Applying Duration

A Bond Hedging Example

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Duration: A Definition

- Duration is defined as a weighted average of the maturities of the individual payments:

$$D = \frac{C_1}{B(1+r)} + 2\frac{C_2}{B(1+r)^2} + \ldots + t\frac{C_t}{B(1+r)^t} + \ldots + n\frac{C_n + F}{B(1+r)^n}$$

  » This definition of duration is sometimes also referred to as *Macaulay Duration*.

- The duration of a zero coupon bond is equal to its maturity.

Duration

Approximating the maturity of a bond

- Calculate the average maturity of a bond:
  - » Coupon bond is like portfolio of zero coupon bonds
  - » Compute average maturity of this portfolio
  - » Give each zero coupon bond a weight equal to the proportion in the total value of the portfolio
- Write value of the bond as:

$$B = \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + ... + \frac{C_t}{(1+r)^t} + ... + \frac{C_n + F}{(1+r)^n} = \sum_{t=1}^{t=n} PV(C_t) + PV(F)$$

The factor:

$$\frac{C_t}{B(1+r)^t}$$

is the proportion of the t-th coupon payment in the total value of the bond: *PV(C_t)/B*

Calculating Duration

- Calculate the duration of the 6% 5-year bond:

Time	Payment	PV(Payment)	% of PV	Time*%PV
1	60	55.56	6.04%	0.06
2	60	51.44	5.59%	0.11
3	60	47.63	5.18%	0.16
4	60	44.10	4.79%	0.19
5	1060	721.42	78.40%	3.92
		920.15	100.00%	4.44

- Calculate the duration of the 10% 5-year bond:

Time	Payment	PV(Payment)	% of PV	Time*%PV
1	100	92.59	8.57%	0.09
2	100	85.73	7.94%	0.16
3	100	79.38	7.35%	0.22
4	100	73.50	6.81%	0.27
5	1100	748.64	69.33%	3.47
		1079.85	100.00%	4.20

- The duration of the bond with the *lower* coupon is *higher*
  - » Why?

Duration: An Exercise

- What is the interest rate sensitivity of the following two bonds. Assume coupons are paid annually.

	Bond A	Bond B
Coupon rate	10%	0%
Face value	$1,000	$1,000
Maturity	5 years	10 years
YTM	10%	10%
Price	$1,000	$385.54

Duration Exercise (cont.)

- Percentage change in bond price for a small increase in the interest rate:

Pct. Change = - [1/(1.10)][4.17] = - 3.79%
Bond A

Pct. Change = - [1/(1.10)][10.00] = - 9.09%
Bond B

Duration Exercise (cont.)

Year (t)	PV(A)	PV(A) x t	PV(B)	PV(B)xt
1	$90.91	$90.91	0	0
2	$82.64	$165.89	0	0
3	$75.13	$225.39	0	0
4	$68.30	$273.21	0	0
5	$683.01	$3,415.07	0	0
6	0	0	0	0
7	0	0	0	0
8	0	0	0	0
9	0	0	0	0
10	0	0	$385.54	$3,855.43
Totals	$1000.00	$4,170.47	$385.54	$3,855.43
Duration		4.17		10.00

7

Duration of Bonds: Data

Market Segment	Coupon	Years to Maturity	Quality	Price	Duration	Modified Duration
Government Bonds	7.205	8.293	Treasury	106.723	4.994	4.833
Corporate Bonds	7.722	12.353	A2	104.117	6.660	6.423
Mortgage bonds	7.385	21.845		100.669	4.863	4.691
High Yield bonds	9.900	8.828	B1	104.025	5.503	5.528
Total/Average	7.432	12.634		104.569	5.248	5.078

The Difference between Duration and Term to Maturity can be substantial

Only Duration gives the correct answer for assessing price volatility

8

Duration and Volatility

- For a zero-coupon bond with maturity *n* we have derived:

$$\frac{\partial B}{\partial r}\frac{1}{B} = \frac{\text{Percentage Change in bond price}}{\text{1\% increase in yield}} = -\frac{n}{1+r}$$

- For a coupon-bond with maturity *n* we can show:

$$\frac{\partial B}{\partial r}\frac{1}{B} = \frac{\text{Percentage Change in bond price}}{\text{1\% increase in yield}} = -\frac{D}{1+r}$$

  » The right hand side is sometimes also called *modified duration*.
- Hence, in order to analyze bond volatility, *duration,* and not maturity is the appropriate measure.
  » Duration and maturity are the same only for zero-coupon bonds!

Duration and Volatility

The example reconsidered

- Compute the right hand side for the two 5-year bonds in the previous example:
  » 6%-coupon bond:
  D/(1+r) = 4.44/1.08=4.11
  » 10%-coupon bond:
  D/(1+r) = 4.20/1.08=3.89
- But these are exactly the average price responses we found before!
  » Hence, differences in duration explain variation of price responses across bonds with the same maturity.

Is Duration always Exact?

- Consider the two 5-year bonds (6% and 10%) from the example before, but interest rates can change by moving 3% up or down:

Yield	6%-Bond	5-year bond
8%	$920.15	$1,079.85
11%	$815.21	$963.04
% Change	-11.40%	-10.82%
5%	$1,043.29	$1,216.47
% Change	13.38%	12.65%
Average	12.39%	11.73%

- This is different from the duration calculation which gives:
  - » 6% coupon bond: 3*4.11%=12.33%<12.39%
  - » 10% coupon bond: 3*3.89%=11.67%<11.73%
- Result is imprecise for larger interest rate movements
  - » Relationship between bond price and yield is convex, but
  - » Duration is a linear approximation

Hedging a Payment Using Duration

- Suppose you have a fixed liability exactly two years from now of $10,000,
  - » can choose from two bonds to invest in with face value of $1,000:
    - – A zero-coupon bond with maturity one year
    - – A zero-coupon bond with maturity 3 years
  - » Which bond should you invest in (or portfolio)?
  - » Current yields to maturity are 8% on both bonds
  - » Need to invest today:

$$\frac{\$10{,}000}{(1.08)^2} = \$8{,}573$$

- Suppose interest rates can change immediately after investment.

Unhedged Portfolios I:

Reinvestment risk

- Strategy 1: Invest only in 1 year bond, then reinvest:
  - » Duration of portfolio = 1
  - » Bond price today = $926
  - » Invest in $8,753/$926=9.259 of these bonds
  - » Receive $9,259 exactly one year from now
- Portfolio value at year 2:

Interest rate	Portfolio at t=2
7.0%	9907
7.5%	9954
8.0%	10000
8.5%	10046
9.0%	10093

  - » Reinvestment risk if interest rates change

Unhedged Portfolios II:

Capital Risk

- Strategy 2: Invest only in 3 year bond, sell in 2 years:
  - » Duration of portfolio = 3
  - » Bond price today = $794
  - » Invest in $8,753/$794=10.8 of these bonds
  - » Mature at $10,800 exactly 3 years from now:
- Portfolio value at year 2:

Interest rate	Portfolio at t=2
7.0%	10093
7.5%	10047
8.0%	10000
8.5%	9954
9.0%	9908

- Capital risk of selling bond at year 2 if interest rates change
  - » Works in opposite direction!

Hedged Portfolios

An Application of Duration

- Construct portfolio that matches duration of liability = 2
  - » Invest so that:

$$\frac{1*(Value\ of\ 1\text{-}year\ bonds)+3*(Value\ of\ 3\text{-}year\ bonds)}{Value\ of\ 1\text{-}year\ bonds+Value\ of\ 3\text{-}year\ bonds}=2$$

  - » Hence Value of 1-year bonds = Value of 3-year bonds =$8573/2=$4287
    - – 4287/926=4.63 1-year bonds
      Mature at t=1 with $4,630, then reinvest
    - – 4287/794=5.40 3-year bonds
      Mature at t=3 with $5,400, sell in 2 years

Hedged Portfolio Results

- Results for portfolio with matched duration

Interest rate	Bond 1 at t=2	Bond 2 at t=2	Portfolio
6.5%	4930.56	5070.42	10000.98
7.0%	4953.70	5046.73	10000.43
7.5%	4976.85	5023.26	10000.11
8.0%	5000.00	5000.00	10000.00
8.5%	5023.15	4976.96	10000.11
9.0%	5046.30	4954.13	10000.42
9.5%	5069.44	4931.51	10000.95

Observations:

- » Still reinvestment risk with short bond, price risk with long bond, cancel in duration-matched portfolio
- » Error increases with higher fluctuations due to convexity