ORF555 / FIN555:
Fixed Income Models

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Contents

Chapter 1

Introduction

The number of books on fixed income models is growing, yet it is difficult to find a convenient textbook for a one-semester course like this. There are several reasons for this:

- Until recently, many textbooks on mathematical finance have treated stochastic interest rates as an appendix to the elementary arbitrage pricing theory, which usually requires constant (zero) interest rates.
- Interest rate theory is not standardized yet: there is no well-accepted "standard" general model such as the Black–Scholes model for equities.
- The very nature of fixed income instruments causes difficulties, other than for stock derivatives, in implementing and calibrating models. These issues should therefore not been left out.

I will frequently refer to the following books:

B[3]: Björk (98) [3]. A pedagogically well written introduction to mathematical finance. Chapters 15–20 are on interest rates.

BM[6]: Brigo–Mercurio (01) [6]. This is a book on interest rate modelling written by two quantitative analysts in financial institutions. Much emphasis is on the practical implementation and calibration of selected models.

JW[11]: James–Webber (00) [11]. An encyclopedic treatment of interest rates and their related financial derivatives.

J[12]: Jarrow (96) [12]. Introduction to fixed-income securities and interest rate options. Discrete time only.

MR[16]: Musiela–Rutkowski (97) [16]. A comprehensive book on financial mathematics with a large part (Part II) on interest rate modelling. Much emphasis is on market pricing practice.

R[19]: Rebonato (98) [19]. Written by a practitionar. Much emphasis on market practice for pricing and handling interest rate derivatives.

Z[22]: Zagst (02) [22]. A comprehensive textbook on mathematical finance, interest rate modelling and risk management.

I did not intend to write an entire text but rather collect fragments of the material that can be found in the above books and further references.

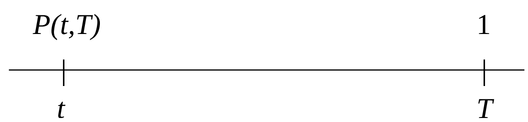
Chapter 2

Interest Rates and Related Contracts

Literature: B[3](Chapter 15), BM[6](Chapter 1), and many more

2.1 Zero-Coupon Bonds

A dollar today is worth more than a dollar tomorrow. The time t value of a dollar at time $T \geq t$ is expressed by the *zero-coupon bond* with *maturity* T, $P(t, T)$, for briefty also T-*bond*. This is a contract which guarantees the holder one dollar to be paid at the maturity date T.



$\rightarrow$ future cashflows can be discounted, such as coupon-bearing bonds

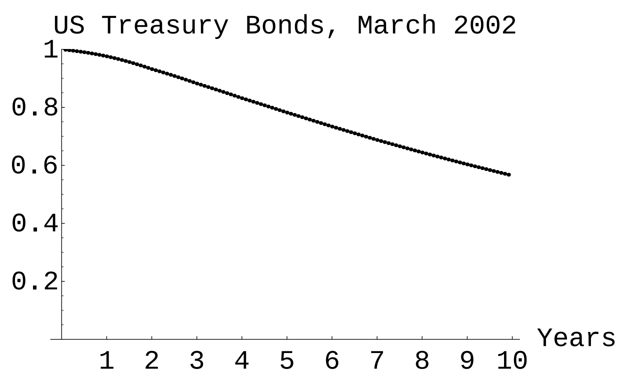
$$C_1 P(t, t_1) + \cdots + C_{n-1} P(t, t_{n-1}) + (1 + C_n) P(t, T).$$

In theory we will assume that

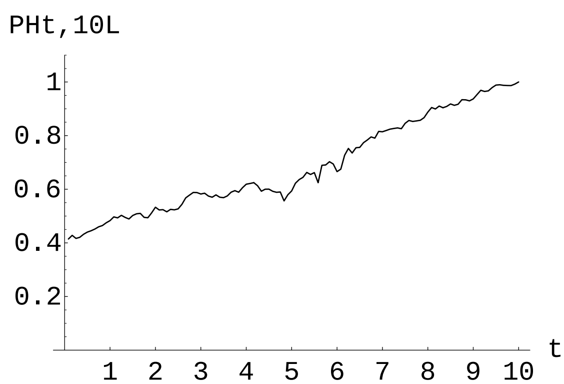
- there exists a frictionless market for T-bonds for every $T > 0$.
- $P(T, T) = 1$ for all T.
- $P(t, T)$ is continuously differentiable in T.

In reality this assumptions are not always satisfied: zero-coupon bonds are not traded for all maturities, and $P(T,T)$ might be less than one if the issuer of the T-bond defaults. Yet, this is a good starting point for doing the mathematics. More realistic models will be introduced and discussed in the sequel.

The third condition is purely technical and implies that the *term structure* of zero-coupon bond prices $T \mapsto P(t,T)$ is a smooth (decreasing) curve.



Note that $t \mapsto P(t,T)$ is a stochastic process since bond prices $P(t,T)$ are not known with certainty before t.



A reasonable assumption would also be that $P(t,T) \leq 1$ (which is equivalent to positivity of interest rates). However, already classical interest rate models imply zero-coupon bond prices greater than 1. Therefore we leave away this requirement.

2.2 Interest Rates

The term structure of zero-coupon bond prices does not contain much visual information (strictly speaking it does). A better measure is given by the implied interest rates. There is a variety of them.

A prototypical *forward rate agreement (FRA)* is a contract involving three time instants $t < T < S$: the current time t, the expiry time $T > t$, and the maturity time $S > T$.

- At t: sell one T-bond and buy $\frac{P(t,T)}{P(t,S)}$ S-bonds = zero net investment.
- At T: pay one dollar.
- At S: obtain $\frac{P(t,T)}{P(t,S)}$ dollars.

The net effect is a forward investment of one dollar at time T yielding $\frac{P(t,T)}{P(t,S)}$ dollars at S with certainty.

We are led to the following definitions.

- The *simple (simply-compounded) forward rate* for $[T,S]$ prevailing at t is given by

$$1+(S-T)F(t;T,S) := \frac{P(t,T)}{P(t,S)} \Leftrightarrow F(t;T,S) = \frac{1}{S-T}\left(\frac{P(t,T)}{P(t,S)} - 1\right).$$

- The *simple spot rate* for $[t,T]$ is

$$F(t,T) := F(t;t,T) = \frac{1}{T-t}\left(\frac{1}{P(t,T)} - 1\right).$$

- The *continuously compounded forward rate* for $[T,S]$ prevailing at t is given by

$$e^{R(t;T,S)(S-T)} := \frac{P(t,T)}{P(t,S)} \Leftrightarrow R(t;T,S) = -\frac{\log P(t,S) - \log P(t,T)}{S-T}.$$

- The *continuously compounded spot rate* for $[T,S]$ is

$$R(t,T) := R(t;t,T) = -\frac{\log P(t,T)}{T-t}.$$

- The *instantaneous forward rate* with maturity T prevailing at time t is defined by

$$f(t,T) := \lim_{S \downarrow T} R(t;T,S) = -\frac{\partial \log P(t,T)}{\partial T}. \tag{2.1}$$

The function $T \mapsto f(t,T)$ is called the *forward curve* at time t.

- The *instantaneous short rate* at time t is defined by

$$r(t) := f(t,t) = \lim_{T \downarrow t} R(t,T).$$

Notice that (2.1) together with the requirement $P(T,T) = 1$ is equivalent to

$$P(t,T) = \exp\left(-\int_t^T f(t,u)\,du\right).$$

2.2.1 Market Example: LIBOR

"Interbank rates" are rates at which deposits between banks are exchanged, and at which swap transactions (see below) between banks occur. The most important interbank rate usually considered as a reference for fixed income contracts is the *LIBOR (London InterBank Offered Rate)*[1] for a series of possible maturities, ranging from *overnight* to 12 months. These rates are quoted on a simple compounding basis. For example, the three-months forward LIBOR for the period $[T, T+1/4]$ at time t is given by

$$L(t,T) = F(t;T,T+1/4).$$

2.2.2 Simple vs. Continuous Compounding

One dollar invested for one year at an interest rate of R per annum growths to $1+R$. If the rate is compounded twice per year the terminal value is $(1+R/2)^2$, etc. It is a mathematical fact that

$$\left(1 + \frac{R}{m}\right)^m \to e^R \quad \text{as } m \to \infty.$$

[1] To be more precise: this is the rate at which high-credit financial institutions can *borrow* in the interbank market.

Moreover,

$$e^R = 1 + R + o(R) \quad \text{for } R \text{ small.}$$

Example: $e^{0.04} = 1.04081$.

Since the exponential function has nicer analytic properties than power functions, we often consider continuously compounded interest rates. This makes the theory more tractable.

2.2.3 Forward vs. Future Rates

Can forward rates predict the future spot rates?

Consider a deterministic world. If markets are efficient (i.e. no arbitrage = no riskless, systematic profit) we have necessarily

$$P(t, S) = P(t, T)P(T, S), \quad \forall t \le T \le S. \tag{2.2}$$

Proof. Suppose that $P(t, S) > P(t, T)P(T, S)$ for some $t \le T \le S$. Then we follow the strategy:

- At t: sell one S-bond, and buy $P(T, S)$ T-bonds.

  Net cost: $-P(t, S) + P(t, T)P(T, S) < 0$.

- At T: receive $P(T, S)$ dollars and buy one S-bond.

- At S: pay one dollar, receive one dollar.

(Where do we use the assumption of a deterministic world?)

The net is a riskless gain of $-P(t, S)+P(t, T)P(T, S)$ $(\times 1/P(t, S))$. This is a pure arbitrage opportunity, which contradicts the assumption.

If $P(t, S) < P(t, T)P(T, S)$ the same profit can be realized by changing sign in the strategy. □

Taking logarithm in (2.2) yields

$$\int_T^S f(t, u)\, du = \int_T^S f(T, u)\, du, \quad \forall t \le T \le S.$$

This is equivalent to

$$f(t, S) = f(T, S) = r(S), \quad \forall t \le T \le S$$

(as time goes by we walk along the forward curve: the forward curve is shifted). In this case, the forward rate with maturity S prevailing at time $t \le S$ is exactly the future short rate at S.

The real world is not deterministic though. We will see that in general the forward rate $f(t,T)$ is the conditional expectation of the short rate $r(T)$ under a particular probability measure (forward measure), depending on T.

Hence the forward rate is a biased estimator for the future short rate. Forecasts of future short rates by forward rates have little or no predictive power.

2.3 Bank Account and Short Rates

The return of a one dollar investment today ($t = 0$) over the period $[0, \Delta t]$ is given by

$$\frac{1}{P(0,\Delta t)} = \exp\left(\int_0^{\Delta t} f(0,u)\,du\right) = 1 + r(0)\Delta t + o(\Delta t).$$

Instantaneous reinvestment in $2\Delta t$-bonds yields

$$\frac{1}{P(0,\Delta t)}\frac{1}{P(\Delta t, 2\Delta t)} = (1 + r(0)\Delta t)(1 + r(\Delta t)\Delta t) + o(\Delta t)$$

at time $2\Delta t$, etc. This strategy of "rolling over"[2] just maturing bonds leads in the limit to the *bank account (money-market account)* $B(t)$. Hence $B(t)$ is the asset which growths at time t instantaneously at short rate $r(t)$

$$B(t + \Delta t) = B(t)(1 + r(t)\Delta t) + o(\Delta t).$$

For $\Delta t \to 0$ this converges to

$$dB(t) = r(t)B(t)dt$$

and with $B(0) = 1$ we obtain

$$B(t) = \exp\left(\int_0^t r(s)\,ds\right).$$

[2]This limiting process is made rigorous in [4].

B is a risk-free asset insofar as its future value at time $t + \Delta t$ is known (up to order Δt) at time t. In stochastic terms we speak of a predictable process. For the same reason we speak of $r(t)$ as the *risk-free rate of return* over the infinitesimal period $[t, t + dt]$.

B is important for relating amounts of currencies available at different times: in order to have one dollar in the bank account at time T we need to have

$$\frac{B(t)}{B(T)} = \exp\left(- \int_t^T r(s)\, ds \right)$$

dollars in the bank account at time $t \leq T$. This *discount factor* is stochastic: it is not known with certainty at time t. There is a close connection to the deterministic (=known at time t) discount factor given by $P(t, T)$. Indeed, we will see that the latter is the conditional expectation of the former under the risk neutral probability measure.

Proxies for the Short Rate

→ JW[11](Chapter 3.5)

The short rate $r(t)$ is a key interest rate in all models and fundamental to no-arbitrage pricing. But it cannot be directly observed.

The overnight interest rate is not usually considered to be a good proxy for the short rate, because the motives and needs driving overnight borrowers are very different from those of borrowers who want money for a month or more.

The overnight fed funds rate is nevertheless comparatively stable and perhaps a fair proxy, but empirical studies suggest that it has low correlation with other spot rates.

The best available proxy is given by one- or three-month spot rates since they are very liquid.

2.4 Coupon Bonds, Swaps and Yields

In most bond markets, there is only a relatively small number of zero-coupon bonds traded. Most bonds include coupons.

2.4.1 Fixed Coupon Bonds

A *fixed coupon bond* is a contract specified by

- a number of future dates $T_1 < \cdots < T_n$ (the *coupon dates*) (T_n is the *maturity* of the bond),
- a sequence of (deterministic) coupons $c_1, \dots, c_n$,
- a nominal value N,

such that the owner receives c_i at time T_i, for $i = 1, \dots, n$, and N at terminal time T_n. The price $p(t)$ at time $t \le T_1$ of this coupon bond is given by the sum of discounted cashflows

$$p(t) = \sum_{i=1}^{n} P(t, T_i) c_i + P(t, T_n) N.$$

Typically, it holds that $T_{i+1} - T_i \equiv \delta$, and the coupons are given as a fixed percentage of the nominal value: $c_i \equiv K\delta N$, for some fixed interest rate K. The above formula reduces to

$$p(t) = \left(K\delta \sum_{i=1}^{n} P(t, T_i) + P(t, T_n) \right) N.$$

2.4.2 Floating Rate Notes

There are versions of coupon bonds for which the value of the coupon is not fixed at the time the bond is issued, but rather reset for every coupon period. Most often the resetting is determined by some market interest rate (e.g. LIBOR).

A *floating rate note* is specified by

- a number of future dates $T_0 < T_1 < \cdots < T_n$,
- a nominal value N.

The deterministic coupon payments for the fixed coupon bond are now replaced by

$$c_i = (T_i - T_{i-1}) F(T_{i-1}, T_i) N,$$

where $F(T_{i-1}, T_i)$ is the prevailing simple market interest rate, and we note that $F(T_{i-1}, T_i)$ is determined already at time T_{i-1} (this is why here we have T_0 in addition to the coupon dates $T_1, \ldots, T_n$), but that the cash-flow c_i is at time T_i.

The value $p(t)$ of this note at time $t \leq T_0$ is obtained as follows. Without loss of generality we set $N = 1$. By definition of $F(T_{i-1}, T_i)$ we then have

$$c_i = \frac{1}{P(T_{i-1}, T_i)} - 1.$$

The time t value of -1 paid out at T_i is $-P(t, T_i)$. The time t value of $\frac{1}{P(T_{i-1}, T_i)}$ paid out at T_i is $P(t, T_{i-1})$:

- At t: buy a T_{i-1}-bond. Cost: $P(t, T_{i-1})$.
- At T_{i-1}: receive one dollar and buy $1/P(T_{i-1}, T_i)$ T_i-bonds. Zero net investment.
- At T_i: receive $1/P(T_{i-1}, T_i)$ dollars.

The the time t value of c_i therefore is

$$P(t, T_{i-1}) - P(t, T_i).$$

Summing up we obtain the (surprisingly easy) formula

$$p(t) = P(t, T_n) + \sum_{i=1}^{n} (P(t, T_{i-1}) - P(t, T_i)) = P(t, T_0).$$

In particular, for $t = T_0$: $p(T_0) = 1$.

2.4.3 Interest Rate Swaps

An interest rate swap is a scheme where you exchange a payment stream at a *fixed* rate of interest for a payment stream at a *floating* rate (typically LIBOR).

There are many versions of interest rate swaps. A *payer interest rate swap* settled in arrears is specified by

- a number of future dates $T_0 < T_1 < \cdots < T_n$ with $T_i - T_{i-1} \equiv \delta$ (T_n is the *maturity* of the swap),

- a fixed rate K,
- a nominal value N.

Of course, the equidistance hypothesis is only for convenience of notation and can easily be relaxed. Cashflows take place only at the coupon dates $T_1, \ldots, T_n$. At T_i, the holder of the contract

- pays fixed $K\delta N$,
- and receives floating $F(T_{i-1}, T_i)\delta N$.

The net cashflow at T_i is thus

$$(F(T_{i-1}, T_i) - K)\delta N,$$

and using the previous results we can compute the value at $t \leq T_0$ of this cashflow as

$$N(P(t, T_{i-1}) - P(t, T_i) - K\delta P(t, T_i)). \tag{2.3}$$

The total value $\Pi_p(t)$ of the swap at time $t \leq T_0$ is thus

$$\Pi_p(t) = N\left(P(t, T_0) - P(t, T_n) - K\delta \sum_{i=1}^{n} P(t, T_i)\right).$$

A *receiver interest rate swap* settled in arrears is obtained by changing the sign of the cashflows at times $T_1, \ldots, T_n$. Its value at time $t \leq T_0$ is thus

$$\Pi_r(t) = -\Pi_p(t).$$

The remaining question is how the "fair" fixed rate K is determined. The *forward swap rate* $R_{swap}(t)$ at time $t \leq T_0$ is the fixed rate K above which gives $\Pi_p(t) = \Pi_r(t) = 0$. Hence

$$R_{swap}(t) = \frac{P(t, T_0) - P(t, T_n)}{\delta \sum_{i=1}^{n} P(t, T_i)}.$$

The following alternative representation of $R_{swap}(t)$ is sometimes useful. Since $P(t, T_{i-1}) - P(t, T_i) = F(t; T_{i-1}, T_i)\delta P(t, T_i)$, we can rewrite (2.3) as

$$N\delta P(t, T_i)\left(F(t; T_{i-1}, T_i) - K\right).$$

Summing up yields

$$\Pi_p(t) = N\delta \sum_{i=1}^{n} P(t,T_i)\left(F(t;T_{i-1},T_i) - K\right),$$

and thus we can write the swap rate as weighted average of simple forward rates

$$R_{swap}(t) = \sum_{i=1}^{n} w_i(t) F(t;T_{i-1},T_i),$$

with weights

$$w_i(t) = \frac{P(t,T_i)}{\sum_{j=1}^{n} P(t,T_j)}.$$

These weights are random, but there seems to be empirical evidence that the variability of $w_i(t)$ is small compared to that of $F(t;T_{i-1},T_i)$. This is used for approximations of swaption (see below) price formulas in LIBOR market models: the swap rate volatility is written as linear combination of the forward LIBOR volatilities ("Rebonato's formula" → BM[6], p.248).

Swaps were developed because different companies could borrow at different rates in different markets.

Example

→ JW[11](p.11)

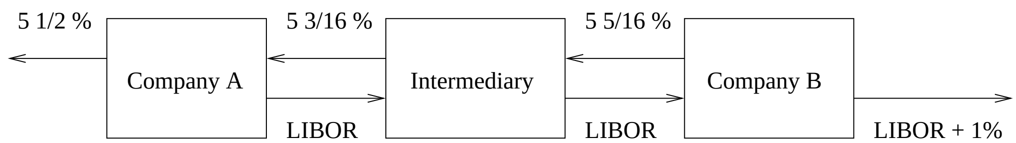
- Company A: is borrowing fixed for five years at 5 1/2%, but could borrow floating at LIBOR plus 1/2%.
- Company B: is borrowing floating at LIBOR plus 1%, but could borrow fixed for five years at 6 1/2%.

By agreeing to swap streams of cashflows both companies could be better off, and a mediating institution would also make money.

- Company A pays LIBOR to the intermediary in exchange for fixed at 5 3/16% (receiver swap).
- Company B pays the intermediary fixed at 5 5/16% in exchange for LIBOR (payer swap).

Net:

- Company A is now paying LIBOR plus 5/16% instead of LIBOR plus 1/2%.
- Company B is paying fixed at 6 5/16% instead of 6 1/2%.
- The intermediary receives fixed at 1/8%.



Everyone seems to be better off. But there is implicit credit risk; this is why Company B had higher borrowing rates in the first place. This risk has been partly taken up by the intermediary, in return for the money it makes on the spread.

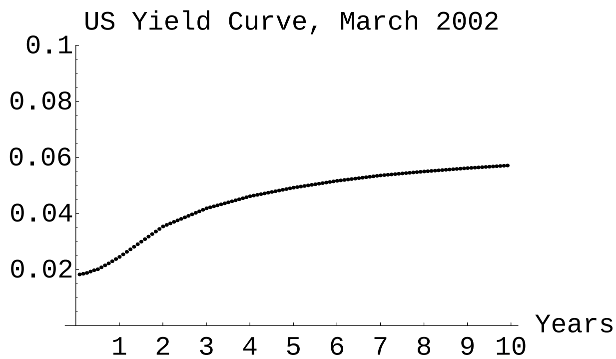
2.4.4 Yield and Duration

For a zero-coupon bond $P(t,T)$ the *zero-coupon yield* is simply the continuously compounded spot rate $R(t,T)$. That is,

$$P(t,T) = e^{-R(t,T)(T-t)}.$$

Accordingly, the function $T \mapsto R(t,T)$ is referred to as *(zero-coupon) yield curve*.

The term "yield curve" is ambiguous. There is a variety of other terminologies, such as zero-rate curve (Z[22]), zero-coupon curve (BM[6]). In JW[11] the yield curve is is given by simple spot rates, and in BM[6] it is a combination of simple spot rates (for maturities up to 1 year) and annually compounded spot rates (for maturities greater than 1 year), etc.



Now let $p(t)$ be the time t market value of a fixed coupon bond with coupon dates $T_1 < \cdots < T_n$, coupon payments $c_1, \dots, c_n$ and nominal value N (see Section 2.4.1). For simplicity we suppose that c_n already contains N, that is,

$$p(t) = \sum_{i=1}^{n} P(t, T_i) c_i, \quad t \le T_1.$$

Again we ask for the bond's "internal rate of interest"; that is, the constant (over the period $[t, T_n]$) continuously compounded rate which generates the market value of the coupon bond: the *(continuously compounded) yield-to-maturity* $y(t)$ of this bond at time $t \le T_1$ is defined as the unique solution to

$$p(t) = \sum_{i=1}^{n} c_i e^{-y(t)(T_i - t)}.$$

Remark 2.4.1. → *R[19](p.21). It is argued by Schaefer (1977) that the yield-to-maturity is an inadequate statistics for the bond market:*

- *coupon payments occurring at the* same point in time *are discounted by* different discount factors, *but*
- *coupon payments at* different points in time *from the same bond are* discounted by the same rate.

To simplify the notation we assume now that $t = 0$, and write $p = p(0)$, $y = y(0)$, etc. The *Macaulay duration* of the coupon bond is defined as

$$D_{Mac} := \frac{\sum_{i=1}^{n} T_i c_i e^{-y T_i}}{p}.$$

The duration is thus a weighted average of the coupon dates $T_1, \ldots, T_n$, and it provides us in a certain sense with the "mean time to coupon payment". As such it is an important concept for interest rate risk management: it acts as a measure of the first order sensitivity of the bond price w.r.t. changes in the yield-to-maturity (see Z[22](Chapter 6.1.3) for a thourough treatment). This is shown by the obvious formula

$$\frac{dp}{dy} = \frac{d}{dy}\left(\sum_{i=1}^{n} c_i e^{-yT_i}\right) = -D_{Mac}p.$$

A first order sensitivity measure of the bond price w.r.t. *parallel shifts* of the entire zero-coupon yield curve $T \mapsto R(0,T)$ is given by the *duration* of the bond

$$D := \frac{\sum_{i=1}^{n} T_i c_i e^{-y_i T_i}}{p} = \sum_{i=1}^{n} \frac{c_i P(0,T_i)}{p} T_i,$$

with $y_i := R(0,T_i)$. In fact, we have

$$\frac{d}{ds}\left(\sum_{i=1}^{n} c_i e^{-(y_i+s)T_i}\right)|_{s=0} = -Dp.$$

Hence duration is essentially for bonds (w.r.t. parallel shift of the yield curve) what delta is for stock options. The bond equivalent of the gamma is *convexity*:

$$C := \frac{d^2}{ds^2}\left(\sum_{i=1}^{n} c_i e^{-(y_i+s)T_i}\right)|_{s=0} = \sum_{i=1}^{n} c_i e^{-y_i T_i}(T_i)^2.$$

2.5 Market Conventions

2.5.1 Day-count Conventions

Time is measured in years.

If t and T denote two dates expressed as day/month/year, it is not clear what $T-t$ should be. The market evaluates the year fraction between t and T in different ways.

The *day-count convention* decides upon the time measurement between two dates t and T.

Here are three examples of day-count conventions:

- Actual/365: a year has 365 days, and the day-count convention for $T-t$ is given by

$$\frac{\text{actual number of days between } t \text{ and } T}{365}.$$

- Actual/360: as above but the year counts 360 days.

- 30/360: months count 30 and years 360 days. Let $t=(d_1,m_1,y_1)$ and $T=(d_2,m_2,y_2)$. The day-count convention for $T-t$ is given by

$$\frac{\min(d_2,30)+(30-d_1)^+}{360}+\frac{(m_2-m_1-1)^+}{12}+y_2-y_1.$$

  Example: The time between t=January 4, 2000 and T=July 4, 2002 is given by

$$\frac{4+(30-4)}{360}+\frac{7-1-1}{12}+2002-2000=2.5.$$

When extracting information on interest rates from data, it is important to realize for which day-count convention a specific interest rate is quoted.
→ BM[6](p.4), Z[22](Sect. 5.1)

2.5.2 Coupon Bonds

→ MR[16](Sect. 11.2), Z[22](Sect. 5.2), J[12](Chapter 2)
Coupon bonds issued in the American (European) markets typically have semi-annual (annual) coupon payments.
Debt securities issued by the U.S. Treasury are divided into three classes:

- *Bills:* zero-coupon bonds with time to maturity less than one year.

- *Notes:* coupon bonds (semi-annual) with time to maturity between 2 and 10 years.

- *Bonds:* coupon bonds (semi-annual) with time to maturity between 10 and 30 years[3].

[3]Recently, the issuance of 30 year treasury bonds has been stopped.

In addition to bills, notes and bonds, Treasury securities called *STRIPS* (separate trading of registered interest and principal of securities) have traded since August 1985. These are the coupons or principal (=nominal) amounts of Treasury bonds trading separately through the Federal Reserve's book-entry system. They are *synthetically* created zero-coupon bonds of longer maturities than a year. They were created in response to investor demands.

2.5.3 Accrued Interest, Clean Price and Dirty Price

Remember that we had for the price of a coupon bond with coupon dates $T_1, \ldots, T_n$ and payments $c_1, \ldots, c_n$ the price formula

$$p(t) = \sum_{i=1}^{n} c_i P(t, T_i), \quad t \le T_1.$$

For $t \in (T_1, T_2]$ we have

$$p(t) = \sum_{i=2}^{n} c_i P(t, T_i),$$

etc. Hence there are systematic discontinuities of the price trajectory at $t = T_1, \ldots, T_n$ which is due to the coupon payments. This is why prices are differently quoted at the exchange.

The *accrued interest* at time $t \in (T_{i-1}, T_i]$ is defined by

$$AI(i; t) := c_i \frac{t - T_{i-1}}{T_i - T_{i-1}}$$

(where now time differences are taken according to the day-count convention). The quoted price, or *clean price*, of the coupon bond at time t is

$$p_{clean}(t) := p(t) - AI(i; t), \quad t \in (T_{i-1}, T_i].$$

That is, whenever we buy a coupon bond quoted at a clean price of $p_{clean}(t)$ at time $t \in (T_{i-1}, T_i]$, the cash price, or *dirty price*, we have to pay is

$$p(t) = p_{clean}(t) + AI(i; t).$$

2.5.4 Yield-to-Maturity

The quoted *(annual) yield-to-maturity* $\hat{y}(t)$ on a Treasury bond at time $t = T_i$ is defined by the relationship

$$p_{clean}(T_i) = \sum_{j=i+1}^{n} \frac{r_c N/2}{(1+\hat{y}(T_i)/2)^{j-i}} + \frac{N}{(1+\hat{y}(T_i)/2)^{n-i}},$$

and at $t \in [T_i, T_{i+1})$

$$p_{clean}(t) = \sum_{j=i+1}^{n} \frac{r_c N/2}{(1+\hat{y}(t)/2)^{j-i-1+\tau}} + \frac{N}{(1+\hat{y}(t)/2)^{n-i-1+\tau}},$$

where r_c is the (annualized) coupon rate, N the nominal amount and

$$\tau = \frac{T_{i+1} - t}{T_{i+1} - T_i}$$

is again given by the day-count convention, and we assume here that

$$T_{i+1} - T_i \equiv 1/2 \quad \text{(semi-annual coupons)}.$$

2.6 Caps and Floors

→ BM[6](Sect. 1.6), Z[22](Sect. 5.6.2)

Caps

A *caplet* with reset date T and settlement date $T + \delta$ pays the holder the difference between a simple market rate $F(T, T+\delta)$ (e.g. LIBOR) and the strike rate κ. Its cashflow at time $T + \delta$ is

$$\delta(F(T, T+\delta) - \kappa)^+.$$

A *cap* is a strip of caplets. It thus consists of

- a number of future dates $T_0 < T_1 < \cdots < T_n$ with $T_i - T_{i-1} \equiv \delta$ (T_n is the *maturity* of the cap),
- a *cap rate* κ.

Cashflows take place at the dates $T_1, \dots, T_n$. At T_i the holder of the cap receives

$$\delta(F(T_{i-1}, T_i) - \kappa)^+. \tag{2.4}$$

Let $t \leq T_0$. We write

$$Cpl(i;t), \quad i = 1, \dots, n,$$

for the time t price of the ith caplet with reset date T_{i-1} and settlement date T_i, and

$$Cp(t) = \sum_{i=1}^{n} Cpl(i;t)$$

for the time t price of the cap.

A cap gives the holder a protection against rising interest rates. It guarantees that the interest to be paid on a floating rate loan never exceeds the predetermined cap rate κ.

It can be shown (→ exercise) that the cashflow (2.4) at time T_i is the equivalent to $(1 + \delta\kappa)$ times the cashflow at date T_{i-1} of a put option on a T_i-bond with strike price $1/(1 + \delta\kappa)$ and maturity T_{i-1}, that is,

$$(1 + \delta\kappa) \left(\frac{1}{1 + \delta\kappa} - P(T_{i-1}, T_i) \right)^+.$$

This is an important fact because many interest rate models have explicit formulae for bond option values, which means that caps can be priced very easily in those models.

Floors

A *floor* is the converse to a cap. It protects against low rates. A floor is a strip of *floorlets*, the cashflow of which is – with the same notation as above – at time T_i

$$\delta(\kappa - F(T_{i-1}, T_i))^+.$$

Write $Fll(i;t)$ for the price of the ith floorlet and

$$Fl(t) = \sum_{i=1}^{n} Fll(i;t)$$

for the price of the floor.

Caps, Floors and Swaps

Caps and floors are strongly related to swaps. Indeed, one can show the parity relation (→ exercise)

$$Cp(t) - Fl(t) = \Pi_p(t),$$

where $\Pi_p(t)$ is the value at t of a payer swap with rate κ, nominal one and the same tenor structure as the cap and floor.

Let $t = 0$. The cap/floor is said to be *at-the-money (ATM)* if

$$\kappa = R_{swap}(0) = \frac{P(0,T_0) - P(0,T_n)}{\delta \sum_{i=1}^n P(0,T_i)},$$

the forward swap rate. The cap (floor) is *in-the-money (ITM)* if $\kappa < R_{swap}(0)$ ($\kappa > R_{swap}(0)$), and *out-of-the-money (OTM)* if $\kappa > R_{swap}(0)$ ($\kappa < R_{swap}(0)$).

Black's Formula

It is market practice to price a cap/floor according to *Black's formula*. Let $t \leq T_0$. Black's formula for the value of the ith caplet is

$$Cpl(i;t) = \delta P(t,T_i) \left(F(t;T_{i-1},T_i)\Phi(d_1(i;t)) - \kappa\Phi(d_2(i;t)) \right),$$

where

$$d_{1,2}(i;t) := \frac{\log\left(\frac{F(t;T_{i-1},T_i)}{\kappa}\right) \pm \frac{1}{2}\sigma(t)^2(T_{i-1}-t)}{\sigma(t)\sqrt{T_{i-1}-t}}$$

(Φ stands for the standard Gaussian cumulative distribution function), and $\sigma(t)$ is the *cap volatility* (it is the same for all caplets).

Correspondingly, Black's formula for the value of the ith floorlet is

$$Fll(i;t) = \delta P(t,T_i) \left(\kappa\Phi(-d_2(i;t)) - F(t;T_{i-1},T_i)\Phi(-d_1(i;t)) \right).$$

Cap/floor prices are quoted in the market in term of their implied volatilities. Typically, we have $t = 0$, and T_0 and $\delta = T_i - T_{i-1}$ being equal to three months.

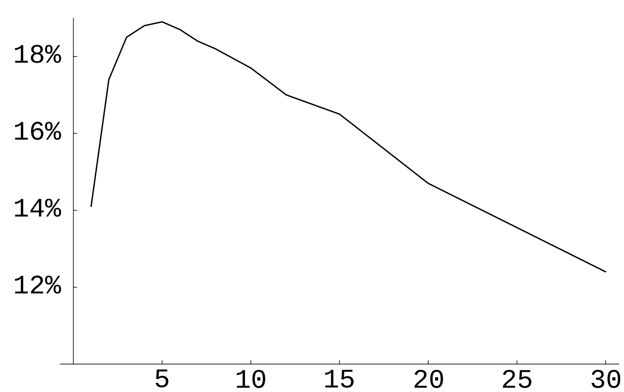
An example of a US dollar ATM market cap volatility curve is shown in Table 2.1 and Figure 2.1 (→ JW[11](p.49)).

It is a challenge for any market realistic interest rate model to match the given volatility curve.

Table 2.1: US dollar ATM cap volatilities, 23 July 1999

Maturity (in years)	ATM vols (in %)
1	14.1
2	17.4
3	18.5
4	18.8
5	18.9
6	18.7
7	18.4
8	18.2
10	17.7
12	17.0
15	16.5
20	14.7
30	12.4

Figure 2.1: US dollar ATM cap volatilities, 23 July 1999



2.7 Swaptions

A European *payer (receiver) swaption* with *strike rate* K is an option giving the right to enter a payer (receiver) swap with fixed rate K at a given future date, the *swaption maturity*. Usually, the swaption maturity coincides with the first reset date of the underlying swap. The underlying swap lenght $T_n - T_0$ is called the *tenor* of the swaption.

Recall that the value of a payer swaption with fixed rate K at its first reset date, T_0, is

$$\Pi_p(T_0, K) = N \sum_{i=1}^{n} P(T_0, T_i)\delta(F(T_0; T_{i-1}, T_i) - K).$$

Hence the payoff of the swaption with strike rate K at maturity T_0 is

$$N \left(\sum_{i=1}^{n} P(T_0, T_i)\delta(F(T_0; T_{i-1}, T_i) - K) \right)^{+}. \tag{2.5}$$

Notice that, contrary to the cap case, this payoff cannot be decomposed into more elementary payoffs. This is a fundamental difference between caps/floors and swaptions. Here the correlation between different forward rates will enter the valuation procedure.

Since $\Pi_p(T_0, R_{swap}(T_0)) = 0$, one can show ($\rightarrow$ exercise) that the payoff (2.5) of the payer swaption at time T_0 can also be written as

$$N\delta(R_{swap}(T_0) - K)^{+} \sum_{i=1}^{n} P(T_0, T_i),$$

and for the receiver swaption

$$N\delta(K - R_{swap}(T_0))^{+} \sum_{i=1}^{n} P(T_0, T_i).$$

Accordingly, at time $t \leq T_0$, the payer (receiver) swaption with strike rate K is said to be *ATM*, *ITM*, *OTM*, if

$$K = R_{swap}(t), \quad K < (>) R_{swap}(t), \quad K > (<) R_{swap}(t),$$

respectively.

Black's Formula

Black's formula for the price at time $t \leq T_0$ of the payer ($Swpt_p(t)$) and receiver ($Swpt_r(t)$) swaption is

$$Swpt_p(t) = N\delta \left(R_{swap}(t)\Phi(d_1(t)) - K\Phi(d_2(t))\right) \sum_{i=1}^{n} P(t, T_i),$$
$$Swpt_r(t) = N\delta \left(K\Phi(-d_2(t)) - R_{swap}(t)\Phi(-d_1(t))\right) \sum_{i=1}^{n} P(t, T_i),$$

with

$$d_{1,2}(t) := \frac{\log\left(\frac{R_{swap}(t)}{K}\right) \pm \frac{1}{2}\sigma(t)^2(T_0 - t)}{\sigma(t)\sqrt{T_0 - t}},$$

and $\sigma(t)$ is the prevailing Black's swaption volatility.

Swaption prices are quoted in terms of implied volatilities in matrix form. An $x \times y$-swaption is the swaption with maturity in x years and whose underlying swap is y years long.

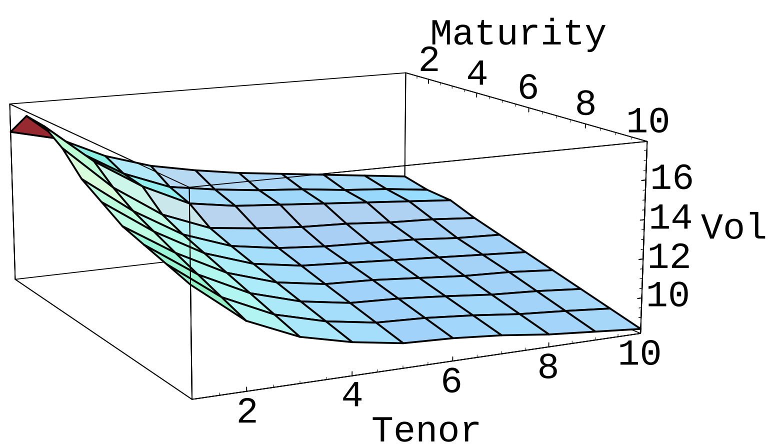
A typical example of implied swaption volatilities is shown in Table 2.2 and Figure 2.2 ($\rightarrow$ BM[6](p.253)).

An interest model for swaptions valuation must fit the given today's volatility surface.

Table 2.2: Black's implied volatilities (in %) of ATM swaptions on May 16, 2000. Maturities are 1,2,3,4,5,7,10 years, swaps lengths from 1 to 10 years.

	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y
1y	16.4	15.8	14.6	13.8	13.3	12.9	12.6	12.3	12.0	11.7
2y	17.7	15.6	14.1	13.1	12.7	12.4	12.2	11.9	11.7	11.4
3y	17.6	15.5	13.9	12.7	12.3	12.1	11.9	11.7	11.5	11.3
4y	16.9	14.6	12.9	11.9	11.6	11.4	11.3	11.1	11.0	10.8
5y	15.8	13.9	12.4	11.5	11.1	10.9	10.8	10.7	10.5	10.4
7y	14.5	12.9	11.6	10.8	10.4	10.3	10.1	9.9	9.8	9.6
10y	13.5	11.5	10.4	9.8	9.4	9.3	9.1	8.8	8.6	8.4

Figure 2.2: Black's implied volatilities (in %) of ATM swaptions on May 16, 2000.



Chapter 3

Some Statistics of the Yield Curve

3.1 Principal Component Analysis (PCA)

→ JW[11](Chapter 16.2), [18]

- Let $x(1), \dots, x(N)$ be a sample of a random $n \times 1$ vector x.
- Form the empirical $n \times n$ covariance matrix $\hat{\Sigma}$,

$$\begin{aligned}\hat{\Sigma}_{ij} &= \frac{\sum_{k=1}^{N}(x_i(k) - \mu[x_i])(x_j(k) - \mu[x_j])}{N-1} \\ &= \frac{\sum_{k=1}^{N} x_i(k)x_i(k) - N\mu[x_i]\mu[x_j]}{N-1},\end{aligned}$$

  where

$$\mu[x_i] := \frac{1}{N}\sum_{k=1}^{N} x_i(k) \quad \text{(mean of } x_i\text{)}.$$

  We assume that $\hat{\Sigma}$ is non-degenerate (otherwise we can express an x_i as linear combination of the other x_js).

- There exists a unique orthogonal matrix $A = (p_1, \dots, p_n)$ (that is, $A^{-1} = A^T$ and $A_{ij} = p_{j;i}$) consisting of orthonormal $n \times 1$ Eigenvectors p_i of $\hat{\Sigma}$ such that

$$\hat{\Sigma} = ALA^T,$$

where $L = diag(\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \geq \cdots \geq \lambda_n > 0$ (the Eigenvalues of $\hat{\Sigma}$).

- Define $z := A^T x$. Then

$$Cov[z_i, z_j] = \sum_{k,l=1}^{n} A^T_{ik} Cov[x_k, x_l] A^T_{jl} = \left(A^T \hat{\Sigma} A\right)_{ij} = \lambda_i \delta_{ij}.$$

Hence the z_is are uncorrelated.

- The *principal components (PCs)* are the $n \times 1$ vectors $p_1, \ldots, p_n$:

$$x = Az = z_1 p_1 + \cdots z_n p_n.$$

The importance of component p_i is determined by the size of the corresponding Eigenvalue, λ_i, which indicates the amount of variance explained by p_i. The key statistics is the proportion

$$\frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j},$$

the *explained variance* by p_i.

- Normalization: let $\tilde{w} := (L^{1/2})^{-1} z$, where $L^{1/2} := diag(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})$, and $w = \tilde{w} - \mu[\tilde{w}]$ ($\mu[\tilde{w}]$=mean of $\tilde{w}$). Then

$$\mu[w] = 0, \qquad Cov[w_i, w_j] = Cov[\tilde{w}_i, \tilde{w}_j] = \delta_{ij},$$

and

$$x = \mu[x] + AL^{1/2} w = \mu[x] + \sum_{j=1}^{n} p_j \sqrt{\lambda_j} w_j.$$

In components

$$x_i = \mu[x_i] + \sum_{j=1}^{n} A_{ij} \sqrt{\lambda_j} w_j.$$

- Sometimes the following view is useful ($\rightarrow$ R[19](Chapter 3)): set

$$\sigma_i := Var[x_i]^{1/2} = \left(\hat{\Sigma}_{ii}\right)^{1/2} = \left(\sum_{j=1}^{n} A_{ij}^2 \lambda_j\right)^{1/2}$$

$$v_i := \frac{x_i - \mu[x_i]}{\sigma_i} = \frac{\sum_{j=1}^{n} A_{ij} \sqrt{\lambda_j} w_j}{\sigma_i}, \quad i = 1, \ldots, n.$$

Then we have $\mu[v_i] = 0$, $\mu[v_i^2] = 1$ and

$$x_i = \mu[x_i] + \sigma_i v_i.$$

It can be appropriate to assume a parametric functional form (→ reduction of parameters) of the correlation structure of x,

$$Corr[x_i, x_j] = Cov[v_i, v_j] = \frac{\hat{\Sigma}_{ij}}{\sigma_i \sigma_j} = \frac{\sum_{k=1}^n A_{ik} A_{jk} \lambda_k}{\sigma_i \sigma_j} = \rho(\pi; i, j),$$

where π is some low-dimensional parameter (this is adapted to the calibration of market models → BM[6](Chapter 6.9)).

3.2 PCA of the Yield Curve

Now let $x = (x_1, \ldots, x_n)^T$ be the increments of the forward curve, say

$$x_i = R(t + \Delta t; t + \Delta t + \tau_{i-1}, t + \Delta t + \tau_i) - R(t; t + \tau_{i-1}, t + \tau_i),$$

for some maturity spectrum $0 = \tau_0 < \cdots < \tau_n$.

PCA typically leads to the following picture (→ R[19]p.61): UK market in the years 1989-1992 (the original maturity spectrum has been divided into eight distinct buckets, i.e. $n = 8$).

The first three principal components are

$$p_1 = \begin{pmatrix} 0.329 \\ 0.354 \\ 0.365 \\ 0.367 \\ 0.364 \\ 0.361 \\ 0.358 \\ 0.352 \end{pmatrix}, \quad p_2 = \begin{pmatrix} -0.722 \\ -0.368 \\ -0.121 \\ 0.044 \\ 0.161 \\ 0.291 \\ 0.316 \\ 0.343 \end{pmatrix}, \quad p_3 = \begin{pmatrix} 0.490 \\ -0.204 \\ -0.455 \\ -0.461 \\ -0.176 \\ 0.176 \\ 0.268 \\ 0.404 \end{pmatrix}.$$

- The first PC is roughly flat (parallel shift → average rate),
- the second PC is upward sloping (tilt → slope),
- the third PC hump-shaped (flex → curvature).

Figure 3.1: First Three PCs.

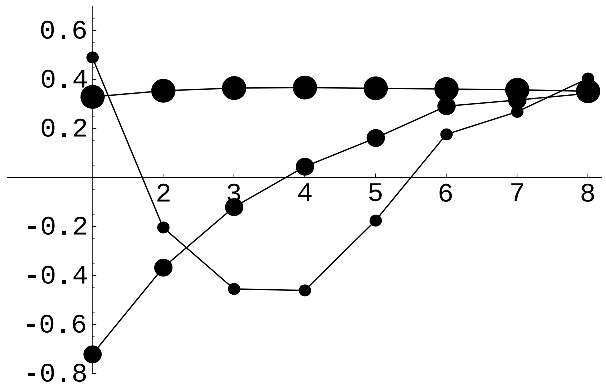


Table 3.1: Explained Variance of the Principal Components (PCs).

PC	Explained Variance (%)
1	92.17
2	6.93
3	0.61
4	0.24
5	0.03
6–8	0.01

The first three PCs explain more than 99 % of the variance of x (→ Table 3.1).

PCA of the yield curve goes back to the seminal paper by Litterman and Scheinkman (88) [?] (Prof. J. Scheinkman is at the Department of Economics, Princeton University).

3.3 Correlation

→ R[19](p.58)

A typical example of correlation among forward rates is provided by

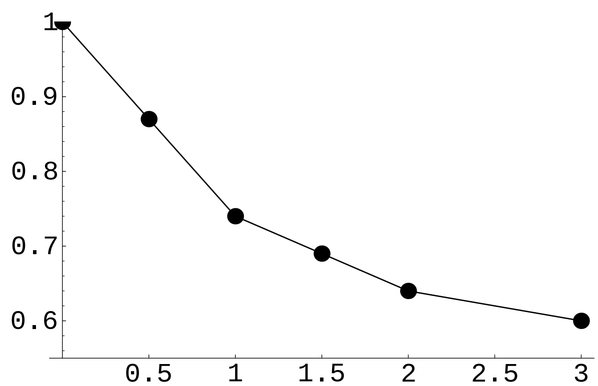
Brown and Schaefer (1994). The data is from the US Treasury yield curve 1987–1994. The following matrix (→ Figure 3.2)

$$\begin{pmatrix} 1 & 0.87 & 0.74 & 0.69 & 0.64 & 0.6 \\ & 1 & 0.96 & 0.93 & 0.9 & 0.85 \\ & & 1 & 0.99 & 0.95 & 0.92 \\ & & & 1 & 0.97 & 0.93 \\ & & & & 1 & 0.95 \\ & & & & & 1 \end{pmatrix}$$

shows the correlation for changes of forward rates of maturities

$$0, \quad 0.5, \quad 1, \quad 1.5, \quad 2, \quad 3 \quad \text{years.}$$

Figure 3.2: Correlation between the short rate and instantaneous forward rates for the US Treasury curve 1987–1994



→ Decorrelation occurs quickly.

→ Exponentially decaying correlation structure is plausible.

Chapter 4

Estimating the Yield Curve

4.1 A Bootstrapping Example

→ JW[11](p.129–136)

This is a naive bootstrapping method of fitting to a money market yield curve. The idea is to build up the yield curve

from shorter maturities to longer maturities.

We take Yen data from 9 January, 1996 (→ JW[11](Section 5.4)). The spot date t_0 is 11 January, 1996. The day-count convention is Actual/360,

$$\delta(T,S) = \frac{\text{actual number of days between } T \text{ and } S}{360}.$$

Table 4.1: Yen data, 9 January 1996.

LIBOR (%)		Futures		Swaps (%)	
o/n	0.49	20 Mar 96	99.34	2y	1.14
1w	0.50	19 Jun 96	99.25	3y	1.60
1m	0.53	18 Sep 96	99.10	4y	2.04
2m	0.55	18 Dec 96	98.90	5y	2.43
3m	0.56			7y	3.01
				10y	3.36

- The first column contains the LIBOR (=simple spot rates) $F(t_0, S_i)$ for maturities

$$\{S_1, \ldots, S_5\} = \{12/1/96,\ 18/1/96,\ 13/2/96,\ 11/3/96,\ 11/4/96\}$$

hence for 1, 7, 33, 60 and 91 days to maturity, respectively. The zero-coupon bonds are

$$P(t_0, S_i) = \frac{1}{1 + F(t_0, S_i)\,\delta(t_0, S_i)}.$$

- The futures are quoted as

$$\text{futures price for settlement day } T_i = 100(1 - F_F(t_0; T_i, T_{i+1})),$$

where $F_F(t_0; T_i, T_{i+1})$ is the futures rate for period $[T_i, T_{i+1}]$ prevailing at t_0, and

$$\{T_1, \ldots, T_5\} = \{20/3/96,\ 19/6/96,\ 18/9/96,\ 18/12/96,\ 19/3/97\},$$

hence $\delta(T_i, T_{i+1}) \equiv 91/360$.

We treat futures rates as if they were simple forward rates, that is, we set

$$F(t_0; T_i, T_{i+1}) = F_F(t_0; T_i, T_{i+1}).$$

To calculate zero-coupon bond from futures prices we need $P(t_0, T_1)$. We use geometric interpoliation

$$P(t_0, T_1) = P(t_0, S_4)^q\, P(t_0, S_5)^{1-q},$$

which is equivalent to using linear interpolation of continuously compounded spot rates

$$R(t_0, T_1) = q\, R(t_0, S_4) + (1 - q)\, R(t_0, S_5),$$

where

$$q = \frac{\delta(T_1, S_5)}{\delta(S_4, S_5)} = \frac{22}{31} = 0.709677.$$

Then we use the relation

$$P(t_0, T_{i+1}) = \frac{P(t_0, T_i)}{1 + \delta(T_i, T_{i+1})\, F(t_0; T_i, T_{i+1})}$$

to derive $P(t_0, T_2), \ldots, P(t_0, T_5)$.

- Yen swaps have semi-annual cashflows at dates

$$\{U_1, \ldots, U_{20}\} = \left\{ \begin{array}{ll} 11/7/96, & 13/1/97, \\ 11/7/97, & 12/1/98, \\ 13/7/98, & 11/1/99, \\ 12/7/99, & 11/1/00, \\ 11/7/00, & 11/1/01, \\ 11/7/01, & 11/1/02, \\ 11/7/02, & 13/1/03, \\ 11/7/03, & 12/1/04, \\ 12/7/04, & 11/1,05, \\ 11/7/05, & 11/1/06 \end{array} \right\}.$$

For a swap with maturity U_n the swap rate at t_0 is given by

$$R_{swap}(t_0, U_n) = \frac{1 - P(t_0, U_n)}{\sum_{i=1}^{n} \delta(U_{i-1}, U_i)\, P(t_0, U_i)}, \quad (U_0 := t_0).$$

From the data we have $R_{swap}(t_0, U_i)$ for $i = 4, 6, 8, 10, 14, 20$.

We obtain $P(t_0, U_1)$, $P(t_0, U_2)$ (and hence $R_{swap}(t_0, U_1), R_{swap}(t_0, U_2)$) by linear interpolation of the continuously compounded spot rates

$$\begin{aligned} R(t_0, U_1) &= \frac{69}{91} R(t_0, T_2) + \frac{22}{91} R(t_0, T_3) \\ R(t_0, U_2) &= \frac{65}{91} R(t_0, T_4) + \frac{26}{91} R(t_0, T_5). \end{aligned}$$

All remaining swap rates are obtained by linear interpolation. For maturity U_3 this is

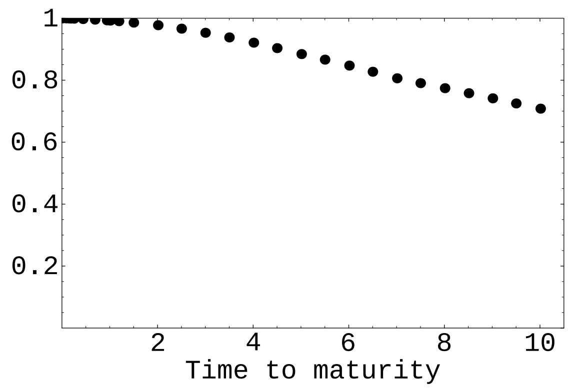
$$R_{swap}(t_0, U_3) = \frac{1}{2}(R_{swap}(t_0, U_2) + R_{swap}(t_0, U_4)).$$

We have ($\rightarrow$ exercise)

$$P(t_0, U_n) = \frac{1 - R_{swap}(t_0, U_n) \sum_{i=1}^{n-1} \delta(U_{i-1}, U_i)\, P(t_0, U_i)}{1 + R_{swap}(t_0, U_n) \delta(U_{n-1}, U_n)}.$$

This gives $P(t_0, U_n)$ for $n = 3, \ldots, 20$.

Figure 4.1: Zero-coupon bond curve



In Figure 4.1 is the implied zero-coupon bond price curve

$$P(t_0, t_i), \quad i = 0, \dots, 29$$

(we have 29 points and set $P(t_0, t_0) = 1$).

The spot and forward rate curves are in Figure 4.2. Spot and forward rates are continuously compounded

$$R(t_0, t_i) = -\frac{\log P(t_0, t_i)}{\delta(t_0, t_i)}$$
$$R(t_0, t_i, t_{i+1}) = -\frac{\log P(t_0, t_{i+1}) - \log P(t_0, t_i)}{\delta(t_i, t_{i+1})}, \quad i = 1, \dots, 29.$$

The forward curve, reflecting the derivative of $T \mapsto -\log P(t_0, T)$, is very unsmooth and sensitive to slight variations (errors) in prices.

Figure 4.3 shows the spot rate curves from LIBOR, futures and swaps. It is evident that the three curves are not coincident to a common underlying curve. Our naive method made no attempt to meld the three curves together.

→ The entire yield curve is constructed from relatively few instruments. The method exactly reconstructs market prices (this is desirable for interest rate option traders). But it produces an unstable, non-smooth forward curve.

Figure 4.2: Spot rates (lower curve), forward rates (upper curve)

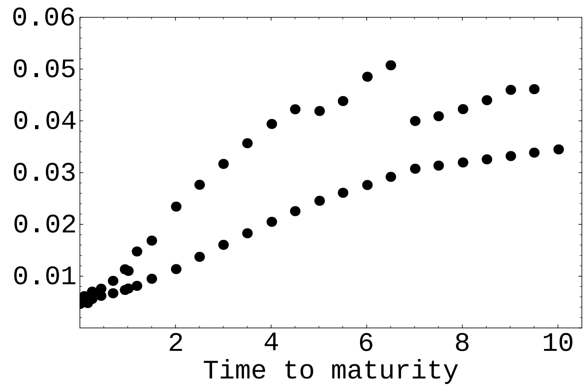
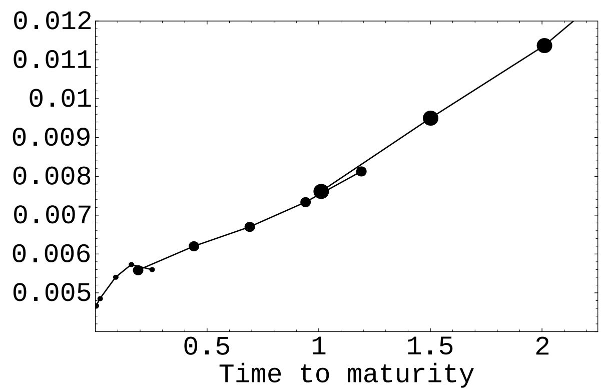


Figure 4.3: Comparison of money market curves



→ Another method would be to estimate a smooth yield curve parametrically from the market rates (for fund managers, long term strategies).

The main difficulties with our method are:

- Futures rates are treated as forward rates. In reality futures rates are greater than forward rates. The amount by which the futures rate is above the forward rate is called the convexity adjustment, which is model dependend. An example is

$$\text{forward rate} = \text{futures rate} - \frac{1}{2}\sigma^2\tau^2,$$

where τ is the time to maturity of the futures contract, and σ is the volatility parameter.

- LIBOR rates beyond the "stup date" $T_1 = 20/3/96$ (that is, at $S_5 = 11/4/96$) are ignored once $P(t_0, T_1)$ is found. In general, the segments of LIBOR, futures and swap markets overlap.

- Swap rates are inappropriately interpolated. The linear interpolation produces a "sawtooth" in the forward rate curve. However, in some markets intermediate swaps are indeed priced as if their prices were found by linear interpolation.

4.2 General Case

The general problem of finding today's (t_0) term structure of zero-coupon bond prices (or the *discount function*)

$$x \mapsto D(x) := P(t_0, t_0 + x)$$

can be formulated as

$$p = C \cdot d + \epsilon,$$

where p is a vector of n market prices, C the related cashflow matrix, and $d = (D(x_1), \ldots, D(x_N))$ with cashflow dates $t_0 < T_1 < \cdots < T_N$,

$$T_i - t_0 = x_i,$$

and ϵ a vector of pricing errors. Reasons for including errors are

- prices are never exactly simultaneous,

- round-off errors in the quotes (bid-ask spreads, etc),

- liquidity effects,

- tax effects (high coupons, low coupons),

- allows for smoothing.

4.2.1 Bond Markets

Data:

- vector of quoted/market bond prices $p = (p_1, \ldots, p_n)$,
- dates of all cashflows $t_0 < T_1 < \cdots < T_N$,
- bond i with cashflows (coupon and principal payments) $c_{i,j}$ at time T_j (may be zero), forming the $n \times N$ cashflow matrix

$$C = (c_{i,j})_{\substack{1\leq i\leq n \\ 1\leq j\leq N}}.$$

Example (→ JW[11], p.426): UK government bond (gilt) market, September 4, 1996, selection of nine gilts. The coupon payments are semiannual. The spot date is 4/9/96, and the day-count convention is actual/365.

Table 4.2: Market prices for UK gilts, 4/9/96.

	coupon (%)	next coupon	maturity date	dirty price (p_i)
bond 1	10	15/11/96	15/11/96	103.82
bond 2	9.75	19/01/97	19/01/98	106.04
bond 3	12.25	26/09/96	26/03/99	118.44
bond 4	9	03/03/97	03/03/00	106.28
bond 5	7	06/11/96	06/11/01	101.15
bond 6	9.75	27/02/97	27/08/02	111.06
bond 7	8.5	07/12/96	07/12/05	106.24
bond 8	7.75	08/03/97	08/09/06	98.49
bond 9	9	13/10/96	13/10/08	110.87

Hence $n = 9$ and $N = 1 + 3 + 6 + 7 + 11 + 12 + 19 + 20 + 25 = 104$,

$$T_1 = 26/09/96, \quad T_2 = 13/10/96, \quad T_3 = 06/11/97, \ldots.$$

No bonds have cashflows at the same date. The 9×104 cashflow matrix is

$$C = \begin{pmatrix} 0 & 0 & 0 & 105 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 4.875 & 0 & 0 & 0 & 0 & \dots \\ 6.125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6.125 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4.5 & 0 & 0 & \dots \\ 0 & 0 & 3.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 4.875 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 4.25 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.875 & 0 & \dots \\ 0 & 4.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \end{pmatrix}$$

4.2.2 Money Markets

Money market data can be put into the same price–cashflow form as above.

LIBOR (rate L, maturity T): $p = 1$ and $c = 1 + (T - t_0)L$ at T.

FRA (forward rate F for $[T, S]$): $p = 0$, $c_1 = -1$ at $T_1 = T$, $c_2 = 1+(S-T)F$ at $T_2 = S$.

Swap (receiver, swap rate K, tenor $t_0 \le T_0 < \cdots < T_n$, $T_i - T_{i-1} \equiv \delta$): since

$$0 = -D(T_0 - t_0) + \delta K \sum_{j=1}^{n} D(T_j - t_0) + (1 + \delta K)D(T_n - t_0),$$

- if $T_0 = t_0$: $p = 1$, $c_1 = \cdots = c_{n-1} = \delta K$, $c_n = 1 + \delta K$,
- if $T_0 > t_0$: $p = 0$, $c_0 = -1$, $c_1 = \cdots = c_{n-1} = \delta K$, $c_n = 1 + \delta K$.

$\to$ at t_0: LIBOR and swaps have notional price 1, FRAs and forward swaps have notional price 0.

Example ($\to$ JW[11], p.428): US money market on October 6, 1997. The day-count convention is Actual/360. The spot date t_0 is 8/10/97.

LIBOR is for o/n (1/365), 1m (33/360), and 3m (92/360).

Futures are three month rates ($\delta = 91/360$). We take them as forward rates. That is, the quote of the futures contract with maturity date (settlement day) T is

$$100(1 - F(t_0; T, T + \delta)).$$

Swaps are annual ($\delta = 1$). The first payment date is 8/10/98.

Table 4.3: US money market, October 6, 1997.

	Period	Rate	Maturity Date
LIBOR	o/n	5.59375	9/10/97
	1m	5.625	10/11/97
	3m	5.71875	8/1/98
Futures	Oct-97	94.27	15/10/97
	Nov-97	94.26	19/11/97
	Dec-97	94.24	17/12/97
	Mar-98	94.23	18/3/98
	Jun-98	94.18	17/6/98
	Sep-98	94.12	16/9/98
	Dec-98	94	16/12/98
Swaps	2	6.01253	
	3	6.10823	
	4	6.16	
	5	6.22	
	7	6.32	
	10	6.42	
	15	6.56	
	20	6.56	
	30	6.56	

Here $n = 3 + 7 + 9 = 19$, $N = 3 + 14 + 30 = 47$, $T_1 = 9/10/97$, $T_2 = 15/10/97$ (first future), $T_3 = 10/11/97$, ... The first 14 columns of the 19×47 cashflow matrix C are

$$
\begin{array}{cccccccccccccc}
c_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c_{36} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & -1 & 0 & 0 & 0 & 0 & c_{47} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & c_{58} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & c_{69} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & c_{7,10} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & c_{8,11} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & c_{9,13} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & c_{10,14} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{11,12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{12,12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{13,12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{14,12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{15,12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{16,12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{17,12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{18,12} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{19,12} & 0 & 0
\end{array}
$$

with

$$
\begin{array}{c}
c_{11} = 1.00016, \quad c_{23} = 1.00516, \quad c_{36} = 1.01461, \\
\hline
c_{47} = 1.01448, \quad c_{58} = 1.01451, \quad c_{69} = 1.01456, \quad c_{7,10} = 1.01459, \\
c_{8,11} = 1.01471, \quad c_{9,13} = 1.01486, \quad c_{10,14} = 1.01517 \\
\hline
c_{11,12} = 0.060125, \quad c_{12,12} = 0.061082, \quad c_{13,12} = 0.0616, \\
c_{14,12} = 0.0622, \quad c_{15,12} = 0.0632, \quad c_{16,12} = 0.0642, \\
c_{17,12} = c_{18,12} = c_{19,12} = 0.0656.
\end{array}
$$

4.2.3 Problems

Typically, we have $n \ll N$. Moreover, many entries of C are zero (different cashflow dates). This makes ordinary least square (OLS) regression

$$\min_{d \in \mathbb{R}^N} \{\|\epsilon\|^2 \mid \epsilon = p - C \cdot d\} \quad (\Rightarrow C^T p = C^T C d^*)$$

unfeasible.

One could chose the data set such that cashflows are at same points in time (say four dates each year) and the cashflow matrix C is not entirely full of zeros (Carleton–Cooper (1976)). Still regression only yields values $D(x_i)$ at the payment dates $t_0 + x_i$

$\rightarrow$ interpolation technics necessary.

But there is nothing to regularize the discount factors (discount factors of similar maturity can be very different). As a result this leads to a ragged spot rate (yield) curve, and even worse for forward rates.

4.2.4 Parametrized Curve Families

Reduction of parameters and smooth yield curves can be achieved by using parametrized families of smooth curves

$$D(x) = D(x; z) = \exp\left(-\int_0^x \phi(u; z)\, du\right), \quad z \in \mathcal{Z},$$

with state space $\mathcal{Z} \subset \mathbb{R}^m$.

For regularity reasons (see below) it is best to estimate the forward curve

$$\mathbb{R}_+ \ni x \mapsto f(t_0, t_0 + x) = \phi(x) = \phi(x; z).$$

This leads to a nonlinear optimization problem

$$\min_{z \in \mathcal{Z}} \|p - C \cdot d(z)\|,$$

with

$$d_i(z) = \exp\left(-\int_0^{x_i} \phi(u; z)\, du\right)$$

for some payment tenor $0 < x_1 < \cdots < x_N$.

Linear Families

Fix a set of basis functions $\psi_1, \ldots, \psi_m$ (preferably *with compact support*), and let

$$\phi(x; z) = z_1\psi_1(x) + \cdots + z_m\psi_m(x).$$

Cubic B-splines A cubic spline is a piecewise cubic polynomial that is everywhere twice differentiable. It interpolates values at $m+1$ *knot points* $\xi_0 < \cdots < \xi_m$. Its general form is

$$\sigma(x) = \sum_{i=0}^{3} a_i x^i + \sum_{j=1}^{m-1} b_j (x - \xi_j)_+^3,$$

hence it has $m+3$ parameters $\{a_0, \dots, a_4, b_1, \dots, b_{m-1}\}$ (a kth degree spline has $m+k$ parameters). The spline is uniquely characterized by specification of σ' or σ'' at ξ_0 and ξ_m.

Introduce six extra knot points

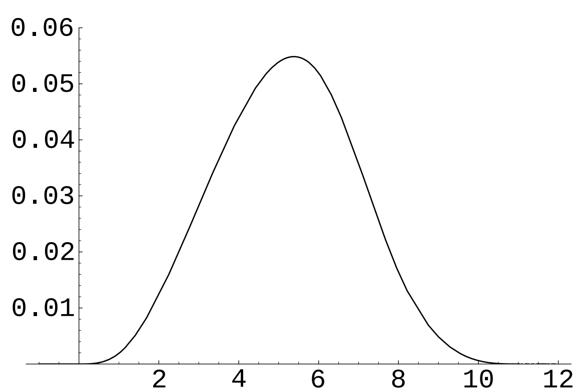
$$\xi_{-3} < \xi_{-2} < \xi_{-1} < \xi_0 < \cdots < \xi_m < \xi_{m+1} < \xi_{m+2} < \xi_{m+3}.$$

A basis for the cubic splines on $[\xi_0, \xi_m]$ is given by the $m+3$ *B-splines*

$$\psi_k(x) = \sum_{j=k}^{k+4} \left(\prod_{i=k, i \neq j}^{k+4} \frac{1}{\xi_i - \xi_j} \right) (x - \xi_j)_+^3, \quad k = -3, \dots, m-1.$$

The B-spline ψ_k is zero outside $[\xi_k, \xi_{k+4}]$.

Figure 4.4: B-spline with knot points $\{0, 1, 6, 8, 11\}$.



Estimating the Discount Function B-splines can also be used to estimate the discount function directly (Steeley (1991)),

$$D(x; z) = z_1\psi_1(x) + \cdots + z_m\psi_m(x).$$

With

$$d(z) = \begin{pmatrix} D(x_1; z) \\ \vdots \\ D(x_N; z) \end{pmatrix} = \begin{pmatrix} \psi_1(x_1) & \cdots & \psi_m(x_1) \\ \vdots & & \vdots \\ \psi_1(x_N) & \cdots & \psi_m(x_N) \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix} =: \Psi \cdot z$$

this leads to the linear optimization problem

$$\min_{z \in \mathbb{R}^m} \|p - C\Psi z\|.$$

If the $n \times m$ matrix $A := C\Psi$ has full rank m, the unique unconstrained solution is

$$z^* = (A^T A)^{-1} A^T p.$$

A reasonable constraint would be

$$D(0; z) = \psi_1(0)z_1 + \cdots + \psi_m(0)z_m = 1.$$

Example We take the UK government bond market data from the last section (Table 4.2). The maximum time to maturity, x_{104}, is 12.11 [years]. Notice that the first bond is a zero-coupon bond. Its exact yield is

$$y = -\frac{365}{72} \log \frac{103.822}{105} = -\frac{1}{0.197} \log 0.989 = 0.0572.$$

- As a basis we use the 8 (resp. first 7) B-splines with the 12 knot points

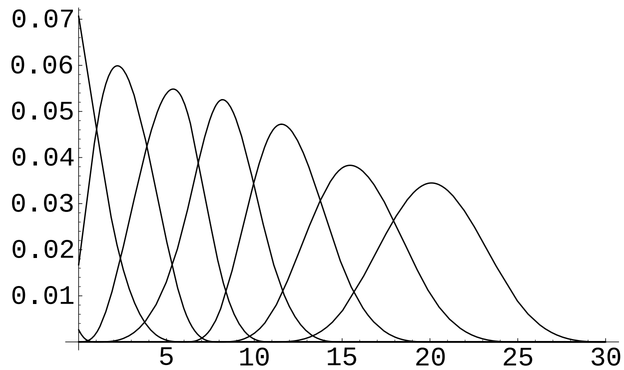
  $$\{-20, -5, -2, 0, 1, 6, 8, 11, 15, 20, 25, 30\}$$

  (see Figure 4.5).

  The estimation with all 8 B-splines leads to

  $$\min_{z \in \mathbb{R}^8} \|p - C\Psi z\| = \|p - C\Psi z^*\| = 0.23$$

Figure 4.5: B-splines with knots $\{-20, -5, -2, 0, 1, 6, 8, 11, 15, 20, 25, 30\}$.



with

$$z^* = \begin{pmatrix} 13.8641 \\ 11.4665 \\ 8.49629 \\ 7.69741 \\ 6.98066 \\ 6.23383 \\ -4.9717 \\ 855.074 \end{pmatrix},$$

and the discount function, yield curve (cont. comp. spot rates), and forward curve (cont. comp. 3-monthly forward rates) shown in Figure 4.7.

The estimation with only the first 7 B-splines leads to

$$\min_{z \in \mathbb{R}^7} \|p - C\Psi z\| = \|p - C\Psi z^*\| = 0.32$$

with

$$z^* = \begin{pmatrix} 17.8019 \\ 11.3603 \\ 8.57992 \\ 7.56562 \\ 7.28853 \\ 5.38766 \\ 4.9919 \end{pmatrix},$$

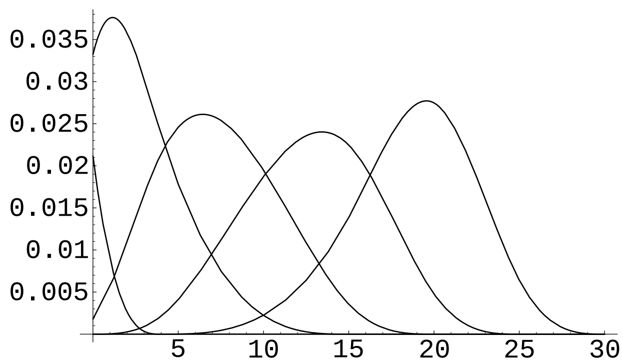
and the discount function, yield curve (cont. comp. spot rates), and forward curve (cont. comp. 3-month forward rates) shown in Figure 4.8.

- Next we use only 5 B-splines with the 9 knot points

$$\{-10, -5, -2, 0, 4, 15, 20, 25, 30\}$$

(see Figure 4.6).

Figure 4.6: Five B-splines with knot points $\{-10, -5, -2, 0, 4, 15, 20, 25, 30\}$.



The estimation with this 5 B-splines leads to

$$\min_{z\in\mathbb{R}^5} \|p - C\Psi z\| = \|p - C\Psi z^*\| = 0.39$$

with

$$z^* = \begin{pmatrix} 15.652 \\ 19.4385 \\ 12.9886 \\ 7.40296 \\ 6.23152 \end{pmatrix},$$

and the discount function, yield curve (cont. comp. spot rates), and forward curve (cont. comp. 3-monthly forward rates) shown in Figure 4.9.

Figure 4.7: Discount function, yield and forward curves for estimation with 8 B-splines. The dot is the exact yield of the first bond.

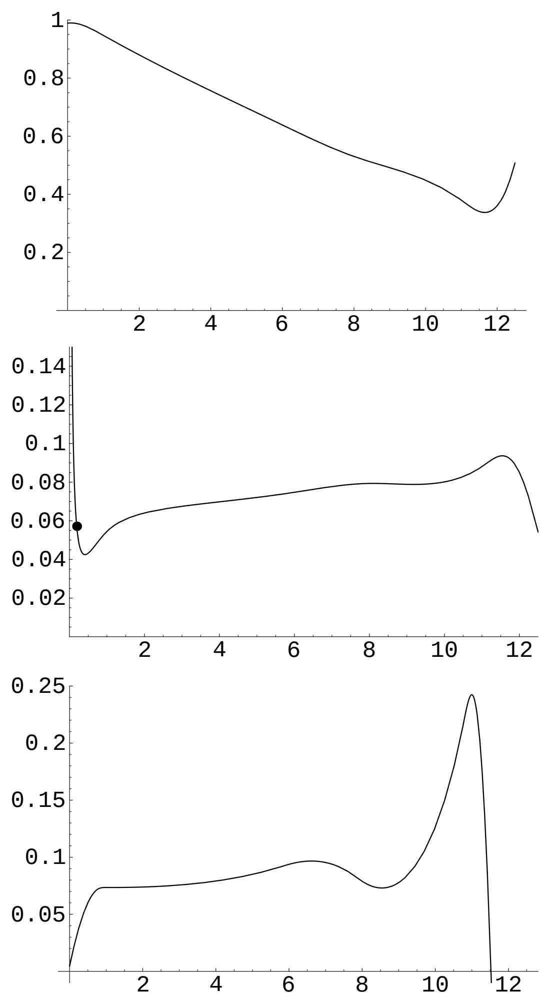


Figure 4.8: Discount function, yield and forward curves for estimation with 7 B-splines. The dot is the exact yield of the first bond.

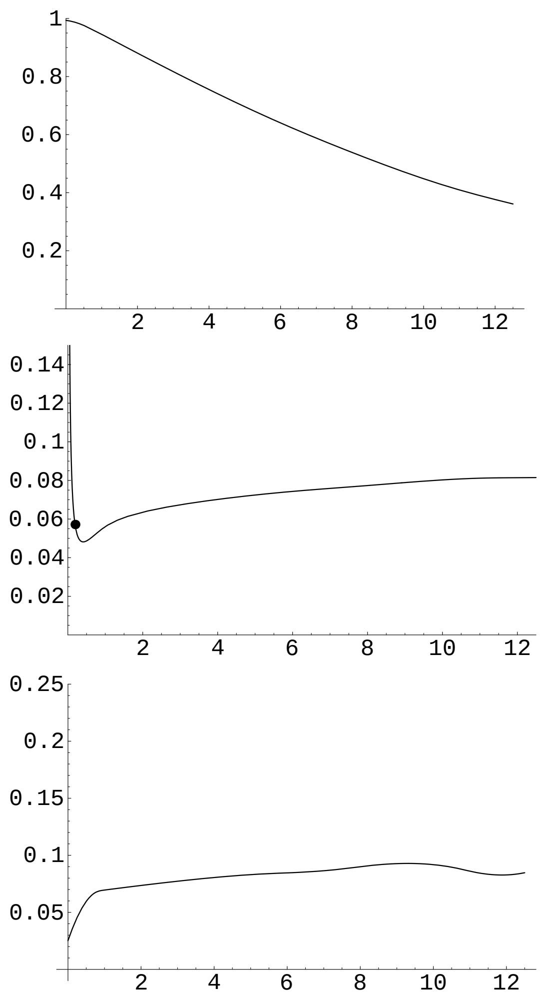
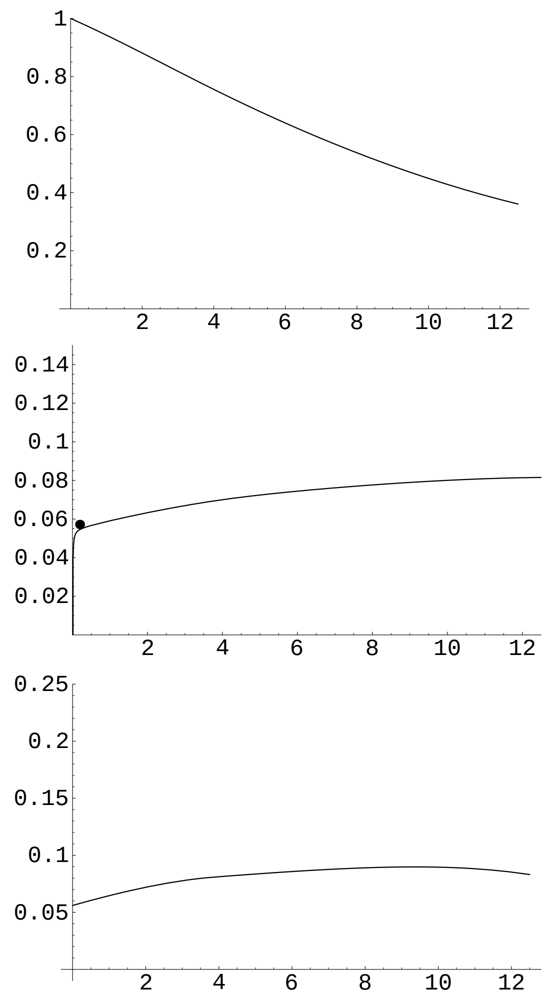


Figure 4.9: Discount function, yield and forward curves for estimation with 5 B-splines. The dot is the exact yield of the first bond.



Discussion

- In general, splines can produce bad fits.
- Estimating the discount function leads to unstable and non-smooth yield and forward curves. Problems mostly at short and long term maturities.
- Splines are not useful for extrapolating to long term maturities.
- There is a trade-off between the quality (or regularity) and the correctness of the fit. The curves in Figures 4.8 and 4.9 are more regular than those in Figure 4.7, but their correctness criteria (0.32 and 0.39) are worse than for the fit with 8 B-splines (0.23).
- The B-spline fits are extremely sensitive to the number and location of the knot points.

→ Need criterions asserting smooth yield and forward curves that do not fluctuate too much and flatten towards the long end.

→ Direct estimation of the yield or forward curve.

→ Optimal selection of number and location of knot points for splines.

→ Smoothing splines.

Smoothing Splines The least squares criterion

$$\min_z \|p - C \cdot d(z)\|^2$$

has to be replaced/extended by criterions for the smoothness of the yield or forward curve.

Example: Lorimier (95). In her PhD thesis 1995, Sabine Lorimier suggests a spline method where the number and location of the knots are determined by the observed data itself.

For ease of notation we set $t_0 = 0$ (today). The data is given by N observed zero-coupon bonds $P(0, T_1), \ldots, P(0, T_N)$ at $0 < T_1 < \cdots < T_N \equiv T$, and consequently the N yields

$$Y_1, \ldots, Y_N, \quad P(0, T_i) = \exp(-T_i Y_i).$$

Let $f(u)$ denote the forward curve. The fitting requirement now is for the forward curve

$$\int_0^{T_i} f(u)\,du + \epsilon_i/\sqrt{\alpha} = T_i Y_i, \tag{4.1}$$

with an arbitrary constant $\alpha > 0$. The aim is to minimize $\|\epsilon\|^2$ as well as the smoothness criterion

$$\int_0^T (f'(u))^2\,du. \tag{4.2}$$

Introduce the Sobolev space

$$H = \{g \mid g' \in L^2[0,T]\}$$

with scalar product

$$\langle g, h\rangle_H = g(0)h(0) + \int_0^T g'(u)h'(u)\,du,$$

and the nonlinear functional on H

$$F(f) := \left[\int_0^T (f'(u))^2\,du + \alpha \sum_{i=1}^N \left(Y_i T_i - \int_0^{T_i} f(u)\,du\right)^2\right].$$

The optimization problem then is

$$\min_{f\in H} F(f). \tag{*}$$

The parameter α tunes the trade-off between smoothness and correctness of the fit.

Theorem 4.2.1. *Problem (*) has a unique solution f, which is a second order spline characterized by*

$$f(u) = f(0) + \sum_{k=1}^N a_k h_k(u) \tag{4.3}$$

where $h_k \in C^1[0,T]$ is a second order polynomial on $[0,T_k]$ with

$$h_k'(u) = (T_k - u)^+, \quad h_k(0) = T_k, \quad k = 1,\dots,N, \tag{4.4}$$

and $f(0)$ and a_k solve the linear system of equations

$$\sum_{k=1}^{N} a_k T_k = 0, \tag{4.5}$$

$$\alpha \left(Y_k T_k - f(0) T_k - \sum_{l=1}^{N} a_l \langle h_l, h_k \rangle_H \right) = a_k, \quad k = 1, \ldots, N. \tag{4.6}$$

Proof. Integration by parts yields

$$\begin{aligned} \int_0^{T_k} g(u)\,du &= T_k g(T_k) - \int_0^{T_k} u g'(u)\,du \\ &= T_k g(0) + T_k \int_0^{T_k} g'(u)\,du - \int_0^{T_k} u g'(u)\,du \\ &= T_k g(0) + \int_0^{T} (T_k - u)^+ g'(u)\,du = \langle h_k, g \rangle_H, \end{aligned}$$

for all $g \in H$. In particular,

$$\int_0^{T_k} h_l\,du = \langle h_l, h_k \rangle_H.$$

A (local) minimizer f of F satisfies

$$\frac{d}{d\epsilon} F(f + \epsilon g)|_{\epsilon=0} = 0$$

or equivalently

$$\int_0^T f' g'\,du = \alpha \sum_{k=1}^{N} \left(Y_k T_k - \int_0^{T_k} f\,du \right) \int_0^{T_k} g\,du, \quad \forall g \in H. \tag{4.7}$$

In particular, for all $g \in H$ with $\langle g, h_k \rangle_H = 0$ we obtain

$$\langle f - f(0), g \rangle_H = \int_0^T f'(u) g'(u)\,du = 0.$$

Hence

$$f - f(0) \in span\{h_1, \ldots, h_N\}$$

what proves (4.3), (4.4) and (4.5) (set $u = 0$). Hence we have

$$\int_0^T f'(u)g'(u)\,du = \sum_{k=1}^N a_k \left(-T_k g(0) + \int_0^{T_k} g(u)\,du\right) = \sum_{k=1}^N a_k \int_0^{T_k} g(u)\,du,$$

and (4.7) can be rewritten as

$$\sum_{k=1}^N \left(a_k - \alpha\left(Y_k T_k - f(0)T_k - \sum_{l=1}^N a_l \langle h_l, h_k\rangle_H\right)\right) \int_0^{T_k} g(u)\,du = 0$$

for all $g \in H$. This is true if and only if (4.6) holds.

Thus we have shown that (4.7) is equivalent to (4.3)–(4.6).

Next we show that (4.7) is a sufficient condition for f to be a global minimizer of F. Let $g \in H$, then

$$\begin{aligned} F(g) &= \int_0^T \left((g' - f') + f'\right)^2\,du + \alpha \sum_{k=1}^N \left(Y_k T_k - \int_0^{T_k} g\,du\right)^2 \\ &\overset{(4.7)}{=} F(f) + \int_0^T (g' - f')^2\,du + \alpha \sum_{k=1}^N \left(\int_0^{T_k} f\,du - \int_0^{T_k} g\,du\right)^2 \\ &\geq F(f), \end{aligned}$$

where we used (4.7) with $g - f \in H$.

It remains to show that f exists and is unique; that is, that the linear system (4.5)–(4.6) has a unique solution $(f(0), a_1, \ldots, a_N)$. The corresponding $(N+1) \times (N+1)$ matrix is

$$A = \begin{pmatrix} 0 & T_1 & T_2 & \cdots & T_N \\ \alpha T_1 & \alpha\langle h_1, h_1\rangle_H + 1 & \alpha\langle h_1, h_2\rangle_H & \cdots & \alpha\langle h_1, h_N\rangle_H \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \alpha T_N & \alpha\langle h_N, h_1\rangle_H & \alpha\langle h_N, h_2\rangle_H & \cdots & \alpha\langle h_N, h_N\rangle_H + 1 \end{pmatrix}. \tag{4.8}$$

Let $\lambda = (\lambda_0, \ldots, \lambda_N)^T \in \mathbb{R}^{N+1}$ such that $A\lambda = 0$, that is,

$$\sum_{k=1}^N T_k \lambda_k = 0$$

$$\alpha T_k \lambda_0 + \alpha \sum_{l=1}^N \langle h_k, h_l\rangle_H \lambda_l + \lambda_k = 0, \quad k = 1, \ldots, N.$$

Multiplying the latter equation with λ_k and summing up yields

$$\alpha \left\| \sum_{k=1}^{N} \lambda_k h_k \right\|_H^2 + \sum_{k=1}^{N} \lambda_k^2 = 0.$$

Hence $\lambda = 0$, whence A is non-singular. □

The role of α is as follows:

- If $\alpha \to 0$ then by (4.3) and (4.6) we have $f(u) \equiv f(0)$, a constant function. That is, maximal regularity

  $$\int_0^T (f'(u))^2 \, du = 0$$

  but no fitting of data, see (4.1).

- If $\alpha \to \infty$ then (4.7) implies that

  $$\int_0^{T_k} f(u) \, du = Y_k T_k, \quad k = 1, \ldots, N, \tag{4.9}$$

  a perfect fit. That is, f minimizes (4.2) subject to the constraints (4.9).

To estimate the forward curve from N zero-coupon bonds—that is, yields $Y = (Y_1, \ldots, Y_N)^T$—one has to solve the linear system

$$A \cdot \begin{pmatrix} f(0) \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ Y \end{pmatrix} \quad \text{(see (4.8))}.$$

Of course, if coupon bond prices are given, then the above method has to be modified and becomes nonlinear. With $p \in \mathbb{R}^n$ denoting the market price vector and c_{kl} the cashflows at dates T_l, $k = 1, \ldots, n$, $l = 1, \ldots, N$, this reads

$$\min_{f \in H} \left\{ \int_0^T (f')^2 \, du + \alpha \sum_{k=1}^{n} \left(\log p_k - \log \left[\sum_{l=1}^{N} c_{kl} \exp \left[- \int_0^{T_l} f \, du \right] \right] \right)^2 \right\}.$$

If the coupon payments are small compared to the nominal (=1), then this problem has a unique solution. This and much more is carried out in Lorimier's thesis.

Exponential-Polynomial Families

Exponential-polynomial functions

$$p_1(x)e^{-\alpha_1 x} + \cdots + p_n(x)e^{-\alpha_n x} \quad (p_i\text{=polynomial of degree } n_i)$$

form non-linear families of functions. Popular examples are:

Nelson–Siegel (87) [17] There are 4 parameters $z_1, \ldots, z_4$ and

$$\phi_{NS}(x; z) = z_1 + (z_2 + z_3 x)e^{-z_4 x}.$$

Svensson (94) [21] (Prof. L. E. O. Svensson is at the Economics Department, Princeton University) This is an extension of Nelson–Siegel, including 6 parameters $z_1, \ldots, z_6$,

$$\phi_S(x; z) = z_1 + (z_2 + z_3 x)e^{-z_4 x} + z_5 e^{-z_6 x}.$$

Figure 4.10: Nelson–Siegel curves for $z_1 = 7.69$, $z_2 = -4.13$, $z_4 = 0.5$ and 7 different values for $z_3 = 1.76$, 0.77, -0.22, -1.21, -2.2, -3.19, -4.18.

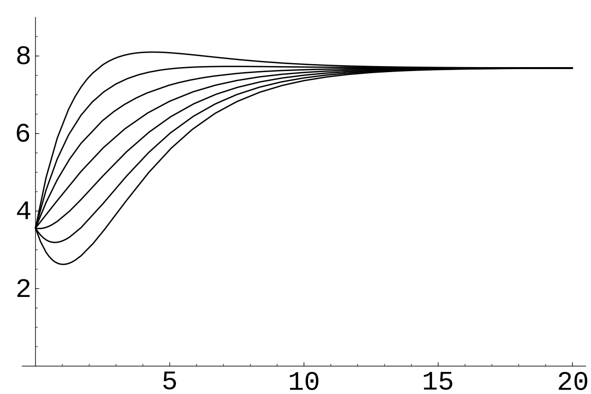


Table 4.4 is taken from a document of the Bank for International Settlements (BIS) 1999 [2].

Table 4.4: Overview of estimation procedures by several central banks. BIS 1999 [2]. NS is for Nelson–Siegel, S for Svensson, wp for weighted prices.

Central Bank	Method	Minimized Error
Belgium	S or NS	wp
Canada	S	sp
Finland	NS	wp
France	S or NS	wp
Germany	S	yields
Italy	NS	wp
Japan	smoothing splines	prices
Norway	S	yields
Spain	S	wp
Sweden	S	yields
UK	S	yields
USA	smoothing splines	bills: wp bonds: prices

Criteria for Curve Families

- Flexibility (do the curves fit a wide range of term structures?)
- Number of factors not too large (curse of dimensionality).
- Regularity (smooth yield or forward curves that flatten out towards the long end).
- Consistency: do the curve families go well with interest rate models? $\rightarrow$ this point will be exploited in the sequel.

Chapter 5

Why Yield Curve Models?

→ R[19](Chapter 5)

Why modelling the entire term structure of interest rates? There is no need when pricing a single European call option on a bond.

But: the payoffs even of "plain-vanilla" fixed income products such as caps, floors, swaptions consist of a sequence of cashflows at $T_1, \dots, T_n$, where n may be 20 (e.g. a 10y swap with semi-annual payments) or more.

→ The valuation of such products requires the modelling of the entire covariance structure. Historical estimation of such large covariance matrices is statistically not tractable anymore.

→ Need strong structure to be imposed on the co-movements of financial quantities of interest.

→ Specify the dynamics of a small number of variables (e.g. PCA).

→ Correlation structure among observable quantities can now be obtained analytically or numerically.

→ Simultaneous pricing of different options and hedging instruments in a consistent framework.

This is exactly what interest rate (curve) models offer:

- reduction of fitting degrees of freedom → makes problem manageable.

$\Longrightarrow$ It is practically and intellectually rewarding to consider no-arbitrage conditions in much broader generality.

Chapter 6

No-Arbitrage Pricing

This chapter briefly recalls the basics about pricing and hedging in a Brownian motion driven market. Reference is B[3], MR[16](Chapter 10), and many more.

6.1 Self-Financing Portfolios

The stochastic basis is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a d-dimensional Brownian motion $W = (W_1, \dots, W_d)$, and the filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by W. We shall assume that $\mathcal{F} = \mathcal{F}_\infty = \vee_{t\geq 0}\mathcal{F}_t$, and do not a priori fix a finite time horizon. This is not a restriction since always one can set a stochastic process to be zero after a finite time T if this were the ultimate time horizon (as in the Black–Scholes model).

The background for stochastic analysis can be found in many textbooks, such as [13], [**?**], [20], etc. From time to time we recall some of the fundamental results without proof.

Financial Market We consider a financial market with n traded assets, following strictly positive Itô processes

$$dS_i(t) = S_i(t)\mu_i(t)\,dt + \sum_{j=1}^{d} S_i(t)\sigma_{ij}(t)\,dW_j(t), \quad S_i > 0, \quad i = 1, \dots, n$$

and the risk-free asset

$$dS_0(t) = r(t)S_0(t)\,dt, \quad S_0(0) = 1 \quad \left(\Leftrightarrow S_0(t) = e^{\int_0^t r(s)\,ds}\right).$$

The drift $\mu = (\mu_1, \dots, \mu_n)$, volatility $\sigma = (\sigma_{ij})$, and short rates r are assumed to form adapted processes which meet the required integrability conditions such that all of the above (stochastic) integrals are well-defined.

Remark 6.1.1. *It is always understood that for a random variable "$X \geq 0$" means "$X \geq 0$ a.s." (that is, $\mathbb{P}[X \geq 0] = 1$), etc.*

Theorem 6.1.2 (Stochastic Integrals). *Let $h = (h_1, \dots, h_d)$ be a measurable adapted process. If*

$$\int_0^t \|h(s)\|^2 \, ds < \infty \quad \text{for all } t > 0$$

(the class of such processes is denoted by $\mathcal{L}$) one can define the stochastic integral

$$(h \cdot W)_t \equiv \int_0^t h(s) \, dW(s) \equiv \sum_{j=1}^d \int_0^t h_j(s) \, dW_j(s).$$

If moreover

$$\mathbb{E}\left[\int_0^\infty \|h(s)\|^2 \, ds\right] < \infty$$

(the class of such processes is denoted by $\mathcal{L}^2$) then $h \cdot W$ is a martingale and the Itô isometry *holds*

$$\mathbb{E}\left[\left(\int_0^t h(s) \, dW(s)\right)^2\right] = \mathbb{E}\left[\int_0^t \|h(s)\|^2 \, ds\right].$$

Self-financing Portfolios A *portfolio*, or trading *strategy*, is any adapted process

$$\phi = (\phi_0, \dots, \phi_n).$$

Its corresponding *value process* is

$$V(t) = V(t; \phi) := \sum_{i=0}^n \phi_i(t) S_i(t).$$

The portfolio ϕ is called *self-financing* (for S) if the stochastic integrals

$$\int_0^t \phi_i(u) \, dS_i(u), \quad i = 0, \dots, n$$

are well defined and

$$dV(t;\phi) = \sum_{i=0}^{n} \phi_i(t)\, dS_i(t).$$

Numeraires All prices are interpreted as being given in terms of a *numeraire*, which typically is a local currency such as US dollars. But we may and will express from time to time the prices in terms of other numeraires, such as S_p for some $0 \le p \le n$. The *discounted price process* vector

$$Z(t) := \frac{S(t)}{S_p(t)}$$

implies the *discounted value process*

$$\tilde{V}(t;\phi) := \sum_{i=0}^{n} \phi_i(t) Z_i(t) = \frac{V(t;\phi)}{S_p(t)}.$$

Up to integrability, the self-financing property does not depend on the choice of the numeraire.

Lemma 6.1.3. *Suppose that a portfolio ϕ satisfies the integrability conditions for S and Z. Then ϕ is self-financing for S if and only if it is self-financing for Z, in particular*

$$d\tilde{V}(t;\phi) = \sum_{i=0}^{n} \phi_i(t)\, dZ_i(t) = \sum_{\substack{i=0 \\ i \neq p}}^{n} \phi_i(t)\, dZ_i(t). \tag{6.1}$$

Since Z_p is constant, the number of terms in (6.1) reduces to n.

Often (but not always) we chose S_0 as the numeraire.

6.2 Arbitrage and Martingale Measures

Contingent Claims Related to any option (such as a cap, floor, swaption, etc) is an uncertain future payoff, say at date T, hence an $\mathcal{F}_T$-measurable random variable X (a *contingent (T-)claim*). Two main problems now are:

- What is a “fair” price for a contingent claim X?
- How can one hedge against the financial risk involved in trading contingent claims?

Arbitrage An *arbitrage portfolio* is a self-financing portfolio ϕ with value process satisfying

$$V(0) = 0 \quad \text{and} \quad V(T) \geq 0 \quad \text{and} \quad \mathbb{P}[V(T) > 0] > 0$$

for some $T > 0$. If no arbitrage portfolios exist for any $T > 0$ we say the model is *arbitrage-free*.

An example of arbitrage is the following.

Lemma 6.2.1. *Suppose there exists a self-financing portfolio with value process*

$$dU(t) = k(t)U(t)\,dt,$$

for some measurable adapted process k. If the market is arbitrage-free then necessarily

$$r = k, \quad dt \otimes d\mathbb{P}\text{-}a.s.$$

Proof. Indeed, after discounting with S_0 we obtain

$$\tilde{U}(t) := \frac{U(t)}{S_0(t)} = U(0) \exp\left(\int_0^t (k(s) - r(s))\,ds \right).$$

Then ($\rightarrow$ exercise)

$$\psi(t) := 1_{\{k(t) > r(t)\}}$$

yields a self-financing strategy with discounted value process

$$\tilde{V}(t) = \int_0^t \psi(s)\,d\tilde{U}(s) = \int_0^t \left(1_{\{k(s)>r(s)\}}(k(s) - r(s))\tilde{U}(s) \right) ds \geq 0.$$

Hence absence of arbitrage requires

$$0 = \mathbb{E}[\tilde{V}(T)] = \int_{\mathcal{N}} \underbrace{\left(1_{\{k(t,\omega)>r(t,\omega)\}}(k(t,\omega) - r(t,\omega))\tilde{U}(t,\omega) \right)}_{>0 \text{ on } \mathcal{N}} dt \otimes d\mathbb{P}$$

where

$$\mathcal{N} := \{(t,\omega) \mid k(t,\omega) > r(t,\omega)\}$$

is a measurable subset of $[0,T] \times \Omega$. But this can only hold if $\mathcal{N}$ is a $dt \otimes d\mathbb{P}$-nullset. Using the same arguments with changed signs proves the lemma. $\square$

Martingale Measures We now investigate when a given model is arbitrage-free. To simplify things in the sequel

- we fix S_0 as a numeraire, and
- $\tilde{V}$ will express the discounted value process V/S_0.

But the following can be made valid for any choice of numeraire.

An equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$ is called an *equivalent (local) martingale measure (E(L)MM)* if the discounted price processes

$$Z_i = S_i/S_0 \text{ are } \mathbb{Q}\text{-(local) martingales.}$$

Theorem 6.2.2 (Girsanov's Change of Measure Theorem). *Let* $\mathbb{Q} \sim \mathbb{P}$ *be an equivalent probability measure. Then there exists* $\gamma \in \mathcal{L}$ *such that the* density process $d\mathbb{Q}/d\mathbb{P}$ *is the* stochastic exponential $\mathcal{E}(\gamma \cdot W)$ *of* $\gamma \cdot W$

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = \mathcal{E}_t\left(\gamma \cdot W\right) := \exp\left(\int_0^t \gamma(s)\, dW(s) - \frac{1}{2}\int_0^t \|\gamma(s)\|^2\, ds\right). \tag{6.2}$$

Moreover, the process

$$\tilde{W}(t) := W(t) - \int_0^t \gamma(s)\, ds \tag{6.3}$$

is a $\mathbb{Q}$*-Brownian motion.*

Conversely, if $\gamma \in \mathcal{L}$ *is such that* $\mathcal{E}\left(\gamma \cdot W\right)$ *is a uniformly integrable martingale with* $\mathcal{E}_\infty(\gamma \cdot W) > 0$ *— sufficient is the* Novikov condition

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^\infty \|\gamma(s)\|^2\, ds\right)\right] < \infty \tag{6.4}$$

(see [20, Proposition (1.26), Chapter IV]) — then (6.2) *defines an equivalent probability measure* $\mathbb{Q} \sim \mathbb{P}$*.*

Market Price of Risk Let $\mathbb{Q}$ be an ELMM and γ (the *stochastic logarithm* of the density process) and $\tilde{W}$ given by (6.2) and (6.3). Integration by parts yields the Z-dynamics

$$\begin{aligned} dZ_i(t) &= Z_i(t)\left(\mu_i(t) - r(t)\right) dt + Z_i(t)\sigma_i(t)\, dW(t) \\ &= Z_i(t)\left(\mu_i(t) - r(t) + \sigma_i(t)\cdot\gamma(t)\right) dt + Z_i(t)\sigma_i(t)\, d\tilde{W}(t). \end{aligned}$$

Hence necessarily γ satisfies

$$\mu_i - r + \sigma_i \cdot \gamma = 0 \quad dt \otimes d\mathbb{Q}\text{-a.s. for all } i = 1, \ldots, n. \tag{6.5}$$

If σ is non-degenerate (in particular $d \leq n$ and $rank[\sigma] = d$) then γ is uniquely specified by

$$-\gamma = \sigma^{-1} \cdot (\mu - r\mathbf{1})$$

where $\mathbf{1} := (1, \ldots, 1)^T$, and vice versa. This is why $-\gamma$ is called the *market price of risk*.

Conversely, if (6.5) has a solution $\gamma \in \mathcal{L}$ such that $\mathcal{E}(\gamma \cdot W)$ is a uniformly integrable martingale (the Novikov condition (6.4) is sufficient) then (6.2) defines an ELMM $\mathbb{Q}$. If γ is unique then $\mathbb{Q}$ is the unique ELMM.

Notice that, by Itô's formula, Z_i can be written as stochastic exponential

$$Z_i = \mathcal{E}(\sigma_i \cdot \tilde{W}).$$

Hence if σ_i satisfies the Novikov condition (6.4) for all $i = 1, \ldots, n$ then the ELMM $\mathbb{Q}$ is in fact an EMM.

Admissible Strategies In the presence of local martingales one has to be alert to pitfalls. For example it is possible to construct a local martingale M with $M(0) = 0$ and $M(1) = 1$. Even worse, M can be chosen to be of the form

$$M(t) = \int_0^t \phi(s)\, dW(s)$$

(Dudley's Representation Theorem), which looks like the (discounted) value process of a self-financing strategy. This would certainly be a money-making machine, say arbitrage. In the same way "suicide strategies" (e.g. $M(0) = 1$ and $M(1) = 0$) can be constructed. To rule out such examples we have to impose additional constraints on the choice of strategies. There are several ways to do so. Here are two typical examples:

A self-financing strategy ϕ is *admissible* if

1. $\tilde{V}(t; \phi) \geq -a$ for some $a \in \mathbb{R}$, OR
2. $\tilde{V}(t; \phi)$ is a true $\mathbb{Q}$-martingale, for some ELMM $\mathbb{Q}$.

Condition 1 is more universal (it does not depend on a particular $\mathbb{Q}$) and implies that $V(t; \phi)$ is a $\mathbb{Q}$-supermartingale for every ELMM $\mathbb{Q}$. Yet, "suicide strategies" remain (however, they do not introduce arbitrage).

Both conditions 1 and 2, however, are sensitive with respect to the choice of numeraire!

The Fundamental Theorem of Asset Pricing The existence of an ELMM rules out arbitrage.

Lemma 6.2.3. *Suppose there exists an ELMM $\mathbb{Q}$. Then the model is arbitrage-free, in the sense that there exists no admissible (either Condition 1 or 2) arbitrage strategy.*

Proof. Indeed, let $\tilde{V}$ be the discounted value process of an admissible strategy, with $\tilde{V}(0) = 0$ and $\tilde{V}(T) \geq 0$. Since $\tilde{V}$ is a $\mathbb{Q}$-supermartingale in any case (for some ELMM $\mathbb{Q}$), we have

$$0 \leq \mathbb{E}_{\mathbb{Q}}[\tilde{V}(T)] \leq \tilde{V}(0) = 0,$$

whence $\tilde{V}(T) = 0$. □

It is folklore (Delbaen and Schachermayer 1994, etc) that also the converse holds true: if arbitrage is defined in the right way ("No Free Lunch with Vanishing Risk"), then its absence implies the existence of an ELMM $\mathbb{Q}$. This is called the *Fundamental Theorem of Asset Pricing*.

It has become a custom (and we will follow this tradition) to consider the existence of an ELMM as synonym for the absence of arbitrage:

absence of arbitrage = existence of an ELMM;

→ the existence of an ELMM is now a standing assumption.

6.3 Hedging and Pricing

Attainable Claims A contingent claim X due at T is *attainable* if the exists an admissible strategy ϕ which *replicates/hedges* X; that is,

$$V(T; \phi) = X.$$

A simple example: suppose S_1 is the price process of the T-bond. Then the contingent claim $X = 1$ due at T is attainable by an obvious buy and hold strategy with value process $V(t) = S_1(t)$.

Complete Markets The main problem is to determine which claims are attainable. This is most conveniently carried out in terms of discounted prices.

Suppose that σ is non-degenerate; that is

$$d \leq n \quad \text{and} \quad rank[\sigma] = d, \tag{6.6}$$

and that the unique market price of risk $-\gamma$ given by (6.5) yields a uniformly integrable martingale $\mathcal{E}(\gamma \cdot W)$ and hence a unique ELMM $\mathbb{Q}$.

Lemma 6.3.1. *Then the model is* complete *in the sense that any contingent claim X with*

$$X/S_0(T) \in L^1(\mathcal{F}_T; \mathbb{Q}) \tag{6.7}$$

is attainable.

Proof. Define the $\mathbb{Q}$-martingale

$$Y(t) := \mathbb{E}_{\mathbb{Q}}\left[X/S_0(T) \mid \mathcal{F}_t\right], \quad t \in [0,T].$$

Then

$$Y(t)D(t) = D(t)\mathbb{E}_{\mathbb{Q}}[Y(T) \mid \mathcal{F}_t] \overset{Bayes}{=} \mathbb{E}[Y(T)D(T) \mid \mathcal{F}_t],$$

with the density process $D(t) = d\mathbb{Q}/d\mathbb{P}|_{\mathcal{F}_t} = \mathcal{E}_t(\gamma \cdot W)$. Hence YD is a $\mathbb{P}$-martingale and by the representation theorem 6.3.3 we can find $\psi \in \mathcal{L}$ such that

$$Y(t)D(t) = Y(0) + \int_0^t \psi(s)\, dW(s).$$

Applying Itô's formula yields

$$d\left(\frac{1}{D}\right) = -\frac{1}{D}\gamma\, dW + \frac{1}{D}\|\gamma\|^2\, dt,$$

and

$$\begin{aligned} dY &= d\left((YD)\frac{1}{D}\right) = YD\, d\left(\frac{1}{D}\right) + \frac{1}{D}\, d(YD) + d\left\langle YD, \frac{1}{D}\right\rangle \\ &= \left(\frac{1}{D}\psi - Y\gamma\right) dW - \left(\frac{1}{D}\psi - Y\gamma\right) \cdot \gamma\, dt \\ &= \underbrace{\left(\frac{1}{D}\psi - Y\gamma\right)}_{=:\tilde{\psi}} d\tilde{W}. \end{aligned}$$

Now define

$$\phi_i = \frac{((\sigma^{-1})^T \tilde{\psi})_i}{Z_i}, \tag{6.8}$$

then it follows that

$$\sum_{i=1}^{n} \phi_i \, dZ_i = \sum_{i=1}^{n} \phi_i Z_i \sigma_i \, d\tilde{W} = (\sigma^{-1})^T \tilde{\psi} \cdot \sigma \, d\tilde{W} = \tilde{\psi} \cdot \sigma^{-1} \sigma \, d\tilde{W} = \tilde{\psi} \, d\tilde{W} = dY.$$

Hence ϕ yields an admissible strategy with discounted value process satisfying

$$\tilde{V}(T;\phi) = Y(T) = \mathbb{E}_{\mathbb{Q}}\left[X/S_0(T)\right] + \sum_{i=1}^{n} \int_0^T \phi_i(s) \, dZ_i(s) = X/S_0(T). \tag{6.9}$$

□

Hence non-degeneracy of σ (see (6.6) and (6.8)) implies uniqueness of $\mathbb{Q}$ and completeness of the model. These conditions are in fact equivalent (see for example MR[16](Chapter 10)).

Theorem 6.3.2 (Completeness). *The following are equivalent:*

1. *the model is complete;*
2. *σ is non-degenerate, see* (6.6)*;*
3. *there exists a unique ELMM $\mathbb{Q}$.*

Theorem 6.3.3 (Representation Theorem). *Every $\mathbb{P}$-local martingale M has a continuous version and there exists $\psi \in \mathcal{L}$ such that*

$$M(t) = M(0) + \int_0^t \psi(s) \, dW(s).$$

(This theorem requires the filtration $(\mathcal{F}_t)$ to be generated by W.)

Pricing In the above complete model the fair price prevailing at $t \le T$ of a T-claim X which satisfies (6.7) is given by (6.9)

$$V(t,\phi) = S_0(t)\tilde{V}(t;\phi) = S_0(t)\mathbb{E}_{\mathbb{Q}}\left[X/S_0(T) \mid \mathcal{F}_t\right]. \tag{6.10}$$

We shall often encounter complete models. However, models can be generically incomplete (as real markets are), and then the pricing becomes a difficult issue. The literature on incomplete markets is huge, and the topic beyond the scope of this course.

State-price Density It is a custom (e.g. for short rate models) to exogenously specify a particular ELMM $\mathbb{Q}$ (or equivalently, *the* market price of risk) and then price a T-claim X satisfying (6.7) according to (6.10)

$$\text{price of } X \text{ at } t =: Y(t) = S_0(t)\mathbb{E}_{\mathbb{Q}}\left[X/S_0(T) \mid \mathcal{F}_t\right].$$

This is a consistent pricing rule in the sense that the enlarged market

$$Y, S_0, \ldots, S_n$$

is still arbitrage-free (why?).

Now define

$$\pi(t) := \frac{1}{S_0(t)} \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t}.$$

By Bayes formula we then have

$$\begin{aligned} Y(t) = S_0(t)\mathbb{E}_{\mathbb{Q}}\left[X/S_0(T) \mid \mathcal{F}_t\right] &= S_0(t)\frac{\mathbb{E}\left[\frac{X}{S_0(T)}\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_T} \mid \mathcal{F}_t\right]}{\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t}} \\ &= \frac{\mathbb{E}\left[X\pi(T) \mid \mathcal{F}_t\right]}{\pi(t)}, \end{aligned}$$

and, in particular, for the price at $t = 0$

$$Y(0) = \mathbb{E}[X\pi(T)].$$

This is why π is called the *state-price density* process.

The price of a T-bond for example is (if $1/S_0(T) \in L^1(\mathbb{Q})$, $\rightarrow$ exercise)

$$P(t,T) = \mathbb{E}\left[\frac{\pi(T)}{\pi(t)} \mid \mathcal{F}_t\right] = \mathbb{E}_{\mathbb{Q}}\left[\frac{S_0(t)}{S_0(T)} \mid \mathcal{F}_t\right].$$

Also one can check ($\rightarrow$ exercise) that if $\mathbb{Q}$ is an EMM then

$$S_i\pi \quad \text{are } \mathbb{P}\text{-martingales.}$$

Chapter 7

Short Rate Models

$\rightarrow$ B[3](Chapters 16–17), MR[16](Chapter 12), etc

7.1 Generalities

Short rate models are the classical interest rate models. As in the last section we fix a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is considered as objective probability measure. The filtration $(\mathcal{F}_t)_{t\geq 0}$ is generated by a d-dimensional Brownian motion W.

We assume that

- the short rates follow an Itô process

$$dr(t) = b(t)\,dt + \sigma(t)\,dW(t)$$

  determining the savings account $B(t) = \exp\left(\int_0^t r(s)\,ds\right)$,

- all zero-coupon bond prices $(P(t,T))_{t\in[0,T]}$ are adapted processes (with $P(T,T) = 1$ as usual),

- no-arbitrage: there exists an EMM $\mathbb{Q}$, such that

$$\frac{P(t,T)}{B(t)}, \quad t \in [0,T],$$

  is a $\mathbb{Q}$-martingale for all $T > 0$.

According to the last chapter, the existence of an ELMM for *all* T-bonds excludes arbitrage among every *finite selection* of zero-coupon bonds, say $P(t,T_1),\dots,P(t,T_n)$. To be more general one would have to consider strategies involving a continuum of bonds. This can be done (see [4] or Mike Tehranchi's PhD thesis 2002) but is beyond the scope of this course.

For convenience we require $\mathbb{Q}$ to be an EMM (and not merely an ELMM) because then we have

$$P(t,T) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(s)\,ds} \mid \mathcal{F}_t\right] \tag{7.1}$$

(compare this to the last section). Let $-\gamma$ denote the corresponding market price of risk

$$\mathcal{E}_t(\gamma \cdot W) = \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t}$$

and $\tilde{W} = W - \int \gamma\,dt$ the implied $\mathbb{Q}$-Brownian motion.

Proposition 7.1.1. *Under the above assumptions, the process r satisfies under $\mathbb{Q}$*

$$dr(t) = (b(t) + \sigma(t)\cdot\gamma(t))\;dt + \sigma(t)\,d\tilde{W}(t). \tag{7.2}$$

Moreover, for any $T>0$ there exists an adapted $\mathbb{R}^d$-valued process $\sigma^\gamma(t,T)$, $t\in[0,T]$, such that

$$\frac{dP(t,T)}{P(t,T)} = r(t)\,dt + \sigma^\gamma(t,T)\,d\tilde{W}(t) \tag{7.3}$$

and hence

$$\frac{P(t,T)}{B(t)} = P(0,T)\mathcal{E}_t\left(\sigma^\gamma\cdot\tilde{W}\right).$$

Proof. Exercise (proceed as in the Completeness Lemma 6.3.1). □

It follows from (7.3) that the T-bond price satisfies under the objective probability measure $\mathbb{P}$

$$\frac{dP(t,T)}{P(t,T)} = (r(t) - \gamma(t)\cdot\sigma^\gamma(t,T))\;dt + \sigma^\gamma\,dW(t).$$

This illustrates again the role of the market price of risk $-\gamma$ as the excess of instantaneous return over $r(t)$ in units of volatility.

In a general equilibrium framework, the market price of risk is given endogenously (as it is carried out in the seminal paper by Cox, Ingersoll and Ross (85) [7]). Since our arguments refer only to the absence of arbitrage between primary securities (bonds) and derivatives, we are unable to identify the market price of risk. In other words, we started by specifying the $\mathbb{P}$-dynamics of the short rates, and hence the savings account $B(t)$. However, the savings account alone cannot be used to replicate bond payoffs: the model is incomplete. According to the Completeness Theorem 6.3.2, this is also reflected by the non-uniqueness of the EMM (the market price of risk). A priori, $\mathbb{Q}$ can be any equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$.

> A short rate model is not fully determined without the exogenous specification of the market price of risk.

It is custom (and we follow this tradition) to postulate the $\mathbb{Q}$-dynamics ($\mathbb{Q}$ being the EMM) of r which implies the $\mathbb{Q}$-dynamics of all bond prices by (7.1), see also (7.3). All contingent claims can be priced by taking $\mathbb{Q}$-expectations of their discounted payoffs. The market price of risk (and hence the objective measure $\mathbb{P}$) can be inferred by statistical methods from historical observations of price movements.

7.2 Diffusion Short Rate Models

We fix a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$, where now $\mathbb{Q}$ is considered as martingale measure. We let W denote a d-dimensional $(\mathbb{Q}, \mathcal{F}_t)$-Brownian motion.

Let $\mathcal{Z} \subset \mathbb{R}$ be a closed interval, and b and σ continuous functions on $\mathbb{R}_+ \times \mathcal{Z}$. We assume that for any $\rho \in \mathcal{Z}$ the stochastic differential equation (SDE)

$$dr(t) = b(t, r(t))\, dt + \sigma(t, r(t))\, dW(t) \tag{7.4}$$

admits a unique $\mathcal{Z}$-valued solution $r = r^\rho$ with

$$r(t) = \rho + \int_0^t b(u, r(u))\, du + \int_0^t \sigma(u, r(u))\, dW(u)$$

and such that

$$\exp\left(-\int_t^T r(u)\, du \right) \in L^1(\mathbb{Q}) \tag{7.5}$$

for all $0 \leq t \leq T$. Notice that (7.5) is always satisfied if $\mathcal{Z} \subset \mathbb{R}_+$.

Sufficient for the existence and uniqueness is Lipschitz continuity of $b(t, r)$ and $\sigma(t, r)$ in r, uniformly in t. If $d = 1$ then Hölder continuity of order 1/2 of σ in r, uniformly in t, is enough. A good reference for SDEs is the book of Karatzas and Shreve [13] on Brownian motion and stochastic calculus.

Condition (7.5) allows us to define the T-bond prices

$$P(t, T) = \mathbb{E}_{\mathbb{Q}} \left[\exp \left(-\int_t^T r(u)\, du \right) \mid \mathcal{F}_t \right].$$

It turns out that $P(t, T)$ can be written as a function of $r(t)$, t and T. This is a general property of certain functionals of Markov process, usually referred to as Feynman–Kac formula. In the following we write

$$a(t, r) := \frac{\|\sigma(t, r)\|^2}{2}$$

for the diffusion term of $r(t)$.

Lemma 7.2.1. *Let $T > 0$ and Φ be a continuous function on $\mathcal{Z}$, and assume that $F = F(t, r) \in C^{1,2}([0, T] \times \mathcal{Z})$ is a solution to the boundary value problem on $[0, T] \times \mathcal{Z}$*

$$\begin{cases} \partial_t F(t, r) + b(t, r)\partial_r F(t, r) + a(t, r)\partial_r^2 F(t, r) - rF(t, r) = 0 \\ \hfill F(T, r) = \Phi(r). \end{cases} \tag{7.6}$$

Then

$$M(t) = F(t, r(t))e^{-\int_0^t r(u)\, du}, \quad t \in [0, T],$$

is a local martingale. If in addition either

1. *$\partial_r F(t, r(t))e^{-\int_0^t r(u)\, du}\sigma(t, r(t)) \in \mathcal{L}^2[0, T]$, or*
2. *M is uniformly bounded,*

then M is a true martingale, and

$$F(t, r(t)) = \mathbb{E}_{\mathbb{Q}} \left[\exp \left(-\int_t^T r(u)\, du \right) \Phi(r(T)) \mid \mathcal{F}_t \right], \quad t \leq T. \tag{7.7}$$

Proof. We can apply Itô's formula to M and obtain

$$\begin{aligned} dM(t) = \Big(&\partial_t F(t,r(t)) + b(t,r(t))\partial_r F(t,r(t)) \\ &+ a(t,r)\partial_r^2 F(t,r(t)) - r(t)F(t,r(t))\Big) e^{-\int_0^t r(u)\,du}\,dt \\ &+ \partial_r F(t,r(t)) e^{-\int_0^t r(u)\,du} \sigma(t,r(t))\,dW(t) \\ = {}& \partial_r F(t,r(t)) e^{-\int_0^t r(u)\,du} \sigma(t,r(t))\,dW(t). \end{aligned}$$

Hence M is a local martingale.

It is now clear that either Condition 1 or 2 imply that M is a true martingale. Since

$$M(T) = \Phi(r(T)) e^{-\int_0^T r(u)\,du}$$

we get

$$F(t,r(t)) e^{-\int_0^t r(u)\,du} = M(t) = \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_0^T r(u)\,du\right)\Phi(r(T)) \mid \mathcal{F}_t\right].$$

Multiplying with $e^{\int_0^t r(u)\,du}$ yields the claim. □

We call (7.6) the *term structure equation* for Φ. Its solution F gives the price of the T-claim $\Phi(r(T))$. In particular, for $\Phi \equiv 1$ we get the T-bond price $P(t,T)$ as a function of t, $r(t)$ (and T)

$$P(t,T) = F(t,r(t);T).$$

Remark 7.2.2. *Strictly speaking, we have only shown that* if *a smooth solution F of* (7.6) *exists and satisfies some additional properties (Condition 1 or 2) then the time t price of the claim $\Phi(r(T))$ (which is the right hand side of* (7.7)*) equals $F(t,r(t))$. One can also show the converse that the expectation on the right hand side of* (7.7) conditional on $r(t) = r$ *can be written as $F(t,r)$ where F solves the term structure equation* (7.6) *but usually only in a* weak sense*, which in particular means that F may not be in $C^{1,2}([0,T]\times\mathcal{Z})$. This is general Markov theory and we will not prove this here.*

In any case, we have found a pricing algorithm. Is it computationally efficient? Solving PDEs numerically in more than three dimensions causes difficulties. PDEs in less than three space dimensions are numerically feasible, and the dimension of $\mathcal{Z}$ is one. The nuisance is that we have to solve a

PDE for every single zero-coupon bond price function $F(\cdot,\cdot;T)$, $T > 0$. From that we might want to derive the yield or even forward curve. If we do not impose further structural assumptions we may run into regularity problems. Hence

> short rate models that admit closed form solutions to the term structure equation (7.6), at least for $\Phi \equiv 1$, are favorable.

7.2.1 Examples

This is a (far from complete) list of the most popular short rate models. For all examples we have $d = 1$. If not otherwise stated, the parameters are real-valued.

1. Vasicek (1977): $\mathcal{Z} = \mathbb{R}$,
$$dr(t) = (b + \beta r(t))\,dt + \sigma\,dW(t),$$
2. Cox–Ingersoll–Ross (CIR, 1985): $\mathcal{Z} = \mathbb{R}_+$, $b \geq 0$,
$$dr(t) = (b + \beta r(t))\,dt + \sigma\sqrt{r(t)}\,dW(t),$$
3. Dothan (1978): $\mathcal{Z} = \mathbb{R}_+$,
$$dr(t) = \beta r(t)\,dt + \sigma r(t)\,dW(t),$$
4. Black–Derman–Toy (1990): $\mathcal{Z} = \mathbb{R}_+$,
$$dr(t) = \beta(t)r(t)\,dt + \sigma(t)r(t)\,dW(t),$$
5. Black–Karasinski (1991): $\mathcal{Z} = \mathbb{R}_+$, $\ell(t) = \log r(t)$,
$$d\ell(t) = (b(t) + \beta(t)\ell(t))\,dt + \sigma(t)\,dW(t),$$
6. Ho–Lee (1986): $\mathcal{Z} = \mathbb{R}$,
$$dr(t) = b(t)\,dt + \sigma\,dW(t),$$
7. Hull–White (extended Vasicek, 1990): $\mathcal{Z} = \mathbb{R}$,
$$dr(t) = (b(t) + \beta(t)r(t))\,dt + \sigma(t)\,dW(t),$$
8. Hull–White (extended CIR, 1990): $\mathcal{Z} = \mathbb{R}_+$, $b(t) \geq 0$,
$$dr(t) = (b(t) + \beta(t)r(t))\,dt + \sigma(t)\sqrt{r(t)}\,dW(t).$$

7.3 Inverting the Yield Curve

Once the short rate model is chosen, the initial term structure

$$T \mapsto P(0,T) = F(0,r(0);T)$$

and hence the initial yield and forward curve are fully specified by the term structure equation (7.6).

Conversely, one may want to invert the term structure equation (7.6) to match a given initial yield curve. Say we have chosen the Vasicek model. Then the implied T-bond price is a function of the current short rate level and the three model parameters b, β and σ

$$P(0,T) = F(0,r(0);T,b,\beta,\sigma).$$

But $F(0,r(0);T,b,\beta,\sigma)$ is just a parametrized curve family with three degrees of freedom. It turns out that it is often too restrictive and will provide a poor fit of the current data in terms of accuracy (least squares criterion).

Therefore the class of time-inhomogeneous short rate models (such as the Hull–White extensions) was introduced. By letting the parameters depend on time one gains infinite degree of freedom and hence a perfect fit of any given curve. Usually, the functions $b(t)$ etc are fully determined by the empirical initial yield curve.

7.4 Affine Term Structures

Short rate models that admit closed form expressions for the implied bond prices $F(t,r;T)$ are favorable.

The most tractable models are those where bond prices are of the form

$$F(t,r;T) = \exp(-A(t,T) - B(t,T)r),$$

for some smooth functions A and B. Such models are said to provide an *affine term structure (ATS)*. Notice that $F(T,r;T) = 1$ implies

$$A(T,T) = B(T,T) = 0.$$

The nice thing about ATS models is that they can be completely characterized.

Proposition 7.4.1. *The short rate model* (7.4) *provides an ATS only if its diffusion and drift terms are of the form*

$$a(t,r) = a(t) + \alpha(t)r \quad \text{and} \quad b(t,r) = b(t) + \beta(t)r, \tag{7.8}$$

for some continuous functions a, α, b, β. *The functions* A *and* B *in turn satisfy the system*

$$\partial_t A(t,T) = a(t)B^2(t,T) - b(t)B(t,T), \quad A(T,T) = 0, \tag{7.9}$$

$$\partial_t B(t,T) = \alpha(t)B^2(t,T) - \beta(t)B(t,T) - 1, \quad B(T,T) = 0. \tag{7.10}$$

Proof. We insert $F(t,r;T) = \exp(-A(t,T) - B(t,T)r)$ in the term structure equation (7.6) and obtain

$$a(t,r)B^2(t,T) - b(t,r)B(t,T) = \partial_t A(t,T) + (\partial_t B(t,T) + 1)r. \tag{7.11}$$

The functions $B(t,\cdot)$ and $B^2(t,\cdot)$ are linearly independent since otherwise $B(t,\cdot) \equiv B(t,t) = 0$, which trivially would lead to be above results with $a(t) = \alpha(t) \equiv 0$. Hence we can find $T_1 > T_2 > t$ such that the matrix

$$\begin{pmatrix} B^2(t,T_1) & -B(t,T_1) \\ B^2(t,T_2) & -B(t,T_2) \end{pmatrix}$$

is invertible. Hence we can solve (7.11) for $a(t,r)$ and $b(t,r)$, which yields (7.8). Replace $a(t,r)$ and $b(t,r)$ by (7.8), so the left hand side of (7.11) reads

$$a(t)B^2(t,T) - b(t)B(t,T) + \left(\alpha(t)B^2(t,T) - \beta(t)B(t,T)\right) r.$$

Terms containing r must match. This proves the claim. □

The functions a, α, b, β in (7.8) can be further specified. They have to be such that $a(t,r) \geq 0$ and $r(t)$ does not leave the state space $\mathcal{Z}$. In fact, it can be shown that every ATS model can be transformed via affine transformation into one of the two cases

1. $\mathcal{Z} = \mathbb{R}$: necessarily $\alpha(t) = 0$ and $a(t) \geq 0$, and b, β are arbitrary. This is the (Hull–White extension of the) Vasicek model.

2. $\mathcal{Z} = \mathbb{R}_+$: necessarily $a(t) = 0$, $\alpha(t) \geq 0$ and $b(t) \geq 0$ (otherwise the process would cross zero), and β is arbitrary. This is the (Hull–White extension of the) CIR model.

Looking at the list in Section 7.2.1 we see that all short rate models except the Dothan, Black–Derman–Toy and Black–Karasinski models have an ATS.

7.5 Some Standard Models

We discuss some of the most common short rate models.

→ B[3](Section 17.4), BM[6](Chapter 3)

7.5.1 Vasicek Model

The solution to

$$dr = (b + \beta r)\, dt + \sigma\, dW$$

is explicitly given by (→ exercise)

$$r(t) = r(0)e^{\beta t} + \frac{b}{\beta}\left(e^{\beta t} - 1\right) + \sigma e^{\beta t} \int_0^t e^{-\beta s}\, dW(s).$$

It follows that $r(t)$ is a Gaussian process with mean

$$\mathbb{E}\left[r(t)\right] = r(0)e^{\beta t} + \frac{b}{\beta}\left(e^{\beta t} - 1\right)$$

and variance

$$Var[r(t)] = \sigma^2 e^{2\beta t} \int_0^t e^{-2\beta s}\, ds = \frac{\sigma^2}{2\beta}\left(e^{2\beta t} - 1\right).$$

Hence

$$\mathbb{Q}[r(t) < 0] > 0,$$

which is not satisfactory (although this probability is usually very small).

Vasicek assumed the market price of risk to be constant, so that also the objective $\mathbb{P}$-dynamics of $r(t)$ is of the above form.

If $\beta < 0$ then $r(t)$ is mean-reverting with mean reversion level $b/|\beta|$, see Figure 7.1, and $r(t)$ converges to a Gaussian random variable with mean $b/|\beta|$ and variance $\sigma^2/(2|\beta|)$, for $t \to \infty$.

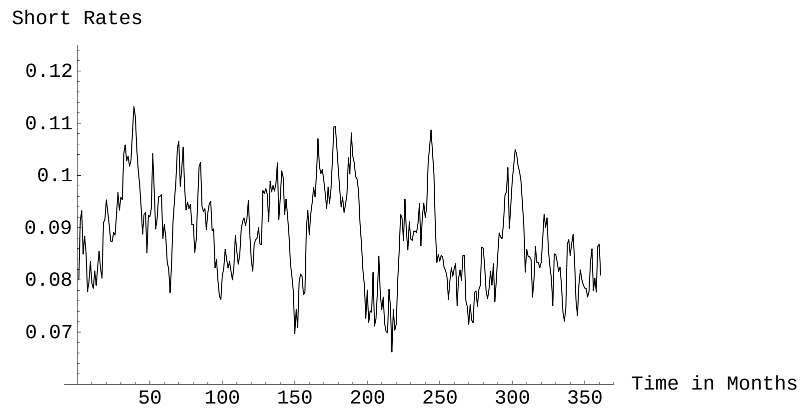
Equations (7.9)–(7.10) become

$$\partial_t A(t,T) = \frac{\sigma^2}{2} B^2(t,T) - bB(t,T), \quad A(T,T) = 0,$$
$$\partial_t B(t,T) = -\beta B(t,T) - 1, \quad B(T,T) = 0.$$

The explicit solution is

$$B(t,T) = \frac{1}{\beta}\left(e^{\beta(T-t)} - 1\right)$$

Figure 7.1: Vasicek short rate process for $\beta = -0.86$, $b/|\beta| = 0.09$ (mean reversion level), $\sigma = 0.0148$ and $r(0) = 0.08$.



and A is given as ordinary integral

$$\begin{aligned} A(t,T) &= A(T,T) - \int_t^T \partial_s A(s,T)\,ds \\ &= -\frac{\sigma^2}{2}\int_t^T B^2(s,T)\,ds + b\int_t^T B(s,T)\,ds \\ &= \frac{\sigma^2\left(4e^{\beta(T-t)} - e^{2\beta(T-t)} - 2\beta(T-t) - 3\right)}{4\beta^3} + b\frac{e^{\beta(T-t)} - 1 - \beta(T-t)}{\beta^2}. \end{aligned}$$

We recall that zero-coupon bond prices are given in closed form by

$$P(t,T) = \exp\left(-A(t,T) - B(t,T)r(t)\right).$$

It is possible to derive closed form expression also for bond options (see Section 7.6).

7.5.2 Cox–Ingersoll–Ross Model

It is worth to mention that, for $b \geq 0$,

$$dr(t) = (b + \beta r(t))\,dt + \sigma\sqrt{r(t)}\,dW(t), \quad r(0) \geq 0,$$

has a unique strong solution $r \geq 0$, for every $r(0) \geq 0$. This also holds when the coefficients depend continuously on t, as it is the case for the Hull–White extension. Even more, if $b \geq \sigma^2/2$ then $r > 0$ whenever $r(0) > 0$.

The ATS equation (7.10) now becomes non-linear

$$\partial_t B(t,T) = \frac{\sigma^2}{2} B^2(t,T) - \beta B(t,T) - 1, \quad B(T,T) = 0.$$

This is called a *Riccati equation*. It is good news that the explicit solution is known

$$B(t,T) = \frac{2\left(e^{\gamma(T-t)} - 1\right)}{(\gamma - \beta)\left(e^{\gamma(T-t)} - 1\right) + 2\gamma}$$

where $\gamma := \sqrt{\beta^2 + 2\sigma^2}$. Integration yields

$$A(t,T) = -\frac{2b}{\sigma^2} \log\left(\frac{2\gamma e^{(\gamma-\beta)(T-t)/2}}{(\gamma - \beta)\left(e^{\gamma(T-t)} - 1\right) + 2\gamma}\right).$$

Hence also in the CIR model we have closed form expressions for the bond prices. Moreover, it can be shown that also bond option prices are explicit(!) Together with the fact that it yields positive interest rates, this is mainly the reason why the CIR model is so popular.

7.5.3 Dothan Model

Dothan (78) starts from a drift-less geometric Brownian motion under the objective probability measure $\mathbb{P}$

$$dr(t) = \sigma r(t)\, dW^{\mathbb{P}}(t).$$

The market price of risk is chosen to be constant, which yields

$$dr(t) = \beta r(t)\, dt + \sigma r(t)\, dW(t)$$

as $\mathbb{Q}$-dynamics. This is easily integrated

$$r(t) = r(s) \exp\left(\left(\beta - \sigma^2/2\right)(t-s) + \sigma(W(t) - W(s))\right), \quad s \leq t.$$

Thus the $\mathcal{F}_s$-conditional distribution of $r(t)$ is lognormal with mean and variance (→ exercise)

$$\mathbb{E}[r(t) \mid \mathcal{F}_s] = r(s) e^{\beta(t-s)}$$
$$Var[r(t) \mid \mathcal{F}_s] = r^2(s) e^{2\beta(t-s)}\left(e^{\sigma^2(t-s)} - 1\right).$$

The Dothan and all lognormal short rate models (Black–Derman–Toy and Black–Karasinski) yield positive interest rates. But no closed form expressions for bond prices or options are available (with one exception: Dothan admits an "semi-explicit" expression for the bond prices, see BM[6]).

A major drawback of lognormal models is the explosion of the bank account. Let Δt be small, then

$$\mathbb{E}[B(\Delta t)] = \mathbb{E}\left[\exp\left(\int_0^{\Delta t} r(s)\,ds\right)\right] \approx \mathbb{E}\left[\exp\left(\frac{r(0)+r(\Delta t)}{2}\Delta t\right)\right].$$

We face an expectation of the type

$$\mathbb{E}[\exp(\exp(Y))]$$

where Y is Gaussian distributed. But such an expectation is infinite. This means that in arbitrarily small time the bank account growths to infinity in average. Similarly, one shows that the price of a Eurodollar future is infinite for all lognormal models.

The idea of lognormal rates is taken up later by Sandmann and Sondermann (1997) and many others, which finally led to the so called market models with lognormal LIBOR or swap rates.

7.5.4 Ho–Lee Model

For the Ho–Lee model

$$dr(t) = b(t)\,dt + \sigma\,dW(t)$$

the ATS equations (7.9)–(7.10) become

$$\begin{aligned}
\partial_t A(t,T) &= \frac{\sigma^2}{2}B^2(t,T) - b(t)B(t,T), \quad A(T,T)=0,\\
\partial_t B(t,T) &= -1, \quad B(T,T)=0.
\end{aligned}$$

Hence

$$\begin{aligned}
B(t,T) &= T-t,\\
A(t,T) &= -\frac{\sigma^2}{6}(T-t)^3 + \int_t^T b(s)(T-s)\,ds.
\end{aligned}$$

The forward curve is thus

$$f(t,T) = \partial_T A(t,T) + \partial_T B(t,T) r(t) = -\frac{\sigma^2}{2}(T-t)^2 + \int_t^T b(s)\,ds + r(t).$$

Let $f^*(0,T)$ be the observed (estimated) initial forward curve. Then

$$b(s) = \partial_s f^*(0,s) + \sigma^2 s.$$

gives a perfect fit of $f^*(0,T)$. Plugging this back into the ATS yields

$$f(t,T) = f^*(0,T) - f^*(0,t) + \sigma^2 t(T-t) + r(t).$$

We can also integrate this expression to get

$$P(t,T) = e^{-\int_t^T f^*(0,s)\,ds + f^*(0,t)(T-t) - \frac{\sigma^2}{2}t(T-t)^2 - (T-t)r(t)}.$$

It is interesting to see that

$$r(t) = r(0) + \int_0^t b(s)\,ds + \sigma W(t) = f^*(0,t) + \frac{\sigma^2 t^2}{2} + \sigma W(t).$$

That is, $r(t)$ fluctuates along the modified initial forward curve, and we have

$$f^*(0,t) = \mathbb{E}[r(t)] - \frac{\sigma^2 t^2}{2}.$$

7.5.5 Hull–White Model

The Hull–White (1990) extensions of Vasicek and CIR can be fitted to the initial yield and volatility curve. However, this flexibility has its price: the model cannot be handled analytically in general. We therefore restrict ourself to the following extension of the Vasicek model that was analyzed by Hull and White 1994

$$dr(t) = (b(t) + \beta r(t))\,dt + \sigma\,dW(t).$$

In this model we choose the constants β and σ to obtain a nice volatility structure whereas $b(t)$ is chosen in order to match the initial yield curve.

Equation (7.10) for $B(t,T)$ is just as in the Vasicek model

$$\partial_t B(t,T) = -\beta B(t,T) - 1, \quad B(T,T) = 0$$

with explicit solution

$$B(t,T) = \frac{1}{\beta}\left(e^{\beta(T-t)} - 1\right).$$

Equation (7.9) for $A(t,T)$ now reads

$$A(t,T) = -\frac{\sigma^2}{2}\int_t^T B^2(s,T)\,ds + \int_t^T b(s)B(s,T)\,ds$$

We consider the initial forward curve (notice that $\partial_T B(s,T) = -\partial_s B(s,T)$)

$$\begin{aligned} f^*(0,T) &= \partial_T A(0,T) + \partial_T B(0,T)r(0) \\ &= \frac{\sigma^2}{2}\int_0^T \partial_s B^2(s,T)\,ds + \int_0^T b(s)\partial_T B(s,T) + \partial_T B(0,T)r(0) \\ &= \underbrace{-\frac{\sigma^2}{2\beta^2}\left(e^{\beta T} - 1\right)^2}_{=:g(T)} + \underbrace{\int_0^T b(s)e^{\beta(T-s)}\,ds + e^{\beta T}r(0)}_{=:\phi(T)}. \end{aligned}$$

The function ϕ satisfies

$$\partial_T\phi(T) = \beta\phi(T) + b(T), \quad \phi(0) = r(0).$$

It follows that

$$\begin{aligned} b(T) &= \partial_T\phi(T) - \beta\phi(T) \\ &= \partial_T(f^*(0,T) + g(T)) - \beta(f^*(0,T) + g(T)). \end{aligned}$$

Plugging in and performing performing some calculations eventually yields

$$\begin{aligned} f(t,T) = f^*(0,T) - e^{\beta(T-t)}f^*(0,t) - \frac{\sigma^2}{2\beta^2}\left(e^{\beta(T-t)} - 1\right)\left(e^{\beta(T-t)} - e^{\beta(T+t)}\right) \\ + e^{\beta(T-t)}r(t). \end{aligned}$$

7.6 Option Pricing in Affine Models

We show how to price bond options in the affine framework. The discussion is informal, we do not worry about integrability conditions. The procedure has to be carried out rigorously from case to case.

Let $r(t)$ be a diffusion short rate model with drift

$$b(t) + \beta(t)r,$$

diffusion term

$$a(t) + \alpha(t)r$$

and ATS

$$P(t,T) = e^{-A(t,T)-B(t,T)r(t)}.$$

Let $\lambda \in \mathbb{C}$, and ϕ and ψ be given as solutions to

$$\begin{aligned}
\partial_t \phi(t,T,\lambda) &= a(t)\psi^2(t,T,\lambda) - b(t)\psi(t,T,\lambda)\\
\phi(T,T,\lambda) &= 0\\
\partial_t \psi(t,T,\lambda) &= \alpha(t)\psi^2(t,T,\lambda) - \beta(t)\psi(t,T,\lambda) - 1\\
\psi(T,T,\lambda) &= \lambda.
\end{aligned}$$

This looks much like the ATS equations (7.9)–(7.10), and indeed, by plugging the right hand side below in the term structure equation (7.6), one sees that

$$\mathbb{E}\left[e^{-\int_t^T r(s)\,ds} e^{-\lambda r(T)} \mid \mathcal{F}_t\right] = e^{-\phi(t,T,\lambda)-\psi(t,T,\lambda)r(t)}.$$

In fact, we have

$$\phi(t,T,0) = A(t,T) \quad \text{and} \quad \psi(t,T,0) = B(t,T).$$

Now let $t = 0$ (for simplicity only). Since discounted zero-coupon bond prices are martingales we obtain for $T \le S$ ($\to$ exercise)

$$\begin{aligned}
\mathbb{E}\left[e^{-\int_0^S r(s)\,ds} e^{-\lambda r(T)}\right] &= \mathbb{E}\left[e^{-\int_0^T r(s)\,ds} e^{-A(T,S)-B(T,S)r(T)} e^{-\lambda r(T)}\right]\\
&= e^{-A(T,S)} \mathbb{E}\left[e^{-\int_0^T r(s)\,ds} e^{-(\lambda+B(T,S))r(T)}\right]\\
&= e^{-A(T,S)-\phi(0,T,\lambda+B(T,S))-\psi(0,T,\lambda+B(T,S))r(0)}.
\end{aligned}$$

But

$$\frac{d\mathbb{Q}^S}{d\mathbb{Q}} = \frac{e^{-\int_0^S r(s)\,ds}}{P(0,S)}$$

defines an equivalent probability measure $\mathbb{Q}^S \sim \mathbb{Q}$ on $\mathcal{F}_S$, the so called S-*forward measure*. Hence we have shown that the (extended) Laplace transform of $r(T)$ with respect to $\mathbb{Q}^S$ is

$$\mathbb{E}_{\mathbb{Q}^S}\left[e^{-\lambda r(T)}\right] = e^{A(0,S)-A(T,S)-\phi(0,T,\lambda+B(T,S))+(B(0,S)-\psi(0,T,\lambda+B(T,S)))r(0)}.$$

By Laplace (or Fourier) inversion, one gets the distribution of $r(T)$ under $\mathbb{Q}^S$. In some cases (e.g. Vasicek or CIR) this distribution is explicitly known (e.g. Gaussian or chi-square). In general, this is done numerically.

We now consider a European call option on a S-bond with expiry date $T < S$ and strike price K. Its price today ($t = 0$) is

$$\pi = \mathbb{E}\left[e^{-\int_0^T r(s)\,ds}\left(e^{-A(T,S)-B(T,S)r(T)} - K\right)^+\right].$$

The payoff can be decomposed according to

$$\left(e^{-A(T,S)-B(T,S)r(T)} - K\right)^+ = e^{-A(T,S)-B(T,S)r(T)} 1_{\{r(T)\le r^*\}} - K 1_{\{r(T)\le r^*\}}$$

where

$$r^* = r^*(T,S,K) := -\frac{A(T,S) + \log K}{B(T,S)}.$$

Hence

$$\begin{aligned}\pi &= \mathbb{E}\left[e^{-\int_0^S r(s)\,ds} 1_{\{r(T)\le r^*\}}\right] - K\mathbb{E}\left[e^{-\int_0^T r(s)\,ds} 1_{\{r(T)\le r^*\}}\right] \\ &= P(0,S)\mathbb{Q}^S[r(T)\le r^*] - KP(0,T)\mathbb{Q}^T[r(T)\le r^*].\end{aligned}$$

The pricing of the option boils down to the computation of the probability of the event $\{r(T)\le r^*\}$ under the S- and T-forward measure.

7.6.1 Example: Vasicek Model (a, b, β const, $\alpha = 0$).

We obtain ($\rightarrow$ exercise)

$$\pi = P(0,S)\Phi\left(\frac{r^* - \ell_1(T,S,r(0))}{\sqrt{\ell_2(T)}}\right) - KP(0,T)\Phi\left(\frac{r^* - \ell_1(T,T,r(0))}{\sqrt{\ell_2(T)}}\right)$$

where

$$\begin{aligned}\ell_1(T,S,r) &:= \frac{1}{\beta^2}\left(\beta\left(e^{\beta T}(b+\beta r) - b\right) - a\left(2 - e^{\beta(S-T)} - 2e^{\beta T} + e^{\beta(S+T)}\right)\right) \\ \ell_2(T) &:= \frac{a}{\beta}\left(e^{2\beta T} - 1\right)\end{aligned}$$

and $\Phi(x)$ is the cumulative standard Gaussian distribution function.

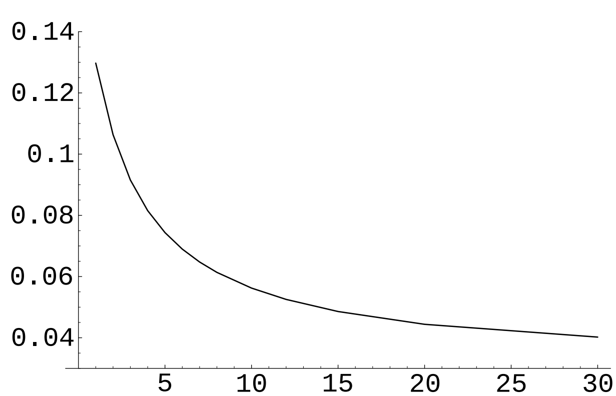
A similar closed form expression is available for the price of a put option, and hence an explicit price formula for caps. For $\beta = -0.86$, $b/|\beta| = 0.09$

(mean reversion level), $\sigma = 0.0148$ and $r(0) = 0.08$, as in Figure 7.1, one gets the ATM cap prices and Black volatilities shown in Table 7.1 and Figure 7.2 (→ exercise). In contrast to Figure 2.1, the Vasicek model cannot produce humped volatility curves.

Table 7.1: Vasicek ATM cap prices and Black volatilities.

Maturity	ATM prices	ATM vols
1	0.00215686	0.129734
2	0.00567477	0.106348
3	0.00907115	0.0915455
4	0.0121906	0.0815358
5	0.01503	0.0743607
6	0.017613	0.0689651
7	0.0199647	0.0647515
8	0.0221081	0.0613624
10	0.025847	0.0562337
12	0.028963	0.0525296
15	0.0326962	0.0485755
20	0.0370565	0.0443967
30	0.0416089	0.0402203

Figure 7.2: Vasicek ATM cap Black volatilities.



Chapter 8

Heath–Jarrow–Morton (HJM) Methodology

$\rightarrow$ original article by Heath, Jarrow and Morton (HJM, 1992) [9].

Chapter 9

Forward Measures

We consider the HJM setup (Chapter 8) and directly focus on the (unique) EMM $\mathbb{Q} \sim \mathbb{P}$ under which all discounted bond price processes

$$\frac{P(t,T)}{B(t)}, \quad t \in [0,T],$$

are strictly positive martingales.

9.1 T-Bond as Numeraire

Fix $T > 0$. Since

$$\frac{1}{P(0,T)B(T)} > 0 \quad \text{and} \quad \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{P(0,T)B(T)}\right] = 1$$

we can define an equivalent probability measure $\mathbb{Q}^T \sim \mathbb{Q}$ on $\mathcal{F}_T$ by

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}} = \frac{1}{P(0,T)B(T)}.$$

For $t \leq T$ we have

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}}|_{\mathcal{F}_t} = \mathbb{E}_{\mathbb{Q}}\left[\frac{d\mathbb{Q}^T}{d\mathbb{Q}} \mid \mathcal{F}_t\right] = \frac{P(t,T)}{P(0,T)B(t)}.$$

This probability measure has already been introduced in Section 7.6. It is called the T-*forward measure*.

Lemma 9.1.1. *For any $S > 0$,*

$$\frac{P(t,S)}{P(t,T)}, \quad t \in [0, S \wedge T],$$

is a $\mathbb{Q}^T$-martingale.

Proof. Let $s \leq t \leq S \wedge T$. Bayes' rule gives

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}^T}\left[\frac{P(t,S)}{P(t,T)} \mid \mathcal{F}_s\right] &= \frac{\mathbb{E}_{\mathbb{Q}}\left[\frac{P(t,T)}{P(0,T)B(t)}\frac{P(t,S)}{P(t,T)} \mid \mathcal{F}_s\right]}{\frac{P(s,T)}{P(0,T)B(s)}} \\ &= \frac{\frac{P(s,S)}{B(s)}}{\frac{P(s,T)}{B(s)}} = \frac{P(s,S)}{P(s,T)}.\end{aligned}$$

□

We thus have an entire collection of EMMs now! Each $\mathbb{Q}^T$ corresponds to a different numeraire, namely the T-bond. Since $\mathbb{Q}$ is related to the risk-free asset, one usually calls $\mathbb{Q}$ the *risk neutral measure*.

T-forward measures give simpler pricing formulas. Indeed, let X be a T-claim such that

$$\frac{X}{B(T)} \in L^1(\mathbb{Q}, \mathcal{F}_T). \tag{9.1}$$

Its fair price at time $t \leq T$ is then given by

$$\pi(t) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(s)\,ds} X \mid \mathcal{F}_t\right].$$

To compute $\pi(t)$ we have to know the joint distribution of $\exp\left[-\int_t^T r(s)\,ds\right]$ and X, and integrate with respect to that distribution. Thus we have to compute a double integral, which in most cases turns out to be rather hard work. *If $B(T)/B(t)$ and X were independent under* $\mathbb{Q}$ (which is not realistic! it holds, for instance, if r is deterministic) we would have

$$\pi(t) = P(t,T)\mathbb{E}_{\mathbb{Q}}\left[X \mid \mathcal{F}_t\right],$$

a much nicer formula, since

- we only have to compute the single integral $\mathbb{E}_{\mathbb{Q}}[X \mid \mathcal{F}_t]$;

- the bond price $P(t,T)$ can be observed at time t and does not have to be computed.

The good news is that the above formula holds — not under $\mathbb{Q}$ though, but under $\mathbb{Q}^T$:

Proposition 9.1.2. *Let X be a T-claim such that* (9.1) *holds. Then*

$$\mathbb{E}_{\mathbb{Q}^T}\left[|X|\right] < \infty \tag{9.2}$$

and

$$\pi(t) = P(t,T)\mathbb{E}_{\mathbb{Q}^T}\left[X \mid \mathcal{F}_t\right]. \tag{9.3}$$

Proof. Bayes's rule yields

$$\mathbb{E}_{\mathbb{Q}^T}\left[|X|\right] = \mathbb{E}_{\mathbb{Q}}\left[\frac{|X|}{P(0,T)B(T)}\right] < \infty \quad \text{(by (9.1))},$$

whence (9.2). And

$$\begin{aligned}\pi(t) &= P(0,T)B(t)\mathbb{E}_{\mathbb{Q}}\left[\frac{X}{P(0,T)B(T)} \mid \mathcal{F}_t\right] \\ &= P(0,T)B(t)\frac{P(t,T)}{P(0,T)B(t)}E_{\mathbb{Q}^T}\left[X \mid \mathcal{F}_t\right] \\ &= P(t,T)\mathbb{E}_{\mathbb{Q}^T}\left[X \mid \mathcal{F}_t\right],\end{aligned}$$

which proves (9.3). □

9.2 An Expectation Hypothesis

Under the forward measure the expectation hypothesis holds. That is, the expression of the forward rates $f(t,T)$ as conditional expectation of the future short rate $r(T)$.

To see that, we write W for the driving $\mathbb{Q}$-Brownian motion. The forward rates then follow the dynamics

$$f(t,T) = f(0,T) + \int_0^t \left(\sigma(s,T)\cdot\int_s^T \sigma(s,u)\,du\right)ds + \int_0^t \sigma(s,T)\,dW(s). \tag{9.4}$$

The $\mathbb{Q}$-dynamics of the discounted bond price process is

$$\frac{P(t,T)}{B(t)} = P(0,T) + \int_0^t \frac{P(s,T)}{B(s)} \left(-\int_s^T \sigma(s,u)\, du \right) dW(s). \tag{9.5}$$

This equation has a unique solution

$$\frac{P(t,T)}{B(t)} = P(0,T)\mathcal{E}_t \left(\left(-\int_\cdot^T \sigma(\cdot,u)\, du \right) \cdot W \right).$$

We thus have

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}}|_{\mathcal{F}_t} = \mathcal{E}_t \left(\left(-\int_\cdot^T \sigma(\cdot,u)\, du \right) \cdot W \right). \tag{9.6}$$

Girsanov's theorem applies and

$$W^T(t) = W(t) + \int_0^t \left(\int_s^T \sigma(s,u)\, du \right) ds, \quad t \in [0,T],$$

is a $\mathbb{Q}^T$-Brownian motion. Equation (9.4) now reads

$$f(t,T) = f(0,T) + \int_0^t \sigma(s,T)\, dW^T(s).$$

Hence, if

$$\mathbb{E}_{\mathbb{Q}^T} \left[\int_0^T \|\sigma(s,T)\|^2\, ds \right] < \infty$$

then

$$(f(t,T))_{t\in[0,T]} \quad \text{is a } \mathbb{Q}^T\text{-martingale.}$$

Summarizing we have thus proved

Lemma 9.2.1. *Under the above assumptions, the expectation hypothesis holds under the forward measures*

$$f(t,T) = \mathbb{E}_{\mathbb{Q}^T} \left[r(T) \mid \mathcal{F}_t \right].$$

9.3 Option Pricing in Gaussian HJM Models

We consider a European call option on an S-bond with expiry date $T < S$ and strike price K. Its price at time $t = 0$ (for simplicity only) is

$$\pi = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^T r(s)\,ds}\,(P(T,S)-K)^+\right].$$

We proceed as in Section 7.6 and decompose

$$\begin{aligned}\pi &= \mathbb{E}_{\mathbb{Q}}\left[B(T)^{-1}P(T,S)\,1(P(T,S)\geq K)\right] - K\mathbb{E}_{\mathbb{Q}}\left[B(T)^{-1}\,1(P(T,S)\geq K)\right]\\ &= P(0,S)\mathbb{Q}^S\left[P(T,S)\geq K\right] - KP(0,T)\mathbb{Q}^T\left[P(T,S)\geq K\right].\end{aligned}$$

This option pricing formula holds in general.

We already know that

$$\frac{dP(t,T)}{P(t,T)} = r(t)\,dt + v(t,T)\,dW(t)$$

and hence

$$P(t,T) = P(0,T)\exp\left[\int_0^t v(s,T)\,dW(s) + \int_0^t\left(r(s)-\frac{1}{2}\|v(s,T)\|^2\right)ds\right]$$

where

$$v(t,T) := -\int_t^T \sigma(t,u)\,du. \tag{9.7}$$

We also know that $\left(\frac{P(t,T)}{P(t,S)}\right)_{t\in[0,T]}$ is a $\mathbb{Q}^S$-martingale and $\left(\frac{P(t,S)}{P(t,T)}\right)_{t\in[0,T]}$ is a $\mathbb{Q}^T$-martingale. In fact ($\to$ exercise)

$$\begin{aligned}\frac{P(t,T)}{P(t,S)} &= \frac{P(0,T)}{P(0,S)}\\ &\qquad\times\exp\left[\int_0^t \sigma_{T,S}(s)\,dW(s) - \frac{1}{2}\int_0^t\left(\|v(s,T)\|^2-\|v(s,S)\|^2\right)ds\right]\\ &= \frac{P(0,T)}{P(0,S)}\exp\left[\int_0^t \sigma_{T,S}(s)\,dW^S(s) - \frac{1}{2}\int_0^t \|\sigma_{T,S}(s)\|^2\,ds\right]\end{aligned}$$

where

$$\sigma_{T,S}(s) := v(s,T)-v(s,S) = \int_T^S \sigma(s,u)\,du, \tag{9.8}$$

and

$$\begin{aligned}\frac{P(t,S)}{P(t,T)} &= \frac{P(0,S)}{P(0,T)} \\ &\quad \times \exp\left[-\int_0^t \sigma_{T,S}(s)\,dW(s) - \frac{1}{2}\int_0^t \left(\|v(s,S)\|^2 - \|v(s,T)\|^2\right) ds\right] \\ &= \frac{P(0,S)}{P(0,T)} \exp\left[-\int_0^t \sigma_{T,S}(s)\,dW^T(s) - \frac{1}{2}\int_0^t \|\sigma_{T,S}(s)\|^2\,ds\right].\end{aligned}$$

Now observe that

$$\begin{aligned}\mathbb{Q}^S[P(T,S) \geq K] &= \mathbb{Q}^S\left[\frac{P(T,T)}{P(T,S)} \leq \frac{1}{K}\right] \\ \mathbb{Q}^T[P(T,S) \geq K] &= \mathbb{Q}^T\left[\frac{P(T,S)}{P(T,T)} \geq K\right].\end{aligned}$$

This suggests to look at those models for which $\sigma_{T,S}$ is deterministic, and hence $\frac{P(T,T)}{P(T,S)}$ and $\frac{P(T,S)}{P(T,T)}$ are log-normally distributed under the respective forward measures.

We thus assume that $\sigma(t,T) = (\sigma_1(t,T), \dots, \sigma_d(t,T))$ are *deterministic* functions of t and T, and hence forward rates $f(t,T)$ are Gaussian distributed.

We obtain the following closed form option price formula.

Proposition 9.3.1. *Under the above Gaussian assumption, the option price is*

$$\pi = P(0,S)\Phi[d_1] - KP(0,T)\Phi[d_2],$$

where

$$d_{1,2} = \frac{\log\left[\frac{P(0,S)}{KP(0,T)}\right] \pm \frac{1}{2}\int_0^T \|\sigma_{T,S}(s)\|^2\,ds}{\sqrt{\int_0^T \|\sigma_{T,S}(s)\|^2\,ds}},$$

$\sigma_{T,S}(s)$ *is given in* (9.8) *and* Φ *is the standard Gaussian CDF.*

Proof. It is enough to observe that

$$\frac{\log\frac{P(T,T)}{P(T,S)} - \log\frac{P(0,T)}{P(0,S)} + \frac{1}{2}\int_0^T \|\sigma_{T,S}(s)\|^2\,ds}{\sqrt{\int_0^T \|\sigma_{T,S}(s)\|^2\,ds}}$$

and

$$\frac{\log \frac{P(T,S)}{P(T,T)} - \log \frac{P(0,S)}{P(0,T)} + \frac{1}{2}\int_0^T \|\sigma_{T,S}(s)\|^2\,ds}{\sqrt{\int_0^T \|\sigma_{T,S}(s)\|^2\,ds}}$$

are standard Gaussian distributed under $\mathbb{Q}^S$ and $\mathbb{Q}^T$, respectively. □

Of course, the Vasicek option price formula from Section 7.6.1 can now be obtained as a corollary of Proposition 9.3.1 (→ exercise).

Chapter 10

Forwards and Futures

→ B[3](Chapter 20), or Hull (2002) [10]

We discuss two common types of term contracts: forwards, which are mainly traded OTC, and futures, which are actively traded on many exchanges.

The underlying is in both cases a T-claim $\mathcal{Y}$, for some fixed future date T. This can be an exchange rate, an interest rate, a commodity such as copper, any traded or non-traded asset, an index, etc.

10.1 Forward Contracts

A *forward contract* on $\mathcal{Y}$, contracted at t, with time of delivery $T > t$, and with the *forward price* $f(t; T, \mathcal{Y})$ is defined by the following payment scheme:

- at T, the holder of the contract (long position) pays $f(t; T, \mathcal{Y})$ and receives $\mathcal{Y}$ from the underwriter (short position);
- at t, the forward price is chosen such that the present value of the forward contract is zero, thus

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(s)\,ds}\left(\mathcal{Y} - f(t; T, \mathcal{Y})\right) \mid \mathcal{F}_t\right] = 0.$$

This is equivalent to

$$\begin{aligned} f(t; T, \mathcal{Y}) &= \frac{1}{P(t,T)}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(s)\,ds}\mathcal{Y} \mid \mathcal{F}_t\right] \\ &= \mathbb{E}_{\mathbb{Q}^T}\left[\mathcal{Y} \mid \mathcal{F}_t\right]. \end{aligned}$$

Examples The forward price at t of

1. a dollar delivered at T is 1;
2. an S-bond delivered at $T \leq S$ is $\frac{P(t,S)}{P(t,T)}$;
3. any traded asset S delivered at T is $\frac{S(t)}{P(t,T)}$.

The forward price $f(s;T,\mathcal{Y})$ has to be distinguished from the (spot) price at time s of the forward contract entered at time $t \leq s$, which is

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_s^T r(u)\,du}\left(\mathcal{Y} - f(t;T,\mathcal{Y})\right) \mid \mathcal{F}_s\right] = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_s^T r(u)\,du}\mathcal{Y} \mid \mathcal{F}_s\right] - P(t,T)f(t;T,\mathcal{Y}).$$

10.2 Futures Contracts

A *futures contract* on $\mathcal{Y}$ with time of delivery T is defined as follows:

- at every $t \leq T$, there is a market quoted *futures price* $F(t;T,\mathcal{Y})$, which makes the futures contract on $\mathcal{Y}$, if entered at t, equal to zero;
- at T, the holder of the contract (long position) pays $F(T;T,\mathcal{Y})$ and receives $\mathcal{Y}$ from the underwriter (short position);
- during any time interval $(s,t]$ the holder of the contract receives (or pays, if negative) the amount $F(t;T,\mathcal{Y}) - F(s;T,\mathcal{Y})$ (this is called *marking to market*).

So there is a continuous cash-flow between the two parties of a futures contract. They are required to keep a certain amount of money as a safety margin.

The volumes in which futures are traded are huge. One of the reasons for this is that in many markets it is difficult to trade (hedge) directly in the underlying object. This might be an index which includes many different (illiquid) instruments, or a commodity such as copper, gas or electricity, etc. Holding a (short position in a) futures does not force you to physically deliver the underlying object (if you exit the contract before delivery date), and selling short makes it possible to hedge against the underlying.

Suppose $\mathcal{Y} \in L^1(\mathbb{Q})$. Then the futures price process is given by the $\mathbb{Q}$-martingale

$$F(t;T,\mathcal{Y}) = \mathbb{E}_{\mathbb{Q}}\left[\mathcal{Y} \mid \mathcal{F}_t\right]. \tag{10.1}$$

Often, this is just how futures prices are *defined*. We now give a heuristic argument for (10.1) based on the above characterization of a futures contract.

First, our model economy is driven by Brownian motion and changes in a continuous way. Hence there is no reason to believe that futures prices evolve discontinuously, and we may assume that

$$F(t) = F(t;T,\mathcal{Y}) \text{ is a continuous semimartingale (or Itô process).}$$

Now suppose we enter the futures contract at time $t < T$. We face a continuum of cashflows in the interval $(t, T]$. Indeed, let $t = t_0 < \cdots < t_N = t$ be a partition of $[t, T]$. The present value of the corresponding cashflows $F(t_i) - F(t_{i-1})$ at t_i, $i = 1, \dots, N$, is given by $\mathbb{E}_{\mathbb{Q}}[\Sigma \mid \mathcal{F}_t]$ where

$$\Sigma := \sum_{i=1}^{N} \frac{1}{B(t_i)} \left(F(t_i) - F(t_{i-1})\right).$$

But the futures contract has present value zero, hence

$$\mathbb{E}_{\mathbb{Q}}[\Sigma \mid \mathcal{F}_t] = 0.$$

This has to hold for any partition (t_i). We can rewrite Σ as

$$\sum_{i=1}^{N} \frac{1}{B(t_{i-1})} \left(F(t_i) - F(t_{i-1})\right) + \sum_{i=1}^{N} \left(\frac{1}{B(t_i)} - \frac{1}{B(t_{i-1})}\right) \left(F(t_i) - F(t_{i-1})\right).$$

If we let the partition become finer and finer this expression converges in probability towards

$$\int_t^T \frac{1}{B(s)}\, dF(s) + \int_t^T d\left\langle \frac{1}{B}, F \right\rangle_s = \int_t^T \frac{1}{B(s)}\, dF(s),$$

since the quadratic variation of $1/B$ (finite variation) and F (continuous) is zero. Under the appropriate integrability assumptions (uniform integrability) we conclude that

$$\mathbb{E}_{\mathbb{Q}}\left[\int_t^T \frac{1}{B(s)}\, dF(s) \mid \mathcal{F}_t\right] = 0,$$

and that

$$M(t) = \int_0^t \frac{1}{B(s)}\, dF(s) = \mathbb{E}_{\mathbb{Q}}\left[\int_0^T \frac{1}{B(s)}\, dF(s) \mid \mathcal{F}_t\right], \quad t \in [0,T],$$

is a $\mathbb{Q}$-martingale. If, moreover

$$\mathbb{E}_{\mathbb{Q}}\left[\int_0^T \frac{1}{B(s)^2}\, d\langle M, M\rangle_s\right] = \mathbb{E}_{\mathbb{Q}}\left[\langle F, F\rangle_T\right] < \infty$$

then

$$F(t) = \int_0^t \frac{1}{B(s)}\, dM(s), \quad t \in [0,T],$$

is a $\mathbb{Q}$-martingale, which implies (10.1).

10.3 Interest Rate Futures

→ Z[22](Section 5.4)

Interest rate futures contracts may be divided into futures on short term instruments and futures on coupon bonds. We only consider an example from the first group.

Eurodollars are deposits of US dollars in institutions outside of the US. LIBOR is the interbank rate of interest for Eurodollar loans. The *Eurodollar futures* contract is tied to the LIBOR. It was introduced by the International Money Market (IMM) of the Chicago Mercantile Exchange (CME) in 1981, and is designed to protect its owner from fluctuations in the 3-months (=1/4 years) LIBOR. The maturity (delivery) months are March, June, September and December.

Fix a maturity date T and let $L(T)$ denote the 3-months LIBOR for the period $[T, T+1/4]$, prevailing at T. The market *quote* of the Eurodollar futures contract on $L(T)$ at time $t \le T$ is

$$1 - L_F(t,T) \quad [100 \text{ per cent}]$$

where $L_F(t,T)$ is the corresponding *futures rate* (compare with the example in Section 4.2.2). As t tends to T, $L_F(t,T)$ tends to $L(T)$. The *futures price*, used for the marking to market, is defined by

$$F(t; T, L(T)) = 1 - \frac{1}{4} L_F(t,T) \quad [\text{Mio. dollars}].$$

Consequently, a change of 1 basis point (0.01%) in the futures rate $L_F(t,T)$ leads to a cashflow of

$$10^6 \times 10^{-4} \times \frac{1}{4} = 25 \quad \text{[dollars]}.$$

We also see that the final price $F(T;T,L(T)) = 1 - \frac{1}{4}L(T) = \mathcal{Y}$ is not $P(T,T+1/4) = 1 - \frac{1}{4}L(T)P(T,T+1/4)$ as one might suppose. In fact, the underlying $\mathcal{Y}$ is a synthetic value. At maturity there is no physical delivery. Instead, settlement is made in cash.

On the other hand, since

$$\begin{aligned} 1 - \frac{1}{4}L_F(t,T) &= F(t;T,L(T)) \\ &= \mathbb{E}_{\mathbb{Q}}\left[F(T;T,L(T)) \mid \mathcal{F}_t\right] = 1 - \frac{1}{4}\mathbb{E}_{\mathbb{Q}}\left[L(T) \mid \mathcal{F}_t\right], \end{aligned}$$

we obtain an explicit formula for the futures rate

$$L_F(t,T) = \mathbb{E}_{\mathbb{Q}}\left[L(T) \mid \mathcal{F}_t\right].$$

10.4 Forward vs. Futures in a Gaussian Setup

Let S be the price process of a traded asset. Hence the $\mathbb{Q}$-dynamics of S is of the form

$$\frac{dS(t)}{S(t)} = r(t)\,dt + \rho(t)\,dW(t),$$

for some volatility process ρ. Fix a delivery date T. The forward and futures prices of S for delivery at T are

$$f(t;T,S(T)) = \frac{S(t)}{P(t,T)}, \quad F(t;T,S(T)) = \mathbb{E}_{\mathbb{Q}}[S(T) \mid \mathcal{F}_t].$$

Under Gaussian assumption we can establish the relationship between the two prices.

Proposition 10.4.1. *Suppose $\rho(t)$ and $v(t,T)$ are deterministic functions in t, where*

$$v(t,T) = -\int_t^T \sigma(t,u)\,du$$

is the volatility of the T-bond (see (9.7)). Then

$$F(t;T,S(T)) = f(t;T,S(T)) \exp\left(\int_t^T (v(s,T)-\rho(s))\cdot v(s,T)\,ds\right)$$

for $t \le T$.

Hence, if the instantaneous correlation of $dS(t)$ and $dP(t,T)$ is negative

$$\frac{d\langle S, P(\cdot,T)\rangle_t}{dt} = S(t)P(t,T)\rho(t)\cdot v(t,T) \le 0$$

then the futures price dominates the forward price.

Proof. Write $\mu(s) := v(s,T) - \rho(s)$. It is clear that

$$\begin{aligned} f(t;T,S(T)) = \frac{S(0)}{P(0,T)} &\exp\left(-\int_0^t \mu(s)\,dW(s) - \frac{1}{2}\int_0^t \|\mu(s)\|^2\,ds\right) \\ &\times \exp\left(\int_0^t \mu(s)\cdot v(s,T)\,ds\right), \end{aligned}$$

and hence

$$\begin{aligned} f(T;T,S(T)) = f(t;T,S(T)) &\exp\left(-\int_t^T \mu(s)\,dW(s) - \frac{1}{2}\int_t^T \|\mu(s)\|^2\,ds\right) \\ &\times \exp\left(\int_t^T \mu(s)\cdot v(s,T)\,ds\right). \end{aligned}$$

By assumption $\mu(s)$ is deterministic. Consequently,

$$\mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^T \mu(s)\,dW(s) - \frac{1}{2}\int_t^T \|\mu(s)\|^2\,ds\right) \mid \mathcal{F}_t\right] = 1$$

and

$$\begin{aligned} F(t;T,S(T)) &= \mathbb{E}_{\mathbb{Q}}[f(T;T,S(T)) \mid \mathcal{F}_t] \\ &= f(t;T,S(T)) \exp\left(\int_t^T \mu(s)\cdot v(s,T)\,ds\right), \end{aligned}$$

as desired. □

Similarly, one can show (→ exercise)

Lemma 10.4.2. *In a Gaussian HJM framework ($\sigma(t,T)$ deterministic) we have the following relations* (convexity adjustments) *between instantaneous and simple futures and forward rates*

$$\begin{aligned} f(t,T) &= \mathbb{E}_{\mathbb{Q}}[r(T) \mid \mathcal{F}_t] - \int_t^T \left(\sigma(s,T) \cdot \int_s^T \sigma(s,u)\, du \right) ds, \\ F(t;T,S) &= \mathbb{E}_{\mathbb{Q}}[F(T,S) \mid \mathcal{F}_t] \\ &\quad - \frac{P(t,T)}{(S-T)P(t,S)} \left(e^{\int_t^T \left(\int_T^S \sigma(s,v)\, dv \cdot \int_s^S \sigma(s,u)\, du \right) ds} - 1 \right) \end{aligned}$$

for $t \leq T < S$.

Hence, if

$$\sigma(s,v) \cdot \sigma(s,u) \geq 0 \quad \text{for all } s \leq \min(u,v)$$

then futures rates are always greater than the corresponding forward rates.

Chapter 11

Multi-Factor Models

We have seen that every time-homogeneous diffusion short rate model $r(t)$ induces forward rates of the form

$$f(t,T) = H(T-t, r(t)),$$

for some deterministic function H. This a one-factor model, since the driving (Markovian) factor, $r(t)$, is one-dimensional. This is too restrictive from two points of view:

- statistically: the evolution of the entire yield curve is explained by a single variable. The infinitesimal increments of all bond prices are perfectly correlated

$$\frac{d\langle P(\cdot,T), P(\cdot,S)\rangle_t}{\sqrt{d\langle P(\cdot,T), P(\cdot,T)\rangle_t}\sqrt{d\langle P(\cdot,S), P(\cdot,S)\rangle_t}} = \frac{\int_t^T \sigma(t,u)du \int_t^S \sigma(t,u)du}{\int_t^T \sigma(t,u)du \int_t^S \sigma(t,u)du} = 1.$$

- analytically: the family of attainable forward curves

$$\mathcal{H} = \{H(\cdot, r) \mid r \in \mathbb{R}\}$$

  is only one-dimensional.

To gain more flexibility, we now allow for multiple factors. Fix $m \geq 1$ and a closed set $\mathcal{Z} \subset \mathbb{R}^m$ (state space). A (m-)factor model is an interest rate model of the form

$$f(t,T) = H(T-t, Z(t))$$

where H is a deterministic function and Z (state process) is a $\mathcal{Z}$-valued diffusion process,

$$\begin{aligned} dZ(t) &= b(Z(t))\,dt + \rho(Z(t))\,dW(t) \\ Z(0) &= z_0. \end{aligned}$$

Here W is a d-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q})$, satisfying the usual conditions. We assume that

(A1) $H \in C^{1,2}(\mathbb{R}_+ \times \mathcal{Z})$;

(A2) $b : \mathcal{Z} \to \mathbb{R}^m$ and $\rho : \mathcal{Z} \to \mathbb{R}^{m\times d}$ are continuous functions;

(A3) the above SDE has a unique $\mathcal{Z}$-valued solution $Z = Z^{z_0}$, for every $z_0 \in \mathcal{Z}$;

(A4) $\mathbb{Q}$ is the risk neutral local martingale measure for the induced bond prices

$$P(t,T) = \Pi(T-t, Z^{z_0}(t)),$$

for all $z_0 \in \mathcal{Z}$, where

$$\Pi(x,z) := \exp\left(-\int_0^x H(s,z)\,ds\right).$$

Notice that the short rates are now given by $r(t) = H(0, Z(t))$. Hence the assumption **(A4)** is equivalent to

(A4')

$$\left(\frac{\Pi(T-t, Z^{z_0}(t))}{e^{\int_0^t H(0,Z^{z_0}(s))\,ds}}\right)_{t\in[0,T]}$$

is a $\mathbb{Q}$-local martingale, for all $z_0 \in \mathcal{Z}$.

Time-inhomogeneous models are included in the above setup. Simply set $Z_1(t) = t$ (that is, $b_1 \equiv 1$ and $\rho_{1j} \equiv 0$ for $j = 1, \ldots, d$).

11.1 No-Arbitrage Condition

Since the function $(x,z) \mapsto H(x,z)$ is in $C^{1,2}(\mathbb{R}_+ \times \mathcal{Z})$ we can apply Itô's formula and obtain

$$\begin{aligned} df(t,T) = \Big(&- \partial_x H(T-t, Z(t)) + \sum_{i=1}^m b_i(Z(t)) \partial_{z_i} H(T-t, Z(t)) \\ &+ \frac{1}{2} \sum_{i,j=1}^m a_{ij}(Z(t)) \partial_{z_i} \partial_{z_j} H(T-t, Z(t)) \Big) \, dt \\ &+ \sum_{i=1}^m \sum_{j=1}^d \partial_{z_i} H(T-t, Z(t)) \rho_{ij}(Z(t)) \, dW_j(t), \end{aligned}$$

where

$$a(z) := \rho(z) \rho^T(z). \tag{11.1}$$

Hence the induced forward rate model is of the HJM type with

$$\sigma_j(t,T) = \sum_{i=1}^m \partial_{z_i} H(T-t, Z(t)) \rho_{ij}(Z(t)), \quad j = 1, \dots, d.$$

The HJM drift condition now reads

$$\begin{aligned} &- \partial_x H(T-t, Z(t)) + \sum_{i=1}^m b_i(Z(t)) \partial_{z_i} H(T-t, Z(t)) \\ &\qquad + \frac{1}{2} \sum_{i,j=1}^m a_{ij}(Z(t)) \partial_{z_i} \partial_{z_j} H(T-t, Z(t)) \\ &= \sum_{j=1}^d \sum_{k,l=1}^m \rho_{kj}(Z(t)) \rho_{lj}(Z(t)) \partial_{z_i} H(T-t, Z(t)) \int_t^T \partial_{z_i} H(u-t, Z(t)) \, du \\ &= \sum_{k,l=1}^m a_{kl}(Z(t)) \partial_{z_i} H(T-t, Z(t)) \int_t^T \partial_{z_i} H(u-t, Z(t)) \, du. \end{aligned}$$

This has to hold a.s. for all $t \le T$ and initial points $z_0 = Z(0)$. Letting $t \to 0$ we thus get the following result.

Proposition 11.1.1 (Consistency Condition). *Under the above assumptions* **(A1)–(A3)**, *there is equivalence between* **(A4)** *and*

$$\partial_x H(x,z) = \sum_{i=1}^m b_i(z)\partial_{z_i} H(x,z) + \sum_{i,j=1}^m a_{ij}(z)\left(\frac{1}{2}\partial_{z_i}\partial_{z_j} H(x,z) - \partial_{z_i} H(x,z)\int_0^x \partial_{z_i} H(u,z)\,du\right) \tag{11.2}$$

for all $(x,z) \in \mathbb{R}_+ \times \mathcal{Z}$, *where* a *is defined in* (11.1).

Remark 11.1.2. *Notice that, by symmetry, the last expression in* (11.2) *can be written as*

$$\sum_{i,j=1}^m a_{ij}(z)\partial_{z_i} H(x,z)\int_0^x \partial_{z_i} H(u,z)\,du = \frac{1}{2}\partial_x \sum_{i,j=1}^m a_{ij}(z)\left(\int_0^x \partial_{z_i} H(u,z)\,du \int_0^x \partial_{z_j} H(u,z)\,du\right).$$

There are two ways to approach equation (11.2). First, one takes b and ρ (and hence a) as given and looks for a solution H for the PDE (11.2). Or, one takes H as given (an estimation method for the yield curve) and tries to find b and a such that (11.2) is satisfied for all (x,z). This is an *inverse problem*. It turns out that the latter approach is quite restrictive on possible choices of b and a.

Proposition 11.1.3. *Suppose that the functions*

$$\partial_{z_i} H(\cdot,z) \quad \textit{and} \quad \frac{1}{2}\partial_{z_i}\partial_{z_j} H(\cdot,z) - \partial_{z_i} H(\cdot,z)\int_0^{\cdot} \partial_{z_i} H(u,z)\,du,$$

for $1 \le i \le j \le m$, *are linearly independent for all* z *in some dense subset* $\mathcal{D} \subset \mathcal{Z}$. *Then* b *and* a *are uniquely determined by* H.

Proof. Set $M = m + m(m+1)/2$, the number of unknown functions b_k and $a_{kl} = a_{lk}$. Let $z \in \mathcal{D}$. Then there exists a sequence $0 \le x_1 < \cdots < x_M$ such that the $M \times M$-matrix with k-th row vector built by

$$\partial_{z_i} H(x_k,z) \quad \text{and} \quad \frac{1}{2}\partial_{z_i}\partial_{z_j} H(x_k,z) - \partial_{z_i} H(\cdot,z)\int_0^{x_k} \partial_{z_i} H(u,z)\,du,$$

for $1 \leq i \leq j \leq m$, is invertible. Thus, $b(z)$ and $a(z)$ are uniquely determined by (11.2). This holds for each $z \in \mathcal{D}$. By continuity of b and a hence for all $z \in \mathcal{Z}$. □

Remark 11.1.4. *Suppose that the the parametrized curve family*

$$\mathcal{H} = \{H(\cdot, z) \mid z \in \mathcal{Z}\}$$

is used for daily estimation of the forward curve in terms of the state variable z. Then the above proposition tells us that, under the stated assumption, any $\mathbb{Q}$-diffusion model Z for z is fully determined by H.

If $\mathcal{F}_t = \mathcal{F}_t^W$ is the Brownian filtration, then the diffusion coefficient, $a(z)$, of Z is not affected by any Girsanov transformation. Consequently, statistical calibration is only possible for the drift of the model (or equivalently, for the market price of risk), since the observations of z are made under the objective measure $\mathbb{P} \sim \mathbb{Q}$, where $d\mathbb{Q}/d\mathbb{P}$ is left unspecified by our consistency considerations.

11.2 Affine Term Structures

We first look at the simplest, namely the *affine* case:

$$H(x, z) = g_0(x) + g_1(x)z_1 + \cdots g_m(x)z_m.$$

Here the second order z-derivatives vanish, and (11.2) reduces to

$$\partial_x g_0(x) + \sum_{i=1}^m z_i \partial_x g_i(x) = \sum_{i=1}^m b_i(z) g_i(x) - \frac{1}{2}\partial_x \left(\sum_{i,j=1}^m a_{ij}(z) G_i(x) G_j(x) \right), \tag{11.3}$$

where

$$G_i(x) := \int_0^x g_i(u)\, du.$$

Integrating (11.3) yields

$$g_0(x) - g_0(0) + \sum_{i=1}^m z_i (g_i(x) - g_i(0)) = \sum_{i=1}^m b_i(z) G_i(x) - \frac{1}{2} \sum_{i,j=1}^m a_{ij}(z) G_i(x) G_j(x). \tag{11.4}$$

Now if

$$G_1, \dots, G_m,\ G_1G_1,\ G_1G_2, \dots, G_mG_m$$

are linearly independent functions, we can invert and solve the linear equation (11.4) for b and a. Since the left hand side is affine is z, we obtain that also b and a are affine

$$b_i(z) = b_i + \sum_{j=1}^{m} \beta_{ij} z_j$$
$$a_{ij}(z) = a_{ij} + \sum_{k=1}^{m} \alpha_{k;ij} z_k,$$

for some constant vectors and matrices b, β, a and α_k. Plugging this back into (11.4) and matching constant terms and terms containing z_ks we obtain a system of Riccati equations

$$\partial_x G_0(x) = g_0(0) + \sum_{i=1}^{m} b_i G_i(x) - \frac{1}{2} \sum_{i,j=1}^{m} a_{ij} G_i(x) G_j(x) \tag{11.5}$$
$$\partial_x G_k(x) = g_k(0) + \sum_{i=1}^{m} \beta_{ki} G_i(x) - \frac{1}{2} \sum_{i,j=1}^{m} \alpha_{k;ij} G_i(x) G_j(x), \tag{11.6}$$

with initial conditions $G_0(0) = \dots = G_m(0) = 0$. This extends what we have found in Section 7.4 for the one-factor case.

Notice that we have the freedom to choose $g_0(0), \dots, g_m(0)$, which are related to the short rates by

$$r(t) = f(t,t) = g_0(0) + g_1(0)Z_1(t) + \dots + g_m(0)Z_m(t).$$

A typical choice is $g_1(0) = 1$ and all the other $g_i(0) = 0$, whence $Z_1(t)$ is the (non-Markovian) short rate process.

11.3 Polynomial Term Structures

We extend the ATS setup and consider *polynomial term structures (PTS)*

$$H(x,z) = \sum_{|\mathbf{i}|=0}^{n} g_{\mathbf{i}}(x)\,(Z_t)^{\mathbf{i}}, \tag{11.7}$$

where we use the multi-index notation $\mathbf{i} = (i_1, \dots, i_m)$, $|\mathbf{i}| = i_1 + \cdots + i_m$ and $z^{\mathbf{i}} = z_1^{i_1} \cdots z_m^{i_m}$. Here n denotes the *degree* of the PTS; that is, there exists an index $\mathbf{i}$ with $|\mathbf{i}| = n$ and $g_{\mathbf{i}} \neq 0$.

Thus for $n = 1$ we are back to the ATS case.

For $n = 2$ we have a *quadratic term structure (QTS)*, which has also been studied in the literature.

Do we gain something by looking at $n = 3$ and higher degree PTS models? The answer is no. In fact, we now shall show the amazing result that $n > 2$ is not consistent with (11.2).

For $\mu \in \{1, \dots, n\}$ and $k \in \{1, \dots, m\}$ we write $(\mu)_k$ for the multi-index with μ at the k-th position and zeros elsewhere. Let $\mathbf{i}_1, \mathbf{i}_2, \dots, \mathbf{i}_N$ be a numbering of the set of multi-indices

$$I = \{\mathbf{i} = (i_1, \dots, i_m) \mid |\mathbf{i}| \leq n\}, \quad \text{where} \quad N := |I| = \sum_{|\mathbf{i}|=0}^{n} 1.$$

As above, we denote the integral of $g_{\mathbf{i}}$ by

$$G_{\mathbf{i}}(x) := \int_0^x g_{\mathbf{i}}(u)\, du.$$

Theorem 11.3.1 (Maximal Degree Problem I). *Suppose that $G_{\mathbf{i}_\mu}$ and $G_{\mathbf{i}_\mu} G_{\mathbf{i}_\nu}$ are linearly independent functions, $1 \leq \mu \leq \nu \leq N$, and that $\rho \not\equiv 0$.*

Then necessarily $n \in \{1, 2\}$. Moreover, $b(z)$ and $a(z)$ are polynomials in z with $\deg b(z) \leq 1$ in any case (QTS and ATS), and $\deg a(z) = 0$ if $n = 2$ (QTS) and $\deg a(z) \leq 1$ if $n = 1$ (ATS).

Proof. Define the functions

$$B_{\mathbf{i}}(z) := b_k(z) \frac{\partial z^{\mathbf{i}}}{\partial z_k} + \frac{1}{2} \sum_{k,l=1}^{m} a_{kl}(z) \frac{\partial^2 z^{\mathbf{i}}}{\partial z_k \partial z_l} \tag{11.8}$$

$$A_{\mathbf{ij}}(z) = A_{\mathbf{ji}}(z) := \frac{1}{2} \sum_{k,l=1}^{m} a_{kl}(z) \frac{\partial z^{\mathbf{i}}}{\partial z_k} \frac{\partial z^{\mathbf{j}}}{\partial z_l}. \tag{11.9}$$

Equation (11.2) can be rewritten

$$\sum_{\mu=1}^{N} \left(g_{\mathbf{i}_\mu}(x) - g_{\mathbf{i}_\mu}(0) \right) z^{\mathbf{i}_\mu} = \sum_{\mu=1}^{N} G_{\mathbf{i}_\mu}(x) B_{\mathbf{i}_\mu}(z) - \sum_{\mu,\nu=1}^{N} G_{\mathbf{i}_\mu}(x) G_{\mathbf{i}_\nu}(x) A_{\mathbf{i}_\mu \mathbf{i}_\nu}(z). \tag{11.10}$$

By assumption we can solve this linear equation for B and A, and thus $B_{\mathbf{i}}(z)$ and $A_{\mathbf{ij}}(z)$ are polynomials in z of order less than or equal n. In particular, we have

$$\begin{aligned} B_{(1)_k}(z) &= b_k(z), \\ 2A_{(1)_k(1)_l}(z) &= a_{kl}(z), \quad k,l \in \{1,\dots,m\}, \end{aligned} \tag{11.11}$$

hence $b(z)$ and $a(z)$ are polynomials in z with $\deg b(z), \deg a(z) \leq n$. An easy calculation shows that

$$2A_{(n)_k(n)_k}(z) = a_{kk}(z) n^2 z_k^{2n-2}, \quad k \in \{1,\dots,m\}. \tag{11.12}$$

We may assume that $a_{kk} \not\equiv 0$, since $\rho \not\equiv 0$. But then the right hand side of (11.12) cannot be a polynomial in z of order less than or equal n unless $n \leq 2$. This proves the first part of the theorem.

If $n = 1$ there is nothing more to prove. Now let $n = 2$. Notice that by definition

$$\deg_\mu a_{kl}(z) \leq (\deg_\mu a_{kk}(z) + \deg_\mu a_{ll}(z))/2,$$

where $\deg_\mu$ denotes the degree of dependence on the single component z_μ. Equation (11.12) yields $\deg_k a_{kk}(z) = 0$. Hence $\deg_l a_{kl}(z) \leq 1$. Consider

$$2A_{(1)_k+(1)_l,(1)_k+(1)_l}(z) = a_{kk}(z) z_l^2 + 2a_{kl}(z) z_k z_l + a_{ll}(z) z_k^2, \quad k,l \in \{1,\dots,m\}.$$

From the preceding arguments it is now clear that also $\deg_l a_{kk}(z) = 0$, and hence $\deg a(z) = 0$. We finally have

$$B_{(1)_k+(1)_l}(z) = b_k(z) z_l + b_l(z) z_k + a_{kl}(z), \quad k,l \in \{1,\dots,m\},$$

from which we conclude that $\deg b(z) \leq 1$. □

We can relax the hypothesis on G in Theorem 11.3.1 if from now on we make the following standing assumptions: $\mathcal{Z} \subset \mathbb{R}^m$ is a cone, and b and ρ satisfy a linear growth condition

$$\|b(z)\| + \|\rho(z)\| \leq C(1 + \|z\|), \quad \forall z \in \mathcal{Z}, \tag{11.13}$$

for some constant $C \in \mathbb{R}_+$.

Theorem 11.3.2 (Maximal Degree Problem II). *Suppose that*

$$\langle a(z)v, v\rangle \geq k(z)\|v\|^2, \quad \forall v \in \mathbb{R}^m, \tag{11.14}$$

for some function $k : \mathcal{Z} \to \mathbb{R}_+$ *with*

$$\liminf_{z\in\mathcal{Z}, \|z\|\to\infty} k(z) > 0. \tag{11.15}$$

Then necessarily $n \in \{1, 2\}$.

Conditions (11.14) and (11.15) say that $a(z)$ becomes uniformly elliptic for $\|z\|$ large enough.

Proof. We shall make use of the basic inequality

$$|z^{\mathbf{i}}| \leq \|z\|^{|\mathbf{i}|}, \quad \forall z \in \mathbb{R}^m. \tag{11.16}$$

This is immediate, since

$$\frac{|z^{\mathbf{i}}|}{\|z\|^{|\mathbf{i}|}} = \left(\frac{|z_1|}{\|z\|}\right)^{i_1} \cdots \left(\frac{|z_m|}{\|z\|}\right)^{i_m} \leq 1, \quad \forall z \in \mathbb{R}^m \setminus \{0\}.$$

Now define

$$\Gamma_k(x, z) := \sum_{\mu=1}^{N} G_{\mathbf{i}_\mu}(x) \frac{\partial z^{\mathbf{i}_\mu}}{\partial z_k} \tag{11.17}$$

$$\Lambda_{kl}(x, z) = \Lambda_{lk}(x, z) := \sum_{\mu=1}^{N} G_{\mathbf{i}_\mu}(x) \frac{\partial^2 z^{\mathbf{i}_\mu}}{\partial z_k \partial z_l}. \tag{11.18}$$

Then (11.2) can be rewritten as (integration)

$$\begin{aligned}\sum_{|\mathbf{i}|=0}^{n} \left(g_{\mathbf{i}}(x) - g_{\mathbf{i}}(0)\right) z^{\mathbf{i}} = \sum_{k=1}^{m} &b_k(z)\Gamma_k(x, z) \\ &+ \frac{1}{2}\sum_{k,l=1}^{m} a_{kl}(z)\left(\Lambda_{kl}(x, z) - \Gamma_k(x, z)\Gamma_l(x, z)\right),\end{aligned} \tag{11.19}$$

Suppose now that $n > 2$. We have from (11.17)

$$\Gamma_k(x, z) = \sum_{|\mathbf{i}|=n} G_{\mathbf{i}}(x) i_k z^{\mathbf{i}-(1)_k} + \cdots =: P_k(x, z) + \cdots ,$$

where $P_k(x,z)$ is a homogeneous polynomial in z of order $n-1$, and $\cdots$ stands for lower order terms in z. By assumptions there exist $x \in \mathbb{R}_+$ and $k \in \{1,\dots,m\}$ such that $P_k(x,\cdot) \neq 0$. Choose $z^* \in \mathcal{Z}\setminus\{0\}$ with $P_k(x,z^*) \neq 0$ and set $z_\alpha := \alpha z^*$, for $\alpha > 0$. Then we have $z_\alpha \in \mathcal{Z}$ and

$$\Gamma_k(x, z_\alpha) = \alpha^{n-1} P_k(x, z^*) + \cdots,$$

where $\cdots$ denotes lower order terms in α. Consequently,

$$\lim_{\alpha\to\infty} \frac{\Gamma_k(x,z_\alpha)}{\|z_\alpha\|^{n-1}} = \frac{P_k(x,z^*)}{\|z^*\|^{n-1}} \neq 0. \tag{11.20}$$

Combining (11.14) and (11.15) with (11.20) we conclude that

$$\begin{aligned} L &:= \liminf_{\alpha\to\infty} \frac{1}{\|z_\alpha\|^{2n-2}} \langle a(z_\alpha)\Gamma(x,z_\alpha), \Gamma(x,z_\alpha)\rangle \\ &\geq \liminf_{\alpha\to\infty} k(z_\alpha) \frac{\|\Gamma(x,z_\alpha)\|^2}{\|z_\alpha\|^{2n-2}} > 0. \end{aligned} \tag{11.21}$$

On the other hand, by (11.19),

$$\begin{aligned} L &\leq \sum_{|\mathbf{i}|=0}^{n} |g_{\mathbf{i}}(x) - g_{\mathbf{i}}(0)| \frac{|z_\alpha^{\mathbf{i}}|}{\|z_\alpha\|^{2n-2}} \\ &\quad + \frac{\|b(z_\alpha)\|}{\|z_\alpha\|} \frac{\|\Gamma(x,z_\alpha)\|}{\|z_\alpha\|^{2n-3}} + \frac{1}{2} \frac{\|a(z_\alpha)\|}{\|z_\alpha\|^2} \frac{\|\Lambda(x,z_\alpha)\|}{\|z_\alpha\|^{2n-4}}, \end{aligned}$$

for all $\alpha > 0$. In view of (11.17), (11.18), (11.13) and (11.16), the right hand side converges to zero for $\alpha \to \infty$. This contradicts (11.21), hence $n \leq 2$. $\square$

11.4 Exponential-Polynomial Families

We consider the Nelson–Siegel and Svensson families. For a discussion of general exponential-polynomial families see [8].

11.4.1 Nelson–Siegel Family

Recall the form of the Nelson–Siegel curves

$$G_{NS}(x,z) = z_1 + (z_2 + z_3 x) e^{-z_4 x}.$$

Proposition 11.4.1. *There is no non-trivial diffusion process Z that is consistent with the Nelson–Siegel family. In fact, the unique solution to* (11.2) *is*

$$a(z) = 0, \quad b_1(z) = b_4(z) = 0, \quad b_2(z) = z_3 - z_2 z_4, \quad b_3(z) = -z_3 z_4.$$

The corresponding state process is

$$\begin{aligned} Z_1(t) &\equiv z_1, \\ Z_2(t) &= (z_2 + z_3 t)\, e^{-z_4 t}, \\ Z_3(t) &= z_3 e^{-z_4 t}, \\ Z_4(t) &\equiv z_4, \end{aligned}$$

where $Z(0) = (z_1, \dots, z_4)$ denotes the initial point.

Proof. Exercise. □

11.4.2 Svensson Family

Here the forward curve is

$$G_S(x, z) = z_1 + (z_2 + z_3 x) e^{-z_5 x} + z_4 x e^{-z_6 x}.$$

Proposition 11.4.2. *The only non-trivial HJM model that is consistent with the Svensson family is the Hull–White extended Vasicek short rate model*

$$dr(t) = \left(z_1 z_5 + z_3 e^{-z_5 t} + z_4 z^{-2 z_5 t} - z_5 r(t)\right) dt + \sqrt{z_4 z_5} e^{-z_5 t}\, dW^*(t),$$

where $(z_1, \dots, z_5)$ are given by the initial forward curve

$$f(0, x) = z_1 + (z_2 + z_3 x) e^{-z_5 x} + z_4 x e^{-2 z_5 x}$$

and W^* is some Brownian motion. The form of the corresponding state process Z is given in the proof below.

Proof. The consistency equation (11.2) becomes

$$\begin{aligned} q_1(x) + q_2(x) e^{-z_5 x} + q_3(x) e^{-z_6 x} \\ + q_4(x) e^{-2 z_5 x} + q_5(x) e^{-(z_5 + z_6) x} + q_6(x) e^{-2 z_6 x} = 0, \end{aligned} \quad (11.22)$$

for some polynomials $q_1, \ldots, q_6$. Indeed, we assume for the moment that

$$z_5 \neq z_6, \quad z_5 + z_6 \neq 0 \quad \text{and } z_i \neq 0 \text{ for all } i = 1, \ldots, 6. \tag{11.23}$$

Then the terms involved in (11.2) are

$$\partial_x G_S(x,z) = (-z_2 z_5 + z_3 - z_3 z_5 x)e^{-z_5 x} + (z_4 - z_4 z_6 x)e^{-z_6 x},$$

$$\nabla_z G_S(x,z) = \begin{pmatrix} 1 \\ e^{-z_5 x} \\ x e^{-z_5 x} \\ x e^{-z_6 x} \\ (-z_2 x - z_3 x^2)e^{-z_5 x} \\ -z_4 x^2 e^{-z_6 x} \end{pmatrix},$$

$$\partial_{z_i}\partial_{z_j} G_S(x,z) = 0 \quad \text{for } 1 \leq i,j \leq 4,$$

$$\nabla_z \partial_{z_5} G_S(x,z) = \begin{pmatrix} 0 \\ -x e^{-z_5 x} \\ -x^2 e^{-z_5 x} \\ 0 \\ (z_2 x^2 + z_3 x^3)e^{-z_5 x} \\ 0 \end{pmatrix}, \quad \nabla_z \partial_{z_6} G_S(x,z) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -x^2 e^{-z_6 x} \\ 0 \\ z_4 x^3 e^{-z_6 x} \end{pmatrix},$$

$$\int_0^x \nabla_z G_S(u,z)\, du = \begin{pmatrix} x \\ -\frac{1}{z_5} e^{-z_5 x} + \frac{1}{z_5} \\ \left(-\frac{x}{z_5} - \frac{1}{z_5^2}\right) e^{-z_5 x} + \frac{1}{z_5^2} \\ \left(-\frac{x}{z_6} - \frac{1}{z_6^2}\right) e^{-z_6 x} + \frac{1}{z_6^2} \\ \left(\frac{z_3}{z_5} x^2 + \left(\frac{z_2}{z_5} + \frac{2z_3}{z_5^2}\right) x + \frac{z_2}{z_5^2} + \frac{2z_3}{z_5^3}\right) e^{-z_5 x} - \frac{z_2}{z_5^2} - \frac{z_3}{z_5^3} \\ \left(\frac{z_4}{z_6} x^2 + \frac{2z_4}{z_6^2} x + \frac{2z_4}{z_6^3}\right) e^{-z_6 x} - \frac{z_4}{z_6^3} \end{pmatrix}.$$

Straightforward calculations lead to

$$q_1(x) = -a_{11}(z)x + \cdots,$$

$$q_2(x) = a_{55}(z)\frac{z_3^2}{z_5} x^4 + \cdots,$$

$$q_3(x) = a_{66}(z)\frac{z_4^2}{z_6} x^4 + \cdots,$$

$$\deg q_4,\ \deg q_5\ \deg q_6 \leq 3,$$

where $\cdots$ stands for lower order terms in x. Because of (11.23) we conclude that

$$a_{11}(z) = a_{55}(z) = a_{66}(z) = 0.$$

But a is a positive semi-definite symmetric matrix. Hence

$$a_{1j}(z) = a_{j1}(z) = a_{5j}(z) = a_{j5}(z) = a_{6j}(z) = a_{j6}(z) = 0 \quad \forall j = 1\ldots, 6.$$

Taking this into account, expression (11.22) simplifies considerably. We are left with

$$\begin{aligned} q_1(x) &= b_1(z), \\ \deg q_2(x),\, \deg q_3 &\leq 1, \\ q_4(x) &= a_{33}(z)\frac{1}{z_5}x^2 + \cdots, \\ q_5(x) &= a_{34}(z)\left(\frac{1}{z_5} + \frac{1}{z_6}\right)x^2 + \cdots, \\ q_6(x) &= a_{44}(z)\frac{1}{z_6}x^2 + \cdots. \end{aligned}$$

Because of (11.23) we know that the exponents $-2z_5$, $-(z_5 + z_6)$ and $-2z_6$ are mutually different. Hence

$$b_1(z) = a_{3j}(z) = a_{j3}(z) = a_{4j}(z) = a_{j4}(z) = 0 \quad \forall j = 1, \ldots 6.$$

Only $a_{22}(z)$ is left as strictly positive candidate among the components of $a(z)$. The remaining terms are

$$\begin{aligned} q_2(x) &= (b_3(z) + z_3 z_5)x + b_2(z) - z_3 - \frac{a_{22}(z)}{z_5} + z_2 z_5, \\ q_3(x) &= (b_4(z) + z_4 z_6)x - z_4, \\ q_4(x) &= a_{22}(z)\frac{1}{z_5}, \end{aligned}$$

while $q_1 = q_5 = q_6 = 0$.

If $2z_5 \neq z_6$ then also $a_{22}(z) = 0$. If $2z_5 = z_6$ then the condition $q_3 + q_4 = q_2 = 0$ leads to

$$\begin{aligned} a_{22}(z) &= z_4 z_5, \\ b_2(z) &= z_3 + z_4 - 2_5 z_2, \\ b_3(z) &= -z_5 z_3, \\ b_4(z) &= -2 z_5 z_4. \end{aligned}$$

We derived the above results under the assumption (11.23). But the set of z where (11.23) holds is dense $\mathcal{Z}$. By continuity of $a(z)$ and $b(z)$ in z, the above results thus extend for all $z \in \mathcal{Z}$. In particular, all Z_i's but Z_2 are deterministic; Z_1, Z_5 and Z_6 are even constant.

Thus, since

$$a(z) = 0 \quad \text{if } 2z_5 \neq z_6,$$

we only have a non-trivial process Z if

$$Z_6(t) \equiv 2Z_5(t) \equiv 2Z_5(0).$$

In that case we have, writing shortly $z_i = Z_i(0)$,

$$\begin{aligned} Z_1(t) &\equiv z_1, \\ Z_3(t) &= z_3 e^{-z_5 t}, \\ Z_4(t) &= z_4 z^{-2z_5 t} \end{aligned}$$

and

$$dZ_2(t) = \left(z_3 e^{-z_5 t} + z_4 z^{-2z_5 t} - z_5 Z_2(t)\right) dt + \sum_{j=1}^{d} \rho_{2j}(t)\, dW_j(t),$$

where $\rho_{2j}(t)$ (not necessarily deterministic) are such that

$$\sum_{j=1}^{d} \rho_{2j}^2(t) = a_{22}(Z(t)) = z_4 z_5 e^{-2z_5 t}.$$

By Lévy's characterization theorem we have that

$$W^*(t) := \sum_{j=1}^{d} \int_0^t \frac{\rho_{2j}(s)}{\sqrt{z_4 z_5} e^{-z_5 s}}\, dW_j(s)$$

is a real-valued standard Brownian motion (→ exercise). Hence the corresponding short rate process

$$r(t) = G_S(0, Z(t)) = z_1 + Z_2(t)$$

satisfies

$$dr(t) = \left(z_1 z_5 + z_3 e^{-z_5 t} + z_4 z^{-2z_5 t} - z_5 r(t)\right) dt + \sqrt{z_4 z_5} e^{-z_5 t}\, dW^*(t).$$

□

Chapter 12

Market Models

Instantaneous forward rates are not always easy to estimate, as we have seen. One may want to model other rates, such as LIBOR, directly. There has been some effort in the years after the publication of HJM [9] in 1992 to develop arbitrage-free models of other than instantaneous, continuously compounded rates. The breakthrough came 1997 with the publications of Brace–Gatarek–Musiela [5] (BGM), who succeeded to find a HJM type model inducing log-normal LIBOR rates, and Jamshidian [**?**], who developed a framework for arbitrage-free LIBOR and swap rate models not based on HJM. The principal idea of both approaches is to chose a different numeraire than the risk-free account (the latter does not even necessarily have to exist). Both approaches lead to Black's formula for either caps (LIBOR models) or swaptions (swap rate models). Because of this they are usually referred to as "market models".

To start with we consider the HJM setup, as in Chapter 9. Recall that, for a fixed δ (typically $1/4 = 3$ months), the forward δ-period LIBOR for the future date T prevailing at time t is the simple forward rate

$$L(t,T) = F(t;T,T+\delta) = \frac{1}{\delta}\left(\frac{P(t,T)}{P(t,T+\delta)} - 1\right).$$

We have seen in Chapter 9 that $P(t,T)/P(t,T+\delta)$ is a martingale for the $(T+\delta)$-forward measure $\mathbb{Q}^{T+\delta}$. In particular (see (9.8))

$$d\left(\frac{P(t,T)}{P(t,T+\delta)}\right) = \frac{P(t,T)}{P(t,T+\delta)}\sigma_{T,T+\delta}(t)\,dW^{T+\delta}(t).$$

Hence

$$dL(t,T) = \frac{1}{\delta}\, d\left(\frac{P(t,T)}{P(t,T+\delta)}\right) = \frac{1}{\delta}\frac{P(t,T)}{P(t,T+\delta)}\sigma_{T,T+\delta}(t)\, dW^{T+\delta}(t)$$
$$= \frac{1}{\delta}(\delta L(t,T)+1)\sigma_{T,T+\delta}(t)\, dW^{T+\delta}(t).$$

Now suppose there exists a deterministic $\mathbb{R}^d$-valued function $\lambda(t,T)$ such that

$$\sigma_{T,T+\delta}(t) = \frac{\delta L(t,T)}{\delta L(t,T)+1}\lambda(t,T). \tag{12.1}$$

Plugging this in the above formula, we get

$$dL(t,T) = L(t,T)\lambda(t,T)\, dW^{T+\delta}(t),$$

which is equivalent to

$$L(t,T) = L(s,T)\exp\left(\int_s^t \lambda(u,T)\, dW^{T+\delta}(u) - \frac{1}{2}\int_s^t \|\lambda(u,T)\|^2\, du\right),$$

for $s \le t \le T$. Hence the $\mathbb{Q}^{T+\delta}$-distribution of $\log L(T,T)$ conditional on $\mathcal{F}_t$ is Gaussian with mean

$$\log L(t,T) - \frac{1}{2}\int_t^T \|\lambda(s,T)\|^2\, ds$$

and variance

$$\int_t^T \|\lambda(s,T)\|^2\, ds.$$

The time t price of a caplet with reset date T, settlement date $T+\delta$ and strike rate κ is thus

$$\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_0^{T+\delta} r(s)\, ds}\delta(L(T,T)-\kappa)^+ \mid \mathcal{F}_t\right]$$
$$= P(t,T+\delta)\mathbb{E}_{\mathbb{Q}^{T+\delta}}\left[\delta(L(T,T)-\kappa)^+ \mid \mathcal{F}_t\right]$$
$$= \delta P(t,T+\delta)\left(L(t,T)\Phi(d_1(t,T)) - \kappa\Phi(d_2(t,T))\right),$$

where

$$d_{1,2}(t,T) := \frac{\log\left(\frac{L(t,T)}{\kappa}\right) \pm \frac{1}{2}\int_t^T \|\lambda(s,T)\|^2\, ds}{\left(\int_t^T \|\lambda(s,T)\|^2\, ds\right)^{\frac{1}{2}}},$$

and Φ is the standard Gaussian CDF. This is just Black's formula for the caplet price with $\sigma(t)^2$ set equal to

$$\frac{1}{T-t}\int_t^T \|\lambda(s,T)\|^2\, ds,$$

as introduced in Section 2.6!

We have thus shown that any HJM model satisfying (12.1) yields Black's formula for caplet prices. But do such HJM models exist? The answer is yes, but the construction and proof are not easy. The idea is to rewrite (12.1), using the definition of $\sigma_{T,T+\delta}(t)$, as ($\rightarrow$ exercise)

$$\int_T^{T+\delta} \sigma(t,u)\, du = \left(1 - e^{-\int_T^{T+\delta} f(t,u)\, du}\right)\lambda(t,T).$$

Differentiating in T gives

$$\begin{aligned}\sigma(t,T+\delta)& \\ &= \sigma(t,T) + (f(t,T+\delta) - f(t,T))e^{-\int_T^{T+\delta} f(t,u)\, du}\lambda(t,T) \\ &\qquad + \left(1 - e^{-\int_T^{T+\delta} f(t,u)\, du}\right)\partial_T\lambda(t,T).\end{aligned}$$

This is a recurrence relation that can be solved by forward induction, once $\sigma(t,\cdot)$ is determined on $[0,\delta)$ (typically, $\sigma(t,T)=0$ for $T\in[0,\delta)$). This gives a complicated dependence of σ on the forward curve. Now it has to be proved that the corresponding HJM equations for the forward rates have a unique and well-behaved solution. This all has been carried out by BGM [5], see also [8, Section 5.6].

12.1 Models of Forward LIBOR Rates

$\rightarrow$ MR[16](Chapter 14), Z[22](Section 4.7)

There is a more direct approach to LIBOR models without making reference to continuously compounded forward and short rates. In a sense, we place ourselves outside of the HJM framework (although HJM is often implicitly adopted). Instead of the risk neutral martingale measure we will work under forward measures; the numeraires accordingly being bond price processes.

12.1.1 Discrete-tenor Case

We fix a finite time horizon $T_M = M\delta$, for some $M \in \mathbb{N}$, and a probability space

$$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T_M]}, \mathbb{Q}^{T_M}),$$

where $\mathcal{F}_t = \mathcal{F}_t^{W^{T_M}}$ is the filtration generated by a d-dimensional Brownian motion $W^{T_M}(t)$, $t \in [0, T_M]$. The notation already suggests that $\mathbb{Q}^{T_M}$ will play the role of the T_M-forward measure. Write

$$T_m := m\delta, \quad m = 0, \ldots, M.$$

We are going to construct a model for the forward LIBOR rates with maturities $T_1, \ldots, T_{M-1}$. We take as given:

- for every $m \leq M - 1$, an $\mathbb{R}^d$-valued, bounded, deterministic function $\lambda(t, T_m)$, $t \in [0, T_m]$, which represents the volatility of $L(t, T_m)$;
- an initial strictly positive and decreasing discrete term structure

  $$P(0, T_m), \quad m = 0, \ldots, M,$$

  and hence strictly positive initial forward LIBOR rates

  $$L(0, T_m) = \frac{1}{\delta}\left(\frac{P(0, T_m)}{P(0, T_{m+1})} - 1\right), \quad m = 0, \ldots, M - 1.$$

We proceed by backward induction and *postulate* first that

$$\begin{aligned} dL(t, T_{M-1}) &= L(t, T_{M-1})\lambda(t, T_{M-1})\, dW^{T_M}(t), \quad t \in [0, T_{M-1}], \\ L(0, T_{M-1}) &= \frac{1}{\delta}\left(\frac{P(0, T_{M-1})}{P(0, T_M)} - 1\right) \end{aligned}$$

which is of course equivalent to

$$L(t, T_{M-1}) = \frac{1}{\delta}\left(\frac{P(0, T_{M-1})}{P(0, T_M)} - 1\right)\mathcal{E}_t\left(\lambda(\cdot, T_{M-1}) \cdot W^{T_M}\right).$$

Now define the bounded (why?) $\mathbb{R}^d$-valued process

$$\sigma_{T_{M-1},T_M}(t) := \frac{\delta L(t, T_{M-1})}{\delta L(t, T_{M-1}) + 1}\lambda(t, T_{M-1}), \quad t \in [0, T_{M-1}],$$

compare with (12.1).

This induces an equivalent probability measure $\mathbb{Q}^{T_{M-1}} \sim \mathbb{Q}^{T_M}$ on $\mathcal{F}_{T_{M-1}}$ via

$$\frac{d\mathbb{Q}^{T_{M-1}}}{d\mathbb{Q}^{T_M}} = \mathcal{E}_{T_{M-1}}\left(\sigma_{T_{M-1},T_M} \cdot W^{T_M}\right),$$

and by Girsanov's theorem

$$W^{T_{M-1}}(t) := W^{T_M}(t) - \int_0^t \sigma_{T_{M-1},T_M}(s)\,ds, \quad t \in [0, T_{M-1}],$$

is a $\mathbb{Q}^{T_{M-1}}$-Brownian motion.

Hence we can postulate

$$\begin{aligned} dL(t,T_{M-2}) &= L(t,T_{M-2})\lambda(t,T_{M-2})\,dW^{T_{M-1}}(t), \quad t \in [0,T_{M-2}], \\ L(0,T_{M-2}) &= \frac{1}{\delta}\left(\frac{P(0,T_{M-2})}{P(0,T_{M-1})} - 1\right), \end{aligned}$$

that is,

$$L(t,T_{M-2}) = \frac{1}{\delta}\left(\frac{P(0,T_{M-2})}{P(0,T_{M-1})} - 1\right)\mathcal{E}_t\left(\lambda(\cdot,T_{M-2}) \cdot W^{T_{M-1}}\right),$$

and define the bounded $\mathbb{R}^d$-valued process

$$\sigma_{T_{M-2},T_{M-1}}(t) := \frac{\delta L(t,T_{M-2})}{\delta L(t,T_{M-2}) + 1}\lambda(t,T_{M-2}), \quad t \in [0,T_{M-2}],$$

yielding an equivalent probability measure $\mathbb{Q}^{T_{M-2}} \sim \mathbb{Q}^{T_{M-1}}$ on $\mathcal{F}_{T_{M-2}}$ via

$$\frac{d\mathbb{Q}^{T_{M-2}}}{d\mathbb{Q}^{T_{M-1}}} = \mathcal{E}_{T_{M-2}}\left(\sigma_{T_{M-2},T_{M-1}} \cdot W^{T_{M-1}}\right),$$

and the $\mathbb{Q}^{T_{M-2}}$-Brownian motion

$$W^{T_{M-2}}(t) := W^{T_{M-1}}(t) - \int_0^t \sigma_{T_{M-2},T_{M-1}}(s)\,ds, \quad t \in [0,T_{M-2}].$$

Repeating this procedure leads to a family of log-normal martingales $(L(t,T_m))_{t\in[0,T_m]}$ under their respective measures $\mathbb{Q}^{T_m}$.

Bond Prices

What about bond prices? For all $m = 1, \dots, M$, we then can *define* the forward price process

$$\frac{P(t, T_{m-1})}{P(t, T_m)} := \delta L(t, T_{m-1}) + 1, \quad t \in [0, T_{m-1}].$$

Since

$$\begin{aligned} d\left(\frac{P(t, T_{m-1})}{P(t, T_m)}\right) &= \delta\, dL(t, T_{m-1}) = \delta L(t, T_{m-1})\lambda(t, T_{m-1})\, dW^{T_m}(t) \\ &= \frac{P(t, T_{m-1})}{P(t, T_m)} \sigma_{T_{m-1}, T_m}(t)\, dW^{T_m}(t) \end{aligned}$$

we get that

$$\frac{P(t, T_{m-1})}{P(t, T_m)} = \frac{P(0, T_{m-1})}{P(0, T_m)} \mathcal{E}_t\left(\sigma_{T_{m-1}, T_m} \cdot W^{T_m}\right), \quad t \in [0, T_{m-1}],$$

which is a $\mathbb{Q}^{T_m}$-martingale.

From this we can derive, for $0 \le i < j \le m$,

$$P(T_i, T_j) = \prod_{m=i+1}^{j} \frac{P(T_i, T_m)}{P(T_i, T_{m-1})} = \prod_{m=i+1}^{j} \frac{1}{\delta L(T_i, T_{m-1}) + 1}. \tag{12.2}$$

However, it is not possible to uniquely determine the continuous time dynamics of a bond price $P(t, T_m)$ in the discrete-tenor model of forward LIBOR rates. The knowledge of forward LIBOR rates for all maturities $T \in [0, T_{M-1}]$ is necessary.

LIBOR Dynamics under Different Measures

We are interested in finding the dynamics of $L(t, T_m)$ under any of the forward measures $\mathbb{Q}^{T_k}$.

Lemma 12.1.1. *Let $0 \le m \le M-1$ and $0 \le k \le M$. Then the dynamics of $L(t,T_m)$ under $\mathbb{Q}^{T_k}$ is given according to the three cases*

$$k < m+1: \quad \frac{dL(t,T_m)}{L(t,T_m)} = \lambda(t,T_m) \cdot \sum_{l=k}^{m} \sigma_{T_l,T_{l+1}}(t)\,dt + \lambda(t,T_m)\,dW^{T_k}(t);$$

$$k = m+1: \quad \frac{dL(t,T_m)}{L(t,T_m)} = \lambda(t,T_m)\,dW^{T_{m+1}}(t);$$

$$k > m+1: \quad \frac{dL(t,T_m)}{L(t,T_m)} = -\lambda(t,T_m) \cdot \sum_{l=m+1}^{k-1} \sigma_{T_l,T_{l+1}}(t)dt + \lambda(t,T_m)dW^{T_k}(t),$$

for $t \in [0, T_k \wedge T_m]$.

Proof. This follows from the equality

$$W^{T_i}(t) = W^{T_j}(t) - \sum_{l=i}^{j-1} \int_0^t \sigma_{T_l,T_{l+1}}(s)\,ds, \quad t \in [0,T_i],$$

for all $1 \le i < j \le M$. □

Derivative Pricing

Here is a useful formula, which can be combined with (12.2).

Lemma 12.1.2. *Let $X \in L^1(\mathbb{Q}^{T_m})$ be a T_m-contingent claim, $m \le M$. Then its price $\pi(t)$ at $t \le T_m$ is given by*

$$\begin{aligned}\pi(t) &= P(t,T_m)\mathbb{E}_{\mathbb{Q}^{T_m}}\left[X \mid \mathcal{F}_t\right] \\ &= P(t,T_n)\mathbb{E}_{\mathbb{Q}^{T_n}}\left[\frac{X}{P(T_m,T_n)} \mid \mathcal{F}_t\right],\end{aligned}$$

for all $m < n \le M$ (strictly speaking, this formula makes sense only for $t = T_j$, $0 \le j \le m$, since we know $P(t,T_n)$ only for such t).

Proof. Notice that

$$\frac{d\mathbb{Q}^{T_k}}{d\mathbb{Q}^{T_{k+1}}}|_{\mathcal{F}_t} = \mathcal{E}_t\left(\sigma_{T_k,T_{k+1}} \cdot W^{T_{k+1}}\right) = \frac{P(0,T_{k+1})}{P(0,T_k)}\frac{P(t,T_k)}{P(t,T_{k+1})}, \quad t \in [0,T_k].$$

Hence

$$\frac{d\mathbb{Q}^{T_m}}{d\mathbb{Q}^{T_n}}|_{\mathcal{F}_t} = \prod_{k=m}^{n-1} \frac{d\mathbb{Q}^{T_k}}{d\mathbb{Q}^{T_{k+1}}}|_{\mathcal{F}_t} = \prod_{k=m}^{n-1} \frac{P(0,T_{k+1})}{P(0,T_k)} \frac{P(t,T_k)}{P(t,T_{k+1})}$$
$$= \frac{P(0,T_n)}{P(0,T_m)} \frac{P(t,T_m)}{P(t,T_n)}.$$

Bayes' rule now yields the assertion, since the first equality was derived in Proposition 9.1.2 (strictly speaking, we assumed there the existence of a savings account. But even if there is no risk neutral but only forward measures, the reasoning in Section 9.1 makes it clear that (9.3) is the arbitrage-free price of X). □

Swaptions

Consider a payer swaption with nominal 1, strike rate K, maturity T_μ and underlying tenor T_μ, $T_{\mu+1}, \dots, T_\nu$ (T_μ is the first reset date and T_ν the maturity of the underlying swap), for some positive integers $\mu < \nu \le M$. Its payoff at maturity is

$$\delta \left(\sum_{m=\mu}^{\nu-1} P(T_\mu, T_m)(L(T_\mu, T_m) - K) \right)^+ .$$

The swaption price at $t = 0$ (for simplicity) therefore

$$\pi(0) = \delta P(0, T_\mu) \mathbb{E}_{\mathbb{Q}^{T_\mu}} \left[\left(\sum_{m=\mu}^{\nu-1} P(T_\mu, T_m)(L(T_\mu, T_m) - K) \right)^+ \right].$$

To compute $\pi(0)$ we thus need to know the joint distribution of

$$L(T_\mu, T_\mu),\ L(T_\mu, T_{\mu+1}), \dots, L(T_\mu, T_{\nu-1})$$

under the measure $\mathbb{Q}^{T_\mu}$. This cannot be done analytically anymore, so one has to resort to numerical procedures.

We sketch here the Monte Carlo method. Notice that by Lemma 12.1.1, Itô's formula and the definition of $\sigma_{T_l,T_{l+1}}(t)$ we have

$$d \log L(t, T_m) = \left(\lambda(t, T_m) \cdot \sum_{l=\mu}^{m} \frac{\delta L(t, T_l)}{\delta L(t, T_l) + 1} \lambda(t, T_l) - \frac{1}{2} \|\lambda(t, T_m)\|^2 \right) dt$$
$$+ \lambda(t, T_m)\, dW^{T_\mu}(t),$$

for $t \in [0, T_\mu]$ and $m = \mu, \dots, \nu - 1$. Write $\alpha(t, T_m)$ for the above drift term, and let $t_i = \frac{i}{n} T_\mu$, $i = 0, \dots, n$, $n \in \mathbb{N}$ large enough, be a partition of $[0, T_\mu]$. Then we can approximate

$$\begin{aligned}\log L(t_{i+1}, T_m) &= \log L(t_i, T_m) + \int_{t_i}^{t_{i+1}} \alpha(s, T_m)\, ds + \int_{t_i}^{t_{i+1}} \lambda(s, T_m)\, dW^{T_\mu}(s) \\ &\approx \log L(t_i, T_m) + \alpha(t_i, T_m)\frac{1}{n} + \zeta_m(i),\end{aligned}$$

where

$$\zeta_m(i) := \int_{t_i}^{t_{i+1}} \lambda(s, T_m)\, dW^{T_\mu}(s),$$

such that $\zeta(i) = (\zeta_\mu(i), \dots, \zeta_{\nu-1}(i))$, $i = 0, \dots, n-1$, are independent Gaussian $(\nu - \mu)$-vectors with mean zero and covariance matrix

$$Cov[\zeta_k(i), \zeta_l(i)] = \int_{t_i}^{t_{i+1}} \lambda(s, T_k) \cdot \lambda(s, T_l)\, ds,$$

which can easily be simulated.

Forward Swap Measure

We consider the above payer swap with reset dates $T_\mu, \dots, T_{\nu-1}$ and cashflow dates $T_{\mu+1}, \dots, T_\nu$ (= maturity of the swap). The corresponding forward swap rate at time $t \le T_\mu$ is

$$R_{swap}(t) = \frac{P(t, T_\mu) - P(t, T_\nu)}{\delta \sum_{k=\mu+1}^{\nu} P(t, T_k)} = \frac{1 - \frac{P(t,T_\nu)}{P(t,T_\mu)}}{\delta \sum_{k=\mu+1}^{\nu} \frac{P(t,T_k)}{P(t,T_\mu)}}. \tag{12.3}$$

Since for any $0 \le l < m \le M$

$$\frac{P(t, T_l)}{P(t, T_m)} = \frac{P(t, T_l)}{P(t, T_{l+1})} \cdots \frac{P(t, T_{m-1})}{P(t, T_m)} = \prod_{i=l}^{m-1} (1 + \delta L(t, T_i)),$$

$R_{swap}(t)$ is given in terms of the above constructed LIBOR rates.

Define the positive $\mathbb{Q}^{T_\mu}$-martingale

$$D(t) := \sum_{k=\mu+1}^{\nu} \frac{P(t, T_k)}{P(t, T_\mu)}, \quad t \in [0, T_\mu].$$

This induces an equivalent probability measure $\mathbb{Q}^{swap} \sim \mathbb{Q}^{T_\mu}$, the *forward swap measure*, on $\mathcal{F}_{T_\mu}$ by

$$\frac{d\mathbb{Q}^{swap}}{d\mathbb{Q}^{T_\mu}} = \frac{D(T_\mu)}{D(0)}.$$

Lemma 12.1.3. *The forward swap rate process $R_{swap}(t)$, $t \in [0, T_\mu]$, is a $\mathbb{Q}^{swap}$-martingale.*

Proof. Let $0 \le m \le M$ and $0 \le s \le t \le T_m \wedge T_\mu$. Then

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}^{swap}}\left[\frac{P(t,T_m)}{P(t,T_\mu)D(t)} \mid \mathcal{F}_s\right] &= \frac{1}{D(s)}\mathbb{E}_{\mathbb{Q}^{T_\mu}}\left[\frac{P(t,T_m)}{P(t,T_\mu)D(t)}D(t) \mid \mathcal{F}_s\right] \\
&= \frac{1}{D(s)}\frac{P(s,T_m)}{P(s,T_\mu)}.
\end{aligned}$$

Now the lemma follows (set $m = 0, \mu$) by (12.3). □

The payoff at maturity of the above swaption can be written as

$$\delta D(T_\mu)\left(R_{swap}(T_0) - K\right)^+.$$

Hence the price is

$$\begin{aligned}
\pi(0) &= \delta P(0,T_\mu)\mathbb{E}_{\mathbb{Q}^{T_\mu}}\left[D(T_\mu)\left(R_{swap}(T_0) - K\right)^+\right] \\
&= \delta P(0,T_\mu)D(0)\mathbb{E}_{\mathbb{Q}^{swap}}\left[\left(R_{swap}(T_0) - K\right)^+\right] \\
&= \delta \sum_{k=\mu+1}^{\nu} P(0,T_k)\mathbb{E}_{\mathbb{Q}^{swap}}\left[\left(R_{swap}(T_0) - K\right)^+\right].
\end{aligned}$$

Lemma 12.1.3 tells us that R_{swap} is a positive $\mathbb{Q}^{swap}$-martingale and hence of the form

$$dR_{swap}(t) = R_{swap}(t)\rho^{swap}(t)\,dW^{swap}(t), \quad t \in [0, T_\mu],$$

for some $\mathbb{Q}^{swap}$-Brownian motion W^{swap} and some swap volatility process ρ^{swap}. Hence, under the hypothesis

(H) $\rho^{swap}(t)$ is deterministic, we would have that $\log R_{swap}(T_\mu)$ is Gaussian distributed under $\mathbb{Q}^{swap}$ with mean

$$\log R_{swap}(0) - \frac{1}{2} \int_0^{T_\mu} \|\rho^{swap}(t)\|^2 \, dt$$

and variance

$$\int_0^{T_\mu} \|\rho^{swap}(t)\|^2 \, dt.$$

The swaption price would then be

$$\pi(0) = \delta \sum_{k=\mu+1}^{\nu} P(0, T_k) \left(R_{swap}(0)\Phi(d_1) - K\Phi(d_2) \right),$$

with

$$d_{1,2} := \frac{\log \left(\frac{R_{swap}(0)}{K} \right) \pm \frac{1}{2} \int_0^{T_\mu} \|\rho^{swap}(t)\|^2 \, dt}{\left(\int_0^{T_\mu} \|\rho^{swap}(t)\|^2 \, dt \right)^{\frac{1}{2}}}.$$

This is Black's formula with volatility σ^2 given by

$$\frac{1}{T_\mu} \int_0^{T_\mu} \|\rho^{swap}(t)\|^2 \, dt.$$

However, one can show that ρ^{swap} cannot be deterministic in our log-normal LIBOR setup. So hypothesis **(H)** does not hold. For swaption pricing it would be natural to model the forward swap rates directly and postulate that they are log-normal under the forward swap measures. This approach has been carried out by Jamshidian [**?**] and others. It could be shown, however, that then the forward LIBOR rate volatility cannot be deterministic. So either one gets Black's formula for caps or for swaptions, but not simultaneously for both. Put in other words, when we insist on log-normal forward LIBOR rates then swaption prices have to be approximated. One possibility is to use Monte Carlo methods. Another way (among many others) is now sketched below.

We have seen in Section 2.4.3 that the forward swap rate can be written as weighted sum of forward LIBOR rates

$$R_{swap}(t) = \sum_{m=\mu+1}^{\nu} w_m(t) L(t, T_{m-1}),$$

with weights

$$w_m(t) = \frac{P(t,T_m)}{D(t)P(t,T_\mu)} = \frac{\frac{1}{1+\delta L(t,T_\mu)} \cdots \frac{1}{1+\delta L(t,T_{m-1})}}{\sum_{j=\mu+1}^{\nu} \frac{1}{1+\delta L(t,T_\mu)} \cdots \frac{1}{1+\delta L(t,T_{j-1})}}.$$

According to empirical studies, the variability of the w_m's is small compared to the variability of the forward LIBOR rates. We thus approximate $w_m(t)$ by its deterministic initial value $w_m(0)$. So that

$$R_{swap}(t) \approx \sum_{m=\mu+1}^{\nu} w_m(0) L(t,T_{m-1}),$$

and hence, under the T_μ-forward measure $\mathbb{Q}^{T_\mu}$

$$dR_{swap}(t) \approx (\cdots)\,dt + \sum_{m=\mu+1}^{\nu} w_m(0) L(t,T_{m-1}) \lambda(t,T_{m-1})\,dW^{T_\mu}, \quad t \in [0,T_\mu].$$

We obtain for the forward swap volatility

$$\begin{aligned} \|\rho^{swap}(t)\|^2 &= \frac{d\,\langle \log R_{swap}, \log R_{swap}\rangle_t}{dt} \\ &\approx \sum_{k,l=\mu+1}^{\nu} \frac{w_k(0) w_l(0) L(t,T_{k-1}) L(t,T_{l-1}) \lambda(t,T_{k-1}) \cdot \lambda(t,T_{l-1})}{R_{swap}^2(t)}. \end{aligned}$$

In a further approximation we replace all random variables by their time 0 values, such that the quadratic variation of $\log R_{swap}(t)$ becomes approximatively deterministic

$$\|\rho^{swap}(t)\|^2 \approx \sum_{k,l=\mu+1}^{\nu} \frac{w_k(0) w_l(0) L(0,T_{k-1}) L(0,T_{l-1}) \lambda(t,T_{k-1}) \cdot \lambda(t,T_{l-1})}{R_{swap}^2(0)}.$$

Denote the square root of the right hand side by $\tilde{\rho}^{swap}(t)$, and define the $\mathbb{Q}^{swap}$-Brownian motion (Lévy's characterization theorem)

$$W^*(t) := \int_0^t \sum_{j=1}^{d} \frac{\rho_j^{swap}(s)}{\|\rho^{swap}(s)\|}\,dW_j^{swap}(s), \quad t \in [0,T_\mu].$$

Then we have

$$\begin{aligned} dR_{swap}(t) &= R_{swap}(t)\|\rho^{swap}(t)\|\, dW^*(t) \\ &\approx R_{swap}(t)\tilde{\rho}^{swap}(t)\, dW^*(t). \end{aligned}$$

Hence we can approximate the swaption price in our log-normal forward LIBOR model by Black's swaption price formula where σ^2 is to be replaced by

$$\frac{1}{T_\mu}\int_0^{T_\mu} \sum_{k,l=\mu+1}^{\nu} \frac{w_k(0)w_l(0)L(0,T_{k-1})L(0,T_{l-1})\lambda(t,T_{k-1})\cdot\lambda(t,T_{l-1})}{R_{swap}^2(0)}\, dt.$$

This is "Rebonato's formula", since it originally appears in his book R[19]. The goodness of this approximation has been tested numerically by several authors, see BM[6](Chapter 8). They conclude that "the approximation is satisfactory in general".

Implied Savings Account

Given the LIBOR $L(T_i, T_i)$ for period $[T_i, T_{i+1}]$, for all $i = 0, \ldots, M-1$, we can define the discrete-time, *implied savings account* process

$$\begin{aligned} B^*(0) &:= 1, \\ B^*(T_m) &:= (1 + \delta L(T_{m-1}, T_{m-1}))B^*(T_{m-1}), \quad m = 1, \ldots, M, \end{aligned}$$

that is,

$$B^*(T_n) = B^*(T_m)\prod_{k=m}^{n-1} \frac{1}{P(T_k, T_{k+1})}, \quad m < n \le M.$$

Hence $B^*(T_m)$ can be interpreted as the cash amount accumulated up to time T_m by rolling over a series of zero-coupon bonds with the shortest maturities available.

By construction, B^* is a strictly increasing and predictable process with respect to the discrete-time filtration $(\mathcal{F}_{T_m})$, that is,

$$B^*(T_m) \text{ is } \mathcal{F}_{T_{m-1}}\text{-measurable, for all } m = 1, \ldots, M.$$

Lemma 12.1.4. *For all* $0 \le m \le M$ *we have*

$$\mathbb{E}_{\mathbb{Q}^{T_M}}\left[B^*(T_M) \mid \mathcal{F}_{T_m}\right] = \frac{B^*(T_m)}{P(T_m, T_M)}.$$

Proof. Exercise. □

Lemma 12.1.4 yields in particular

$$\mathbb{E}_{\mathbb{Q}^{T_M}}\left[B^*(T_M)P(0,T_M)\right] = 1 \quad \text{and} \quad B^*(T_M)P(0,T_M) > 0,$$

so that we can define the equivalent probability measure $\mathbb{Q}^* \sim \mathbb{Q}^{T_M}$ on $\mathcal{F}_{T_M}$ by

$$\frac{d\mathbb{Q}^*}{d\mathbb{Q}^{T_M}} = B^*(T_M)P(0,T_M).$$

$\mathbb{Q}^*$ can be interpreted as risk neutral martingale measure since

$$P(T_k,T_l) = \mathbb{E}_{\mathbb{Q}^*}\left[\frac{B^*(T_k)}{B^*(T_l)} \mid \mathcal{F}_{T_k}\right], \quad 0 \le k \le l \le M. \tag{12.4}$$

Indeed, in view of Lemma 12.1.4 we have for $m \le M$

$$\frac{d\mathbb{Q}^*}{d\mathbb{Q}^{T_M}}|_{\mathcal{F}_{T_m}} = B^*(T_m)\frac{P(0,T_M)}{P(T_m,T_M)}.$$

Hence (Bayes again)

$$\mathbb{E}_{\mathbb{Q}^*}\left[\frac{B^*(T_k)}{B^*(T_l)} \mid \mathcal{F}_{T_k}\right] = \frac{\mathbb{E}_{\mathbb{Q}^{T_M}}\left[\frac{B^*(T_k)}{B^*(T_l)}\frac{B^*(T_l)}{P(T_l,T_M)} \mid \mathcal{F}_{T_k}\right]}{\frac{B^*(T_k)}{P(T_k,T_M)}} = P(T_k,T_l),$$

which proves (12.4). Put in other words, (12.4) shows that for any $0 \le l \le M$ the discrete-time process

$$\left(\frac{P(T_k,T_l)}{B^*(T_k)}\right)_{k=0,\dots,l}$$

is a $\mathbb{Q}^*$-martingale with respect to $(\mathcal{F}_{T_k})$.

12.1.2 Continuous-tenor Case

We now specify the continuum of all forward LIBOR rates $L(t,T)$, for $T \in [0,T_{M-1}]$. Given the discrete-tenor skeleton constructed in the previous section, it is enough to fill the gaps between the T_js. Each forward LIBOR rate $L(t,T)$ will follow a lognormal process under the forward measure for the date $T+\delta$.

The stochastic basis is the same as before. In addition, we now need a continuum of initial dates:

- for every $T \in [0, T_{M-1}]$, an $\mathbb{R}^d$-valued, bounded, deterministic function $\lambda(t, T)$, $t \in [0, T]$, which represents the volatility of $L(t, T)$;
- an initial strictly positive and decreasing term structure

$$P(0, T), \quad T \in [0, T_M],$$

and hence an initial strictly positive forward LIBOR curve

$$L(0, T) = \frac{1}{\delta}\left(\frac{P(0, T)}{P(0, T + \delta)} - 1\right), \quad T \in [0, T_{M-1}].$$

First, we construct a discrete-tenor model for $L(t, T_m)$, $m = 0, \ldots, M-1$, as in the previous section.

Second, we focus on the forward measures for dates $T \in [T_{M-1}, T_M]$. We do not have to take into account forward LIBOR rates for these dates, since they are not defined there. However, we are given the values of the implied savings account $B^*(T_{M-1})$ and $B^*(T_M)$ and the probability measure $\mathbb{Q}^*$. By monotonicity there exists a unique deterministic increasing function

$$\alpha : [T_{M-1}, T_M] \to [0, 1]$$

with $\alpha(T_{M-1}) = 0$ and $\alpha(T_M) = 1$, such that

$$\log B^*(T) := (1 - \alpha(T)) \log B^*(T_{M-1}) + \alpha(T) \log B^*(T_M)$$

satisfies

$$P(0, T) = \mathbb{E}_{\mathbb{Q}^*}\left[\frac{1}{B^*(T)}\right], \quad \forall T \in [T_{M-1}, T_M].$$

Let $T \in [T_{M-1}, T_M]$. Since ($\to$ exercise) $B^*(T)$ is $\mathcal{F}_T$-measurable, strictly positive and

$$\mathbb{E}_{\mathbb{Q}^*}\left[\frac{1}{B^*(T)P(0, T)}\right] = 1$$

we can define the T-forward measure $\mathbb{Q}^T \sim \mathbb{Q}^*$ on $\mathcal{F}_T$ by

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}^*} = \frac{1}{B^*(T)P(0, T)}.$$

Then we have

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}^{T_M}} = \frac{d\mathbb{Q}^T}{d\mathbb{Q}^*}\frac{d\mathbb{Q}^*}{d\mathbb{Q}^{T_M}} = \frac{B^*(T_M)P(0, T_M)}{B^*(T)P(0, T)}.$$

By the representation theorem for $\mathbb{Q}^{T_M}$-martingales there exists a unique $\sigma_{T,T_M} \in \mathcal{L}$ such that (→ exercise)

$$
\begin{aligned}
\frac{d\mathbb{Q}^T}{d\mathbb{Q}^{T_M}}|_{\mathcal{F}_t} &= \mathbb{E}_{\mathbb{Q}^{T_M}}\left[\frac{B^*(T_M)P(0,T_M)}{B^*(T)P(0,T)} \mid \mathcal{F}_t\right] \\
&= \exp\left(\int_0^t \sigma_{T,T_M}(s)\, dW^{T_M}(s) - \frac{1}{2}\int_0^t \|\sigma_{T,T_M}(s)\|^2\, ds\right) \\
&= \mathcal{E}_t\left(\sigma_{T,T_M} \cdot W^{T_M}\right),
\end{aligned}
$$

for $t \in [0,T]$. Girsanov's theorem tells us that

$$
W^T(t) := W^{T_M}(t) - \int_0^t \sigma_{T,T_M}(s)\, ds, \quad t \in [0,T],
$$

is a $\mathbb{Q}^T$-Brownian motion.

Third, since $T \in [T_{M-1}, T_M]$ was arbitrary, we can now define the forward LIBOR process $L(t,T)$ for any $T \in [T_{M-2}, T_{M-1}]$ as

$$
\begin{aligned}
dL(t,T) &= L(t,T)\lambda(t,T)\, dW^{T+\delta}(t), \\
L(0,T) &= \frac{1}{\delta}\left(\frac{P(0,T)}{P(0,T+\delta)} - 1\right).
\end{aligned}
$$

This in turn defines the positive and bounded process

$$
\sigma_{T,T+\delta}(t) := \frac{\delta L(t,T)}{\delta L(t,T)+1}\lambda(t,T), \quad t \in [0,T],
$$

for any $T \in [T_{M-2}, T_{M-1}]$. The forward measures for $T \in [T_{M-2}, T_{M-1}]$ are now given by

$$
\frac{d\mathbb{Q}^T}{d\mathbb{Q}^{T+\delta}} = \mathcal{E}_T\left(\sigma_{T,T+\delta} \cdot W^{T+\delta}\right).
$$

Hence we have (→ exercise)

$$
\begin{aligned}
\frac{d\mathbb{Q}^T}{d\mathbb{Q}^{T_M}}|_{\mathcal{F}_t} &= \frac{d\mathbb{Q}^T}{d\mathbb{Q}^{T+\delta}}|_{\mathcal{F}_t} \frac{d\mathbb{Q}^{T+\delta}}{d\mathbb{Q}^{T_M}}|_{\mathcal{F}_t} \\
&= \mathcal{E}_t\left(\sigma_{T,T+\delta} \cdot W^{T+\delta}\right) \mathcal{E}_t\left(\sigma_{T+\delta,T_M} \cdot W^{T_M}\right) \\
&= \mathcal{E}_t\left(\sigma_{T,T_M} \cdot W^{T_M}\right), \quad t \in [0,T],
\end{aligned}
$$

for any $T \in [T_{M-2}, T_{M-1}]$, where

$$\sigma_{T,T_M} := \sigma_{T,T+\delta} + \sigma_{T+\delta,T_M}.$$

Proceeding by backward induction yields the forward measure $\mathbb{Q}^T$ and $\mathbb{Q}^T$-Brownian motion W^T for all $T \in [0, T_M]$, and forward LIBOR rates $L(t,T)$ for all $T \in [0, T_{M-1}]$.

This way, we obtain the zero-coupon bond prices for all maturities. Indeed, for any $0 \leq T \leq S \leq T_M$, it is reasonable to define (why?) the forward price process

$$\begin{aligned}
\frac{P(t,S)}{P(t,T)} &:= \frac{P(0,S)}{P(0,T)} \frac{d\mathbb{Q}^S}{d\mathbb{Q}^T}|_{\mathcal{F}_t} \\
&= \frac{P(0,S)}{P(0,T)} \frac{d\mathbb{Q}^S}{d\mathbb{Q}^{T_M}}|_{\mathcal{F}_t} \frac{d\mathbb{Q}^{T_M}}{d\mathbb{Q}^T}|_{\mathcal{F}_t} \\
&= \frac{P(0,S)}{P(0,T)} \mathcal{E}_t\left(-\sigma_{T,S} \cdot W^T\right), \quad t \in [0,T],
\end{aligned}$$

where ($\rightarrow$ exercise)

$$\sigma_{T,S} := \sigma_{T,T_M} - \sigma_{S,T_M}.$$

In particular, for $t = T$ we get

$$P(T,S) = \frac{P(0,S)}{P(0,T)} \mathcal{E}_T\left(-\sigma_{T,S} \cdot W^T\right).$$

Notice that now $P(T,S)$ may be greater than 1, unless $S - T = m\delta$ for some integer m. Hence even though all δ-period forward LIBOR rates $L(t,T)$ are positive, there may be negative interest rates for other than δ periods.

Chapter 13

Default Risk

$\rightarrow$ [**?**, Chapter 2], [1], etc.

So far bond price processes $P(t,T)$ had the property that $P(T,T) = 1$. That is, the payoff was certain, there was no risk of default of the issuer. This may be the case for treasury bonds. Corporate bonds however may bear a substantial risk of default. Investors should be adequately compensated by a risk premium, which is reflected by a higher yield on the bond.

For the modelling of credit risk we have to consider the following risk elements:

- Default probabilities: probability that the debtor will default on its obligations to repay its debt.
- Recovery rates: proportion of value delivered after default has occurred.
- Transition probabilities: between credit ratings (credit migration).

Usually one has to model objective (for the rating) and risk-neutral (for the pricing) probabilities.

13.1 Transition and Default Probabilities

There are three main approaches to the modelling of transition and default probabilities:

- Historical method: rating agencies determine default and transition probabilities by counting defaults that actually occurred in the past for different rating classes.

- Structural approach: models the value of a firm's assets. Default is when this value hits a certain lower bound. Goes back to Merton (1974) [15].

- Intensity based method: default is specified exogenously by a stopping time with given intensity process.

We briefly discuss the first two approaches in this section. The intensity based method is treated in more detail in Section 13.2 below.

13.1.1 Historical Method

Rating agencies provide timely, objective information and credit analysis of obligors. Usually they operate without government mandate and are independent of any investment banking firm or similar organization. Among the biggest US agencies are Moody's Investors Service and Standard&Poor's (S&P).

After issuance and assignment of the initial obligor's rating, the rating agency regularly checks and adjusts the rating. If there is a tendency observable that may affect the rating, the obligor is set on the Rating Review List (Moody's) or the Credit Watch List (S&P). The number of Moody's rated obligors has increased from 912 in 1960 to 3841 in 1997.

The formal definition of default and transition rates is the following.

Definition 13.1.1. 1. *The historical one-year* default rate*, based on the time frame* $[Y_0, Y_1]$*, for an* R*-rated issuer is*

$$d_R := \frac{\sum_{y=Y_0}^{Y_1} M_R(y)}{\sum_{y=Y_0}^{Y_1} N_R(y)},$$

where $N_R(y)$ *is the number of issuers with rating* R *at beginning of year* y*, and* $M_R(y)$ *is the number of issuers with rating* R *at beginning of year* y *which defaulted in that year.*

2. *The historical one-year* transition rate *from rating* R *to* R'*, based on the time frame* $[Y_0, Y_1]$*, is*

$$tr_{R,R'} := \frac{\sum_{y=Y_0}^{Y_1} M_{R,R'}(y)}{\sum_{y=Y_0}^{Y_1} N_R(y)},$$

Table 13.1: Rating symbols.

S&P	Moody's	Interpretation
Investment-grade ratings		
AAA	Aaa	Highest quality, extremely strong
AA+ AA AA-	Aa1 Aa2 Aa3	High quality
A+ A A-	A1 A2 A3	Strong payment capacity
BBB+ BBB BBB-	Baa1 Baa2 Baa3	Adequate payment capacity
Speculative-grade ratings		
BB+ BB BB-	Ba1 Ba2 Ba3	Likely to fulfill obligations ongoing uncertainty
B+ B B-	B1 B2 B3	High risk obligations
CCC+ CCC CCC- CC	Caa1 Caa2 Caa3	Current vulnerability to default
C D	Ca	In bankruptcy or default or other marked shortcoming

where $N_R(y)$ is as above, and $M_{R,R'}(y)$ is the number of issuers with rating R at beginning of year y and R' at the end of that year.

Transition rates are gathered in a *transition matrix* as shown in Table 13.2.

The historical method has several shortcomings:

- It neglects the default rate volatility. Transition and default probabilities are dynamic and vary over time, depending on economic conditions.

Table 13.2: S&P's one-year transition and default rates, based on the time frame [1980,2000] (Standard&Poor's, Ratings Performance 2000, see http://financialcounsel.com/Articles/Investment/ARTINV0000069-2000Ratings.pdf).

	Rating at end of year (R')							
Initial rating (R)	AAA	AA	A	BBB	BB	B	CCC	D
AAA	93.66	5.83	0.40	0.09	0.03	0.00	0.00	0.00
AA	0.66	91.72	6.94	0.49	0.06	0.09	0.02	0.01
A	0.07	2.25	91.76	5.18	0.49	0.20	0.01	0.04
BBB	0.03	0.26	4.83	89.24	4.44	0.81	0.16	0.24
BB	0.03	0.06	0.44	6.66	83.23	7.46	1.05	1.08
B	0.00	0.10	0.32	0.46	5.72	83.62	3.84	5.94
CCC	0.15	0.00	0.29	0.88	1.91	10.28	61.23	25.26

- It neglects cross-country differences and business cycle effects.
- Rating agencies react too slow to change ratings. There is a systematic overestimation of $tr_{R,R}$ and d_R, and hence underestimation of $tr_{R,R'}$ for some $R \neq R'$.

13.1.2 Structural Approach

Merton [15] proposed a simple capital structure of a firm consisting of equity and one type of zero coupon debt with promised terminal constant payoff $X > 0$ at maturity T. The obligor (=the firm) defaults by T if the total market value of its assets $V(T)$ at T is less than its liabilities X. Thus the probability of default by time T conditional on the information available at $t \leq T$ is

$$p_d(t,T) = \mathbb{P}\left[V(T) < X \mid \mathcal{F}_t\right],$$

with respect to some stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$. The dynamics of $V(t)$ is modelled as geometric Brownian motion

$$\frac{dV(t)}{V(t)} = \mu\, dt + \sigma\, dW(t), \quad t \in [0,T],$$

that is

$$V(T) = V(t)\exp\left(\sigma(W(T)-W(t)) + \left(\mu - \frac{1}{2}\sigma^2\right)(T-t)\right), \quad t \in [0,T].$$

Then we have

$$p_d(t,T) = \Phi\left(\frac{\log\left(\frac{X}{V(t)}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right), \quad t \in [0,T].$$

If the firm value process $V(t)$ is continuous, as in the Merton approach, the instantaneous probability of default $(\partial_T^+ p_d(t,T)|_{T=t})$ is zero. To include "unexpected" defaults one has to consider firm value processes with jumps. Zhou (1997) models $V(t)$ as jump-diffusion process

$$V(T) = V(t)\left(\prod_{j=N(t)+1}^{N(T)} e^{Z_j}\right) e^{\left(\mu - \frac{\sigma^2}{2}\right)(T-t) + \sigma(W(T)-W(t))},$$

where $N(t)$ is a Poisson process with intensity λ and $Z_1, Z_2, \ldots$ is a sequence of i.i.d. Gaussian $\mathcal{N}(m,\rho^2)$ distributed random variables. It is assumed that W, N and Z_j are mutually independent. A dynamic description of V is

$$V(t) = V(0) + \int_0^t V(s)\,(\mu\,ds + \sigma\,dW(s)) + \sum_{j=1}^{N(t)} V(\tau_j-)\left(e^{Z_j} - 1\right),$$

where $\tau_1, \tau_2, \ldots$ are the jump times of N.

It is clear that the distribution of $\log V(T)$ conditional on $\mathcal{F}_t$ and $N(T) - N(t) = n$ is Gaussian with mean

$$\log V(t) + mn + \left(\mu - \frac{\sigma^2}{2}\right)(T-t)$$

and variance

$$n\rho^2 + \sigma^2(T-t).$$

Hence the conditional default probability

$$
\begin{aligned}
p_d(t,T) &= \mathbb{P}\left[\log V(T) < \log X \mid \mathcal{F}_t\right] \\
&= \sum_{n=0}^{\infty} \mathbb{P}\left[\log V(T) < \log X \mid \mathcal{F}_t,\, N(T)-N(t)=n\right] \mathbb{P}\left[N(T)-N(t)=n\right] \\
&= \sum_{n=0}^{\infty} \Phi\left(\frac{\log\left(\frac{X}{V(t)}\right) - mn - \left(\mu - \frac{\sigma^2}{2}\right)(T-t)}{\sqrt{n\rho^2 + \sigma^2(T-t)}}\right) e^{-\lambda(T-t)} \frac{(\lambda(T-t))^n}{n!}
\end{aligned}
$$

First passage time models make this approach more realistic by admitting default at any time $T_d \in [0,T]$, and not just at maturity T. That means, bankruptcy occurs if the firm value $V(t)$ hits a specified stochastic boundary $X(t)$, such that

$$T_d = \inf\{t \mid V(t) \le X(t)\}.$$

In this case the conditional default probability is

$$p_d(t,T) = \mathbb{P}\left[T_d \le T \mid \mathcal{F}_t\right], \quad t \in [0,T],$$

which can be determined by Monte Carlo simulation.

13.2 Intensity Based Method

Default is often a complicated event. The precise conditions under which it must occur (such as hitting a barrier) are easily misspecified. The above structural approach has the additional deficiency that it is usually difficult to determine and trace a firm's value process.

In this section we focus directly on describing the evolution of the default probabilities $p_d(t,T)$ without defining the exact default event. Formally, we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The flow of the complete market information is represented by a filtration $(\mathcal{F}_t)$ satisfying the usual conditions. The default time T_d is assumed to be an $(\mathcal{F}_t)$-stopping time, hence the right-continuous default process

$$H(t) := 1_{\{T_d \le t\}}$$

is $(\mathcal{F}_t)$-adapted. The $\mathcal{F}_t$-conditional default probability is now

$$p_d(t,T) = \mathbb{E}\left[H(T) \mid \mathcal{F}_t\right], \quad t \in [0,T].$$

Obviously, H is a uniformly integrable submartingale. By the Doob–Meyer decomposition ([13, Theorem 1.4.10]) there exists a unique $(\mathcal{F}_t)$-predictable[1] increasing process $A(t)$ such that

$$M(t) := H(t) - A(t)$$

is a (uniformly integrable) martingale (notice that $A(t) = A(t \wedge T_d)$). Hence

$$p_d(t,T) = 1_{\{T_d \le t\}} + \mathbb{E}\left[A(T) - A(t) \mid \mathcal{F}_t\right].$$

This formula is the best we can hope for in general.

We proceed in several steps towards an explicit expression for $p_d(t,T)$ by making more and more restrictive assumptions **(D1)–(D4)**.

(D1) There exists a strict sub-filtration $(\mathcal{G}_t) \subset (\mathcal{F}_t)$ (partial market information) and a $(\mathcal{G}_t)$-adapted process Λ such that

$$A(t) = \Lambda(t \wedge T_d) \quad \text{and} \quad \mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t,$$

where $\mathcal{H}_t := \sigma(H(s) \mid s \le t)$ and $\mathcal{G}_t \vee \mathcal{H}_t$ stands for the smallest σ-algebra containing $\mathcal{G}_t$ and $\mathcal{H}_t$.

A market participant with access to the partial market information $\mathcal{G}_t$ cannot observe whether default has occurred by time t ($T_d \le t$) or not ($T_d > t$). In other words, T_d is not a stopping time for $(\mathcal{G}_t)$. This nicely reflects the aforementioned difficulties to determine the exact default event in practice.

Intuitively speaking, events in $\mathcal{F}_t$ are $\mathcal{G}_t$-observable given that $T_d > t$. The formal statement is as follows.

Lemma 13.2.1. *Let $t \in \mathbb{R}_+$. For every $A \in \mathcal{F}_t$ there exists $B \in \mathcal{G}_t$ such that*

$$A \cap \{T_d > t\} = B \cap \{T_d > t\}. \tag{13.1}$$

Proof. Let

$$\mathcal{F}_t^* := \left\{A \in \mathcal{F}_t \mid \exists B \in \mathcal{G}_t \text{ with property (13.1)}\right\}.$$

[1] The $(\mathcal{F}_t)$-predictable σ-algebra on $\mathbb{R}_+ \times \Omega$ is generated by all left-continuous $(\mathcal{F}_t)$-adapted processes; or equivalently, by the sets $\{0\} \times B$ where $B \in \mathcal{F}_0$ and $(s,t] \times B$ where $s < t$ and $B \in \mathcal{F}_s$.

Clearly $\mathcal{G}_t \subset \mathcal{F}_t^*$. Simply take $B = A$. Moreover $\mathcal{H}_t \subset \mathcal{F}_t^*$. Indeed, for every $A \in \mathcal{H}_t$ the intersection $A \cap \{T_d > t\}$ is either $\emptyset$ or $\{T_d > t\}$, so we can take for B either $\emptyset$ or Ω.

Since $\mathcal{F}_t^*$ is a σ-algebra ($\to$ exercise) and $\mathcal{F}_t$ is defined to be the smallest σ-algebra containing $\mathcal{G}_t$ and $\mathcal{H}_t$, we conclude that $\mathcal{F}_t \subset \mathcal{F}_t^*$. This proves the lemma. □

(D2) The default probability by t as seen by a $\mathcal{G}_t$-informed observer satisfies

$$0 < \mathbb{P}[T_d \le t \mid \mathcal{G}_t] < 1.$$

Hence we can define the positive $(\mathcal{G}_t)$-adapted hazard process Γ by

$$e^{-\Gamma(t)} := \mathbb{P}[T_d > t \mid \mathcal{G}_t].$$

Notice that $X(t) := \mathbb{P}[T_d > t \mid \mathcal{G}_t]$ is a $(\mathcal{G}_t)$-supermartingale and $\mathbb{E}[X(t)]$ is right-continuous in t ($\to$ exercise). Hence $X(t)$, and thus $\Gamma(t)$, admits a right-continuous modification, see e.g. [13, Theorem I.3.13]. We show below (Lemma 13.2.4) the rather surprising fact that if Γ is regular enough then it coincides with Λ on $[0, T_d]$.

A consequence of the next lemma is that for any $\mathcal{F}_t$-measurable random variable Y there exists an $\mathcal{G}_t$-measurable random variable $\tilde{Y}$ such that $Y = \tilde{Y}$ on $\{T_d > t\}$.

Lemma 13.2.2. *Let $t \in \mathbb{R}_+$ and Y a random variable. Then*

$$\mathbb{E}\left[1_{\{T_d>t\}} Y \mid \mathcal{F}_t\right] = 1_{\{T_d>t\}} e^{\Gamma(t)} \mathbb{E}\left[1_{\{T_d>t\}} Y \mid \mathcal{G}_t\right]. \tag{13.2}$$

Proof. Let $A \in \mathcal{F}_t$. By Lemma 13.2.1 there exists a $B \in \mathcal{G}_t$ with (13.1), and so $1_A 1_{\{T_d>t\}} = 1_B 1_{\{T_d>t\}}$. Hence, by the very definition of the $\mathcal{G}_t$-conditional expectation,

$$\begin{aligned}
\int_A 1_{\{T_d>t\}} Y \mathbb{P}[T_d > t \mid \mathcal{G}_t]\, d\mathbb{P} &= \int_B 1_{\{T_d>t\}} Y \mathbb{P}[T_d > t \mid \mathcal{G}_t]\, d\mathbb{P} \\
&= \int_B \mathbb{E}\left[1_{\{T_d>t\}} Y \mid \mathcal{G}_t\right] \mathbb{P}[T_d > t \mid \mathcal{G}_t]\, d\mathbb{P} \\
&= \int_B 1_{\{T_d>t\}} \mathbb{E}\left[1_{\{T_d>t\}} Y \mid \mathcal{G}_t\right] d\mathbb{P} \\
&= \int_A 1_{\{T_d>t\}} \mathbb{E}\left[1_{\{T_d>t\}} Y \mid \mathcal{G}_t\right] d\mathbb{P}.
\end{aligned}$$

This implies

$$\mathbb{E}\left[1_{\{T_d>t\}} Y \mathbb{P}\left[T_d > t \mid \mathcal{G}_t\right] \mid \mathcal{F}_t\right] = 1_{\{T_d>t\}} \mathbb{E}\left[1_{\{T_d>t\}} Y \mid \mathcal{G}_t\right],$$

which proves the lemma. □

As a consequence of the preceding lemmas we may now formulate the following results, which contain an expression for the aforementioned default probabilities.

Lemma 13.2.3. *For any $t \le T$ we have*

$$\mathbb{P}\left[T_d > T \mid \mathcal{F}_t\right] = 1_{\{T_d>t\}} \mathbb{E}\left[e^{\Gamma(t)-\Gamma(T)} \mid \mathcal{G}_t\right], \tag{13.3}$$

$$\mathbb{P}\left[t < T_d \le T \mid \mathcal{F}_t\right] = 1_{\{T_d>t\}} \mathbb{E}\left[1 - e^{\Gamma(t)-\Gamma(T)} \mid \mathcal{G}_t\right]. \tag{13.4}$$

Moreover, the processes

$$L(t) := 1_{\{T_d>t\}} e^{\Gamma(t)} = (1 - H(t)) e^{\Gamma(t)}$$

is an $(\mathcal{F}_t)$-martingale.

Proof. Let $t \le T$. Then $1_{\{T_d>T\}} = 1_{\{T_d>t\}} 1_{\{T_d>T\}}$. Using this and (13.2) we derive

$$\begin{aligned}
\mathbb{P}\left[T_d > T \mid \mathcal{F}_t\right] &= \mathbb{E}\left[1_{\{T_d>t\}} 1_{\{T_d>T\}} \mid \mathcal{F}_t\right] \\
&= 1_{\{T_d>t\}} e^{\Gamma(t)} \mathbb{E}\left[1_{\{T_d>T\}} \mid \mathcal{G}_t\right] \\
&= 1_{\{T_d>t\}} e^{\Gamma(t)} \mathbb{E}\left[\mathbb{E}\left[1_{\{T_d>T\}} \mid \mathcal{G}_T\right] \mid \mathcal{G}_t\right] \\
&= 1_{\{T_d>t\}} e^{\Gamma(t)} \mathbb{E}\left[e^{-\Gamma(T)} \mid \mathcal{G}_t\right],
\end{aligned}$$

which proves (13.3). Equation (13.4) follows since

$$1_{\{t<T_d\le T\}} = 1_{\{T_d>t\}} - 1_{\{T_d>T\}}.$$

For the second statement it is enough to consider

$$\begin{aligned}
\mathbb{E}\left[L(T) \mid \mathcal{F}_t\right] &= \mathbb{E}\left[1_{\{T_d>t\}} 1_{\{T_d>T\}} e^{\Gamma(T)} \mid \mathcal{F}_t\right] \\
&= 1_{\{T_d>t\}} e^{\Gamma(t)} \mathbb{E}\left[1_{\{T_d>T\}} e^{\Gamma(T)} \mid \mathcal{G}_t\right] = L(t),
\end{aligned}$$

since by definition of Γ

$$\mathbb{E}\left[1_{\{T_d>T\}} e^{\Gamma(T)} \mid \mathcal{G}_t\right] = \mathbb{E}\left[\mathbb{E}\left[1_{\{T_d>T\}} \mid \mathcal{G}_T\right] e^{\Gamma(T)} \mid \mathcal{G}_t\right] = 1.$$

□

(D3) There exists a positive, measurable, $(\mathcal{G}_t)$-adapted process λ such that

$$\Gamma(t) = \int_0^t \lambda(s)\, ds.$$

Taking (formally) the right-hand T-derivative at $T = t$ in (13.4) gives $\lambda(t)$. Hence we refer to $\lambda(t)$ as default intensity.

Here is the announced result for Γ.

Lemma 13.2.4. *The process*

$$N(t) := H(t) - \int_0^t \lambda(s) 1_{\{T_d > s\}}\, ds$$

is an $(\mathcal{F}_t)$-martingale. Hence, by the uniqueness of the predictable Doob–Meyer decomposition, we have

$$\Lambda(t \wedge T_d) = \int_0^t \lambda(s) 1_{\{T_d > s\}}\, ds = \Gamma(t \wedge T_d).$$

Proof. Let $t \leq T$. In view of (13.3) we have

$$\begin{aligned}
\mathbb{E}\left[N(T) \mid \mathcal{F}_t\right] &= 1 - \mathbb{E}\left[1_{\{T_d > T\}} \mid \mathcal{F}_t\right] - \int_0^t \lambda(s) 1_{\{T_d > s\}}\, ds \\
&\quad - \int_t^T \mathbb{E}\left[\lambda(s) 1_{\{T_d > s\}} \mid \mathcal{F}_t\right] ds \\
&= 1 - 1_{\{T_d > t\}} \mathbb{E}\left[e^{-\int_t^T \lambda(u)\, du} \mid \mathcal{G}_t\right] - \int_0^t \lambda(s) 1_{\{T_d > s\}}\, ds \\
&\quad - \underbrace{\int_t^T 1_{\{T_d > t\}} e^{\int_0^t \lambda(u)\, du} \mathbb{E}\left[\lambda(s) 1_{\{T_d > s\}} \mid \mathcal{G}_t\right] ds}_{=:I}.
\end{aligned}$$

We have further

$$\begin{aligned}
I &= \int_t^T 1_{\{T_d > t\}} e^{\int_0^t \lambda(u)\, du} \mathbb{E}\left[\lambda(s) \mathbb{E}\left[1_{\{T_d > s\}} \mid \mathcal{G}_s\right] \mid \mathcal{G}_t\right] ds \\
&= 1_{\{T_d > t\}} \mathbb{E}\left[\int_t^T \lambda(s) e^{-\int_t^s \lambda(u)\, du}\, ds \mid \mathcal{G}_t\right] \\
&= 1_{\{T_d > t\}} \mathbb{E}\left[1 - e^{-\int_t^T \lambda(u)\, du} \mid \mathcal{G}_t\right],
\end{aligned}$$

hence

$$\mathbb{E}\left[N(T) \mid \mathcal{F}_t\right] = 1 - 1_{\{T_d > t\}} - \int_0^t \lambda(s) 1_{\{T_d > s\}}\, ds = N(t).$$

□

The next and last assumption leads the way to implement a default risk model.

(D4) $\mathbb{P}\left[T_d > t \mid \mathcal{G}_\infty\right] = \mathbb{P}\left[T_d > t \mid \mathcal{G}_t\right] \quad \forall t \in \mathbb{R}_+$.

It can be shown that **(D4)** is equivalent to the hypothesis

(H) Every square integrable $(\mathcal{G}_t)$-martingale is an $(\mathcal{F}_t)$-martingale.

For more details we refer to [1, Chapter 6]. For the next lemma we only assume **(D1)**, **(D2)** and **(D4)**.

Lemma 13.2.5. *Suppose* Γ *is continuous. Then* $\phi := \Gamma(T_d)$ *is an exponential random variable with parameter 1 and independent of* $\mathcal{G}_\infty$*. Moreover,*

$$T_d = \inf\left\{t \mid \Gamma(t) \geq \phi\right\}.$$

Proof. By assumption,

$$\mathbb{P}\left[T_d > t \mid \mathcal{G}_\infty\right] = e^{-\Gamma(t)}.$$

Hence $\Gamma(t)$ is non-decreasing and continuous. We can define its right inverse

$$C(s) := \inf\{t \mid \Gamma(t) > s\}.$$

Then $\Gamma(t) > s \Leftrightarrow t > C(s)$ and $\Gamma(C(s)) = s$, so

$$\mathbb{P}\left[\Gamma(T_d) > s \mid \mathcal{G}_\infty\right] = \mathbb{P}\left[T_d > C(s) \mid \mathcal{G}_\infty\right] = e^{-\Gamma(C(s))} = e^{-s}.$$

This proves that $\phi = \Gamma(T_d)$ is an exponential random variable with parameter 1 and independent of $\mathcal{G}_\infty$. Moreover,

$$T_d = \inf\{t \mid \Gamma(t) \geq \Gamma(T_d)\} = \inf\{t \mid \Gamma(t) \geq \phi\}.$$

□

13.2.1 Construction of Intensity Based Models

The construction of a model that satisfies **(D1)–(D4)** is straightforward. We start with a filtration $(\mathcal{G}_t)$ satisfying the usual conditions and

$$\mathcal{G}_\infty = \sigma(\mathcal{G}_t \mid t \in \mathbb{R}_+) \subset \mathcal{F}.$$

Let $\lambda(t)$ be a positive, measurable, $(\mathcal{G}_t)$-adapted process with the property

$$\int_0^t \lambda(s)\, ds < \infty \quad \text{a.s. for all } t \in \mathbb{R}_+.$$

We then fix an exponential random variable ϕ with parameter 1 and independent of $\mathcal{G}_\infty$, and define the random time

$$T_d := \inf \left\{ t \mid \int_0^t \lambda(s)\, ds \geq \phi \right\}$$

with values in $(0, \infty]$. Consequently, we have for $t \leq T$

$$\begin{aligned}
\mathbb{P}\left[T_d > T \mid \mathcal{G}_t\right] &= \mathbb{P}\left[\phi > \int_0^T \lambda(u)\, du \mid \mathcal{G}_t\right] \\
&= \mathbb{E}\left[\mathbb{P}\left[\phi > \int_0^T \lambda(u)\, du \mid \mathcal{G}_T\right] \mid \mathcal{G}_t\right] \\
&= \mathbb{E}\left[e^{-\int_0^T \lambda(u)\, du} \mid \mathcal{G}_t\right],
\end{aligned}$$

by the independence of ϕ and $\mathcal{G}_T$ (this is a basic lemma for conditional expectations). And it is an easy exercise to show that

$$0 < \mathbb{P}\left[T_d > t \mid \mathcal{G}_t\right] = e^{-\int_0^t \lambda(u)\, du} < 1 \quad \text{and} \quad \phi = \int_0^{T_d} \lambda(u)\, du.$$

We finally define $\mathcal{F}_t := \mathcal{G}_t \vee \mathcal{H}_t$, where $\mathcal{H}_t = \sigma(H(s) \mid s \leq t)$. Conditions **(D1)–(D3)** are obviously satisfied for

$$\Lambda(t) = \Gamma(t) := \int_0^t \lambda(s)\, ds.$$

As for **(D4)** we notice that

$$\mathbb{P}\left[T_d > t \mid \mathcal{G}_\infty\right] = \mathbb{P}\left[\phi > \int_0^t \lambda(u)\, du \mid \mathcal{G}_\infty\right] = e^{-\int_0^t \lambda(u)\, du} = \mathbb{P}\left[T_d > t \mid \mathcal{G}_t\right].$$

13.2.2 Computation of Default Probabilities

When it comes to computations of the default probabilities (13.3) we need a tractable model for the intensity process λ. But the right-hand side of (13.3) looks just like what we had for the risk-neutral valuation of zero-coupon bonds in terms of a given short rate process (Chapter 7). Notice that $\lambda \geq 0$ is essential. An obvious and popular choice for λ is thus a square root (or affine) process. So let W be a $(\mathcal{G}_t)$-Brownian motion, $b \geq 0$, $\beta \in \mathbb{R}$ and $\sigma > 0$ some constants, and let

$$d\lambda(t) = (b + \beta\lambda(t))\, dt + \sigma\sqrt{\lambda(t)}\, dW(t), \quad \lambda(0) \geq 0. \tag{13.5}$$

The proof of the following lemma is left as an exercise.

Lemma 13.2.6. *For the intensity process* (13.5) *the conditional default probability is*

$$p_d(t,T) = \mathbb{P}\left[T_d \leq T \mid \mathcal{F}_t\right] = \begin{cases} 1 - e^{-A(T-t)-B(T-t)\lambda(t)}, & \text{if } T_d > t \\ 0, & \text{else,} \end{cases}$$

where

$$\begin{aligned} A(u) &:= -\frac{2b}{\sigma^2}\log\left(\frac{2\gamma e^{(\gamma-\beta)u/2}}{(\gamma-\beta)\left(e^{\gamma u}-1\right)+2\gamma}\right), \\ B(u) &:= \frac{2\left(e^{\gamma u}-1\right)}{(\gamma-\beta)\left(e^{\gamma u}-1\right)+2\gamma}, \\ \gamma &:= \sqrt{\beta^2+2\sigma^2}. \end{aligned}$$

13.2.3 Pricing Default Risk

The stochastic setup is as above. In addition, we suppose now that we are given a risk-neutral probability measure $\mathbb{Q} \sim \mathbb{P}$ and a measurable, $(\mathcal{G}_t)$-adapted short rate process $r(t)$. Moreover, we assume that there exists a positive, measurable, $(\mathcal{G}_t)$-adapted process $\lambda_{\mathbb{Q}}$ such that

$$\Gamma_{\mathbb{Q}}(t) := \int_0^t \lambda_{\mathbb{Q}}(s)\, ds < \infty \quad \text{a.s. for all } t \in \mathbb{R}_+,$$

and **(D1)–(D3)** are satisfied for $\mathbb{Q}$, $\Lambda_Q := \Gamma_{\mathbb{Q}}$ and $\Gamma_{\mathbb{Q}}$ replacing $\mathbb{P}$, Λ and Γ, respectively (unfortunately, these conditions are not preserved under an equivalent change of measure in general). So that Lemmas 13.2.1–13.2.4 apply.

We will determine the price $C(t,T)$ of a corporate zero-coupon bond with maturity T, which may default. As for the recovery we fix a constant recovery rate $\delta \in (0,1)$ and distinguish three cases:

- Zero recovery: the cashflow at T is $1_{\{T_d>T\}}$.
- Partial recovery at maturity: the cashflow at T is $1_{\{T_d>T\}} + \delta 1_{\{T_d \leq T\}}$.
- Partial recovery at default: the cashflow is $\begin{cases} 1 \text{ at } T \text{ if } T_d > T, \\ \delta \text{ at } T_d \text{ if } T_d \leq T. \end{cases}$

Zero-Recovery

The arbitrage price of $C(t,T)$ is

$$C(t,T) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(s)\,ds} 1_{\{T_d>T\}} \mid \mathcal{F}_t\right].$$

In view of Lemma 13.2.2 this is

$$\begin{aligned} C(t,T) &= 1_{\{T_d>t\}} e^{\int_0^t \lambda_{\mathbb{Q}}(s)\,ds} \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(s)\,ds} 1_{\{T_d>T\}} \mid \mathcal{G}_t\right] \\ &= 1_{\{T_d>t\}} e^{\int_0^t \lambda_{\mathbb{Q}}(s)\,ds} \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(s)\,ds} \mathbb{E}_{\mathbb{Q}}\left[1_{\{T_d>T\}} \mid \mathcal{G}_T\right] \mid \mathcal{G}_t\right] \\ &= 1_{\{T_d>t\}} \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T (r(s)+\lambda_{\mathbb{Q}}(s))ds} \mid \mathcal{G}_t\right]. \end{aligned} \tag{13.6}$$

Notice that this is a very nice formula. Pricing a corporate bond boils down to the pricing of a non-defaultable zero-coupon bond with the short rate process replaced by

$$r(s) + \lambda_{\mathbb{Q}}(s) \geq r(s).$$

A tractable (hence affine) model is easily found. For the short rates we chose CIR: let W be a $(\mathbb{Q}, \mathcal{G}_t)$-Brownian motion, $b \geq 0$, $\beta \in \mathbb{R}$, $\sigma > 0$ constant parameters and

$$dr(t) = (b + \beta r(t))\,dt + \sigma\sqrt{r(t)}\,dW(t), \quad r(0) \geq 0. \tag{13.7}$$

For the intensity process we chose the affine combination

$$\lambda_{\mathbb{Q}}(t) = c_0 + c_1 r(t), \tag{13.8}$$

for two constants $c_0, c_1 \geq 0$.

Lemma 13.2.7. *For the above affine model we have*

$$C(t,T) = 1_{\{T_d>t\}} e^{-A(T-t)-B(T-t)r(t)},$$

where

$$\begin{aligned} A(u) &:= c_0 u - \frac{2b}{\sigma^2} \log \left(\frac{2\gamma e^{(\gamma-\beta)u/2}}{(\gamma-\beta)\left(e^{\gamma u}-1\right)+2\gamma} \right), \\ B(u) &:= \frac{2\left(e^{\gamma u}-1\right)}{(\gamma-\beta)\left(e^{\gamma u}-1\right)+2\gamma}(1+c_1), \\ \gamma &:= \sqrt{\beta^2+2(1+c_1)\sigma^2}. \end{aligned}$$

Proof. Exercise. □

A special case is $c_1 = 0$ (constant intensity). Here we have

$$C(t,T) = 1_{\{T_d>t\}} e^{-c_0(T-t)} P(t,T),$$

where $P(t,T)$ is the CIR price of a default-free zero-coupon bond.

Partial Recovery at Maturity

This is an easy modification of the preceding case since

$$1_{\{T_d>T\}} + \delta 1_{\{T_d \le T\}} = (1-\delta)\, 1_{\{T_d>T\}} + \delta.$$

In view of (13.6) hence

$$C(t,T) = (1-\delta)\, 1_{\{T_d>t\}} \mathbb{E}_{\mathbb{Q}} \left[e^{-\int_t^T (r(s)+\lambda_{\mathbb{Q}}(s))ds} \mid \mathcal{G}_t \right] + \delta P(t,T),$$

where $P(t,T)$ stands for the price of the default-free zero-coupon bond.

Partial Recovery at Default

A straightforward modification of the proofs of Lemmas 13.2.1 and 13.2.2 shows that

$$\mathbb{E}_{\mathbb{Q}} \left[1_{\{T_d>t\}} Y \mid \mathcal{G}_\infty \vee \mathcal{H}_t \right] = 1_{\{T_d>t\}} e^{\int_0^t \lambda_{\mathbb{Q}}(s)\, ds} \mathbb{E}_{\mathbb{Q}} \left[1_{\{T_d>t\}} Y \mid \mathcal{G}_\infty \right]$$

for every random variable Y. Combining this with Section 13.2.1 we obtain for $t \le u$

$$
\begin{aligned}
\mathbb{Q}[t < T_d \le u \mid \mathcal{G}_\infty \vee \mathcal{H}_t] &= 1_{\{T_d > t\}} e^{\int_0^t \lambda_{\mathbb{Q}}(s)\, ds} \mathbb{E}_{\mathbb{Q}}\left[1_{\{t < T_d \le u\}} \mid \mathcal{G}_\infty\right] \\
&= 1_{\{T_d > t\}} e^{\int_0^t \lambda_{\mathbb{Q}}(s)\, ds} \left(e^{-\int_0^t \lambda_{\mathbb{Q}}(s)\, ds} - e^{-\int_0^u \lambda_{\mathbb{Q}}(s)\, ds}\right) \\
&= 1_{\{T_d > t\}} \left(1 - e^{-\int_t^u \lambda_{\mathbb{Q}}(s)\, ds}\right),
\end{aligned}
$$

which is the regular conditional distribution of T_d conditional on $\{T_d > t\}$ and $\mathcal{G}_\infty \vee \mathcal{H}_t$. Differentiation in with respect to u yields its density function

$$
1_{\{T_d > t\}} \lambda_{\mathbb{Q}}(u) e^{-\int_t^u \lambda_{\mathbb{Q}}(s)\, ds} 1_{\{t \le u\}}.
$$

Hence the arbitrage price of the recovery at default given that $t < T_d \le T$ is given by

$$
\begin{aligned}
\pi(t) &= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^{T_d} r(s)\, ds} \delta 1_{\{t < T_d \le T\}} \mid \mathcal{F}_t\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^{T_d} r(s)\, ds} \delta 1_{\{t < T_d \le T\}} \mid \mathcal{G}_\infty \vee \mathcal{H}_t\right] \mid \mathcal{F}_t\right] \\
&= \delta 1_{\{T_d > t\}} \mathbb{E}_{\mathbb{Q}}\left[\int_t^T e^{-\int_t^u r(s)\, ds} \lambda_{\mathbb{Q}}(u) e^{-\int_t^u \lambda_{\mathbb{Q}}(s)\, ds}\, du \mid \mathcal{F}_t\right] \\
&= \delta 1_{\{T_d > t\}} \int_t^T \mathbb{E}_{\mathbb{Q}}\left[\lambda_{\mathbb{Q}}(u) e^{-\int_t^u (r(s) + \lambda_{\mathbb{Q}}(s))\, ds} \mid \mathcal{F}_t\right] du.
\end{aligned}
$$

For the above affine model (13.7)–(13.8) this expression can be made more explicit (→ exercise). As a result, the price of the corporate bond bond price with recovery at default is

$$
C(t, T) = C_0(t, T) + \pi(t),
$$

where $C_0(t, T)$ is the bond price with zero recovery.

The above calculations and an extension to stochastic recovery go back to Lando [14].

13.2.4 Measure Change

We consider an equivalent change of measure and derive the behavior of the compensator process for the stopping time T_d. Again, we take the above stochastic setup and let **(D1)–(D3)** hold. So that

$$M(t) = H(t) - \int_0^t \lambda(s) 1_{\{T_d > s\}} \, ds$$

is a $(\mathbb{P}, \mathcal{F}_t)$-martingale. Now let μ be a positive $(\mathcal{G}_t)$-predictable process such that

$$\Lambda_{\mathbb{Q}}(t) := \int_0^t \mu(s)\lambda(s) \, ds < \infty \quad \text{a.s. for all } t \in \mathbb{R}_+.$$

We will construct an equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$ such that

$$\Lambda_{\mathbb{Q}}(t \wedge T_d)$$

is the $(\mathbb{Q}, \mathcal{F}_t)$-compensator of H. This does not, however, imply that $\Lambda_{\mathbb{Q}}(t)$ is the $(\mathbb{Q}, \mathcal{G}_t)$-hazard process $\Gamma_{\mathbb{Q}}(t) = -\log \mathbb{Q}[T_d > t \mid \mathcal{G}_t]$ of T_d in general. A counterexample has been constructed by Kusuoka [**?**], see also [1, Section 7.3].

The following analysis involves stochastic calculus for cadlag processes of finite variation (FV), which in a sense is simpler than for Brownian motion since it is a pathwise calculus. We recall the integration by parts formula for two right-continuous FV functions f and g

$$f(t)g(t) = f(0)g(0) + \int_0^t f(s-) \, dg(s) + \int_0^t g(s-) \, df(s) + [f, g](t),$$

where

$$[f, g](t) = \sum_{0 < s \leq t} \Delta f(s) \Delta g(s), \quad \Delta f(s) := f(s) - f(s-).$$

Lemma 13.2.8. *The process*

$$D(t) := C(t)V(t)$$

with

$$C(t) := \exp\left(\int_0^t (1 - \mu(s))\lambda(s) 1_{\{T_d > s\}} \, ds \right)$$

$$V(t) := \left(1_{\{T_d > t\}} + \mu(T_d) 1_{\{T_d \leq t\}} \right) = \begin{cases} 1, & t < T_d \\ \mu(T_d), & t \geq T_d \end{cases}$$

satisfies

$$D(t) = 1 + \int_0^t D(s-)\,(\mu(s) - 1)\,dM(s)$$

and is thus a positive $\mathbb{P}$-local martingale.

Proof. Notice that $[C, V] = 0$ and

$$V(t) = 1 + \int_0^t (\mu(s) - 1)\,dH(s) = 1 + \int_0^t V(s-)\,(\mu(s) - 1)\,dH(s).$$

Hence

$$\begin{aligned} D(t) &= 1 + \int_0^t C(s-)\,dV(s) + \int_0^t V(s-)\,dC(s) \\ &= 1 + \int_0^t C(s-)V(s-)\,(\mu(s) - 1)\,dH(s) \\ &\quad + \int_0^t C(s)V(s-)(1 - \mu(s))\lambda(s)1_{\{T_d > s\}}\,ds \\ &= 1 + \int_0^t D(s-)\,(\mu(s) - 1)\,dM(s). \end{aligned}$$

Since $D(s-)$ is locally bounded and $\Lambda_{\mathbb{Q}}(t) < \infty$ we conclude that D is a $\mathbb{P}$-local martingale. □

Lemma 13.2.9. *Let $T \in \mathbb{R}_+$. Suppose $\mathbb{E}[D(T)] = 1$ (hence $(D(t))_{t\in[0,T]}$ is a martingale), so that we can define an equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$ on $\mathcal{F}_T$ by*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = D(T).$$

Then the process

$$M_{\mathbb{Q}}(t) := H(t) - \Lambda_{\mathbb{Q}}(t \wedge T_d), \quad t \in [0, T], \tag{13.9}$$

is a $\mathbb{Q}$-martingale.

Proof. It is enough to show that $M_{\mathbb{Q}}$ is a $\mathbb{Q}$-local martingale. Indeed, $\Lambda_{\mathbb{Q}}$ is an increasing continuous (and hence predictable) process, (13.9) therefore the unique Doob–Meyer decomposition of H under $\mathbb{Q}$. Since H is uniformly integrable, so is $M_{\mathbb{Q}}$ ([13, Theorem 1.4.10]).

From Bayes' rule we know that $M_{\mathbb{Q}}$ is a $\mathbb{Q}$-local martingale if and only if $DM_{\mathbb{Q}}$ is a $\mathbb{P}$-local martingale. Notice that

$$\begin{aligned}[D, M_{\mathbb{Q}}](t) &= \Delta D(T_d) 1_{\{T_d \geq t\}} = D(T_d-)\,(\mu(T_d) - 1)\, 1_{\{T_d \geq t\}} \\ &= \int_0^t D(s-)\,(\mu(s) - 1)\, dH(s).\end{aligned}$$

Integration by parts gives

$$\begin{aligned}DM_{\mathbb{Q}}(t) &= \int_0^t D(s-)\, dM_{\mathbb{Q}}(s) + \int_0^t M_{\mathbb{Q}}(s-)\, dD(s) + [D, M_{\mathbb{Q}}](t) \\ &= \int_0^t D(s-)\, dH(s) - \int_0^t D(s-)\mu(s)\lambda(s) 1_{\{T_d > s\}}\, ds \\ &\quad + \int_0^t M_{\mathbb{Q}}(s-)\, dD(s) + \int_0^t D(s-)\,(\mu(s) - 1)\, dH(s) \\ &= \int_0^t M_{\mathbb{Q}}(s-)\, dD(s) + \int_0^t D(s-)\mu(s)\, dM(s),\end{aligned}$$

which proves the claim. □

Pricing by the "Martingale Approach"

We remark again that $\Lambda_{\mathbb{Q}}$ is different from $\Gamma_{\mathbb{Q}}$ in general, so that the methods from Section 13.2.3 do not apply for the above situation. Yet, there is a way to derive the pricing formulas from Section 13.2.3 under Assumption **(D4)** for $\mathbb{Q}$. The detailed analysis can be found in [1, Section 8.3].

Bibliography

[1] Tomasz R. Bielecki and Marek Rutkowski, *Credit risk: modelling, valuation and hedging*, Springer Finance, Springer-Verlag, Berlin, 2002.

[2] BIS, *Zero-coupon yield curves: Technical documentation*, Bank for International Settlements, Basle, March 1999.

[3] T. Björk, *Arbitrage theory in continuous time*, Oxford University Press, 1998.

[4] T. Björk, Y. Kabanov, and W. Runggaldier, *Bond market structure in the presence of marked point processes*, Math. Finance **7** (1997), no. 2, 211–239.

[5] A. Brace, D. Gątarek, and M. Musiela, *The market model of interest rate dynamics*, Math. Finance **7** (1997), no. 2, 127–155.

[6] D. Brigo and F. Mercurio, *Interest rate models—theory and practice*, Springer-Verlag, Berlin, 2001.

[7] J. Cox, J. Ingersoll, and S. Ross, *A theory of the term structure of interest rates*, Econometrica **53** (1985), 385–408.

[8] D. Filipović, *Consistency problems for Heath–Jarrow–Morton interest rate models*, Springer-Verlag, Berlin, 2001.

[9] D. Heath, R. Jarrow, and A. Morton, *Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation*, Econometrica **60** (1992), 77–105.

[10] J. C. Hull, *Options, futures, and other derivatives*, fourth ed., Prentice-Hall International, Inc., 2000.

[11] J. James and N. Webber, *Interest rate modelling*, Wiley, 2000.

[12] R. Jarrow, *Modelling fixed income securities and interest rate options*, McGraw-Hill, 1996.

[13] I. Karatzas and S. E. Shreve, *Brownian motion and stochastic calculus*, second ed., Springer-Verlag, New York, 1991.

[14] D. Lando, *On Cox processes and credit-risky securities*, Rev. Derivatives Res. **2** (1998), 99–120.

[15] R. C. Merton, *On the pricing for corporate debt: the risk structure of interest rates*, J. Finance **29** (1974).

[16] M. Musiela and M. Rutkowski, *Martingale methods in financial modelling*, Applications of Mathematics, vol. 36, Springer-Verlag, Berlin-Heidelberg, 1997.

[17] C. Nelson and A. Siegel, *Parsimonious modeling of yield curves*, J. of Business **60** (1987), 473–489.

[18] C. R. Rao, *Principal component and factor analyses*, Statistical methods in finance, North-Holland, Amsterdam, 1996, pp. 489–505.

[19] R. Rebonato, *Interest-rate option models*, second ed., Wiley, 1998.

[20] D. Revuz and M. Yor, *Continuous martingales and Brownian motion*, Grundlehren der mathematischen Wissenschaften, vol. 293, Springer-Verlag, Berlin-Heidelberg-New York, 1994.

[21] L. E. O. Svensson, *Estimating and interpreting forward interest rates: Sweden 1992-1994*, IMF Working Paper No. 114, September 1994.

[22] R. Zagst, *Interest-rate management*, Springer-Verlag, Berlin, 2002.