Present Value (PV) & Future Value (FV)

NOTATION

- t = time
- T = number of time periods
- C = cash-flow
- r = interest rate per period
- RA = regular annuity (at the end of t, loan payments)
- AD = annuity due (at the beginning of t, rent)
- g = growth rate

Relationship between Present Value *(PV)* & Future Value *(FV)*

$$FV = PV(1+r)^T \Leftrightarrow PV = \frac{FV}{(1+r)^T}$$

FV calculates C at the end of T
PV discounts all C at the beginning of T

Example 1:

A player was offered 2 options on his 3-year contract. (a) SEK 2 m. at the end of every year (i.e. year 0, year 1 and year 2). (b) SEK 1.5 m. at the end of year 0 and SEK 5 m. at the end of year 2. Which option is best if r = 10 % ?

(a) $FV = 2(1,1)^2 + 2(1,1) + 2 = 2,42 + 2,2 + 2 = 6,62$ *m.*

(b) $FV = 1,5(1,1)^2 + 5 = 1,815 + 5 = 6,815$ *m.*

(b) is best because *FV* is higher.

Present Value *(PV)* & Future Value *(FV)* of Annuity

Example 2:

Triss highest profit one can earn is SEK 25,000 / month, over 25 years (i.e. 300,000 per year). If r = 8 %, what is the value of that cash-flow today?

$$PV = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \dots + \frac{C}{(1+r)^T} \qquad (1)$$

Rewrite (1) as:

$$PV = \frac{C}{(1+r)}[1 + \frac{1}{(1+r)} + \frac{1}{(1+r)^2} + \dots + \frac{1}{(1+r)^{T-1}}] \qquad (2)$$

Set $\alpha = \frac{C}{(1+r)}$ and $x = \frac{1}{(1+r)}$ i (2)

$$PV = \alpha[1 + x + x^2 + \dots + x^{T-1}] \qquad (3)$$

Multiply (3) by x

$$xPV = \alpha[x + x^2 + x^3 + \dots + x^T] \qquad (4)$$

Subtract (4) from (3)

$$PV - xPV = \alpha[1 - x^T] \Rightarrow PV(1 - x) = \alpha[1 - x^T] \Rightarrow$$

$$PV = \alpha\left[\frac{1-x^T}{1-x}\right] \qquad (5)$$

Substitute α and x by their simplifications in (2)

$$PV = \frac{C}{(1+r)}\left[\frac{1-1/(1+r)^T}{1-1/(1+r)}\right] = \frac{C}{(1+r)}\left[\frac{1-(1+r)^{-T}}{1-(1+r)^{-1}}\right] =$$

$$C\left[\frac{1-(1+r)^{-T}}{(1+r)-(1+r)^1(1+r)^{-1}}\right] = C\left[\frac{1-(1+r)^{-T}}{(1+r)-1}\right] =$$

$$C\left[\frac{1-(1+r)^{-T}}{r}\right] \Rightarrow \mathbf{PV} = C\left[\frac{1}{r} - \frac{1}{r(1+r)^T}\right] \qquad (6)$$

Example 2: $\boldsymbol{PV = 300000}\left[\frac{1}{0{,}08} - \frac{1}{0{,}08(1{,}08)^{25}}\right] = 3202432$

If $T \longrightarrow \infty \Rightarrow PV = \frac{C}{r}$; it is called "**Perpetuity**"

If payments take place **at the beginning of** the period, we multiply (6) by *(1 + r);* it is called "**Annuity Due**".

From (6) and relationship $FV = PV(1+r)^T$, we can estimate *FV*.

$$FV = C\left[\frac{1}{r} - \frac{1}{r(1+r)^T}\right](1+r)^T = C\left[\frac{(1+r)^T}{r} - \frac{(1+r)^T}{r(1+r)^T}\right] \Rightarrow$$

$$\boldsymbol{FV} = C\left[\frac{(1+r)^T}{r} - \frac{1}{r}\right] \qquad (7)$$

<u>Example 3</u>:

An options trader plans to buy a Ferrari in 5 years. The future price of Ferrari in 5 years is estimated to cost SEK 1.2 m. How much must he save per year if r = 10 % ?

$$1200000 = C\left[\frac{(1{,}1)^5}{0{,}1} - \frac{1}{0{,}1}\right] \Rightarrow C = 196557\ kr.$$

Present Value *(PV)* and Future Value *(FV)* of growing Annuity

<u>Example 4</u>: A player was offered 2 options on his 10-year contract.
(a) SEK 1 m. which will grow by 5 % every year. (b) cash of SEK 7 m. now. Which option is best if r = 10 % ?

$$PV = \frac{C}{(1+r)} + \frac{C(1+g)}{(1+r)^2} + \ldots + \frac{C(1+g)^{T-1}}{(1+r)^T} \qquad (1)$$

Rewrite (1) as:

$$PV = \frac{C}{(1+r)}[1 + \frac{(1+g)}{(1+r)} + \ldots + \frac{(1+g)^{T-1}}{(1+r)^{T-1}}] \qquad (2)$$

Set $\alpha = \dfrac{C}{(1+r)}$ and $x = \dfrac{1+g}{1+r}$ i (2)

$$PV = \alpha[1 + x + x^2 + \ldots + x^{T-1}] \qquad (3)$$

Multiply (3) by *x*

$$xPV = \alpha[x + x^2 + x^3 + \ldots + x^T] \qquad (4)$$

Subtract (4) from (3)

$PV - xPV = \alpha[1 - x^T] \Rightarrow PV(1 - x) = \alpha[1 - x^T] \Rightarrow$

$$PV = \alpha\left[\frac{1-x^T}{1-x}\right] \qquad (5)$$

Substitute α and x by their simplifications in (2)

$$PV = \frac{C}{(1+r)}\left[\frac{1-[(1+g)/(1+r)]^T}{1-(1+g)/(1+r)}\right] = C\left[\frac{1-(1+g)^T/(1+r)^T}{(1+r)-(1+g)}\right]$$

$$\Rightarrow \boldsymbol{PV} = C\left[\frac{1}{(r-g)} - \frac{1}{(r-g)}\frac{(1+g)^T}{(1+r)^T}\right], \quad \textit{for } r > g \qquad (6)$$

Example 4:

$$\boldsymbol{PV} = 1000000\left[\frac{1}{(0{,}1-0{,}05)} - \frac{1}{(0{,}1-0{,}05)}\frac{(1{,}05)^{10}}{(1{,}1)^{10}}\right] = SEK\ 7.44\ m.$$

(a) is best, because *PV* is higher than 7 m.

If $T \rightarrow \infty \Rightarrow PV = \dfrac{C}{r-g}$; it is called "**Growing perpetuity**"

From (6) and relationship $FV = PV(1+r)^T$, we can estimate *FV*.

$$FV = C\left[\frac{1}{(r-g)} - \frac{1}{(r-g)}\frac{(1+g)^T}{(1+r)^T}\right](1+r)^T \Rightarrow$$

$$\boldsymbol{FV} = C\left[\frac{(1+r)^T}{r-g} - \frac{(1+g)^T}{r-g}\right] \qquad (7)$$

Compounding and Effective interest rate

<u>Example 5</u>:

Estimate FV of C = SEK 1000 (just one cash-flow), after 10 years if the annual interest rate is r = 8 %.

(i) If the interest rate is calculated once per year

$FV = PV(1 + r)^T = 1000(1{,}08)^{10}$ = SEK 2,158.9

(ii) If the interest rate is calculated once per month

$$FV = PV\left(1+\frac{r}{12}\right)^{12T} = 1000\left(1+\frac{0{,}08}{12}\right)^{120} = \text{SEK } 2{,}219.6$$

(iii) If the interest rate is calculated continuously

$FV = PVe^{rT} = 1000(2{,}71828)^{0{,}8} = 2{,}225.5$

$$e = \lim_{n \to \infty} \left[1+\frac{1}{n}\right]^n = 2{,}71828\ldots$$ *(the base of natural logarithms)*

Effective interest rate = $e^r - 1$

For instance, for an annual interest rate of 8 %, the effective interest rate is:

$2{,}71828^{0{,}08} - 1 = 8{,}329$ %