Duration, convexity and portfolio immunization

Some principles of bonds' prices

As is known, a bond's price is given by:

$$P = \sum_{t=1}^{n} \frac{C}{(1+r)^t} + \frac{F}{(1+r)^n} = \frac{C}{r} - \frac{C}{r(1+r)^n} + \frac{F}{(1+r)^n}$$

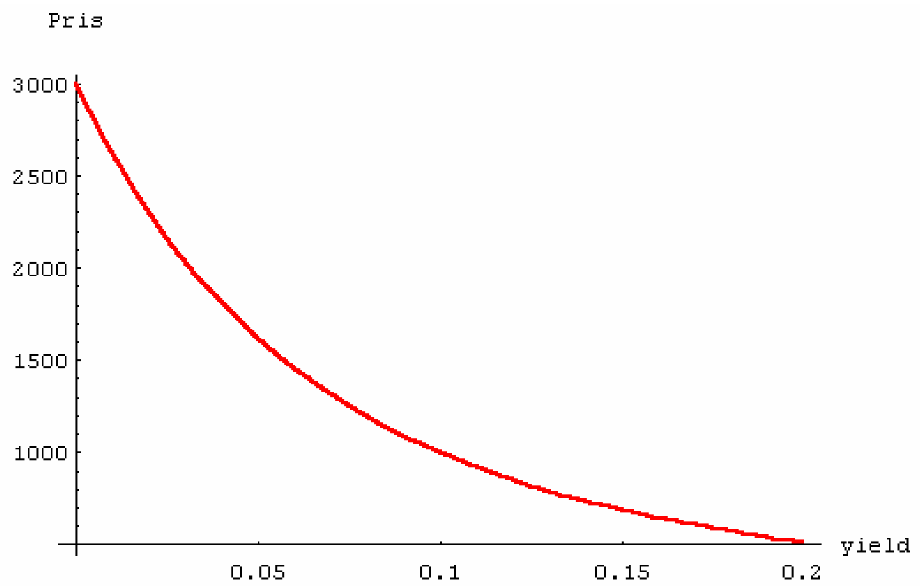
If we derive that with respect to its parameters, *C*, *F*, *r* and *t* or *n*, partially, we can examine how its price is affected. An alternative method is to examine it using numerical values.

Among the principles, we can name the following:

(i) ***The higher the r, the lower its P,*** *because:*

$$\frac{\partial P}{\partial r} = -\sum_{t=1}^{n} \frac{tC}{(1+r)^{t+1}} - \frac{nF}{(1+r)^{n+1}} \prec 0$$

<u>Example (i)</u>: *F = 1,000, C = 100, r= 0.01,...0.20, n = 20*



(ιι) ***The bond prices change over time, if r*** $\neq \frac{C}{F}$***,*** *because:*

$$\frac{\partial P}{\partial n} = \frac{C\ln(1+r)}{r(1+r)^n} - \frac{F\ln(1+r)}{(1+r)^n} = \frac{(C-rF)\ln(1+\gamma)}{r(1+r)^n}$$

<u>Example (ii)</u>: *F = 1,000, C = 100, r(a) = 0.10, r(b) = 0.08, r(c) = 0.12, n = 0, 5, 10, 15, 20*

The following table shows the bond prices for various *n* & *r*.

	n = 0	***n = 5***	***n = 10***	***n = 15***	***n = 20***
r = 0.10	1,000	1,000	1,000	1,000	1,000
r = 0.08	1,000	1,079.85	1,134.19	1,171.18	1,196.36
r = 0.12	1,000	927.904	887	863.782	850.611

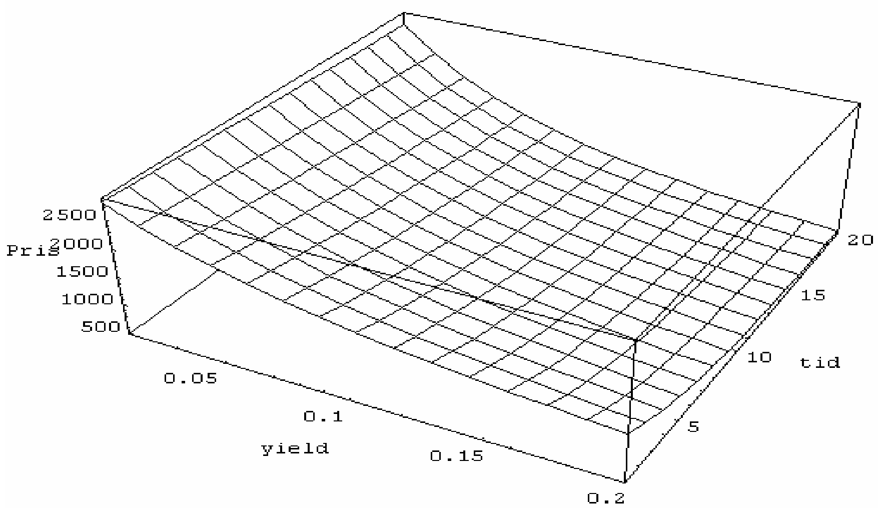
This means that premium bonds fall over time, while discount bonds increase. Both will reach *P = F* at *n = 0*.

(iii) ***For similar interest rate changes, the short bonds are less volatile compared to long bonds***

If a corporation issues longer bonds, it will pay the given coupons per period, irrespectively if the interest rate increases (and therefore the corporation will pay less) or falls (and consequently the corporation will pay more). In addition, if the interest rate falls, it would be better if it could issue new ones at lower coupons. That would be easier if it had short bonds to replace and rather expensive if it had longer bonds to replace. This can explain why the short bonds are, relatively, risk free.

<u>Example (iii)</u>: *F = 1,000, C = 100, r = 0.01,..., 0.20, n = 1,..., 20*

The following three-dimensional graph shows clearly that the shorter bonds are less volatile than the longer ones, for all changes in yield to maturity.



(iv) ***The more frequent the coupon payments, the lower the volatility if the yield changes***

This is due to the fact that the investor who receives *C* often, she can save that to a higher interest rate, if it increases. Such an option does not exist if *C* is paid seldom. A similar argument applies if the interest rate declines.

Example (iv): F *= 1,000,* C_A *= 100 (per year),* C_B *= 100 (every second year),* $n = 2$, $r = 0.05, 0.10, 0.20$.

The following table shows it.

	$r = 0.05$	$r = 0.10$	$r = 0.20$	*Volatility*
A	1,092.97	1,000	847.22	24.575 %
B	1,097.51	1,000	840.28	25.723 %

(v) ***There exist a linear (positive) relationship between a bonds' price and its coupon, as for similar bonds with same time to maturity****, because:*

$$\frac{\partial P}{\partial C} = \sum_{t=1}^{n} \frac{1}{(1+r)^t} = \frac{1}{r} - \frac{1}{r(1+r)^n} > 0$$

Example (v): *F = 1,000, C = 0,...,160, r = 0.10, n= 20*

$$\frac{\partial P}{\partial C} = \sum_{t=1}^{20} \frac{1}{(1+0.1)^t} = \frac{1}{0.1} - \frac{1}{0.1(1+0.1)^{20}} = 8.5136\,,$$

and, if $C = 0 \Rightarrow P = \sum_{t=1}^{n} \frac{0}{(1+r)^t} + \frac{1000}{(1+0.1)^{20}} = 148.644$

This implies that $P = 148.644 + 8.51356\,C$.

(vi) ***The %-change of the large coupon bond is lower than of the small coupon bond.***

Example (vi): $F = 100$, $C_A = 10$, $C_B = 1$, $n = 5$, $r = 0.08, 0.10, 0.12$.

	$r = 0.08$	$r = 0.10$	$r = 0.12$	*Volatility*
A	107.99	100	92.79	15.2 %
B	72.05	65.88	60.35	17.76 %

Why? The answer is found in the bonds' duration.

Duration

From the principle (i) above, it is clear that when the interest rate increases (decreases) the price of a bond decreases (increases), other things being unchanged. This negative relationship though is more complicated. On one hand, the price of a bond falls as the interest rate increases, because the coupon payments are worth less. On the other hand, from principle (vi) we can argue that these payments can be reinvested at the higher interest rate. Thus, at some point, these two effects will cancel each other. The time where these effects cancel each other is called a bond's duration (Dur).

There are various methods to define and estimate the duration. Below we will define it with two alternative (and identical) formulas. Both definitions are plausible if coupons are constant and are paid at specific periods (often per year or per half year).

<u>Duration as elasticity</u>

We showed above that:

$$\frac{\partial P}{\partial r} = -\sum_{t=1}^{n} \frac{tC}{(1+r)^{t+1}} - \frac{nF}{(1+r)^{n+1}} \prec 0 \quad (1)$$

We can express that derivative as elasticity, by multiplying it with: $-\frac{(1+r)}{P}$. The product becomes the bond's duration.

$$-\frac{\partial P}{\partial r}\frac{(1+r)}{P} = \left[-\sum_{t=1}^{n}\frac{tC}{(1+r)^{t+1}} - \frac{nF}{(1+r)^{n+1}}\right]\left[-\frac{(1+r)}{P}\right] =$$

$$\left[\sum_{t=1}^{n}\frac{tC}{(1+r)^{t}} + \frac{nF}{(1+r)^{n}}\right]\left[\frac{1}{P}\right] = Dur \qquad (2)$$

This formula shows the percentage change of a bond's price caused by a percentage change of interest rate, and is measured in years. Contrary from the bond's legal life to maturity, i.e. *n*, the duration shows the bond's economic life. The economic life is always shorter than the legal life for all non zero-coupon bonds. The economic and legal lives coincide for the zero-coupon bonds, because both such bonds pay only principal at maturity day.

<u>Example</u>: *F = 1,000, C = 130, r = 0.10, n = 4.*

Its price is *P = 1,095.10*, and $\frac{\partial P}{\partial r}$ = *-3,375.75*

Therefore, its duration is: $Dur = -(-3{,}375.75)\frac{1.1}{1{,}095.10} = 3.39$ years.

If instead, *C = 0*, its price is *P = 683.01*, and $\frac{\partial P}{\partial r}$ = *-2,483.68*, implying duration of: $Dur = -(-2{,}483.68)\frac{1.1}{683.01} = 4$ years.

<u>Duration as average time of all cash flows until maturity</u>

According to this method, the duration is defined as:

$Dur = \sum_{t=1}^{n} tw_t$, where, $w_t = \frac{\frac{(F+C_t)}{(1+r)^t}}{P}$, i.e., weights based on discounted cash flows.

For the same example we obtain the same duration as the table below shows.

Period	$F + C_t$	$(F+C_t)(1.1)^{-t}$	$w_t = (3)/P$	*Dur=(1)*(4)*
(1)	(2)	(3)	(4)	(5)
1	0 + 130	118.18	0.1079	0.1079
2	0 + 130	107.44	0.0981	0.1962
3	0 + 130	97.67	0.0892	0.2676
4	1,000+130	771.80	0.7048	2.8192
Sum		*P = 1,095.10*	*1*	*3.3909*

An important advantage with duration is that it provides the same information as three variables (*r, C* and *n*). Instead of checking all of them, when bond prices are compared, it is sufficient to compare their duration. A disadvantage with duration though is that it requires horizontal yield curve, i.e. short and long interest rates at the same level.

Duration and volatility

The duration can be used to estimate a bond's volatility (σ).

The formula (2) can be rewritten as:

$$-\frac{\partial P}{\partial r}\frac{(1+r)}{P} = Dur \Rightarrow \partial P = -Dur\,\frac{\partial r}{(1+r)}P \qquad (3)$$

That shows how a bond's price changes, if the interest rate changes (given its *Dur* and *P*).

For the same example, we can estimate its volatility if the interest rate <u>increases</u>, or <u>falls</u> with 0.5 percentage units.

$\partial P = -3.39\frac{0.005}{(1.10)}1{,}095.1 = -16.87$, i.e. a new price of *1,095.1 – 16.87 = 1,078.22*, and,
$\partial P = -3.39(-)\frac{0.005}{(1.10)}1{,}095.1 = 16.87$, i.e. a new price of *1,095.1 + 16.87 = 1,111.97*.

Thus the bond's volatility is: $\sigma = \frac{1{,}111.97 - 1{,}078.22}{1{,}095.1} = 0.0308$

Volatility can also be defined as: $\sigma = \frac{Dur}{(1+r)}$ (4)

That formula provides of course the same volatility because, $\sigma = \frac{3.39}{(1.1)} = 3.08\%$.

If we derive (4) with respect to *Dur*, we obtain:

$\frac{\partial V}{\partial Dur} = \frac{1}{(1+r)} > 0$, i.e., the longer the duration, the higher the volatility.

Principles of Duration

(i) $\frac{\partial Dur}{\partial t} < 0, \, and, \, \frac{\partial^2 Dur}{\partial t^2} > 0$

Thus, the duration falls as time passes (which is obvious), at diminishing rate (i.e., it is convex). The bond portfolios must therefore be rebalanced, if one wishes to keep a given duration.

(ii) $\frac{\partial Dur}{\partial r} < 0, \, and, \, \frac{\partial^2 Dur}{\partial r^2} > 0$

The duration falls at higher interest rates, at diminishing rate, because the most recent coupon payments are discounted by higher interest rates and contribute therefore to a lower weight. In our example, if *r* changes from 0.10 to 0.15, the duration falls from 3.39 to approximately 3.32 years.

(iii) $\frac{\partial Dur}{\partial n} > 0, \, and, \, \frac{\partial^2 Dur}{\partial n^2} < 0$

Long bonds have obviously longer duration, even if the duration rate declines. That depends on the fact that duration is influenced by other parameters as well and not only on the time to maturity.

(iv) $\frac{\partial Dur}{\partial C} < 0, \, and, \, \frac{\partial^2 Dur}{\partial C^2} > 0$

The duration falls for higher coupon payments, at diminishing rate, because the higher the coupon, the higher the weight that the first coupons receive. We showed previously that zero-coupon bonds have the longest duration. Thus, if one wishes to keep a long duration, should seek for bonds paying low coupons.

Convexity

When the duration is expressed as elasticity, it would lead to rough approximations for large changes in the interest rates. Additional measures are therefore needed, to take into consideration both the curvature and the sensitivity of interest rate changes. Such a measure which is often used is the convexity (Con).

The second derivate of price with respect to interest rate is given from (5):

$$\frac{\partial^2 P}{\partial r^2} = \sum_{t=1}^{n} \frac{t(t+1)C}{(1+r)^{t+2}} + \frac{n(n+1)F}{(1+r)^{n+2}} \succ 0 \qquad (5)$$

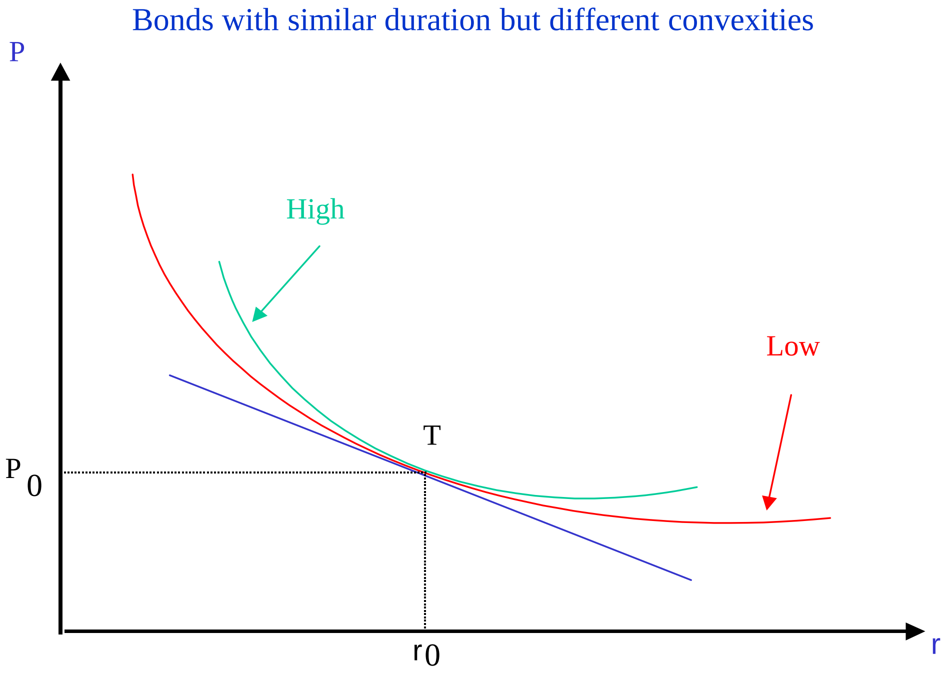
As with the case of duration, we can also express this derivative as elasticity, by multiplying it with $\frac{(1+r)^2}{P}$. The outcome, see below, is convexity.

$$\frac{\partial^2 P}{\partial r^2}\frac{(1+r)^2}{P} = \left[\sum_{t=1}^{n} \frac{t(t+1)C}{(1+r)^{t+2}} + \frac{n(n+1)F}{(1+r)^{n+2}}\right]\left[\frac{(1+r)^2}{P}\right] =$$

$$\left[\sum_{t=1}^{n} \frac{t(t+1)C}{(1+r)^{t}} + \frac{n(n+1)F}{(1+r)^{n}}\right]\left[\frac{1}{P}\right] = Con \qquad (6)$$

Since convexity and duration are related positively to each other, the same principles apply as for duration. For instance higher interest rates or coupon payments would lead to lower convexity, while longer bonds would lead to higher convexity.

Higher convexity should be preferred to lower convexity, if two bonds have the same duration, as the following figure shows. If the interest rate increases, the bond's price with the higher convexity will fall less. Similarly, if the interest rate declines the same bond will increase more, compared to the bond with lower convexity.



Immunization of interest rate changes

Very often, pension funds and insurance companies seek after bonds with longer duration, because their liabilities (obligation payments) take place in the future. On the other hand, bonds with long maturities are more volatile in interest rate changes. They must therefore find a bond portfolio that immunizes (neutralises) these effects.

Obviously, a bond portfolio's future value depends on the yield curve that prevails until the maturity day. If the future value of liabilities is equal to the future value of bond portfolio at every point of time, the obligations are fully immunized, no matter if the yield curve shifts upward or downward. A simple immunization strategy to achieve that is to equate the bond portfolio's duration (and convexity) with the liabilities' duration (and convexity).

Let us examine this strategy, with the help of duration only.

<u>Example</u>: Company Alfa owns a machine that rents it to Beta, over a period of 8 years. Beta pays yearly rent payments (coupons) of 2 m. SEK; r = 12 %.

Alfa's $PV = \sum_{t=1}^{8} \frac{2\,milllion}{(1.12)^t} = 9.94\ million$

Because, $\frac{\partial PV}{\partial r} = -\sum_{t=1}^{n} \frac{tC}{(1+r)^{t+1}} = -\sum_{t=1}^{8} \frac{t\,(2\,million)}{(1.12)^{t+1}} = -34.6125$,

its duration is: $Dur = -\ (-34.6125)\frac{1.12}{9.94} = 3.9$ *years*.

Alfa can secure his PV = *9,94* m. SEK, by issuing, for instance, two bonds, a zero-coupon for 1 year (with a PV equal to X) and a 6-year level coupon bond (with a PV equal to Y), so that the weighted average duration will be equal to 3.9 years. For simplicity, assume that the yield of the 6-year bond is also equal to12%.

Obviously, $X + Y = 9.94$. Any linear combination of X, such as *1.91 m.* and Y, such as *8.03 m.*, must therefore lead to the following weighted average duration of 3.9 years, i.e., $\frac{1.91}{9.94}Dur_X + \frac{8.03}{9.94}Dur_Y = 3.9$ years.

Since $Dur_x = 1$, Alfa must issue a 6-year level coupon bond Y, with $Dur_y = 4.6$ years. Other combinations than the above, for instance 25 % at 1-year bond (i.e. 2,485 m. SEK) and 75 % at 6-year bond (i.e. 7,455 m. SEK.), would increase Dur_Y to 4.86 years.

If the interest rate increases to 13 %, from (3) we obtain:

$\partial P = -D\frac{\partial r}{(1+r)}P = (-3.9)\frac{0.01}{1.12}9.94 = -0.3461$, i.e. a new PV = *9.594*.

If the 6-year bond has the same duration as before (*4.6* years), and the proportions remained unchanged (i.e. no rebalance of portfolio takes place), these bonds, immunize almost perfectly.

Alternatively, small changes are needed to immunize perfectly. For instance, the new proportions must be:

$$\frac{X}{9.594}1+\frac{Y}{9.594}(4.6)=\frac{X}{9.594}1+\frac{(9.597-X)}{9.594}(4.6)=3.9\Rightarrow X=1.866$$

and $Y = 7.728$.

Notice that these proportions are almost equal to the previous ones, since:

$$\frac{1.91}{9.94}=0.192\approx\frac{1.866}{9.594}=0.194$$

Another possibility is of course to decrease the duration, or to try to renegotiate the contract and rent the machine to less than 8 years.