero coupon bonds* paying one unit at their time of maturity. Their price processes will be denoted by $B^s = (B^s_t)_{0 \le t \le s}$ where $0 \le s \le T$ denotes maturity. So we have $B^s_s = 1$ $(0 \le s \le T)$.

- a *money market account* with *interest rate* process $r = (r_t)_{0 \le t \le T}$ such that investing one unit at time 0 will produce a return of size

$$\beta_t \triangleq \exp\left(\int_0^t r_s \, ds\right) \tag{1}$$

up to time $0 \le t \le T$.

Sometimes we may wish to consider two national financial markets (called the "American" and the "British" one). To distinguish the dollar- and pound-denominated assets the corresponding processes are indexed by "\$" and "£" respectively. Furthermore we assume that at any time $0 \le t \le T$ one can change pounds into dollars and vice versa according to the *exchange rate* of X_t dollars for one pound.

We suppose frictionless markets, i.e. unlimited short sales are possible and investors can trade any friction of every asset at any time, even continuously. We will also neglect transaction costs, dividends and taxes. In order to ensure that the gain from trade is well defined as stochastic integral we assume that all price processes including the exchange rate X are strictly positive, continuous semimartingales on a common stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$. The filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ is supposed to satisfy the usual hypothesis of right-continuity and completeness; $\mathcal{F}_0$ is $\mathbb{P}$-a.s. trivial. The interest rates $r = (r_t)_{0 \le t \le T}$ are only assumed to form an adapted, $\mathcal{B}([0,T]) \otimes \mathcal{F}_T$-measurable stochastic process for which the integral in (1) is well defined.

2.1 No Arbitrage and Martingale Measures

A basic assumption in the theory of financial markets is that there are no arbitrage opportunities, i.e. there is no possibility of gaining money without bearing some risk. The impact of this condition on models of stock prices has been studied for a long time beginning with the seminal papers by Harrison-Kreps [7] and Harrison-Pliska [8]. These papers revealed a connection between the no arbitrage assumption and the existence of an *equivalent martingale measure*. Finally, F. Delbaen and W. Schachermayer [5] have been able to show the following fairly general result:

Theorem 2.1 [1] *Given a bounded $\mathbb{R}^d$-valued semimartingale S there is an equivalent martingale measure for S if and only if S satisfies the condition "no free lunch with vanishing risk" (NFLVR)* [2]*. For a locally bounded S the condition (NFLVR) is equivalent to the existence of an equivalent local martingale measure.* [3]

□

With exception of Section 3.3 on continuous differential swaps we will only consider a finite number of all the assets described above. So let us fix a finite set $\mathcal{S} = \{S^1, \ldots, S^d\}$

[1] see Delbaen-Schachermayer [5] for a proof

[2] For an exact definition of this condition see Delbaen-Schachermayer [5]. Informally it says that even asymptotically there shall be no arbitrage-possibilities. Especially it implies the absence of arbitrage.

[3] i.e. a measure under which discounted stock prices are at least local martingales

of primary assets. Then the above theorem suggests to assume the existence of at least one equivalent martingale measure $\mathbb{P}_\beta$ for all discounted asset prices S/β ($S \in \mathcal{S}$). Here we index the martingale measure by β because we use the money market account as numéraire.

As pointed out in the introduction we are going to deal with cross-currency derivatives. So we will consider two national financial markets (called the "U.S.-American" and the "British" one) with given finite sets of primary assets ${}^\$\mathcal{S}$, ${}^£\mathcal{S}$. Clearly, from the American point of view a foreign British asset can be considered as a "domestic American" one after its price has been converted into dollars. Thus, instead of considering ${}^\$\mathcal{S}$ and ${}^£\mathcal{S}$, we can again consider just one set $\mathcal{S} \triangleq {}^\$S \cup \{X\,{}^£S |\, {}^£S \in {}^£\mathcal{S}\}$ of primary ("American") assets. Another consequence of this observation is the following. Let $\mathbb{P}_{{}^\$\beta}$ denote a martingale measure for the U.S.-market. Under the American martingale measure $\mathbb{P}_{{}^\$\beta}$ not only the discounted primary American asset prices (e.g. ${}^\$B^T/{}^\$\beta$, ${}^\$P/{}^\$\beta$) have to be martingales, but also the discounted British asset prices converted into dollar prices (e.g. $X\,{}^£B^T/{}^\$\beta$, $X\,{}^£P/{}^\$\beta$, $X\,{}^£\beta/{}^\$\beta$). Clearly, a martingale measure $\mathbb{P}_{{}^£\beta}$ for the British market fulfills a symmetric property. Finally, a given British martingale measure can be converted into an American martingale measure in the following manner:

Theorem 2.2 *Let $\mathbb{P}_{{}^£\beta}$ be a martingale measure for the British market. Then the measure*

$$d\mathbb{P}_{{}^\$\beta} \triangleq \frac{X_0\,{}^\$\beta_T}{X_T\,{}^£\beta_T} d\mathbb{P}_{{}^£\beta} \tag{2}$$

is a martingale measure for the U.S.-market.

Proof: We only have to note that for any dollar-denomatied asset ${}^\$S$ the process $X^{-1}\,{}^\$S/{}^£\beta$ has to be a $\mathbb{P}_{{}^£\beta}$-martingale and then use the transformation rule for conditional expectations for a change of measure (see e.g. Karatzas-Shreve [14], Chapter 3, Lemma 5.3). □

2.2 Price Functionals

Martingale measures for $\mathcal{S}$ can be viewed as price functionals in the following way. Let $H_T \geq 0$ be a *contingent claim*, i.e. an $\mathcal{F}_T$-measurable random variable. For any martingale measure $\mathbb{P}_\beta$ we define the $\mathbb{P}_\beta$*-price* at time $0 \leq t \leq T$ of H_T by

$$^{\mathbb{P}_\beta}\boldsymbol{\pi}_t^T[H_T] \triangleq \beta_t \mathbb{E}_{\mathbb{P}_\beta}\left[\frac{H_T}{\beta_T} \,|\, \mathcal{F}_t\right]. \tag{3}$$

Of course, it is by no means obvious why this definition should make sense: Why should this give a reasonable price for a time T payment of size H_T? And what about the dependency on the chosen martingale measure $\mathbb{P}_\beta$? Are there several prices for the same contingent claim? So some clarification is needed. For this we have to formalize the notions "portfolio strategy", "gain from trade", "selffinancing" and "admissible".

Definition 2.3 *A* portfolio strategy *for the assets $\mathcal{S} = \{S^1, \ldots, S^d\}$ is an $\mathbb{F}$-predictable process $\xi = (\xi_t^1, \ldots, \xi_t^d)_{0 \le t \le T}$ satisfying*

$$\int_0^T (\xi_s^i)^2 \, d[S^i, S^i]_s < \infty \quad (i = 1, \ldots, d) \quad \mathbb{P}\text{-a.s.}$$

For such a strategy ξ the gain from trade *is defined by the stochastic integral*

$$G(\xi)_t \stackrel{\Delta}{=} \int_0^t \xi_s' \, dS_s \qquad (0 \le t \le T).$$

A portfolio strategy ξ is called selffinancing *if at any time $0 \le t \le T$ the actual* wealth *of the position*

$$V(\xi)_t \stackrel{\Delta}{=} \xi_t' S_t = \sum_{i=1}^d \xi_t^i S_t^i$$

results from some deterministic initial investment $V(\xi)_0$ and the gains from trade:

$$V(\xi)_t = V(\xi)_0 + G(\xi)_t \qquad (0 \le t \le T). \tag{4}$$

As already mentioned in the introduction we have to restrict the set of selffinancing trading strategies to preclude arbitrage possibilities from continuous trading.

Definition 2.4 *Let $\mathbb{P}_\beta$ be an equivalent martingale measure for the assets $\mathcal{S}$. We call a selffinancing strategy* admissible with respect to $\mathbb{P}_\beta$ *($\mathbb{P}_\beta$-admissible for short) if its discounted wealth process $V(\xi)/\beta$ is a true $\mathbb{P}_\beta$-martingale. A contingent claim H_T is called* attainable with respect to $\mathbb{P}_\beta$ *if it coincides with the terminal value $V(\xi)_T$ of some $\mathbb{P}_\beta$-admissible trading strategy ξ.*

Now we are able to give a first argument why one might want to view (3) as definition of a price functional:

Proposition 2.5 *If a contingent claim H_T is attained by some $\mathbb{P}_\beta$-admissible trading strategy ξ then we have*

$$^{\mathbb{P}_\beta}\boldsymbol{\pi}_t^T[H_T] = V(\xi)_t \qquad (0 \le t \le T), \tag{5}$$

i.e. at any time $0 \le t \le T$ the $\mathbb{P}_\beta$-price for H_T is exactly the wealth needed to start the selffinancing trading strategy ξ at time t and to duplicate the desired payoff H_T thereby.

Proof: Just notice that both $V(\xi)/\beta$ and $^{\mathbb{P}_\beta}\boldsymbol{\pi}^T[H_T]/\beta$ are $\mathbb{P}_\beta$-martingales with the same terminal value. □

From a mathematical point of view the duplication argument in Proposition 2.5 is not enough to guarantee that the $\mathbb{P}_\beta$-price in (3) does not depend on the specific choice of the martingale measure $\mathbb{P}_\beta$. In fact, the claim H_T might be duplicated by two different selffinancing strategies ξ and $\bar{\xi}$, which are both admissible, but with respect to different martingale measures $\mathbb{P}_\beta$ and $\overline{\mathbb{P}}_\beta$. In this situation we can not conclude $V(\xi) = V(\bar{\xi})$ since the martingale property need not necessarily be preserved under a change of martingale measure. This suggests to analyze the behaviour of contingent claims under such a change of measure. This was done in Jacka [11] from where we adopt the following

Theorem 2.6 *Let H_T be a nonnegative (!) contingent claim, and let $\mathbb{P}_\beta$ denote a fixed martingale measure. Then the following conditions are equivalent:*

(i) *H_T is attainable with respect to $\mathbb{P}_\beta$.*

(ii) *For any martingale measure $\overline{\mathbb{P}}_\beta$ satisfying*

$$\text{ess sup}\left\{\frac{d\mathbb{P}_\beta}{d\overline{\mathbb{P}}_\beta} \vee \frac{d\overline{\mathbb{P}}_\beta}{d\mathbb{P}_\beta}\right\} < \infty$$

we have

$$^{\mathbb{P}_\beta}\boldsymbol{\pi}_t^T[H_T] = {}^{\overline{\mathbb{P}}_\beta}\boldsymbol{\pi}_t^T[H_T] \qquad (0 \le t \le T).$$

(iii) *For any martingale measure $\overline{\mathbb{P}}_\beta$ we have*

$$^{\mathbb{P}_\beta}\boldsymbol{\pi}_t^T[H_T] \ge {}^{\overline{\mathbb{P}}_\beta}\boldsymbol{\pi}_t^T[H_T] \qquad (0 \le t \le T).$$

If H_T is simultaneously attainable with respect to two martingale measures $\mathbb{P}_\beta$ and $\overline{\mathbb{P}_\beta}$, it follows that the corresponding price processes in the sense of (3) coincide.

Proof: The proof consists of a slight extension of Jacka [11], Theorems 3.1 and 3.4. See Bank [1] for details on this extension. □

If H_T is attainable with respect to some martingale measure $\mathbb{P}_\beta$ and is *nonnegative* then we can define the price process of H_T as

$$\boldsymbol{\pi}_t^T[H_T] \triangleq {}^{\mathbb{P}_\beta}\boldsymbol{\pi}_t^T[H_T] = \beta_t \mathbb{E}_{\mathbb{P}_\beta}\left[\frac{H_T}{\beta_T} \,|\, \mathcal{F}_t\right] \qquad (0 \le t \le T). \tag{6}$$

By the above theorem, this definition solves the problems mentioned after the definition of the $\mathbb{P}_\beta$-price functional (3), but only for *nonnegative* contingent claims. Some of the contingent claims we are going to discuss later (e.g. forwards) do not satisfy this property, and so we need one further extension of our definition of the price functional.

Definition 2.7 *Assume that the contingent claim H_T can be written as the difference of two other contingent claims $J_T \ge 0$ and $K_T \ge 0$ where J_T is attainable with respect to some martingale measure $\mathbb{P}_\beta$ and K_T is attainable with respect to* every *(!) martingale measure. Then we set*

$$\boldsymbol{\pi}_t^T[H_T] \triangleq {}^{\mathbb{P}_\beta}\boldsymbol{\pi}_t^T[J_T] - {}^{\mathbb{P}_\beta}\boldsymbol{\pi}_t^T[K_T] \qquad (0 \le t \le T). \tag{7}$$

Contingent claims allowing a decomposition as above will be called attainable*.*

The following lemma explains to which extent this definition is independent of the specific choice of the martingale measure and of the decomposition of H_T.

Lemma 2.8 *Assume that H_T satisfies the assumption of Definition 2.7 with respect to some martingale measure $\mathbb{P}_\beta$. Let $\overline{\mathbb{P}}_\beta$ be any other martingale measure such that H_T can be written as the difference of two nonnegative claims $\bar{J}_T$, $\bar{K}_T$ which are both attainable under $\overline{\mathbb{P}}_\beta$. Then we have*

$$
{}^{\mathbb{P}_\beta}\boldsymbol{\pi}_t^T[H_T] = {}^{\overline{\mathbb{P}}_\beta}\boldsymbol{\pi}_t^T[H_T] \qquad (0 \leq t \leq T). \tag{8}
$$

PROOF: First we observe that with $\bar{J}_T$, $\bar{K}_T$ and K_T also $0 \leq J_T = \bar{J}_T - \bar{K}_T + K_T$ is attainable with respect to $\overline{\mathbb{P}}_\beta$. Theorem 2.6 implies that then J_T has the same $\mathcal{F}_t$-conditional expectation under $\mathbb{P}_\beta$ as under $\overline{\mathbb{P}}_\beta$. This and the analogous properties of K_T easily lead to equation (8). □

2.3 Change of Numéraire

Obviously, most of the notions (e.g. attainability, price functional) introduced so far depend on the money market account as choice of numéraire. But often it will be convenient to use other processes (e.g. bond prices) as numéraires as well. Thus we would like to examine more closely to which extent the above notions really depend on the specific choice of numéraire. To begin with we give

Definition 2.9 *A* numéraire *is a strictly positive, continuous semimartingale N. In analogy to the notion "martingale measure" we define a* numéraire measure *for N to be any probability measure $\mathbb{P}_N \approx \mathbb{P}$ such that S/N is a $\mathbb{P}_N$-martingale for any primary asset $S \in \mathcal{S}$. The set of all those measures $\mathbb{P}_N$ will be denoted by $\mathcal{P}_N$. A numéraire $\bar{N}$ is said to be* compatible *with another numéraire N if there is a numéraire measure $\mathbb{P}_N$ for N such that $\bar{N}/N$ becomes a martingale under $\mathbb{P}_N$.*

Our main tool will be the following theorem which we adopt from El Karoui-Geman-Rochet [6]:

Theorem 2.10 *Let $\bar{N}$ be a numéraire compatible with the numéraire N under the numéraire measure $\mathbb{P}_N$ for N. Then we can define a numéraire measure for $\bar{N}$ by*

$$
d\mathbb{P}_{\bar{N}} \triangleq \frac{N_0}{\bar{N}_0}\frac{\bar{N}_T}{N_T} d\mathbb{P}_N. \tag{9}
$$

PROOF: This is an easy application of the transformation rule for conditional expectations (see e.g. Karatzas-Shreve [14], Chapter 3, Lemma 5.3). □

Definition 2.11 *Let H_T be a contingent claim which can be written as the difference of two nonnegative contingent claims J_T, K_T satisfying:*

(i) $J_T = V(\xi)_T$ for some trading strategy ξ such that $V(\xi)/N$ is a true martingale under some $\mathbb{P}_N \in \mathcal{P}_N$.

(ii) $K_T = V(\eta)_T$ *for some trading strategy* η *such that* $V(\eta)/N$ *is a true martingale under* all $\mathbb{P}_N \in \mathcal{P}_N$.

Then H_T *is called* N-attainable *and there is a well defined*[4] *price functional*

$$(10) \qquad {}^N\boldsymbol{\pi}_t^T[H_T] \triangleq N_t \mathbb{E}_{\mathbb{P}_N}\left[\frac{H_T}{N_T} \,|\, \mathcal{F}_t\right] \; \left(= {}^N\boldsymbol{\pi}_t^T[J_T] - {}^N\boldsymbol{\pi}_t^T[K_T]\right) \qquad (0 \le t \le T).$$

The resulting prices for N-attainable assets will be uniquely determined in the class of numéraires compatible with β.

Theorem 2.12 *Let* $\bar{N}$ *be a numéraire compatible with the numéraire* N *under* $\mathbb{P}_N \in \mathcal{P}_N$. *An asset* H_T *is* N*-attainable if and only if it is* $\bar{N}$*-attainable, and in that case the price functionals defined in (10) coincide:*

$$(11) \qquad {}^N\boldsymbol{\pi}_t^T[H_T] = {}^{\bar{N}}\boldsymbol{\pi}_t^T[H_T] \qquad (0 \le t \le T).$$

Proof: Let ξ be a trading strategy so that $V(\xi)/N$ is a true martingale under some numéraire measure $\mathbb{P}_N$. First we wish to show that $V(\xi)/\bar{N}$ is a true martingale under $\mathbb{P}_{\bar{N}}$ defined by (9). By the transformation rule for conditional expectations this is equivalent to

$$\frac{V(\xi)_t}{\bar{N}_t}\frac{d\mathbb{P}_{\bar{N}}}{d\mathbb{P}_N}|\mathcal{F}_t = \frac{V(\xi)_t}{\bar{N}_t}\frac{N_0}{\bar{N}_0}\frac{\bar{N}_t}{N_t} = \frac{N_0}{\bar{N}_0}\frac{V(\xi)_t}{N_t} \qquad (0 \le t \le T)$$

being a martingale under $\mathbb{P}_N$, which is true by assumption. Equation (11) follows easily from $V(\xi)_T = H_T$ for some admissible strategy and the martingale property of $V(\xi)/N$, $V(\xi)/\bar{N}$ under $\mathbb{P}_N$, $\mathbb{P}_{\bar{N}}$ respectively. □

The theorem implies that the "standard" numéraire β can be replaced by any other numéraire N which is given by the price process of some attainable contingent claim. More precisely, we have the following

Corollary 2.13 *Let* N *be a numéraire compatible to* β *under some martingale measure* $\mathbb{P}_\beta$ *in the sense of Section 2.2. Then the notions "attainability" of Definition 2.7 and "*N*-attainability" coincide, and for any attainable contingent claim* H_T *we get the valuation formula*

$$(12) \qquad \boldsymbol{\pi}_t^T[H_T] = N_t \mathbb{E}_{\mathbb{P}_N}\left[\frac{H_T}{N_T} \,|\, F_t\right] \qquad (0 \le t \le T)$$

where $\boldsymbol{\pi}_t^T[.]$ *is defined by (6) and*

$$d\mathbb{P}_N \triangleq \frac{1}{N_0}\frac{N_T}{\beta_T} d\mathbb{P}_\beta.$$

□

[4] Note that the results in Jacka [11] are presented in terms of *discounted* price processes. Thus Theorem 2.6 remains valid for any numéraire N replacing β.

2.4 An Approach to Pricing and Hedging via "Change of Numéraire"

In this section we describe the valuation method used in the following examples. This method proceeds in four steps:

Let H_T be some contingent claim defined in terms of some given primary assets $\mathcal{S} = \{S^1, \ldots, S^d\}$.

STEP 1: We first express H_T as a combination of terminal values of some continuous processes X which are (local) martingales under a common class $\mathcal{P}_N$ of numéraire measures obtained by a suitable "change of numéraire".

STEP 2: The processes X obtained in Step 1 will be strictly positive and so they can be expressed as stochastic exponentials[5], i.e., $X = X_0 \mathcal{E}(X^{-1} \cdot X)$.[6]

STEP 3: We calculate the conditional $\mathbb{P}_N$-expectations of H_T along the filtration $\mathbb{F}$ for some numéraire measure $\mathbb{P}_N \in \mathcal{P}_N$; this will give us the price process for H_T as explained in the preceding sections. In most cases[7] the calculation is done as follows. The expression for H_T found in Step 1 will be multiplicative. It induces an analogous combination of the (local) $\mathbb{P}_N$-martingales. The resulting process is then rewritten as the product of a local martingale and a predictable process of finite total variation, i.e., we consider its "multiplicative Doob-Meyer decomposition". Then are introduced the rather restrictive assumptions that the process of finite variation has a *deterministic terminal value*[8], and that the corresponding local martingale is a true martingale under some suitable measure $\mathbb{P}_N$ of the class $\mathcal{P}_N$. This allows us to compute the conditional $\mathbb{P}_N$-expectations directly from the multiplicative Doob-Meyer decomposition.

STEP 4: Having found the price process for H_T we finally compute the hedging strategy by Itô's formula.

Remark 2.14 It is clear that the first three steps of the above scheme will lead to a price process which does not depend on the particular choice of the martingale measure $\mathbb{P}_N$ in Step 3. Thus, due to Theorem 2.6, the contingent claim H_T will be attainable. This is the main reason why Step 4 will always lead to a *selffinancing* hedging strategy. This strategy will clearly be admissible by construction.

[5]For a semimartingale X the corresponding exponential $\mathcal{E}(X)$ is defined as the unique solution Y of the SDE $dY_t = Y_{t-}\, dX_t$, $Y_0 = 1$. For continuous X we have $\mathcal{E}(X)_t = \exp(X_t - \frac{1}{2}[X,X]_t)$ (cf. PROTTER [15], Chapter II, p. 77). Later we will wish to "start" the exponential at a prescribed time $0 \le t_0 \le T$. Therefore we introduce the notation

$$\mathcal{E}(X)_{t_0,t} = \begin{cases} 1 & (0 \le t \le t_0) \\ \mathcal{E}(X)_t / \mathcal{E}(X)_{t_0} & (t_0 \le t \le T). \end{cases}$$

[6]Of course, "$\cdot$" denotes stochastic integration.

[7]The only exception is Section 3.1.3 on "An Option on a Forward".

[8]In the section on swaps we will even suppose that this process is completely deterministic.

3 Examples

In this section we consider specific derivatives, e.g. forwards on ratios, quanto futures, and continuous differential swaps, which were treated by JAMSHIDIAN [12]. Our purpose is to show that all the results of [12] can be derived in a systematic manner using the methodology proposed in Section 2.4.

3.1 Forwards

A *forward contract* is the obligation to purchase a specified good on a specific date (the maturity date of the contract) at an exercise price agreed upon at the inception of the contract. A forward contract is mandatorily exercisable; that is, once the purchaser has entered into a forward contract, he is obliged to honour that contract, to acquire the good at the agreed price (the *forward price*) upon maturity of the forward. This price is determined such that the forward contract itself does not cost anything at the initial time when the contract is entered.

3.1.1 A General Forward Price Formula

Let again $\mathcal{S}$ be a finite set of primary assets containing a bond B^T with maturity T.

Theorem 3.1 [9] *Let $H_T \geq 0$ be an attainable contingent claim. Then the forward on this underlying with the same maturity T must be priced by*

$$\mathbf{Fwd}_t^T[H_T] \triangleq \frac{\boldsymbol{\pi}_t^T[H_T]}{B_t^T} \qquad (0 \leq t \leq T) \tag{13}$$

in order to preclude arbitrage possibilities.

PROOF: Let F_t denote the forward price at time t. By definition this should be determined such that the payoff $H_T - F_t$ of the forward contract at time T is worthless at time t, i.e.

$$0 = \boldsymbol{\pi}_t^T[H_T - F_t]^{10} = \boldsymbol{\pi}_t^T[H_T] - F_t\boldsymbol{\pi}_t^T[1] = \boldsymbol{\pi}_t^T[H_T] - F_tB_t^T.$$

Here the second equality holds because the time t forward price has to be known in this moment. □

Formula (13) states the well-known connection between forwards and bonds of the same maturity. This suggests to use B^T as numéraire and thereby motivates the

[9] cf. COX-INGERSOLL-ROSS [4], Proposition 1

[10] At time t one easily attains a time T payoff F_t by buying and holding a suitable amount of bonds maturing at time T. Therefore, from time t on, the contingent claim $F_t \geq 0$ is attainable under *every* martingale measure and so by Lemma 2.7 the contingent claim $H_T - F_t$ has a well defined price.

Corollary 3.2 [11] *Under the conditions of Theorem 3.1 we can rewrite equation (13) as*

$$\mathbf{Fwd}_t^T[H_T] = \mathbb{E}_{\mathbb{P}_{B^T}}[H_T \,|\, \mathcal{F}_t] \qquad (0 \le t \le T) \tag{14}$$

where $\mathbb{P}_{B^T}$ is the numéraire measure induced as in (9) by any martingale measure $\mathbb{P}_\beta$ with respect to which H_T is attainable. In particular, under $\mathbb{P}_{B^T}$ the forward price process $\mathbf{Fwd}^T[H_T]$ is a true martingale.

Proof: First we note that $N \triangleq \beta$ and $\bar{N} \triangleq B^T$ are a pair of compatible numéraires under any martingale measure. By Corollary 2.13 we get

$$\boldsymbol{\pi}_t^T[H_T] = B_t^T \mathbb{E}_{\mathbb{P}_{B^T}}\left[\frac{H_T}{B_T^T} \,|\, \mathcal{F}_t\right] \qquad (0 \le t \le T).$$

This together with (13) and $B_T^T = 1$ implies (14). □

Remark 3.3 So far we have considered situations where there are only some "classical" primary assets (bonds, stocks, money market account). Now we are going to deal with financial markets which in addition contain certain forward contracts. This means that there are additional constraints on martingale measures. More precisely, under any numéraire measure $\mathbb{P}_{B^T}$ not only all processes S/B^T ($S \in \mathcal{S}$) have to be martingales, but also the price processes of all forward contracts contained in the financial market. This can be shown by the same arguments as in the proofs of Theorem 3.1 and its corollary. Alternatively, one can interpret the addition of forwards as an extension of the set of primary assets $\mathcal{S}$ by all processes $S \triangleq FB^T$ where F denotes the price process of some forward contract to be added.

Due to their close relation to forward prices, numéraire measures for bonds are often called *forward measures*.

3.1.2 Hedging Ratios

In this section we price a so called *"ratio"*. This is a forward contract whose terminal price H_T is given by the ratio of two primary asset prices P_T, $\tilde{P}_T$ at maturity T, i.e. $H_T \triangleq P_T/\tilde{P}_T$. We will treat this first example in more detail in order to show how the method of Section 2 works in the present context.

Step 1: Corollary 3.2 suggests to choose as numéraire measures the class $\mathcal{P}_{B^T}$, since we wish to price a forward contract. Letting $F \triangleq P/B^T$ and $\tilde{F} \triangleq \tilde{P}/B^T$ denote the forward price processes for P and $\tilde{P}$ respectively, we know that both F and $\tilde{F}$ are martingales under every $\mathbb{P}_{B^T} \in \mathcal{P}_{B^T}$. Moreover, we obviously have $H_T = F_T/\tilde{F}_T$.

Step 2: We may write $F = F_0 \mathcal{E}\left(F^{-1} \cdot F\right)$ and $\tilde{F} = \tilde{F}_0 \mathcal{E}\left(\tilde{F}^{-1} \cdot \tilde{F}\right)$.

[11] cf. Jamshidian [13]

Step 3: Using Lemma 3.5 below we get the multiplicative decomposition

$$\frac{F}{\tilde{F}} = \frac{F_0}{\tilde{F}_0}\mathcal{E}\left(F^{-1}\cdot F - \tilde{F}^{-1}\cdot\tilde{F}\right)\exp\left(\int_0^\cdot d[F^{-1}\cdot F - \tilde{F}^{-1}\cdot\tilde{F}, \tilde{F}^{-1}\cdot\tilde{F}]_s\right).$$

The $\mathcal{E}(\ldots)$-term is a local martingale because the same is true for F and $\tilde{F}$, and the last factor is of bounded variation. Now, if we assume that the local martingale is even a true martingale under a suitable numéraire measure $\mathbb{P}_{B^T}$ and if the above process of bounded variation has a deterministic terminal value, we may compute explicitly

$$\mathbb{E}_{\mathbb{P}_{B^T}}[H_T \mid \mathcal{F}_t] = \frac{F_t}{\tilde{F}_t}\exp\left(\int_t^T d[F^{-1}\cdot F - \tilde{F}^{-1}\cdot\tilde{F}, \tilde{F}^{-1}\cdot\tilde{F}]_s\right). \tag{15}$$

Note that this does not depend on the particular choice of the measure $\mathbb{P}_{B^T}$ as long as the martingale property of the above stochastic exponential holds. In this sense the forward price (14) is uniquely determined.

Step 4: An easy application of Itô's formula yields the dynamics

$$d\mathbf{Fwd}_t^T[H_T] = \mathbf{Fwd}_t^T[H_T]\left(\frac{dF_t}{F_t} - \frac{d\tilde{F}_t}{\tilde{F}_t}\right),$$

Thus, at time $0 \le t \le T$ one has to be long $\frac{1}{F_t}\mathbf{Fwd}_t^T[H_T]$ forwards on P and be short $\frac{1}{\tilde{F}_t}\mathbf{Fwd}_t^T[H_T]$ forwards on $\tilde{P}$ in order to hedge the ratio on P, $\tilde{P}$.

All this gives us

Theorem 3.4 *Let P and $\tilde{P}$ be two primary assets and let $F \triangleq P/B^T$ and $\tilde{F} \triangleq \tilde{P}/B^T$ denote the corresponding forward prices. Suppose there exists a numéraire measure $\mathbb{P}_{B^T}$ under which the stochastic exponential $\mathcal{E}(F^{-1}\cdot F - \tilde{F}^{-1}\cdot\tilde{F})$ is a true martingale. Assume furthermore that the quadratic covariation process*

$$C_t \triangleq [F^{-1}\cdot F - \tilde{F}^{-1}\cdot\tilde{F}, \tilde{F}^{-1}\cdot\tilde{F}]_t \qquad (0 \le t \le T)$$

is deterministic at time T. Then there is a unique forward price for the ratio $H_T \triangleq P_T/\tilde{P}_T$ given by

$$\mathbf{Fwd}_t^T[H_T] = \frac{P_t}{\tilde{P}_t}\exp(C_T - C_t) \qquad (0 \le t \le T)$$

and one can replicate the forward contract on H_T by being long $\frac{1}{F_t}\mathbf{Fwd}_t^T[H_T]$ forwards on P and short $\frac{1}{\tilde{F}_t}\mathbf{Fwd}_t^T[H_T]$ forwards on $\tilde{P}$ at any time $0 \le t \le T$.

□

In Step 3 above we used some arithmetic for exponentials which we summarize in

Lemma 3.5 *Let X and Y be two semimartingales starting at zero. Then we have*

$$\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X + Y + [X,Y])$$

as well as

$$\mathcal{E}(X)^{-1} = \mathcal{E}(-X + [X,X]) = \mathcal{E}(-X)\exp([X,X]).$$

Proof: See Protter [15], Chapter II, p. 79. □

3.1.3 An Option on a Forward Contract

Let us now consider a *European call option on a forward contract.* The holder of such an option has the right (not the duty) to buy at a prescribed time T (maturity of the *option*) a certain forward contract maturing at some time $\tau \geq T$. The price to be paid at time τ is prescribed as some fixed strike K. Thus the holder of the option can buy at time T a forward contract which specifies the price K instead of the forward price F_T^τ determined by the market at time T. Clearly, the holder will exercise his option only if $F_T^\tau > K$, because then he can realize a sure profit $F_T^\tau - K > 0$ at time τ by selling at time T the delivered forward at the actual market price F_T^τ. Therefore the option represents a contingent claim promising the payment $H_T \triangleq B_T^\tau (F_T^\tau - K)^+$ at time T. Now we can use the previously developed theory to price this claim.

The forward contract refers to some underlying price process S, i.e. $F_t^\tau = \mathbf{Fwd}_t^\tau[S_\tau]$ $(0 \leq t \leq \tau)$. By the general formula for forward prices (13), we have $F_t^\tau = S_t / B_t^\tau$ $(0 \leq t \leq \tau)$. Now, granting for the moment that H_T is attainable with respect to some martingale measure $\mathbb{P}_\beta$[12], we may write

$$\begin{aligned}
\boldsymbol{\pi}_t^T[H_T] &= \boldsymbol{\pi}_t^T[(B_T^\tau F_T^\tau - B_T^\tau K)^+] \\
&= \beta_t \mathbb{E}_{\mathbb{P}_\beta}\left[\frac{S_T 1_{\{F_T^\tau > K\}}}{\beta_T} \,\middle|\, \mathcal{F}_t\right] - K\beta_t \mathbb{E}_{\mathbb{P}_\beta}\left[\frac{B_T^\tau 1_{\{F_T^\tau > K\}}}{\beta_T} \,\middle|\, \mathcal{F}_t\right] \\
&= S_t \mathbb{P}_S[F_T^\tau > K \,|\, \mathcal{F}_t] - K B_t^\tau \mathbb{P}_{B^\tau}[F_T^\tau > K \,|\, \mathcal{F}_t]
\end{aligned} \tag{16}$$

where $\mathbb{P}_S$ and $\mathbb{P}_{B^\tau}$ are numéraire measures obtained from $\mathbb{P}_\beta$ as in Corollary 2.13. This application of change of numéraire can be found e.g. in El Karoui-Geman-Rochet [6]. In order to compute the conditional distributions of F_T^τ appearing in (16), we first state

Lemma 3.6 *Let $X = (X_t)_{0 \leq t \leq T}$ be a continuous local martingale on some stochastic basis $(\Omega, \mathcal{F}_T, \mathbb{P}^*, \mathbb{F})$. Assume further that its quadratic variation process $[X,X]$ is deterministic at time T. Then we have for $0 \leq t \leq T$*

$$\mathbb{P}^*[X_T \in A \,|\, \mathcal{F}_t] = N_{X_t, [X,X]_T - [X,X]_t}[A] \qquad (A \in \mathcal{B}(\mathbb{R})),$$

i.e. the terminal value of X conditioned on $\mathcal{F}_t$ is normally distributed with mean $m_t \triangleq X_t$ and variance $\sigma_t^2 \triangleq [X,X]_T - [X,X]_t$.

[12] Later we will be able to justify this assumption by Theorem 2.6.

Proof: For fixed $\lambda \in \mathbb{R}$ let us consider the exponential $\mathcal{E}(\lambda X)$. By Novikov's criterion[13] this exponential is a true $\mathbb{P}^*$-martingale under our assumption on $[X,X]_T$. Thus we have

$$\mathcal{E}(\lambda X)_t = \mathbb{E}_{\mathbb{P}^*}\left[\exp\left(\lambda X_T - \frac{\lambda^2}{2}[X,X]_T\right) \mid \mathcal{F}_t\right] \qquad (0 \le t \le T)$$

and because $[X,X]_T$ is deterministic:

$$\mathbb{E}_{\mathbb{P}^*}[\exp(\lambda X_T) \mid \mathcal{F}_t] = \exp\left(\lambda X_t + \frac{\lambda^2}{2}([X,X]_T - [X,X]_t)\right).$$

So the conditional Laplace transform of X_T coincides with the Laplace transform of a normally distributed random variable with mean m_t and variance σ_t^2 as above. $\square$

Now we can calculate $\mathbb{P}_{B^\tau}[F_T^\tau > K \mid \mathcal{F}_t]$ as follows. From Corollary 3.2 we know that the forward price process F^τ is a continuous martingale under any $\mathbb{P}_{B^\tau} \in \mathcal{P}_{B^\tau}$. Thus the process $X \triangleq (F^\tau)^{-1} \cdot F^\tau$ is a continuous local $\mathbb{P}_{B^\tau}$-martingale such that $F^\tau = F_0^\tau \mathcal{E}(X)$. Assuming that

$$[X,X]_T = \int_0^T \frac{d[F^\tau, F^\tau]_s}{(F_s^\tau)^2} \quad \text{is deterministic} \tag{17}$$

Lemma 3.6 (with $\mathbb{P}^* \triangleq \mathbb{P}_{B^\tau}$) yields

$$\begin{aligned}
\mathbb{P}_{B^\tau}[F_T^\tau > K \mid \mathcal{F}_t] &= \mathbb{P}_{B^\tau}\left[X_T > \log(K/F_0^\tau) + \frac{1}{2}[X,X]_T \mid \mathcal{F}_t\right] \\
&= N_{X_t, [X,X]_T - [X,X]_t}\left[\left(\log(K/F_0^\tau) + \frac{1}{2}[X,X]_T, +\infty\right)\right] \\
&= N_{0,1}\left[\left(\frac{K/F_t^\tau/K}{\sigma_t} + \frac{\sigma_t}{2}, +\infty\right)\right] \\
&= \Phi\left(\frac{\log(F_t^\tau/K)}{\sigma_t} - \frac{\sigma_t}{2}\right),
\end{aligned}$$

where

$$\sigma_t^2 \triangleq [X,X]_T - [X,X]_t = \int_t^T \frac{d[F^\tau, F^\tau]_s}{(F_s^\tau)^2} \tag{18}$$

and Φ denotes the distribution function of the standard normal distribution $N_{0,1}$.

To calculate $\mathbb{P}_S[\mathcal{F}_T^\tau > K \mid \mathcal{F}_t]$ we proceed similarly. By definition of $\mathbb{P}_S$ the process $(F^\tau)^{-1} = B^\tau/S$ is a $\mathbb{P}_S$-martingale. So $Y \triangleq F^\tau \cdot (F^\tau)^{-1}$ is a continuous local $\mathbb{P}_S$-martingale such that $(F^\tau)^{-1} = (F_0^\tau)^{-1}\mathcal{E}(Y)$ and

$$[Y,Y]_T = \int_0^T (F_s^\tau)^2 \, d[(F^\tau)^{-1}, (F^\tau)^{-1}]_s = \int_0^T \frac{d[F^\tau, F^\tau]_s}{(F_s^\tau)^2} = [X,X]_T.$$

[13] see e.g. Karatzas-Shreve [14], Chapter 3, Proposition 5.12

Thus the quadratic variation $[Y, Y]_T$ is deterministic, too. Now, Lemma 3.6 (this time with $\mathbb{P}^* \triangleq \mathbb{P}_S$) yields

$$\begin{aligned}
\mathbb{P}_S[F_T^\tau > K \mid \mathcal{F}_t] &= \mathbb{P}_S[(F_T^\tau)^{-1} < K^{-1} \mid \mathcal{F}_t] \\
&= \mathbb{P}_S\left[Y_T < \log(F_0^\tau/K) + \frac{1}{2}[Y,Y]_T \mid \mathcal{F}_t\right] \\
&= N_{Y_t, [Y,Y]_T - [Y,Y]_t}\left[\left(-\infty, \log(F_0^\tau/K) + \frac{1}{2}[Y,Y]_T\right)\right] \\
&= \Phi\left(\frac{\log(F_t^\tau/K}{\sigma_t} + \frac{\sigma_t}{2}\right)
\end{aligned}$$

with the same parameter σ_t as in (18).

We can now deduce that, in fact, H_T has to be attainable under condition (17), because the conditional probabilities appearing in (16) do not depend on the specific choice of the martingale measure $\mathbb{P}_\beta$.

We summarize this in

Theorem 3.7 [14] *Let F^τ be the price process of some forward maturing at time τ. Suppose the quadratic variation*

$$\int_0^T \frac{d[F^\tau, F^\tau]_s}{(F_s^\tau)^2}$$

is deterministic. Then an option $H_T \triangleq B_T^\tau(F_T^\tau - K)^+$ on this forward with strike K maturing at time T is worth

$$\text{(19)} \qquad \boldsymbol{\pi}_t^T[H_T] = B_t^\tau\left\{F_t^\tau \Phi\left(\frac{\log(F_t^\tau/K)}{\sigma_t} + \frac{\sigma_t}{2}\right) - K\Phi\left(\frac{\log(F_t^\tau/K)}{\sigma_t} - \frac{\sigma_t}{2}\right)\right\}$$

in $0 \le t < T$ where σ_t is defined by (18)[15]*. Moreover, at each time $0 \le t_0 < T$ it is possible to duplicate the payoff of this option by purchasing $\boldsymbol{\pi}_{t_0}^T[H_T]/B_{t_0}^\tau$ bonds maturing at τ and holding exactly $\Phi\left(\frac{\log(F_t^\tau/K)}{\sigma_t} + \frac{\sigma_t}{2}\right)$ forward contracts in $t_0 \le t \le T$.*

Proof: We only have to derive the hedging strategy. For this we first note that it is enough to duplicate a forward contract maturing at time $\tau \ge T$ (!) with terminal price $(F_T^\tau - K)^+$. Indeed, having done this, one will realize the profit or loss $B_T^\tau\{(F_T^\tau - K)^+ - \mathbf{Fwd}_{t_0}^\tau[(F_T^\tau - K)^+]\}$ at time T. Thus, additionally purchasing at time t_0

$$\mathbf{Fwd}_{t_0}^\tau[(F_T^\tau - K)^+] = \boldsymbol{\pi}_{t_0}^\tau[(F_T^\tau - K)^+]/B_{t_0}^\tau = \boldsymbol{\pi}_{t_0}^T[B_T^\tau(F_T^\tau - K)^+]/B_{t_0}^\tau$$

bonds maturing at time τ, one can perfectly hedge the contingent claim H_T.

So we have to study the dynamics of the process $\mathbf{Fwd}^\tau[(F_T^\tau - K)^+]$ in terms of its underlying process F^τ. Both of these processes are local martingales under every

[14] cf. Jamshidian [12], Section 9

[15] Without loss of generality we may suppose $\sigma_t > 0$ for $0 \le t < T$ because otherwise the model would be degenerate, i.e. the process F^τ would be constant from the first time σ becomes zero.

numéraire measure $\mathbb{P}_{B^\tau}$ by Corollary 3.2 and Remark 3.3. It follows that calculating $d\mathbf{Fwd}^\tau[(F_T^\tau - K)^+]$ one only has to pay attention to increments resulting from processes of unbounded variation — the other increments will have to cancel out because of the uniqueness of the Doob-Meyer decomposition. So in the following calculation we may and will leave out increments of processes of bounded variation denoting them by "…". For convenience we set

$$h_\pm(t) \triangleq \frac{\log(F_t^\tau/K)}{\sigma_t} \pm \frac{\sigma_t}{2} \qquad (0 \le t \le T).$$

After all these preliminaries we can now calculate:

(i) $dh_\pm \;=\; \frac{dF^\tau}{F^\tau \sigma} + \dots$

(ii)
$$\begin{aligned} d(\Phi(h_\pm)) &= \tfrac{1}{\sqrt{2\pi}} \exp(-h_\pm^2/2)\, dh_\pm + \dots \\ &= \tfrac{1}{\sqrt{2\pi}} \tfrac{\exp(-h_\pm^2/2)}{F^\tau \sigma}\, dF^\tau + \dots \end{aligned}$$

(iii) $d\mathbf{Fwd}^\tau[(F_T^\tau - K)^+]$
$$\begin{aligned} &= d(F^\tau \Phi(h_+) - K\Phi(h_-)) \\ &= \Phi(h_+)\, dF^\tau + F^\tau\, d(\Phi(h_+)) - K\, d(\Phi(h_-)) + \dots \\ &= \left\{ \Phi(h_+) + \tfrac{1}{\sqrt{2\pi\sigma^2}} \underbrace{\left[\exp(-h_+^2/2) - \frac{K}{F^\tau} \exp(-h_-^2/2) \right]}_{=0} \right\} dF^\tau + \dots \\ &= \Phi(h_+)\, dF^\tau. \end{aligned}$$

where the last equality follows from the definition of $h_\pm$ and the already noted fact that increments of processes of bounded variation have to cancel out.

We see that in order to duplicate the forward with terminal value $(F_T^\tau - K)^+$ one has to hold $\Phi(h_+(t))$ forwards F^τ in $t_0 \le t \le T$. □

3.2 Futures

As an alternative to forwards one can use *futures* to fix prices of a certain good in advance. But while forwards impose payments only at time of maturity future contracts guarantee prices via a continuous payment stream. In our (idealized) model the instantaneous payments of a future contract result from a stochastic process called the *future price* process of the underlying. This process is specified by the following properties:

(i) The process of future prices $F = (F_t)_{0 \le t \le T}$ is a continuous semimartingale. The instantaneous payment resulting from a future over the time period $[t, t + dt]$ is given by the infinitesimal increment dF_t of this semimartingale. More precisely, we suppose that the net profit or loss $V = (V_t)_{t_0 \le t \le T}$ from a long position in one future contract written at time $t_0 \ge 0$ follows the stochastic differential equation

$$V_{t_0} = 0, \qquad dV_t = dF_t + V_t r_t\, dt \qquad (t_0 \le t \le T) \tag{20}$$

provided the holder uses his money market account for settlement.

(ii) At time of maturity T the future price F_T coincides with the price $H_T \in \mathcal{F}_T$ of the underlying good at that time. For example, H_T could be the price of some primary asset at time T.

(iii) At *any* time $0 \leq t \leq T$ a future contract is *worthless* — changes in value of the underlying are reflected by changes of the future price F which cause an instantaneous settlement.

Of course, it is not clear at all that conditions (i)-(iii) give a complete characterization of the future price process F. In fact — as we will see in Theorem 3.8 below — in general one is only able to determine F uniquely in a complete financial market.

3.2.1 The Future Price Formula

Let us start with the construction of a price functional $\mathbf{Fut}^T[.]$ for future prices which is similar to the forward price functional $\mathbf{Fwd}^T[.]$ of the preceding section.

Theorem 3.8 [16] *Let $H_T \geq 0$ be a contingent claim. Then the future price process $(\mathbf{Fut}_t^T[H_T])_{0 \leq t \leq T}$ with terminal value H_T is given by*

$$\text{(21)} \qquad \mathbf{Fut}_t^T[H_T] = \boldsymbol{\pi}_t^T \left[\exp\left(\int_t^T r_s\, ds \right) H_T \right] = \mathbb{E}_{\mathbb{P}_\beta}[H_T \,|\, \mathcal{F}_t] \qquad (0 \leq t \leq T)$$

provided the above price is well defined, i.e. the contingent claim $\exp\left(\int_0^T r_s\, ds\right) H_T$ is attainable with respect to some martingale measure $\mathbb{P}_\beta$. In particular, we see that future prices — as soon as they are well defined — form a martingale under a suitable martingale measure $\mathbb{P}_\beta$.

Proof: Let $\xi = (\xi_t)_{t_0 \leq t \leq T}$ be any trading strategy for futures starting at time $0 \leq t_0 \leq T$; settlement shall be done by using the money market account. Let further denote $F = (F_t)_{0 \leq t \leq T}$ the price process of the considered future. Then by equation (20) the resulting profit or loss $V = (V_t)_{t_0 \leq t \leq T}$ is governed by the SDE

$$V_{t_0} = 0, \qquad dV_t = \xi_t\, dF_t + V_t r_t\, dt \qquad (t_0 \leq t \leq T).$$

Thus, under weak regularity conditions on ξ, we will have

$$\text{(22)} \qquad V_t = \exp\left(\int_{t_0}^t r_s\, ds \right) \left\{ \int_{t_0}^t \exp\left(- \int_{t_0}^s r_u\, du \right) \xi_s\, dF_s \right\} \qquad (t_0 \leq t \leq T).$$

If we now choose $\xi_t \triangleq \exp\left(- \int_{t_0}^t r_u\, du \right)$ $(t_0 \leq t \leq T)$, we get

$$V_T = \exp\left(\int_{t_0}^T r_s\, ds \right) (H_T - F_{t_0}),$$

[16]cf. Cox-Ingersoll-Ross [4], Proposition 2 and Jamshidian [13], Formula (1.3)

since $F_T = H_T$ by definition. So trading in futures allows us to create a payoff of the above size V_T without investing any initial capital. Therefore this payoff has to be worthless at the beginning of the strategy, i.e. we must have

$$0 = \boldsymbol{\pi}_{t_0}^T[V_T] = \boldsymbol{\pi}_{t_0}^T\left[\exp\left(\int_{t_0}^T r_s\,ds\right) H_T\right] - F_{t_0}\boldsymbol{\pi}_{t_0}^T\left[\exp\left(\int_{t_0}^T r_s\,ds\right)\right]$$

implying (21). Here we have to ensure that $\boldsymbol{\pi}_{t_0}^T[V_T]$ is well defined; this is done by Lemma 2.7 and our condition on the contingent claim $\exp\left(\int_0^T r_s\,ds\right) H_T$. $\square$

Remark 3.9 In analogy to forwards the addition of futures to a financial market has an impact on the set of martingale measures for the considered financial market: price processes of *primary* futures have to be true martingales under any martingale measure $\mathbb{P}_{{}^{\$}\beta}$.

3.2.2 Pricing an American Future by its British Analogue

Let us consider the following situation: An American investor wishes to fix a future price for a certain good. Unfortunately there is no such future on the American financial market. Only the British market contains a future for that good, but, of course, it is denominated in pounds. How can the investor use this British future for his purposes without taking an exchange rate risk?

Let us assume that besides the above British future ${}^{£}F = ({}^{£}F_t)_{0\le t\le T} > 0$ there exists an exchange rate future of the same maturity T with price process $F^X \triangleq ({}^{\$}\mathbf{Fut}_t^T[X_T]_{0\le t\le T}) > 0$. Recall that $X = (X_t)_{0\le t\le T}$ denotes the exchange rate from pound to dollar. Inspired by Theorem 3.8, we will try to calculate $\mathbb{E}_{\mathbb{P}_{{}^{\$}\beta}}[X_T\,{}^{£}F_T\,|\,\mathcal{F}_t]$ $(0 \le t \le T)$ where $\mathbb{P}_{{}^{\$}\beta}$ denotes as usual an arbitrary American martingale measure. By Theorem 2.2, each $\mathbb{P}_{{}^{\$}\beta}$ is related to a martingale measure $\mathbb{P}_{{}^{£}\beta}$ for the British market via

$$d\mathbb{P}_{{}^{\$}\beta} = \frac{X_0\,{}^{\$}\beta_T}{X_T\,{}^{£}\beta_T}d\mathbb{P}_{{}^{£}\beta}.$$

We have

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}_{{}^{\$}\beta}}[X_T\,{}^{£}F_T\,|\,\mathcal{F}_t]\\
&= \frac{X_t\,{}^{£}\beta_t}{{}^{\$}\beta_t}\mathbb{E}_{\mathbb{P}_{{}^{£}\beta}}\left[\left(\frac{F^X\,{}^{\$}\beta}{X\,{}^{£}\beta}\right)_T\,{}^{£}F_T\,|\,F_t\right]\\
&= F_t^X\,{}^{£}F_t\mathbb{E}_{\mathbb{P}_{{}^{£}\beta}}\left[\mathcal{E}\left(\frac{X\,{}^{£}\beta}{F^X\,{}^{\$}\beta}\cdot\frac{F^X\,{}^{\$}\beta}{X\,{}^{£}\beta}\right)_{t,T}\mathcal{E}\left({}^{£}F^{-1}\cdot{}^{£}F\right)_{t,T}\,|\,F_t\right]\\
&= F_t^X\,{}^{£}F_t\mathbb{E}_{\mathbb{P}_{{}^{£}\beta}}\left[\mathcal{E}\left(\frac{X\,{}^{£}\beta}{F^X\,{}^{\$}\beta}\cdot\frac{F^X\,{}^{\$}\beta}{X\,{}^{£}\beta} + {}^{£}F^{-1}\cdot{}^{£}F\right)_{t,T}\right.\\
&\qquad\left.\cdot\,\exp\left(\int_t^T d[(F^X)^{-1}\cdot F^X - X^{-1}\cdot X,\,{}^{£}F^{-1}\cdot{}^{£}F]_s\right)\,|\,F_t\right].
\end{aligned}\tag{23}$$

Note that F^X has to be a $\mathbb{P}_{\$\beta}$-martingale by Remark 3.9. Thus $\frac{F^X\,{}^{\$}\beta}{X\,{}^{£}\beta}$ is a $\mathbb{P}_{£\beta}$-martingale. Since also ${}^{£}F$ is a $\mathbb{P}_{£\beta}$-martingale, the above exponential has to be a local martingale, at least. If we now assume that it is even a true $\mathbb{P}_{£\beta}$-martingale, and if we further suppose that the covariation $[(F^X)^{-1}\cdot F^X - X^{-1}\cdot X,\ {}^{£}F^{-1}\cdot{}^{£}F]_T$ is *deterministic*, the above formula simplifies to

$$(24)\quad \mathbb{E}_{\mathbb{P}_{\$\beta}}[X_T\,{}^{£}F_T \,|\, \mathcal{F}_t] = F_t^X\,{}^{£}F_t \exp\left(\int_t^T d[(F^X)^{-1}\cdot F^X - X^{-1}\cdot X,\ {}^{£}F^{-1}\cdot{}^{£}F]_s\right).$$

Thus we have

Theorem 3.10 [17] *Assume that the exponential*

$$\mathcal{E}\left(\frac{X\,{}^{£}\beta}{F^X\,{}^{\$}\beta}\cdot\frac{F^X\,{}^{\$}\beta}{X\,{}^{£}\beta} + {}^{£}F^{-1}\cdot{}^{£}F\right)$$

is a true martingale under some martingale measure $\mathbb{P}_{£\beta}$, and that the quadratic co-variation

$$[(F^X)^{-1}\cdot F^X - X^{-1}\cdot X,\ {}^{£}F^{-1}\cdot{}^{£}F]_T$$

is deterministic. Then the dollar- and the pound-denominated future price (denoted by ${}^{\$}F$, ${}^{£}F$ respectively) on the same good maturing at time T are related by

$$(25)\quad {}^{\$}F_t = F_t^X\,{}^{£}F_t \exp\left(\int_t^T d[(F^X)^{-1}\cdot F^X - X^{-1}\cdot X,\ {}^{£}F^{-1}\cdot{}^{£}F]_s\right) \qquad (0 \le t \le T).$$

Furthermore we can duplicate the dollar future by going long $\frac{{}^{\$}F_t}{X_t\,{}^{£}F_t}$ British futures ${}^{£}F$ and long $\frac{{}^{\$}F_t}{{}^{£}F_t}$ exchange rate futures F^X at time $0 \le t \le T$, continually converting the whole instantaneous profit or loss from this position into dollars.

Proof: We only have to check the trading strategy. We first observe that, to duplicate a future, one only has to duplicate the instantaneous profits and losses given by its future price process. Since we already have determined the only possible candidate by equation (24), we only have to calculate its infinitesimal increment and try to express it in terms of the instantaneous profit or loss of a position in ${}^{£}F$-type futures and in exchange rate futures. If we hold a British future and continually convert the instantaneous profit or loss into dollars, then we produce infinitesimal dollar payments of size

$$X_t\, d\,{}^{£}F_t + d[X,\ {}^{£}F]_t \qquad (0 \le t \le T);$$

this is clear in discrete time, and in continuous time it follows by passage to the limit as in, e.g., [1]. Applying Itô's formula to the process of equation (24) we may calculate its infinitesimal increment as

$$\frac{{}^{\$}F_t}{F_t^X} dF_t^X + \frac{{}^{\$}F_t}{{}^{£}F_t} d\,{}^{£}F_t + \frac{{}^{\$}F_t}{X_t\,{}^{£}F_t} d[X,\ {}^{£}F]_t \qquad (0 \le t \le T).$$

Comparing this to the above equation we see that the proposed hedging strategy is correct. □

[17] cf. Jamshidian [12], Section 4

3.2.3 Quantos

Let us now consider a slightly different situation: There is a British future ${}^{£}F$ with pound-denominated terminal price ${}^{£}F_T \triangleq H_T \geq 0$, but there is no dollar future for the same terminal price in *dollars*. How can one use the given British future to simulate the dollar-denominated one?

To this end we consider two martingale measures $\mathbb{P}_{{}^{\$}\beta}$, $\mathbb{P}_{{}^{£}\beta}$ related as in the previous section and calculate

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}_{{}^{\$}\beta}}[H_T \,|\, \mathcal{F}_t] &= \frac{X_t\, {}^{£}\beta_t}{{}^{\$}\beta_t} \mathbb{E}_{\mathbb{P}_{{}^{£}\beta}}\left[\left(\frac{{}^{\$}\beta}{X\, {}^{£}\beta}\right)_T {}^{£}F_T \,|\, \mathcal{F}_t\right] \\
&= {}^{£}F_t \mathbb{E}_{\mathbb{P}_{{}^{£}\beta}}\left[\mathcal{E}\left(\frac{X\, {}^{£}\beta}{{}^{\$}\beta}\cdot \frac{{}^{\$}\beta}{X\, {}^{£}\beta}\right)_{t,T} \mathcal{E}\left({}^{£}F^{-1}\cdot {}^{£}F\right)_{t,T} \,|\, \mathcal{F}_t\right] \\
&= {}^{£}F_t \mathbb{E}_{\mathbb{P}_{{}^{£}\beta}}\left[\mathcal{E}\left(\frac{X\, {}^{£}\beta}{{}^{\$}\beta}\cdot \frac{{}^{\$}\beta}{X\, {}^{£}\beta} + {}^{£}F^{-1}\cdot {}^{£}F\right)_{t,T}\right. \\
&\qquad \left. \cdot \exp\left(-\int_t^T \frac{d[X, {}^{£}F]_s}{X_s\, {}^{£}F_s}\right) \,|\, \mathcal{F}_t\right].
\end{aligned}$$

In analogy to the previous section one can now prove

Theorem 3.11 [18] *Let ${}^{£}F$ (resp. ${}^{\$}F$) denote the pound- (resp. dollar-) denominated future price process with terminal value H_T pounds (resp. dollars). Suppose that the exponential*

$$\mathcal{E}\left(\frac{X\, {}^{£}\beta}{{}^{\$}\beta}\cdot \frac{{}^{\$}\beta}{X\, {}^{£}\beta} + {}^{£}F^{-1}\cdot {}^{£}F\right)$$

is a true martingale under some martingale measure $\mathbb{P}_{{}^{£}\beta}$. Let us assume further that the quadratic covariation

$$\int_0^T \frac{d[X, {}^{£}F]_s}{X_s\, {}^{£}F_s}$$

is deterministic. Then we have

$$ {}^{\$}F_t = {}^{£}F_t \exp\left(-\int_t^T \frac{d[X, {}^{£}F]_s}{X_s\, {}^{£}F_s}\right), \tag{26}$$

and one can replicate the dollar-denominated future by holding $\frac{{}^{\$}F_t}{X_t\, {}^{£}F_t}$ pieces of its British analogue continually converting all profits or losses into dollars.

□

[18] cf. Jamshidian [12], Section 5

3.3 Swaps

As a last example from Jamshidian [12] we are going to derive a valuation formula and a hedging strategy for *continuous differential swaps*. A differential swap expiring at time T, say, between the British and the U.S. financial market, promises its holder an infinitesimal dollar payment stream $({}^{\pounds}r_t - {}^{\$}r_t)\,dt$ $(0 \le t \le T)$. More precisely: If an investor enters a differential swap at time t_0 his cumulative return $(V_{t_0,t})_{t_0 \le t \le T}$ from this investment will develop satisfying

$$(27)\qquad V_{t_0,t_0} = 0, \quad dV_{t_0,t} = V_{t_0,t}\,{}^{\$}r_t\,dt + ({}^{\pounds}r_t - {}^{\$}r_t)\,dt \qquad (t_0 \le t \le T)$$

provided he reinvests his funds in the U.S. money market. So at time $0 \le t \le T$ he will have earned the amount

$$(28)\qquad V_{t_0,t} = \exp\left(\int_{t_0}^{t} {}^{\$}r_u\,du\right)\left\{\int_{t_0}^{t} \exp\left(-\int_{t_0}^{s} {}^{\$}r_u\,du\right)({}^{\pounds}r_s - {}^{\$}r_s)\,ds\right\}.$$

Thus, the value of the above continuous differential swap has to coincide with the value of a payment of size $V_{t_0,T}$ at time T. Therefore it is enough to price and hedge this contingent claim. For this we first recall some well-known results on the relation between interest rates and bonds.

3.3.1 Forward Rates and Bonds

Forward rates allow an investor to fix his interest rates in advance. This is done as follows: At time t he agrees to invest a certain amount of his wealth over an infinitesimal future time period $[s, s+ds]$ at interest rate $r_t^s \in \mathcal{F}_t$ which is fixed at time t. Whatever the true interest rate might be at time s, he will have the interest return $r_t^s\,ds$ on his investment. The forward rate r_t^s is determined so that — in analogy with the classical forward contracts — the above agreement itself does not cost anything.

We assume that besides the "usual" stocks and bonds one can trade forward rates for every future date $0 \le s \le T$ at any time $0 \le t \le s$. Since both bonds and forward rates allow to fix future interest rates — although in different ways — there has to be some relation between them. Under weak regularity conditions this is given by the well-known formulae

$$(29)\quad B_t^s = \exp\left(-\int_t^s r_t^u\,du\right) \quad \mathbb{P}\text{-a.s.,} \qquad \text{resp.} \qquad r_t^s = -\frac{1}{B_t^s}\frac{\partial}{\partial s}B_t^s \quad \mathbb{P}\otimes ds\text{-a.e.}$$

So far we have always argued in terms of a given finite number of assets. From now on we will consider the *full continuum* B^s $(0 \le s \le T)$ of bond price processes. From a mathematical point of view, this causes many technical problems whose solution lies far beyond the scope of this paper. For instance, we do not try to reduce the existence of an equivalent martingale measure to the absence of arbitrage as in Theorem 2.1. Instead we simply introduce the

Assumption 3.12 *There is at least one equivalent martingale measure for all stocks and the whole continuum of bonds.*

Under this standing assumption we have the well-known

Theorem 3.13 [19] *At time $t \geq 0$ forward rates r_t^s $(t \leq s \leq T)$ satisfy*

$$r_t^s = \mathbb{E}_{\mathbb{P}_{B^s}}[r_s \,|\, \mathcal{F}_t] \qquad (t \leq s \leq T) \qquad \mathbb{P} \otimes ds\text{-a.e.} \tag{30}$$

for $\mathbb{P}_{B^s} \in \mathcal{P}_{B^s}$. In particular, for a.e. fixed maturity $0 \leq s \leq T$ the forward rates $(r_t^s)_{0 \leq t \leq s}$ form a $\mathbb{P}_{B^s}$-martingale.

Proof: First we note that for any martingale measure $\mathbb{P}_\beta$ and $0 \leq t \leq s \leq T$ we have

$$\begin{aligned}
B_t^s &= \mathbb{E}_{\mathbb{P}_\beta}\left[\exp\left(-\int_t^s r_u \, du\right) \,|\, \mathcal{F}_t\right] \\
&= \mathbb{E}_{\mathbb{P}_\beta}\left[1 + \int_t^s \frac{\partial}{\partial \tau}\left\{\exp\left(-\int_t^\tau r_u \, du\right)\right\} d\tau \,|\, \mathcal{F}_t\right] \\
&= 1 - \int_t^s \mathbb{E}_{\mathbb{P}_\beta}\left[\exp\left(-\int_t^\tau r_u \, du\right) r_\tau \,|\, \mathcal{F}_t\right] d\tau \\
&= 1 - \int_t^s B_t^\tau \mathbb{E}_{\mathbb{P}_{B^\tau}}[r_\tau \,|\, \mathcal{F}_t] \, d\tau
\end{aligned}$$

Now comparing this to

$$\begin{aligned}
B_t^s &= \exp\left(-\int_t^s r_t^u \, du\right) \\
&= 1 + \int_t^s \frac{\partial}{\partial \tau}\left\{\exp\left(-\int_t^\tau r_t^u \, du\right)\right\} d\tau \\
&= 1 - \int_t^s B_t^\tau r_t^\tau \, d\tau
\end{aligned}$$

yields (30). □

Remark 3.14 By the above theorem, we may interpret the forward rates r_t^s $(0 \leq t \leq s)$ as unbiased estimators for r_s under any numéraire measure $\mathbb{P}_{B^s}$. For this reason the above equality is often called *"unbiased expectation hypothesis"*.

In order to analyze the dynamics of bonds and forward rates we introduce the following Heath-Jarrow-Morton [9]-type assumption:

Assumption 3.15 *There is a finite number I of continuous semimartingales M^i $(i = 1, \ldots, I)$ such that for suitable bounded*[20]*, $\mathcal{B}([0,T]) \otimes \mathcal{P}$-measurable $H^i = (H_t^{s,i}\,; 0 \leq s, t \leq T)$ $(i = 1, \ldots, I)$ the dynamics of forward rates are given by*

$$r_t^s = r_0^s + \sum_i \int_0^t H_u^{s,i} \, dM_u^i \qquad (0 \leq t \leq s). \tag{31}$$

[19] cf. e.g. Ingersoll [10], Chapter 18

[20] Of course, the boundedness-assumption is introduced only for convenience and can be relaxed considerably.

The dynamics of forward rates translates into the dynamics of bonds as follows:

Proposition 3.16 *Under Assumption 3.15 bond prices are governed by the SDE*

$$
\begin{aligned}
\frac{dB_t^s}{B_t^s} &= r_t\,dt - \textstyle\sum_i \left(\int_t^s H_t^{v,i}\,dv\right)\,dM_t^i \\
&\quad + \tfrac{1}{2}\textstyle\sum_{i,j}\left(\int_t^s H_t^{v,i}\,dv\right)\left(\int_t^s H_t^{v,j}\,dv\right)\,d[M^i,M^j]_t.
\end{aligned}
\tag{32}
$$

Proof: First (29) implies

$$dB_t^s = -B_t^s\,d\left(\int_\cdot^s r_\cdot^v\,dv\right)_t + \frac{1}{2}B_t^s\,d[\int_\cdot^s r_\cdot^v\,dv, \int_\cdot^s r_\cdot^v\,dv]_t.$$

By Lemma 3.17 below, we may calculate the above stochastic differentials as

$$d\left(\int_\cdot^s r_\cdot^v\,dv\right)_t = -r_t^t\,dt + \sum_i \left(\int_t^s H_t^{v,i}\,dv\right)\,dM_t^i$$

and

$$d[\int_\cdot^s r_\cdot^v\,dv, \int_\cdot^s r_\cdot^v\,dv]_t = \sum_{i,j}\left(\int_t^s H_t^{v,i}\,dv\right)\left(\int_t^s H_t^{v,j}\,dv\right)\,d[M^i,M^j]_t.$$

Employing these equations in the first one yields the result. □

The Fubini-type argument in the preceding proof will be used again in the sequel, and so we state it explicitly:

Lemma 3.17 *Let $Y = (Y_t)_{0\le t\le s}$ be a process of the form*

$$Y_t = \int_t^s Z_t^v\,dv \qquad (0 \le t \le s)$$

where $\{Z^v = (Z_t^v)_{0\le t\le v}; 0 \le v \le s\}$ is a family of semimartingales satisfying

$$Z_t^v = Z_0^v + \int_0^t H_u^v\,dM_u \qquad (0 \le t, v \le s)$$

for some continuous semimartingale M and some bounded $\mathcal{B}([0,T]) \otimes \mathcal{P}$-measurable $H_t^v = H(v,t,\omega)$. Then the dynamics of Y can be read off

$$Y_t = \int_0^s Z_0^v\,dv - \int_0^t Z_u^u\,du + \int_0^t \left(\int_u^s H_u^v\,dv\right)\,dM_u \qquad (0 \le t \le s). \tag{33}$$

Proof: Let λ denote the Lebesgue measure restricted on $(0,s)$. We may write

$$Y_t = \int 1_{[0,v)}(t) Z_t^v \lambda(dv)\ (0 \le t \le s).$$

[20] $\mathcal{P}$ denotes the σ-algebra of $\mathbb{F}$-predictable sets.

Applying Itô's formula to the integrand yields

$$\begin{aligned} 1_{[0,v)}(t)Z_t^v &= 1_{[0,v)}(0)Z_0^v + \int_0^t Z_{u-}^v \, d1_{[0,v)}(u) + \int_0^t 1_{[0,v]}(u)\, dZ_u^v \\ &= Z_0^v - Z_v^v 1_{[0,t]}(v) + \int_0^t 1_{[0,v]}(u)\, dZ_u^v \end{aligned}$$

Employing this in the first equation and using Theorem 46 in Chapter IV of PROTTER [15] we have

$$\begin{aligned} Y_t &= \int Z_0^v \, \lambda(dv) - \int Z_v^v 1_{[0,t]}(v)\, \lambda(dv) \\ &\quad + \int \left(\int_0^t 1_{[0,v]}(u) H_u^v \, dM_u \right) \lambda(dv) \\ &= \int Z_0^v \, \lambda(dv) - \int Z_v^v 1_{[0,t]}(v)\, \lambda(dv) \\ &\quad + \int_0^t \left(\int 1_{[0,v]}(u) H_u^v \, \lambda(dv) \right) dM_u \\ &= \int_0^s Z_0^v \, dv - \int_0^t Z_u^u \, du + \int_0^t \left(\int_u^s H_u^v \, dv \right) dM_u \end{aligned} \tag{34}$$

for $0 \leq t \leq s$. □

3.3.2 Pricing Continuous Differential Swaps

After these preliminaries let us now price the contingent claim $V_{t_0,T}$ given by equation (28). By Assumption 3.12 we have a martingale measure $\mathbb{P}_{\$\beta}$ for the whole (U.S.-) market including the continuum of American and (converted) British bonds.

Of course, we want to calculate

$$\begin{aligned} {}^{\mathbb{P}_{\$\beta}}\boldsymbol{\pi}_t^T[V_{t_0,T}] &= {}^{\$}\beta_t \mathbb{E}_{\mathbb{P}_{\$\beta}}\left[\frac{V_{t_0,T}}{{}^{\$}\beta_T} \,\middle|\, \mathcal{F}_t \right] \\ &= \mathbb{E}_{\mathbb{P}_{\$\beta}}\left[\exp\left(-\int_t^T {}^{\$}r_u \, du \right) V_{t_0,T} \,\middle|\, \mathcal{F}_t \right] \\ &= V_{t_0,t} + \int_t^T \mathbb{E}_{\mathbb{P}_{\$\beta}}\left[\exp\left(-\int_t^s {}^{\$}r_u \, du \right) ({}^{£}r_s - {}^{\$}r_s) \,\middle|\, \mathcal{F}_t \right] ds \\ &= V_{t_0,t} + \int_t^T {}^{\$}B_t^s \mathbb{E}_{\mathbb{P}_{\$B^s}}[{}^{£}r_s - {}^{\$}r_s \,|\, \mathcal{F}_t]\, ds \\ &= V_{t_0,t} + \int_t^T {}^{\$}B_t^s \left(\mathbb{E}_{\mathbb{P}_{\$B^s}}[{}^{£}r_s \,|\, \mathcal{F}_t] - {}^{\$}r_t^s \right) ds \end{aligned}$$

explicitly. For the last but one equality we performed a change of numéraire from ${}^{\$}\beta$ to ${}^{\$}B^s$ as in Theorem 2.10. For the last equality we used the unbiased expectation hypothesis for ${}^{\$}r_s$ under $\mathbb{P}_{\$B^s}$ (cf. Lemma 3.13).

So the only problem to solve is the calculation of $\mathbb{E}_{\mathbb{P}_{\$B^s}}[{}^{£}r_s \,|\, \mathcal{F}_t]$ $(0 \leq t \leq s \leq T)$. This will be done using the unbiased expectation hypothesis for the British market. We note that, in analogy to Theorem 2.2, from $\mathbb{P}_{{}^{\$}B^s}$ we get a numéraire measure $\mathbb{P}_{{}^{£}B^s}$ for the British market by

$$\frac{d\mathbb{P}_{{}^{\$}B^s}}{d\mathbb{P}_{{}^{£}B^s}}\,|_{\mathcal{F}_t} \triangleq \frac{X_0\,{}^{\$}B_t^s}{X_t\,{}^{£}B_t^s} = X_0(F_t^s)^{-1} \qquad (0 \leq t \leq s \leq T). \tag{35}$$

Here $F^s \triangleq (X\,{}^{£}B^s)/\,{}^{\$}B^s$ is obviously a $\mathbb{P}_{{}^{\$}B^s}$-martingale (so its inverse becomes a $\mathbb{P}_{{}^{£}B^s}$-martingale) which may be interpreted as forward exchange rate from pound to dollar with maturity s. Thus

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}_{{}^{\$}B^s}}[\,{}^{£}r_s\,|\,\mathcal{F}_t] &= F_t^s\mathbb{E}_{\mathbb{P}_{{}^{£}B^s}}\left[\,{}^{£}r_s(F_s^s)^{-1}\,|\,\mathcal{F}_t\right] \\
&= \mathbb{E}_{\mathbb{P}_{{}^{£}B^s}}\left[\,{}^{£}r_s^s\mathcal{E}\left(F^s\cdot(F^s)^{-1}\right)_{t,s}\,|\,\mathcal{F}_t\right].
\end{aligned}$$

The exponential is a local martingale with respect to $\mathbb{P}_{{}^{£}B^s}$. If we assume that it is a true martingale, then, noting that the same is true for ${}^{£}r^s$ (cf. Lemma 3.13), we get

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}_{{}^{\$}B^s}}[\,{}^{£}r_s\,|\,\mathcal{F}_t] \\
&= \mathbb{E}_{\mathbb{P}_{{}^{£}B^s}}\left[\left({}^{£}r_t^s + \int_t^s d({}^{£}r^s)_u\right)\left(1 + \int_t^s d\mathcal{E}\left(F^s\cdot(F^s)^{-1}\right)_{t,u}\right)\,|\,\mathcal{F}_t\right] \\
&= {}^{£}r_t^s + \mathbb{E}_{\mathbb{P}_{{}^{£}B^s}}\left[\left(\int_t^s d({}^{£}r^s)_u\right)\left(\int_t^s d\mathcal{E}\left(F^s\cdot(F^s)^{-1}\right)_{t,u}\right)\,|\,\mathcal{F}_t\right] \\
&= {}^{£}r_t^s + \mathbb{E}_{\mathbb{P}_{{}^{£}B^s}}\left[\int_t^s \mathcal{E}\left(F^s\cdot(F^s)^{-1}\right)_{t,u}\,d\left[\,{}^{£}r^s, F^s\cdot(F^s)^{-1}\right]_u\,|\,\mathcal{F}_t\right].
\end{aligned}$$

If we suppose that the quadratic covariation process

$$\left[\,{}^{£}r^s, F^s\cdot(F^s)^{-1}\right]_u = -\int_0^u \frac{d\left[\,{}^{£}r^s, F^s\right]_v}{F_v^s} \qquad (0 \leq u \leq s)$$

is *deterministic*, we may conclude by Fubini's theorem

$$\mathbb{E}_{\mathbb{P}_{{}^{\$}B^s}}[\,{}^{£}r_s\,|\,\mathcal{F}_t] = {}^{£}r_t^s - \int_t^s \frac{d\left[\,{}^{£}r^s, F^s\right]_u}{F_u^s}.$$

Finally, we get the following price process:

$$\begin{aligned}
{}^{\mathbb{P}_{{}^{\$}\beta}}\boldsymbol{\pi}_t^T[V_{t_0,T}] &= V_{t_0,t} + \left\{\int_t^T {}^{\$}B_t^s\left({}^{£}r_t^s - \int_t^s \frac{d\left[\,{}^{£}r^s, F^s\right]_u}{F_u^s} - {}^{\$}r_t^s\right)ds\right\} \\
&= V_{t_0,t} + \left\{\int_t^T {}^{\$}B_t^s\left({}^{£}r_t^s - \int_t^s \frac{d\left[\,{}^{£}r^s, F^s\right]_u}{F_u^s}\right)ds\right\} + {}^{\$}B_t^T - 1,
\end{aligned} \tag{36}$$

where for the second equality we used the definition of dollar-forward rates.

This implies the following $\mathbb{P}_{\$\beta}$-price process $V = (V_{t_0})_{0\le t_0\le T} = ({}^{\mathbb{P}_{\$\beta}}\boldsymbol{\pi}_{t_0}^{T}[V_{t_0,T}])_{0\le t_0\le T}$ for our continuous differential swap:

$$V_{t_0} = \left\{ \int_{t_0}^{T} {}^{\$}B_{t_0}^{s} \left({}^{\pounds}r_{t_0}^{s} - \int_{t_0}^{s} \frac{d\left[{}^{\pounds}r^{s}, F^{s}\right]_u}{F_u^s} \right) ds \right\} + {}^{\$}B_{t_0}^{T} - 1 \qquad (0 \le t_0 \le T). \tag{37}$$

We summarize these results in

Proposition 3.18 *Let $\mathbb{P}_{\$\beta}$ be a martingale measure for the whole financial market including the continuum of American and (converted) British bonds. From this martingale measure we get by Theorem 2.10 a numéraire measure $\mathbb{P}_{\$B^s}$ for each American bond ${}^{\$}B^{s}$ $(0 \le s \le T)$. From each measure $\mathbb{P}_{\$B^s}$ we can construct a numéraire measure $\mathbb{P}_{\pounds B^s}$ for the British market via formula (35).*

Suppose for each $0 \le s \le T$ the exponential $\mathcal{E}\left(F^s \cdot (F^s)^{-1}\right)$ is a true $\mathbb{P}_{\$B^s}$-martingale where $F^s \triangleq X\,{}^{\pounds}B^s / {}^{\$}B^s$. Assume furthermore the quadratic covariation process

$$\left[{}^{\pounds}r^{s}, F^s \cdot (F^s)^{-1}\right]_u = -\int_0^u \frac{d\left[{}^{\pounds}r^{s}, F^s\right]_v}{F_v^s} \qquad (0 \le u \le s)$$

to be deterministic. Then we get the following $\mathbb{P}_{\$\beta}$-price process $V = (V_t)_{0\le t\le T}$ for our continuous differential swap:

$$V_t = \left\{ \int_t^T {}^{\$}B_t^{s} \left({}^{\pounds}r_t^{s} - \int_t^s \frac{d\left[{}^{\pounds}r^{s}, F^s\right]_u}{F_u^s} \right) ds \right\} + {}^{\$}B_t^{T} - 1 \qquad (0 \le t \le T). \tag{38}$$

$\square$

3.3.3 Hedging Continuous Differential Swaps

Let us now derive a hedging strategy against the above continuous differential swap. To hedge a differential swap sold at time t_0 one should try to duplicate the payoff $V_{t_0,T}$ with maturity T which is defined by equation (28); see the beginning of Section 3.3. Starting from formula (36) we will show below that the $\mathbb{P}_{\$\beta}$-price dynamics of this contingent claim is given by

$$\begin{aligned}
{}^{\mathbb{P}_{\$\beta}}\boldsymbol{\pi}_t^{T}[V_{t_0,T}] \quad = \quad & {}^{\mathbb{P}_{\$\beta}}\boldsymbol{\pi}_t^{T}[V_{t_0,T}] \\
+ \quad & \textstyle\int_{t_0}^{t} \frac{V_{t_0,u}-1}{{}^{\$}\beta_u}\, d\,{}^{\$}\beta_u \\
+ \quad & \textstyle\int_{t_0}^{t} d({}^{\$}B^{T})_u \\
+ \quad & \textstyle\int_{t_0}^{T} \left\{ \int_{t_0}^{t} 1_{[0,s]}(u)({}^{\pounds}r_u^{s} - c_u^s)\, d({}^{\$}B^{s})_u \right\} ds \\
- \quad & \textstyle\int_{t_0}^{t} \frac{{}^{\$}B_u^{T}}{X_u\,{}^{\pounds}B_u^{T}}\, d(X\,{}^{\pounds}B^{T})_u \\
- \quad & \textstyle\int_{t_0}^{T} \left\{ \int_{t_0}^{t} 1_{[0,s]}(u) \frac{{}^{\$}r_u^{s}\,{}^{\$}B_u^{s}}{X_u\,{}^{\pounds}B_u^{s}}\, d(X\,{}^{\pounds}B^{s})_u \right\} ds
\end{aligned} \tag{39}$$

provided forward rates satisfy Assumption 3.15. Here we set

$$c_t^s \triangleq \int_{t_0}^{t} \frac{d[\,{}^{\pounds}r^s, F^s]_u}{F_u^s} \qquad (t_0 \le t \le s \le T).$$

Recall that this covariation process is supposed to be *deterministic*.

Remark 3.19 In Section 2 we introduced the notion "portfolio strategy" for a financial market containing only a *finite* number of assets. In an infinite dimensional market, Björk-Di Masi-Kabanov-Runggaldier [2] suggests to interpret a strategy in a continuum of bonds as a predictable measure-valued process. Since we do not wish to discuss the corresponding theory of stochastic integrals in more detail[21], in the sequel we will tacitly assume the occurring integrals to be well defined.

Equation (39) suggests to consider the following strategy:

- invest $V_{t_0,t} - 1$ dollar in the U.S. money market;
- hold one American bond of maturity T;
- be long $(\,{}^{\pounds}r_t^s - c_t^s)\,ds$ American bonds maturing at time $s \in [t, T]$;
- invest 1 dollar in the British money market account;
- go short $\frac{{}^{\$}B_t^T}{X_t\,{}^{\pounds}B_t^T}$ British bonds of maturity T;
- sell $\frac{{}^{\$}r_t^s\,{}^{\$}B_t^s}{X_t\,{}^{\pounds}B_t^s}\,ds$ British zero-bonds maturing in $s \in [t, T]$.

In fact, this strategy is selffinancing and a perfect hedge for $V_{t_0,T}$ because on the one hand, by definition, it duplicates the changes in value of this contingent claim and on the other hand it uses precisely the wealth available, as one can easily see from the following table:

Position in ...	**... has dollar value**
American money market	$V_{t_0,t} - 1$
dollar-denominated bonds	
· maturing at time T	${}^{\$}B_t^T$
· maturing at time $s \in [t, T]$	$\int_t^T (\,{}^{\pounds}r_u^s - c_u^s)\,{}^{\$}B_u^s\,ds$
British money market	1
pound-denominated bonds	
· maturing at time T	$-\,{}^{\$}B_u^T$
· maturing at time $s \in [t, T]$	$-\int_t^T {}^{\$}r_u^s\,{}^{\$}B_u^s = {}^{\$}B_u^T - 1$
whole portfolio	$V_{t_0,t} - 1 + {}^{\$}B_t^T + \int_t^T (\,{}^{\pounds}r_t^s - c_t^s)\,{}^{\$}B_t^s\,ds = {}^{\mathbb{P}^{\$\beta}}\boldsymbol{\pi}_t^T[V_{t_0,T}]$

[21] see [2] for a rigorous treatment of this question

Thus we have

Theorem 3.20 *Under the assumptions of Proposition 3.18 and Assumption 3.15 one can hedge a short position in a continuous differential swap using the above trading strategy.*

□

Remark 3.21 Note that we only priced the considered swap with respect to some fixed martingale measure $\mathbb{P}_{\$\beta}$. Now, the above hedging strategy yields an argument, why one might want to view the right side of (37) as the *only arbitrage free* price for this contingent claim. Unfortunately, we have not introduced a concept of admissibility for strategies in a continuum of bonds that would guarantee the duplication argument to produce unique prices. Björk-Di Masi-Kabanov-Runggaldier [2] define a measure-valued strategy to be admissible, if the resulting wealth process is nonnegative. Since the contingent claim $V_{t_0,T}$ might be negative, this definition, unfortunately, does not fit into our context. So we would need a definition of admissibility which is similar to the one in Definition 2.4. But this would need an extension of Theorem 2.6 to the context of an *infinite* financial market.

Let us now prove equation (39). This will be done using our Fubini-type Lemma 3.17 and Proposition 3.16 on the relation between forward rate- and bond price-dynamics. First, applying Itô's Lemma to the "ds"-integrand in formula (36) allows us to split ${}^{\mathbb{P}_{\$\beta}}\boldsymbol{\pi}_t^T[V_{t_0,T}]$ into six summands:

$$
\begin{aligned}
{}^{\mathbb{P}_{\$\beta}}\boldsymbol{\pi}_t^T[V_{t_0,T}] \; = \; & \quad V_{t_0,t} + {}^{\$}B_t^T - 1 && (\mathrm{I}_t)\\
& + \; \int_t^T {}^{\$}B_{t_0}^s({}^{\pounds}r_{t_0}^s - c_{t_0}^s)\, ds && (\mathrm{II}_t)\\
& + \; \int_t^T \left\{\int_{t_0}^t ({}^{\pounds}r_u^s - c_u^s)\, d({}^{\$}B^s)_u\right\} ds && (\mathrm{III}_t)\\
& + \; \int_t^T \left\{\int_{t_0}^t {}^{\$}B_u^s d({}^{\pounds}r^s)_u\right\} ds && (\mathrm{IV}_t)\\
& - \; \int_t^T \left\{\int_{t_0}^t {}^{\$}B_u^s\, d(c^s)_u\right\} ds && (\mathrm{V}_t)\\
& + \; \int_t^T \left\{\int_{t_0}^t d[{}^{\$}B^s, {}^{\pounds}r^s]_u\right\} ds && (\mathrm{VI}_t).
\end{aligned}
$$

Now we are going to determine one by one the dynamics of (I_t) - (VI_t):

Lemma 3.22 *Under Assumption 3.15 we have*

- $$
\begin{aligned}
(\mathrm{I}_t) \; = \; & \quad {}^{\$}B_{t_0}^T - 1\\
& + \; \int_{t_0}^t \tfrac{V_{t_0,u}-1}{{}^{\$}\beta_u}\, d\,{}^{\$}\beta_u + \int_{t_0}^t {}^{\pounds}r_u\, du + \int_{t_0}^t d({}^{\$}B^T)_u
\end{aligned}
$$

- $$\begin{aligned}(\mathrm{II}_t) \;=\; & \int_{t_0}^{T} {}^{\$}B^{s}_{t_0}(\,{}^{£}r^{s}_{t_0} - c^{s}_{t_0})\,ds \\ - & \int_{t_0}^{t} {}^{\$}B^{\tau}_{t_0}(\,{}^{£}r^{\tau}_{t_0} - c^{\tau}_{t_0})\,d\tau\end{aligned}$$

- $$\begin{aligned}(\mathrm{III}_t) \;=\; - & \int_{t_0}^{t} \left\{\int_{t_0}^{\tau}(\,{}^{£}r^{\tau}_{u} - c^{\tau}_{u})\,d(\,{}^{\$}B^{\tau})_u\right\}\,d\tau \\ + & \int_{t_0}^{T} \left\{\int_{t_0}^{t} 1_{[0,s]}(u)(\,{}^{£}r^{s}_{u} - c^{s}_{u})\,d(\,{}^{\$}B^{s})_u\right\}\,ds\end{aligned}$$

- $$\begin{aligned}(\mathrm{IV}_t) \;=\; - & \int_{t_0}^{t} \left\{\int_{t_0}^{\tau} {}^{\$}B^{\tau}_{u}\,d(\,{}^{£}r^{\tau})_u\right\}\,d\tau \\ + & \int_{t_0}^{t} {}^{£}r_u\,du \\ - & \int_{t_0}^{T} \left\{\int_{t_0}^{t} 1_{[0,s]}(u)\frac{{}^{\$}B^{s}_{u}}{{}^{£}B^{s}_{u}}\,d[\,{}^{£}r^{s},\,{}^{£}B^{s}]_u\right\}\,ds \\ - & \int_{t_0}^{t} \frac{{}^{\$}B^{T}_{u}}{{}^{£}B^{T}_{u}}\,d(\,{}^{£}B^{T})_u \\ - & \int_{t_0}^{T} \left\{\int_{t_0}^{t} 1_{[0,s]}(u)\frac{{}^{\$}r^{s}_{u}\,{}^{\$}B^{s}_{u}}{{}^{£}B^{s}_{u}}\,d(\,{}^{£}B^{s})_u\right\}\,ds\end{aligned}$$

- $$\begin{aligned}(\mathrm{V}_t) \;=\; & \int_{t_0}^{t} \left\{\int_{t_0}^{\tau} {}^{\$}B^{\tau}_{u}\,d(c^{\tau})_u\right\}\,d\tau \\ + & \int_{t_0}^{T} \left\{\int_{t_0}^{t} 1_{[0,s]}(u)\frac{{}^{\$}B^{s}_{u}}{{}^{£}B^{s}_{u}}\,d[\,{}^{£}r^{s},\,{}^{£}B^{s}]_u\right\}\,ds \\ - & \int_{t_0}^{T} \left\{\int_{t_0}^{t} 1_{[0,s]}(u)d[\,{}^{£}r^{s},\,{}^{\$}B^{s}]_u\right\}\,ds \\ - & \int_{t_0}^{t} \frac{{}^{\$}B^{T}_{u}}{X_u\,{}^{£}B^{T}_{u}}\,d[\,{}^{£}B^{T}, X]_u \\ - & \int_{t_0}^{T} \left\{\int_{t_0}^{t} 1_{[0,s]}(u)\frac{{}^{\$}r^{s}_{u}\,{}^{\$}B^{s}_{u}}{X_u\,{}^{£}B^{s}_{u}}\,d[\,{}^{£}B^{s}, X]_u\right\}\,ds\end{aligned}$$

- $$\begin{aligned}(\mathrm{VI}_t) \;=\; - & \int_{t_0}^{t} \left\{\int_{t_0}^{\tau} d[\,{}^{\$}B^{\tau},\,{}^{£}r^{\tau}]_u\,d\tau\right\} \\ + & \int_{t_0}^{T} \left\{\int_{t_0}^{t} 1_{[0,s]}(u)d[\,{}^{£}r^{s},\,{}^{\$}B^{s}]_u\right\}\,ds\end{aligned}$$

PROOF:

(i) Recalling that $V_{t_0,\cdot}$ solves differential equation (27) we may write

$$\begin{aligned}(\mathrm{I}_t) \;=\; & V_{t_0,t_0} + {}^{\$}B^{T}_{t_0} - 1 \\ + & \int_{t_0}^{t} V_{t_0,u}\,{}^{\$}r_u\,du + \int_{t_0}^{t}(\,{}^{£}r_u - {}^{\$}r_u)\,du \\ + & \int_{t_0}^{t} d(\,{}^{\$}B^{T})_u,\end{aligned}$$

whence we obtain the stated equation for (I_t) by $d\,{}^{\$}\beta_t = {}^{\$}\beta_t\,{}^{\$}r_t\,dt$.

(ii) The expression for (II_t) is trivial.

(iii) The above representation of (III_t) can be proved by the same arguments leading to formula (34).

(iv) Considering expression (IV_t) we see that the "essential" part of its dynamics is given by the "$\int \ldots d({}^{\pounds}r^{s})_u$"-term. Since one cannot interpret this integral as a gain from trade directly, we should try to express the "$d({}^{\pounds}r^{s})_u$"-increment by increments of suitable bonds. This will be achieved using Lemma 3.17 and some partial integrations. Using Assumption 3.15 we will first rewrite $(\mathrm{IV_t})$ such that Lemma 3.17 can be applied:

$$
\begin{aligned}
(\mathrm{IV}_t) &= \sum_i \int_t^T \left\{ \int_{t_0}^t {}^{\$}B_u^{s} H_u^{s,i}\, dM_u^i \right\} ds \\
&= -\int_{t_0}^t \left\{ \int_{t_0}^{\tau} {}^{\$}B_u^{\tau}\, d({}^{\pounds}r^{\tau})_u \right\} d\tau \\
&\quad + \sum_i \int_{t_0}^t \left\{ \int_u^T {}^{\$}B_u^{s} H_u^{s,i}\, ds \right\} dM_u^i.
\end{aligned} \tag{40}
$$

Partial integration yields

$$
\begin{aligned}
&\int_u^T {}^{\$}B_u^{s} H_u^{s,i}\, ds \\
= &\int_u^T {}^{\$}B_u^{s} \left(\frac{\partial}{\partial s} \int_u^s H^{v,i}\, dv \right) ds \\
= &\ {}^{\$}B_u^{T} \left(\int_u^T H^{v,i}\, dv \right) - \int_u^T \left(\frac{\partial}{\partial s}\, {}^{\$}B_u^{s} \right) \left(\int_u^s H^{v,i}\, dv \right) ds,
\end{aligned} \tag{41}
$$

such that we may rewrite the second term in (40) as difference between

$$
(\mathrm{IV}_t^1) \triangleq \sum_i \int_{t_0}^t {}^{\$}B_u^{T} \left(\int_u^T H_u^{v,i}\, dv \right) dM_u^i
$$

and

$$
(\mathrm{IV}_t^2) \triangleq \sum_i \int_{t_0}^t \left\{ \int_u^T \left(\frac{\partial}{\partial s} B_u^s \right) \left(\int_u^s H_u^{v,i}\, dv \right) ds \right\} dM_u^i.
$$

Now, considering formula (32) we obtain

$$
\begin{aligned}
(\mathrm{IV}_t^1) = &\quad \textstyle\int_{t_0}^t {}^{\$}B_u^{T}\, {}^{\pounds}r_u\, du \\
&- \textstyle\int_{t_0}^t \frac{{}^{\$}B_u^T}{{}^{\pounds}B_u^T}\, d({}^{\pounds}B^{T})_u \\
&+ \textstyle\frac{1}{2} \sum_{i,j} \int_{t_0}^t {}^{\$}B_u^{T} \left(\int_u^T H^{v,i}\, dv \right) \left(\int_u^T H^{v,j}\, dv \right) d[M^i, M^j]_u.
\end{aligned} \tag{42}
$$

Proceeding analogously for (IV_t^2) gives

$$\begin{aligned}
(\mathrm{IV}_t^2) \quad = \quad & \textstyle\sum_i \int_{t_0}^T \left\{ \int_{t_0}^t 1_{[0,s]}(u) \left(\frac{\partial}{\partial s} {}^{\$}B_u^s \right) \right. \\
& \qquad\qquad \left. \cdot \, \left(\int_u^s H_u^{v,i}\, dv \right) \, dM_u^i \right\} \, ds \\
= \quad & \textstyle\int_{t_0}^T \left\{ \int_{t_0}^t 1_{[0,s]}(u) \left(\frac{\partial}{\partial s} {}^{\$}B_u^s \right) \, {}^{\pounds}r_u \, du \right\} \, ds \\
- \quad & \textstyle\int_{t_0}^T \left\{ \int_{t_0}^t 1_{[0,s]}(u) \frac{\left(\frac{\partial}{\partial s} {}^{\$}B_u^s \right)}{{}^{\pounds}B_u^s} \, d({}^{\pounds}B_u^s) \right\} \, ds \\
+ \quad & \textstyle\frac{1}{2} \sum_{i,j} \int_{t_0}^T \left\{ \int_{t_0}^t 1_{[0,s]}(u) \left(\frac{\partial}{\partial s} {}^{\$}B_u^s \right) \left(\int_u^T H^{v,i}\, dv \right) \right. \\
& \qquad\qquad \left. \cdot \, \left(\int_u^T H^{v,j}\, dv \right) \, d[M^i, M^j]_u \right\} \, ds \\
= \quad & \textstyle\int_{t_0}^t \left\{ \int_u^T \left(\frac{\partial}{\partial s} {}^{\$}B_u^s \right) \, ds \right\} \, {}^{\pounds}r_u \, du \\
+ \quad & \textstyle\int_{t_0}^T \left\{ \int_{t_0}^t 1_{[0,s]}(u) \frac{{}^{\$}r_s \, {}^{\$}B_u^s}{{}^{\pounds}B_u^s} \, d({}^{\pounds}B_u^s) \right\} \, ds \\
+ \quad & \textstyle\frac{1}{2} \sum_{i,j} \int_{t_0}^t \left\{ \int_u^T \left(\frac{\partial}{\partial s} {}^{\$}B_u^s \right) \left(\int_u^T H^{v,i}\, dv \right) \right. \\
& \qquad\qquad \left. \cdot \, \left(\int_u^T H^{v,j}\, dv \right) \, ds \right\} \, d[M^i, M^j]_u .
\end{aligned}$$

We may simplify the first of these last summands to

$$\begin{aligned}
\int_{t_0}^t \left\{ \int_u^T \left(\frac{\partial}{\partial s} {}^{\$}B_u^s \right) \, ds \right\} \, {}^{\pounds}r_u \, du \quad &= \quad \int_{t_0}^t \left\{ {}^{\$}B_u^T - 1 \right\} \, {}^{\pounds}r_u \, du \\
&= \quad \int_{t_0}^t {}^{\$}B_u^T \, {}^{\pounds}r_u \, du - \int_{t_0}^t {}^{\pounds}r_u \, du.
\end{aligned}$$

Here the first integral cancels with the first term in the above representation of (IV_t^1). Furthermore using partial integration and one more time (32) we have

$$\begin{aligned}
& \textstyle\frac{1}{2} \sum_{i,j} \int_{t_0}^t \left\{ \int_u^T \left(\frac{\partial}{\partial s} {}^{\$}B_u^s \right) \left(\int_u^s H^{v,i}\, dv \right) \right. \\
& \qquad\qquad \left. \cdot \, \left(\int_u^s H^{v,j}\, dv \right) \, ds \right\} \, d[M^i, M^j]_u \\
= \quad & \textstyle\frac{1}{2} \sum_{i,j} \int_{t_0}^t {}^{\$}B_u^T \left(\int_u^T H_u^{v,i}\, dv \right) \left(\int_u^T H_u^{v,j}\, dv \right) \, d[M^i, M^j]_u \\
- \quad & \textstyle\sum_{i,j} \int_{t_0}^t \left\{ \int_u^T {}^{\$}B_u^T H_u^{s,i} \left(\int_u^s H_u^{v,i}\, dv \right) \, ds \right\} \, d[M^i, M^j]_u \\
= \quad & \textstyle\frac{1}{2} \sum_{i,j} \int_{t_0}^t {}^{\$}B_u^T \left(\int_u^T H_u^{v,i}\, dv \right) \left(\int_u^T H_u^{v,j}\, dv \right) \, d[M^i, M^j]_u \\
- \quad & \textstyle\sum_{i,j} \int_{t_0}^T \left\{ \int_{t_0}^t 1_{[0,s]}(u) \, {}^{\$}B_u^s H_u^{s,i} \left(\int_u^s H_u^{v,j}\, dv \right) \, d[M^i, M^j]_u \right\} \, ds \\
= \quad & \textstyle\frac{1}{2} \sum_{i,j} \int_{t_0}^t {}^{\$}B_u^T \left(\int_u^T H_u^{v,i}\, dv \right) \left(\int_u^T H_u^{v,j}\, dv \right) \, d[M^i, M^j]_u \\
+ \quad & \textstyle\int_{t_0}^T \left\{ \int_{t_0}^t 1_{[0,s]}(u) \frac{{}^{\$}B_u^s}{{}^{\pounds}B_u^s} \, d[{}^{\pounds}r^s, \, {}^{\pounds}B^s]_u \right\} \, ds.
\end{aligned}$$

The first term in the last equation cancels with the last one in equation (42) and so we receive the stated formula for (IV_t).

(v) By Lemma 3.17 we have for (V_t):

$$
\begin{aligned}
(\mathrm{V}_t) \; = \; & \quad \textstyle\int_{t_0}^{t} \left\{ \int_{t_0}^{\tau} {}^{\$}B_u^{\tau}\, d(c^{\tau})_u \right\} d\tau \\
& + \; \textstyle\int_{t_0}^{T} \left\{ \int_{t_0}^{t} 1_{[0,s]}(u) \frac{{}^{\$}B_u^{s}}{X_u}\, d[\, {}^{\pounds}r^{s}, X]_u \right\} ds \\
& + \; \textstyle\int_{t_0}^{T} \left\{ \int_{t_0}^{t} 1_{[0,s]}(u) \frac{{}^{\$}B_u^{s}}{{}^{\pounds}B_u^{s}}\, d[\, {}^{\pounds}r^{s},\, {}^{\pounds}B^{s}]_u \right\} ds \\
& - \; \textstyle\int_{t_0}^{T} \left\{ \int_{t_0}^{t} 1_{[0,s]}(u) \frac{{}^{\$}B_u^{s}}{{}^{\$}B_u^{s}}\, d[\, {}^{\pounds}r^{s},\, {}^{\$}B^{s}]_u \right\} ds.
\end{aligned}
$$

Here we only wish to adopt the second term:

$$
\begin{aligned}
& \textstyle\int_{t_0}^{T} \left\{ \int_{t_0}^{t} 1_{[0,s]}(u) \frac{{}^{\$}B_u^{s}}{X_u}\, d[\, {}^{\pounds}r^{s}, X]_u \right\} ds \\
= & \quad \textstyle\sum_i \int_{t_0}^{t} \frac{1}{X_u} \left\{ \int_u^T {}^{\$}B_u^{s} H_u^{s,i}\, ds \right\} d[M^i, X]_u \\
= & \quad \textstyle\sum_i \int_{t_0}^{t} \frac{1}{X_u} \left\{ {}^{\$}B_u^{T} \left(\int_u^T H_u^{v,i}\, dv \right) \right\} d[M^i, X]_u \\
& - \; \textstyle\sum_i \int_{t_0}^{t} \frac{1}{X_u} \left\{ \int_u^T \left(\frac{\partial}{\partial s}\, {}^{\$}B_u^{s} \right) \left(\int_u^s H_u^{v,i}\, dv \right) \right\} d[M^i, X]_u \\
= & \; - \; \textstyle\int_{t_0}^{t} \frac{{}^{\$}B_u^{T}}{X_u\, {}^{\pounds}B_u^{T}} d[\, {}^{\pounds}B^{T}, X]_u \\
& - \; \textstyle\int_{t_0}^{T} \left\{ \int_{t_0}^{t} 1_{[0,s]}(u) \frac{{}^{\$}r_u^{s}\, {}^{\$}B_u^{s}}{X_u\, {}^{\pounds}B_u^{s}}\, d[\, {}^{\pounds}B^{s}, X]_u \right\} ds
\end{aligned}
$$

where for the last but one equation we used (41) and the last equation follows from our bond price dynamics (32). Employing this in the above equation for (V_t) yields the stated formula.

(vi) Our formula for (VI_t) can be proved analogously to the expression for (III_t).

□

By Itô's product rule, the "$d\tau$"-integrands in (II_t) - (VI_t) sum up to

$$
-\int_{t_0}^{t} {}^{\$}B_{\tau}^{\tau}(\, {}^{\pounds}r_{\tau}^{\tau} - c_{\tau}^{\tau})\, d\tau = -\int_{t_0}^{t} {}^{\pounds}r_{\tau}\, d\tau
$$

canceling the second term in the above representation of (IV_t). Furthermore the third summand of (IV_t) cancels with the second of (V_t) and the third of (V_t) with the second of (VI_t). Finally, we get

$$
\begin{aligned}
{}^{\mathbb{P}}{}_{{}^{\$}\beta}\boldsymbol{\pi}_t^T[V_{t_0,T}] = \quad & {}^{\mathbb{P}}{}_{{}^{\$}\beta}\boldsymbol{\pi}_{t_0}^T[V_{t_0,T}] \\
+ \quad & \textstyle\int_{t_0}^t \frac{V_{t_0,u}-1}{{}^{\$}\beta_u}\, d\,{}^{\$}\beta_u \\
+ \quad & \textstyle\int_{t_0}^t {}^{£}r_u\, du \\
+ \quad & \textstyle\int_{t_0}^t d({}^{\$}B^T)_u \\
+ \quad & \textstyle\int_{t_0}^T \left\{ \int_{t_0}^t 1_{[0,s]}(u)({}^{£}r_u^s - c_u^s)\, d({}^{\$}B^s)_u \right\} ds \qquad (43)\\
- \quad & \textstyle\int_{t_0}^t \frac{{}^{\$}B_u^T}{{}^{£}B_u^T}\, d({}^{£}B^T)_u \\
- \quad & \textstyle\int_{t_0}^t \frac{{}^{\$}B_u^T}{X_u\,{}^{£}B_u^T}\, d[{}^{£}B^T, X]_u \\
- \quad & \textstyle\int_{t_0}^T \left\{ \int_{t_0}^t 1_{[0,s]}(u)\frac{{}^{\$}r_u^s\,{}^{\$}B_u^s}{{}^{£}B_u^s}\, d({}^{£}B^s)_u \right\} ds \\
- \quad & \textstyle\int_{t_0}^T \left\{ \int_{t_0}^t 1_{[0,s]}(u)\frac{{}^{\$}r_u^s\,{}^{\$}B_u^s}{X_u\,{}^{£}B_u^s}\, d[{}^{£}B^s, X]_u \right\} ds.
\end{aligned}
$$

In order to derive a hedging strategy from this, we have to replace the "pound-denominated" infinitesimal increments $d({}^{£}B^s)$ $(t_0 \le s \le T)$ by some "dollar increments". This can be achieved using

$$X\, d({}^{£}B^s) + d[{}^{£}B^s, X] = d(X\,{}^{£}B^s) - {}^{£}B^s\, dX \qquad (t_0 \le s \le T).$$

By this formula the fifth and sixth term in the above expression sum up to

$$
\begin{aligned}
& -\int_{t_0}^t \frac{{}^{\$}B_u^T}{{}^{£}B_u^T}\, d({}^{£}B^T)_u - \int_{t_0}^t \frac{{}^{\$}B_u^T}{X_u\,{}^{£}B_u^T}\, d[{}^{£}B^T, X]_u \\
& = \int_{t_0}^t \frac{{}^{\$}B_u^T}{X_u}\, dX_u - \int_{t_0}^t \frac{{}^{\$}B_u^T}{X_u\,{}^{£}B_u^T}\, d(X\,{}^{£}B^T)_u.
\end{aligned} \tag{44}
$$

For the same reason the last two summands give

$$
\begin{aligned}
& -\int_{t_0}^T \left\{ \int_{t_0}^t 1_{[0,s]}(u)\frac{{}^{\$}r_u^s\,{}^{\$}B_u^s}{{}^{£}B_u^s}\, d({}^{£}B^s)_u \right\} ds \\
& \quad - \int_{t_0}^T \left\{ \int_{t_0}^t 1_{[0,s]}(u)\frac{{}^{\$}r_u^s\,{}^{\$}B_u^s}{X_u\,{}^{£}B_u^s}\, d[{}^{£}B^s, X]_u \right\} ds \\
= \; & \int_{t_0}^T \left\{ \int_{t_0}^t 1_{[0,s]}(u)\frac{{}^{\$}r_u^s\,{}^{\$}B_u^s}{X_u}\, dX_u \right\} ds \\
& \quad - \int_{t_0}^T \left\{ \int_{t_0}^t 1_{[0,s]}(u)\frac{{}^{\$}r_u^s\,{}^{\$}B_u^s}{X_u\,{}^{£}B_u^s}\, d(X\,{}^{£}B^s)_u \right\} ds.
\end{aligned}
$$

Here we may simplify the first term

$$\int_{t_0}^{T}\left\{\int_{t_0}^{t} 1_{[0,s]}(u)\frac{{}^{\$}r_u^s\,{}^{\$}B_u^s}{X_u}\,dX_u\right\} ds \;=\; \int_{t_0}^{t}\frac{1}{X_u}\underbrace{\left\{\int_u^T {}^{\$}r_u^s\,{}^{\$}B_u^s\,ds\right\}}_{=-\,{}^{\$}B_u^T+1} dX_u$$

$$= \int_{t_0}^{t} X_u^{-1}\,dX_u - \int_{t_0}^{t}\frac{{}^{\$}B_u^T}{X_u}\,dX_u$$

such that the last two summands can be written as

$$\begin{aligned}
& -\int_{t_0}^{T}\left\{\int_{t_0}^{t} 1_{[0,s]}(u)\frac{{}^{\$}r_u^s\,{}^{\$}B_u^s}{{}^{£}B_u^s}\,d({}^{£}B^s)_u\right\} ds \\
& \quad -\int_{t_0}^{T}\left\{\int_{t_0}^{t} 1_{[0,s]}(u)\frac{{}^{\$}r_u^s\,{}^{\$}B_u^s}{X_u\,{}^{£}B_u^s}\,d[{}^{£}B^s, X]_u\right\} ds \\
= & \int_{t_0}^{t} X_u^{-1}\,dX_u - \int_{t_0}^{t}\frac{{}^{\$}B_u^T}{X_u}\,dX_u \\
& \quad -\int_{t_0}^{T}\left\{\int_{t_0}^{t} 1_{[0,s]}(u)\frac{{}^{\$}r_u^s\,{}^{\$}B_u^s}{X_u\,{}^{£}B_u^s}\,d(X\,{}^{£}B^s)_u\right\} ds \\
= & \int_{t_0}^{t} (X\,{}^{£}\beta)_u^{-1}\,d(X\,{}^{£}\beta)_u - \int_{t_0}^{t}{}^{£}r_u\,du - \int_{t_0}^{t}\frac{{}^{\$}B_u^T}{X_u}\,dX_u \\
& \quad -\int_{t_0}^{T}\left\{\int_{t_0}^{t} 1_{[0,s]}(u)\frac{{}^{\$}r_u^s\,{}^{\$}B_u^s}{X_u\,{}^{£}B_u^s}\,d(X\,{}^{£}B^s)_u\right\} ds.
\end{aligned}$$

For the last equality we used

$$dX_u = {}^{£}\beta^{-1}\,d(X\,{}^{£}\beta) - X_u\,{}^{£}r_u\,du.$$

The above "$-\int_{t_0}^{t}{}^{£}r_s\,ds$"-term cancels with the third term in expression (43) and $-\int_{t_0}^{t}\frac{{}^{\$}B_u^T}{X_u}\,dX_u$ with the first summand on the right side of equation (44). So we finally get the desired formula

$$\begin{aligned}
{}^{\mathbb{P}_{\$\beta}}\boldsymbol{\pi}_t^T[V_{t_0,T}] \;=\; & \;{}^{\mathbb{P}_{\$\beta}}\boldsymbol{\pi}_{t_0}^T[V_{t_0,T}] \\
& + \int_{t_0}^{t}\tfrac{V_{t_0,u}-1}{{}^{\$}\beta_u}\,d\,{}^{\$}\beta_u \\
& + \int_{t_0}^{t} d({}^{\$}B^T)_u \\
& + \int_{t_0}^{T}\left\{\int_{t_0}^{t} 1_{[0,s]}(u)({}^{£}r_u^s - c_u^s)\,d({}^{\$}B^s)_u\right\} ds \\
& + \int_{t_0}^{t}\tfrac{1}{X_u\,{}^{£}\beta_u}\,d(X\,{}^{£}\beta)_u \\
& - \int_{t_0}^{t}\tfrac{{}^{\$}B_u^T}{X_u\,{}^{£}B_u^T}\,d(X\,{}^{£}B^T)_u \\
& - \int_{t_0}^{T}\left\{\int_{t_0}^{t} 1_{[0,s]}(u)\tfrac{{}^{\$}r_u^s\,{}^{\$}B_u^s}{X_u\,{}^{£}B_u^s}\,d(X\,{}^{£}B^s)_u\right\} ds.
\end{aligned}$$

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