BEYOND DURATION

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Abstract This paper addresses the problem of hedging a portfolio of fixed-income cash-flows. We first briefly review the traditional duration hedging method, which is heavily used by practitioners. That approach is based on a series of very restrictive and simplistic assumptions, including the assumptions of a small and parallel shift in the yield curve. We know however that large variations can affect the yield-to-maturity curve and that three main factors (level, slope and curvature) have been found to drive the dynamics of the yield curve. This strongly suggests that duration hedging is inefficient in many circumstances. We first show how to relax the assumption of a small shift of the yield-to-maturity curve, by including a convexity component in the Taylor expansion of the value function of the bond. Then, we show how to relax the assumption of a parallel shift of the yield curve in a fairly general framework. In particular, we argue that principal component analysis of interest rates changes allows a portfolio manager to regroup risk in a natural way. We conclude by analyzing the performance of the various hedging techniques in a realistic situation, and we show that satisfying hedging results can be achieved by using a three-factor model for the yield curve.

Keywords: interest rate risk, duration hedging, convexity hedging, principal components \$durations, Nelson and Siegel \$durations.

1 Introduction

A stock risk is usually proxied by its beta, which is a measure of the stock sensitivity to market movements. In the same vein, bond price risk is, most often, measured in terms of the bond interest rate sensitivity, or duration. These is a one-dimensional measure of the bond's sensitivity to interest rate movements. There is one complication, however: the value of a bond, or a bond portfolio, is affected by changes in interest rates of all possible maturities. In other words, there are more than one risk factor, and simple methods based upon a one-dimensional measure of risk will not allow investors to properly manage interest rate risks.

In the world of equity investment, it has long been recognized that there may be more than one rewarded risk factors (see for example the seminal work by Fama and French (1992)). A variety of more general multi-factor models, economically justified either by equilibrium (Merton (1973) or arbitrage (Ross (1976)) arguments, have been applied for risk management and portfolio performance evaluation. In this paper, we survey the multi-factor models used for interest rate risk management. All these models have been designed to better account for the complex nature of interest rate risk. Because it is never easy to hedge the risk associated with too many sources of uncertainty, it is always desirable to try and reduce the number of risk factors, and identify a limited number of common factors. There are several ways in which that can be done and all of them are to some extent arbitrary. In that context, it is important to know the exact assumptions one has to make in the process, and try to evaluate the robustness of these assumptions with respect to the specific scenario an investor has in mind.

We first briefly review the traditional duration hedging method, which is intensively used by practitioners. That approach is based on a series of very restrictive and simplistic assumptions, the assumptions of a small and parallel shift in the yield-to-maturity curve. We know however that large variations can affect the yield to maturity curve and that three main factors (level, slope and curvature) have been found to drive the dynamics of the yield curve (see Litterman and Scheinkman (1991)). This strongly suggests that duration hedging is inefficient in many circumstances. We first show how to relax the assumption of a small shift of the yield-to-maturity curve, by including a convexity term in the Taylor expansion of the value function of the bond. Then, we show how to relax the assumption of a parallel shift of the yield curve in a fairly general framework. In particular, we argue that principal component analysis of interest rates changes allows a portfolio manager to regroup risk in a natural way. We conclude by analyzing the performance of the various hedging techniques in a realistic situation, and we show that satisfying hedging results can be achieved by using a three-factor model for the yield curve.

2 Qualifying Interest Rate Risk

To illustrate the benefits and limits of each model, we consider a simple experiment. A portfolio manager aims at hedging the value of a fixed-income portfolio which delivers certain (or deterministic) cash-flows in the future, typically cash-flows from straight bonds with a fixed coupon rate. Even if these cash-flows are known in advance, the price of this bond changes in time, which leaves an investor exposed to a potentially significant capital loss.

To fix the notation, we consider at date *t* a portfolio of fixed-income securities that delivers *m* certain cash-flows F_i at future dates t_i for *i=1,...,m*. The price *V* of the bond (in % of the face value) can be written as the sum of the future cash-flows discounted with the appropriate zero-coupon rate with maturity corresponding to the maturity of the cash-flow

$$V_t = \sum_{i=1}^{m} F_i B(t,t_i) = \sum_{i=1}^{m} \frac{F_i}{[1+R(t,t_i-t)]^{t_i-t}} \qquad \textbf{(1)}$$

where $B(t,t_i)$ is the price at date *t* of a zero-coupon bond paying $1 at date t_i (also called the discount factor) and $R(t,t_i-t)$ is the associated zero-coupon rate, starting at date *t* for a residual maturity of $t_i - t$ years. We see in equation (1) that the price V_t is a function of *m* interest rate variables $R(t,t_i-t)$ and of the time variable *t*. This suggests that the value of the bond is subject to a potentially large number *m* of risk factors. For example, a bond with annual cash-flows up to a 10-year maturity is affected by potential changes in 10 zero-coupon rates. To hedge a position in this bond, we need to be hedged against a change of all of these 10 factor risks.

In practice, it is not easy to hedge the risk of so many variables. We must create a global portfolio containing the portfolio to hedge in such a way that the portfolio is insensitive to all sources of risk (the *m* interest rate variables and the time variable *t*).[1] One suitable way to simplify the hedging problematic is to reduce the number of risk variables. Our goal is to design an optimal strategy to select a minimum number of variables which can adequately describe the dynamics of the whole term structure.

3 Hedging with Duration

The whole idea behind duration hedging is to bypass the complication of a multi-dimensional interest rate risk by identifying a single risk factor, the yield to maturity of the portfolio, which will serve as a «proxy» for the whole term structure.

3.1 Using a One-Order Taylor Expansion

The first step consists in writing the price of the portfolio V_t (in % of the face value) as a function of a single source of interest rate risk, its yield to maturity y_t (see equation (2)).

$$V_t = V(y_t) = \sum_{i=1}^{m} \frac{F_i}{[1+y_t]^{t_i-t}} \qquad \textbf{(2)}$$

In this case, we see clearly that the interest rate risk is (imperfectly) summarized by changes of the yield to maturity y_t. Of course, this can only be achieved by losing much generality and imposing important, rather arbitrary and simplifying assumptions. The yield to maturity

[1] In this paper, we do not consider the change of value due to time because it is a deterministic term (for details about the time value of a bond, see Chance and Jordan [1996]). We only consider changes in value due to interest rate variations.

is a complex average of the whole term structure, and it can be regarded as the term structure if and only if the term structure is flat.

A second step involves the derivation of a Taylor expansion of the value of the portfolio *V* as an attempt to quantify the magnitude of value changes *dV* that are triggered by small changes *dy* in yield. We get an approximation of the *absolute* change in the value of the portfolio as

$$dV(y) = V(y+dy) - V(y) = V'(y)dy + o(y) \approx \$Dur(V(y))dy \qquad \textbf{(3)}$$

where $V'(y) = -\sum_{i=1}^{m} \frac{(t_i - t)F_i}{(1+y_t)^{t_i - t + 1}}$, the derivative of the bond value function with respect to the yield to maturity is known as the \$duration of portfolio *V*, and *o(y)* a negligible term.

Dividing equation (3) by *V(y)* we obtain an approximation of the *relative* change in value of the portfolio as

$$\frac{dV(y)}{V(y)} = \frac{V'(y)}{V(y)}dy + o_1(y) \approx -MD(V(y))dy \qquad \textbf{(4)}$$

where $MD(V(y)) = -\frac{V'(y)}{V(y)}$ is known as the modified duration of portfolio *V*.

The \$duration and the modified duration enable us to compute the absolute P&L and relative P&L of portfolio *V* for a small change Δy of the yield to maturity

$$\begin{gathered} Absolute\ P\,\&\,L \approx N_V \times \$Dur \times \Delta y \\ Relative\ P\,\&\,L \approx -MD \times \Delta y \end{gathered}$$

where N_V is the face value of the portfolio.

3.2 How to Hedge in Practice ?

We attempt to hedge a bond portfolio with face value N_V, yield to maturity *y* and price denoted by *V(y)*. The idea is to consider one hedging asset with face value N_H, yield to maturity y_1 (a priori different from *y*) whose price is denoted by $H(y_1)$ and build a global portfolio with value V^* invested in the initial portfolio and some quantity f of the hedging instrument.

$$V^* = N_V V(y) + f N_H H(y_1)$$

The goal is to make the global portfolio insensitive to small interest rate variations. Using equation (3) and assuming that the yield to maturity curve is only affected by parallel shifts so that $dy = dy_1$, we obtain

$$dV^* \approx [N_V V'(y) + f N_H H'(y_1)]dy = 0$$

which translates into

$$fN_H \$Dur(H(y_1)) = -N_V \$Dur(V(y))$$

or

$$fN_H H(y_1)MD(H(y_1)) = -N_V V(y)MD(V(y))$$

and we finally get

$$f = -\frac{N_V \$Dur(V(y))}{N_H \$Dur(H(y_1))} = -\frac{N_V V(y)MD(V(y))}{N_H H(y_1)MD(H(y_1))} \qquad \textbf{(5)}$$

The optimal amount invested in the hedging asset is simply equal to the opposite of the ratio of the \$duration of the bond portfolio to hedge by the \$duration of the hedging instrument, when they have the same face value.

Remark 1 *When the yield curve is flat which means that* $y = y_1$, *we can also use the Macaulay Duration to construct the hedge of the instrument. In this particular case, the hedge ratio* f *given by equation (5) is also equal to*

$$f = -\frac{N_V V(y)D(V(y))}{N_H H(y)D(H(y))}$$

where the Macaulay duration D(V(y)) is defined as

$$D(V(y)) = -(1+y)MD(V(y)) = \frac{\sum_{i=1}^{m} \frac{(t_i - t)F_i}{(1+y)^{t_i - t}}}{V(y)}$$

In practice, it is preferable to use futures contracts or swaps instead of bonds to hedge a bond portfolio because of significantly lower costs and higher liquidity. For example, using futures as hedging instruments, the hedge ratio f_f is equal to

$$f_f = -\frac{N_V \$Dur_V}{N_F \$Dur_{CTD}} \times CF \qquad \textbf{(6)}$$

where N_F is the size of the futures contract. $\$Dur_{CTD}$ is the \$duration of the cheapest to deliver as *CF* is the conversion factor.

Using standard swaps, the hedge ratio f_s is

$$f_s = -\frac{N_V \$Dur_V}{N_F \$Dur_S} \qquad \textbf{(7)}$$

where N_S is the nominal amount of the swap and $\$Dur_S$ is the \$duration of the fixed-coupon bond contained in the swap.[2]

Duration hedging is very simple. However, one should be aware that the method is based upon the following, very restrictive, assumptions:

- It is explicitly assumed that the value of the portfolio could be approximated by its first order Taylor expansion. This assumption is all the more critical as the changes of the interest rates are larger. In other words, the method relies on the assumption of small yield to maturity changes. This is why the hedge portfolio should be re-adjusted reasonably often.
- It is also assumed that the yield curve is only affected by parallel shifts. In other words, the interest rate risk is simply considered as a risk on the general level of interest rates.

In what follows, we attempt to relax both assumptions to account for more realistic changes in the term structure of interest rates.

4 Relaxing the Assumption of a Small Shift

We have argued that \$duration provides a convenient way to estimate the impact of a *small* change *dy* in yield on the value of a bond or a portfolio.

4.1 Using a Second-Order Taylor Expansion

Duration hedging only works for small yield changes, because the price of a bond as a function of yield is nonlinear. In other words, the \$duration of a bond changes as the yield changes. When a portfolio manager expects a potentially large shift in the term structure, a convexity term should be introduced; the price approximation can be improved if one can account for such nonlinearity by explicitly introducing a convexity term. Let us take the following example to illustrate that point.

Example 1 *We consider a 10-year maturity and 6% annual coupon bond traded at par. Its modified duration and convexity are equal to 7.36 and 57.95, respectively. We assume that the yield to maturity goes suddenly from 6% to 8% and we re-price the bond after this large change:*

The new price of the bond, obtained by discounting its future cash-flows, is equal to 86.58\$, and the exact change of value amounts to -\$13.42 (=\$86.58-\$100).

Using a first-order Taylor expansion, the change in value is approximated by -\$14.72 (= -$\$100\times 7.36\times 0.02$); we overestimate the decrease in price by \$1.30.

We conclude that the first-order Taylor expansion gives a good approximation of the bond price change when the variation of its yield to maturity is small.

[2] To see concrete examples of hedges constructed with futures contracts and swaps, we refer to Martellini, Priaulet and Priaulet (2003).

If one is concerned about the impact of a larger move *dy* on a bond portfolio value, one needs to write (at least) a second-order version of the Taylor expansion

$$\begin{aligned} dV(y) &= V'(y)dy + \frac{1}{2}V''(y)(dy)^2 + o\left((dy)^2\right) \\ &\approx \$Dur(V(y))dy + \frac{1}{2}\$Conv(V(y))(dy)^2 \end{aligned} \quad \textbf{(8)}$$

where the quantity V'' also denoted *\$Conv(V(y))* is known as the \$convexity of the bond *V*.

Dividing equation (8) by V(y) we obtain an approximation of the relative change in value of the portfolio as

$$\frac{dV(y)}{V(y)} \approx -MD(V(y))dy + \frac{1}{2}RC(V(y))(dy)^2$$

where *RC(V(y))* is called the (relative) convexity of portfolio *V*.

We now reconsider the previous example and approximate the bond price change by using equation (8). The bond price change is now approximated by -\$13.56 (=-14.72+$(100 \times 57.95 \times 0.02^2 / 2)$). We conclude that the second-order approximation is well suited for larger interest rate deviations.

4.2 Hedging Method

One needs to introduce two hedging assets with value denoted by H_1 and H_2, respectively, in order to hedge interest rate risk at the first and second order. The goal is to obtain a portfolio that is both \$duration neutral and \$convexity neutral. The optimal quantity (f_1, f_2) of these two assets to hold is then given by the solution to the following system of equations, at each date, assuming that $dy = dy_1 = dy_2$

$$\begin{cases} f_1 N_{H_1} H_1'(y_1) + f_2 N_{H_2} H_2'(y_2) = -N_V V'(y) \\ f_1 N_{H_1} H_1''(y_1) + f_2 N_{H_2} H_2''(y_2) = -N_V V''(y) \end{cases}$$

which translates into

$$\begin{cases} f_1 N_{H_1} \$Dur(H_1(y_1)) + f_2 N_{H_2} \$Dur(H_2(y_2)) = -N_V \$Dur(V(y)) \\ f_1 N_{H_1} \$Conv(H_1(y_1)) + f_2 N_{H_2} \$Conv(H_2(y_2)) = -N_V \$Conv(V(y)) \end{cases} \quad \textbf{(9)}$$

or

$$\begin{cases} f_1 N_{H_1} H_1(y_1) MD(H_1(y_1)) + f_2 N_{H_2} H_2(y_2) MD(H_2(y_2)) = -N_V V(y) MD(V(y)) \\ f_1 N_{H_1} H_1(y_1) RC(H_1(y_1)) + f_2 N_{H_2} H_2(y_2) MD(H_2(y_2)) = -N_V V(y) RC(V(y)) \end{cases}$$

5 Relaxing the Assumption of a Parallel Shift

5.1 A Common Principle

A major shortcoming of single-factor models is that they imply all possible zero-coupon rates are perfectly correlated, making bonds redundant assets. We know, however, that rates with different maturities do not always change in the same way. In particular, long term rates tend to be less volatile than short term rates. An empirical analysis of the dynamics of the interest rate term structure suggests that two or three factors account for most of the yield curve changes. They can be interpreted, respectively, as a level, slope and curvature factors (see below). This strongly suggests that a multi-factor approach should be used for pricing and hedging fixed-income securities.

There are different ways to generalize the above method to account for non parallel deformations of the term structure. The common principle behind all techniques is the following. Let us express the value of the portfolio using the whole curve of zero-coupon rates, where we now make explicit the time-dependency of the variables

$$V_t = \sum_{i=1}^{m} \frac{F_i}{\left[1 + R(t, t_i - t)\right]^{t_i - t}}$$

Hence, we consider V_t to be a function of the zero-coupon rates $R(t, t_i - t)$, which will be denoted by R_t^i in this section for simplicity of exposition. The risk factor is the yield curve as a whole, *a priori* represented by m components, as opposed to a single variable, the yield to maturity y. The whole point is to narrow down this number of factors in a least arbitrary way. The starting point is, as usual, a (second-order) Taylor expansion of the value of the portfolio. We treat this as a function of different variables $V_t = V(R_t^1, \ldots, R_t^m)$

$$dV_t \approx \sum_{i=1}^{m} \frac{\partial V_t}{\partial R_t^i} dR_t^i + \frac{1}{2} \sum_{i,i'=1}^{m} \frac{\partial^2 V_t}{\partial R_t^i \partial R_t^{i'}} dR_t^i dR_t^{i'}$$

If we merely consider the first-order terms, we get

$$dV_t \approx \sum_{i=1}^{m} \frac{\partial V_t}{\partial R_t^i} dR_t^i$$

Let us further assume that the investor is willing to use as many hedging assets H^j as there are different risk factors, which is m in that case. This assumption is quite restrictive because, as we have already said, it is not very convenient, and may prove to be very expensive, to use more than a few hedging assets, and will be relaxed below. The price of each of these hedging assets will obviously also be a function of the different rates R_t^i. This is precisely why we may use them as hedging assets! We write for *j=1,...,m*

$$dH_t^j \approx \sum_{i=1}^{m} \frac{\partial H_t^j}{\partial R_t^i} dR_t^i$$

Then we construct our global hedge portfolio (for simplicity of exposition, we assume that the hedging assets and the portfolio to hedge have the same face value equal to \$1)

$$V_t^* = V_t + \sum_{j=1}^{m} f_t^j H_t^j$$

such that, up to the first order

$$dV_t^* = 0$$

We have

$$dV_t^* \approx \sum_{i=1}^{m} \frac{\partial V_t}{\partial R_t^i} dR_t^i + \sum_{j=1}^{m} f_t^j \sum_{i=1}^{m} \frac{\partial H_t^j}{\partial R_t^i} dR_t^i$$

or equivalently

$$dV_t^* \approx \sum_{i=1}^{m} \left(\frac{\partial V_t}{\partial R_t^i} + \sum_{j=1}^{m} f_t^j \frac{\partial H_t^j}{\partial R_t^i} \right) dR_t^i$$

A sufficient and necessary condition to have $dV_t^* = 0$ up to a first order approximation for any set of (small) variations dR_t^i is to take for any i

$$\frac{\partial V_t}{\partial R_t^i} + \sum_{j=1}^{m} f_t^j \frac{\partial H_t^j}{\partial R_t^i} = 0$$

Solving this linear system for f_t^j, $j = 1, ..., m$ for each trading date gives the optimal hedging strategy.
If we now denote

$$H_t^{'} = \left(\frac{\partial H_t^j}{\partial R_t^i} \right)_{\substack{i=1,\dots,m \\ j=1,\dots,m}} = \begin{pmatrix} \frac{\partial H_t^1}{\partial R_t^1} & \cdots & \frac{\partial H_t^m}{\partial R_t^1} \\ \vdots & & \vdots \\ \frac{\partial H_t^1}{\partial R_t^m} & \cdots & \frac{\partial H_t^m}{\partial R_t^m} \end{pmatrix}; \Phi_t = \begin{pmatrix} f_t^1 \\ \\ f_t^m \end{pmatrix} \textit{and } V_t^{'} = \begin{pmatrix} -\frac{\partial V_t}{\partial R_t^1} \\ \vdots \\ -\frac{\partial V_t}{\partial R_t^m} \end{pmatrix}$$

we finally have the following system

$$H_t^{'}.\Phi_t = V_t^{'}$$

the solution of which is given by

$$\Phi_t = \left(H_t^{'} \right)^{-1}.V_t^{'}$$

if we further assume that the matrix $H_t^{'}$ is invertible, which means that no hedging asset price may be a linear combination of the other *m-1*.

Remark 2 *It sometimes happens that the hedging assets value depends on risk factors slightly different from those affecting the hedged portfolio which is called correlation risk or cross-hedge risk. Let us assume for the sake of simplicity that there is only one risk factor which we denote R_t. We write*

$$V_t = V(R_t) \text{ and } H_t = H(R_t^{'})$$

where $R_t^{'}$ is a priori (slightly) different from R_t. Ex-ante one should always try to minimize that difference. Ex-post, the question is: once the hedging asset has been selected, what can be done a posteriori} to improve the hedge efficiency ? We have

$$dV_t^* \approx \frac{\partial V_t}{\partial R_t} dR_t + f_t \frac{\partial H_t}{\partial R_t^{'}} dR_t^{'}$$

In that case, the usual prescription

$$f_t = -\frac{\partial V_t}{\partial R_t} / \frac{\partial H_t}{\partial R_t^{'}}$$

will fail to apply successfully because $dR_t^{'}$ may be different from dR_t, which is precisely what correlation risk} is all about. One may handle the situation in the following way. Let us first consider the convenient situation when one could express $R_t^{'}$ as some function of R_t

$$R_t^{'} = f(R_t)$$

In that case, we have

$$dV_t^* \approx \frac{\partial V_t}{\partial R_t} dR_t + f_t \frac{\partial H_t}{\partial R_t^{'}} f'(R_t) dR_t$$

Then

$$dV_t^* \approx 0 \Leftrightarrow f_t = -\frac{\frac{\partial V_t}{\partial R_t}}{\frac{\partial H_t}{\partial R_t^{'}} f'(R_t)}$$

Hence, we may keep the usual prescription provided that we amend it in order to account for the sensitivity of one factor with respect to the other. Unfortunately, it is not generally possible to express $R_t^{'}$ as some function of R_t. However, a satisfying solution may be found using some statistical estimation of the function $f(R_t)$. We may, for example, assume a simple linear relationship[3]

[3] *Because of cointegration and non-stationarity of series, it is better to consider a linear relation in variations rather than in level. We should write*

$$dR_t^{'} = a dR_t + b + e_t$$

$$R_t' = aR_t + b + e_t$$

where e_t *is the usual error term, and the parameters are estimated using standard statistical tools. Then, taking the error term equal to zero, we get*

$$dR_t' = df(R_t) = adR_t$$

Hence, we should amend the hedge ratio in the following way

$$f_t = -\frac{\frac{\partial V_t}{\partial R_t}}{a\frac{\partial H_t}{\partial R_t'}}$$

Of course, the method is as accurate as the quality of the approximation (measured through the squared correlation factor). This will change the hedge strategy and improve the efficiency of the method in case a is significantly different from 1.

In practice, one should consider a more realistic case, namely a situation in which the hedger does not want to use as many hedging assets as there are different risk factors. The principle is invariably to aggregate the risks in the most sensible way. There is actually a systematic method to do so using results from a Principal Components Analysis (PCA) of the interest rates variations, as will now be explained. This is the state-of-the-art technique for dynamic interest rate hedging.

5.2 Regrouping Risk Factors Through a Principal Component Analysis

The purpose of PCA is to explain the behavior of observed variables using a smaller set of unobserved implied variables. From a mathematical standpoint, it consists of transforming a set of *m* dependent variables into a reduced set of orthogonal variables which reproduce the original information present in the dependent structure. This tool can yield interesting results, especially for the pricing and risk management of correlated positions. Using PCA with historical zero-coupon rate curves (both from the Treasury and Interbank markets), it has been observed that the first three principal components of spot curve changes explain the main part of the returns variations on fixed-income securities over time. **Table 1** summarizes the results of some studies on the topic of PCA of spot interest curves led both by academics and practitioners. These include studies by Barber and Copper (BC), Bühler and Zimmermann (BZ), D'Ecclesia and Zenios (DZ), Golub and Tilman (GT), Kärki and Reyes (KR), Lardic, Priaulet and Priaulet (LPP), Litterman and Scheinkman (LS), Martellini and Priaulet (MP).

Table 1 - Some PCA Results

Authors	Country (Period)	Range	Factors	% of Explanation[4]
LS (1991)	USA (1984-88)	6M-18Y	3	88.04/8.38/1.97
DZ (1994)	Italy (1988-92)	6M-7Y	3	93.91/5.49/0.42
KR (1994)	Germ./Switz./USA (1990-94)	3M-10Y	3	total : 97/98/98

[4] For example «88.04/8.38/1.97» means that the first factor explains 88.04% of the yield curve deformations, the second 8.38%, and the third 1.97%. Sometimes, we also provide the total amount by adding up these terms.

BC (1996)	USA (1985-91)	1M-20Y	3	80.93/11.85/4.36
BZ (1996)	Germany Switzerland (1988-96)	1M-10Y	3	71/18/4 75/16/3
GT (1997)	JP Morgan Risk Metrics – 09/30/96	3M-30Y	3	92.8/4.8/1.27
MP (2000)	France (1995-98)	1M-10Y	3	66.64/20.52/6.96
LPP (2001)	Belgium France Germany Italy UK (1998-2000)	1M-30Y	3	62/27/6 62/21/8 61/23/6 59/24/7 60/24/9

These three factors, currently named level, slope and curvature, are believed to drive interest rate dynamics and can be formulated in terms of interest rate shocks, which can be used to compute principal components durations.

We express the change $dR(t,q_k) = R(t+1,q_k) - R(t,q_k)$ of the zero-coupon rate $R(t,q_k)$ with maturity q_k at date t such as

$$dR(t,q_k) = \sum_{l=1}^{m} c_{lk} C_t^l + e_{tk}$$

where c_{lk} is the sensitivity of the k^{th} variable to the l^{th} factor defined as

$$\frac{\Delta(dR(t,q_k))}{\Delta(C_t^l)} = c_{lk}$$

which amounts to individually applying a, say, 1% variation to each factor, and computing the absolute sensitivity of each zero-coupon yield curve with respect to that unit variation. These sensitivities are commonly called the principal component $durations.

C_t^l is the value of the l^{th} factor at date t, and e_{tk} is the residual part of $dR(t,q_k)$ that is not explained by the factor model.

One can easily see why this method has become popular. Its main achievement is that it allows for the reduction of the number of risk factors without losing much information, i.e., to proceed in the least possible arbitrary way. Since these three factors (parallel movement, slope oscillation, and curvature), regarded as risk factors, explain most of the variance in interest rate variation, we may now use not more than three hedging assets. We write now the changes of value of a fixed-income portfolio as

$$dV_t^* \approx \sum_{k=1}^{m} \left(\frac{\partial V_t}{\partial R(t,q_k)} + \sum_{j=1}^{3} f_t^j \frac{\partial H_t^j}{\partial R(t,q_k)} \right) dR(t,q_k)$$

We then use $dR(t,q_k) \approx \sum_{l=1}^{3} c_{lk} C_t^l$

$$dV_t^* \approx \sum_{k=1}^{m}\left(\left(\frac{\partial V_t}{\partial R(t,q_k)} + \sum_{j=1}^{3} f_t^j \frac{\partial H_t^j}{\partial R(t,q_k)}\right)\sum_{l=1}^{3} c_{kl} C_t^l\right)$$

or

$$\begin{aligned} dV_t^* \approx & \sum_{k=1}^{m}\left(c_{1k}\frac{\partial V_t}{\partial R(t,q_k)} + \sum_{j=1}^{3} f_t^j c_{1k}\frac{\partial H_t^j}{\partial R(t,q_k)}\right)C_t^1 \\ & + \sum_{k=1}^{m}\left(c_{2k}\frac{\partial V_t}{\partial R(t,q_k)} + \sum_{j=1}^{3} f_t^j c_{2k}\frac{\partial H_t^j}{\partial R(t,q_k)}\right)C_t^2 \\ & + \sum_{k=1}^{m}\left(c_{3k}\frac{\partial V_t}{\partial R(t,q_k)} + \sum_{j=1}^{3} f_t^j c_{3k}\frac{\partial H_t^j}{\partial R(t,q_k)}\right)C_t^3 \end{aligned}$$

The quantity $\sum_{k=1}^{m}\left(c_{1k}\frac{\partial V_t}{\partial R(t,q_k)} + \sum_{j=1}^{3} f_t^j c_{1k}\frac{\partial H_t^j}{\partial R(t,q_k)}\right)$ is commonly called the principal component \$durations of portfolio V^* with respect to factor 1.

If we want to set the (first order) variations of the hedged portfolio V_t^* to zero for any possible evolution of the interest rates $dR(t,q_k)$, or equivalently for any possible evolution of the C_t^l, a sufficient condition for this is to take for *l = 1,2,3*

$$\sum_{k=1}^{m}\left(c_{lk}\frac{\partial V_t}{\partial R(t,q_k)} + \sum_{j=1}^{3} f_t^j c_{lk}\frac{\partial H_t^j}{\partial R(t,q_k)}\right) = 0$$

that is neutral principal component \$durations.

Finally, at each possible date, we are left with three unknowns f_t^j and three linear equations. The system may be represented in the following way. Let us introduce

$$H_t^{'} = \begin{pmatrix} \sum_{k=1}^{m} c_{1k}\frac{\partial H_t^1}{\partial R(t,q_k)} & \sum_{k=1}^{m} c_{1k}\frac{\partial H_t^2}{\partial R(t,q_k)} & \sum_{k=1}^{m} c_{1k}\frac{\partial H_t^3}{\partial R(t,q_k)} \\ \sum_{k=1}^{m} c_{2k}\frac{\partial H_t^1}{\partial R(t,q_k)} & \sum_{k=1}^{m} c_{2k}\frac{\partial H_t^2}{\partial R(t,q_k)} & \sum_{k=1}^{m} c_{2k}\frac{\partial H_t^3}{\partial R(t,q_k)} \\ \sum_{k=1}^{m} c_{3k}\frac{\partial H_t^1}{\partial R(t,q_k)} & \sum_{k=1}^{m} c_{3k}\frac{\partial H_t^2}{\partial R(t,q_k)} & \sum_{k=1}^{m} c_{3k}\frac{\partial H_t^3}{\partial R(t,q_k)} \end{pmatrix}; \Phi_t = \begin{pmatrix} f_t^1 \\ f_t^2 \\ f_t^3 \end{pmatrix}; V_t^{'} = \begin{pmatrix} -\sum_{k=1}^{m} c_{1k}\frac{\partial V_t}{\partial R(t,q_k)} \\ -\sum_{k=1}^{m} c_{2k}\frac{\partial V_t}{\partial R(t,q_k)} \\ -\sum_{k=1}^{m} c_{3k}\frac{\partial V_t}{\partial R(t,q_k)} \end{pmatrix}$$

We then have the following system

$$H_t^{'}.\Phi_t = V_t^{'}$$

The solution is given by

$$\Phi_t = \left(H_t^{'}\right)^{-1} .V_t^{'}$$

In practice, we need to estimate the principal components $durations used at date *t*. They are derived from a PCA performed on a period prior to *t*, for example [*t-3* months, *t*]. Hence, the result of the method is strongly sample-dependent. In fact and for estimation purposes, it is more convenient to use some functional specification for the zero-coupon yield curve that it is consistent with results from a PCA.

5.3 Hedging Using a Three Factors Model of the Yield Curve

The idea here consists of using a model for the zero-coupon rate function. We detail below the Nelson and Siegel and the Svensson (or extended Nelson and Siegel) models. One may also alternatively use the Vasicek (1977) model, the extended Vasicek model, or the CIR (1985) model, among many others.[5]

5.3.1 Nelson-Siegel and Svensson Models

Nelson and Siegel (1987) have suggested to model the continuously compounded zero-coupon rate $R^C(0,q)$ as

$$R^C(0,q) = b_0 + b_1\left[\frac{1-\exp(-q/t_1)}{q/t_1}\right] + b_2\left[\frac{1-\exp(-q/t_1)}{q/t_1} - \exp(-q/t_1)\right]$$

a form that was later extended by Svensson (1994) as

$$R^C(0,q) = b_0 + b_1\left[\frac{1-\exp(-q/t_1)}{q/t_1}\right] + b_2\left[\frac{1-\exp(-q/t_1)}{q/t_1} - \exp(-q/t_1)\right]$$
$$+ b_3\left[\frac{1-\exp(-q/t_2)}{q/t_2} - \exp(-q/t_2)\right]$$

where

$R^C(0,q)$ is the continuously compounded zero-coupon rate at time zero with maturity q .

b_0 is the limit of $R^C(0,q)$ as q goes to infinity. In practice, b_0 should be regarded as a long-term interest rate.

b_1 is the limit of $b_0 - R^C(0,q)$ as q goes to 0. In practice, b_1 should be regarded as the short to long term spread.

b_2 and b_3 are curvature parameters.

t_1 and t_2 are scale parameters that measures the rate at which the short-term and medium-term components decay to zero.

As shown by Svensson (1994) the extended form allows for more flexibility in yield curve estimation, in particular in the short-term end of the curve, because it allows for more complex shapes such as U-shaped and hump-shaped curves. The parameters b_0, b_1, b_2 *and* b_3 are estimated daily by using an OLS optimization program, which consists, for a basket of bonds, in minimizing the sum of the squared spread between the market price and the theoretical price of the bond as obtained with the model.

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[5] See Martellini and Priaulet (2000) for details about these zero-coupon functions.

We see that the evolution of the zero-coupon rate $R^C(0,q)$ is entirely driven by the evolution of the beta parameters, the scale parameters being fixed. In an attempt to hedge a bond, for example, one should build a global portfolio with the bond and a hedging instrument, so that the portfolio achieves a neutral sensitivity to each of the beta parameters. Before the method can be implemented, one therefore needs to compute the sensitivities of any arbitrary portfolio of bonds to each of the beta parameters.

Consider a bond which delivers principal or coupon and principal denoted F_i at dates q_i. Its price P_0 at date *t=0* is given by the following formula

$$P_0 = \sum_{i=1}^{m} F_i . e^{-q_i R^C(0,q_i)}$$

In the Nelson and Siegel (1987) and Svensson (1994) models, we can calculate at date *t=0* the \$durations $D_i = \partial P_0 / \partial b_i$ for *i = 0,1,2,3* of the bond *P* to the parameters b_0, b_1, b_2 *and* b_3. They are given by the following formulas[6]

$$\begin{cases} D_0 = -\sum_i q_i F_i e^{-q_i R^C(0,q_i)} \\ D_1 = -\sum_i q_i \left[\dfrac{1-\exp(-q_i/t_1)}{q_i/t_1} \right] F_i e^{-q_i R^C(0,q_i)} \\ D_2 = -\sum_i q_i \left[\dfrac{1-\exp(-q_i/t_1)}{q_i/t_1} - \exp(-q_i/t_1) \right] F_i e^{-q_i R^C(0,q_i)} \\ D_3 = -\sum_i q_i \left[\dfrac{1-\exp(-q_i/t_2)}{q_i/t_2} - \exp(-q_i/t_2) \right] F_i e^{-q_i R^C(0,q_i)} \end{cases} \qquad \textbf{(10)}$$

An example of calculation of the level, slope and curvature \$durations is given in Martellini, Priaulet and Priaulet (2003).[7]

7.2 Hedging Method

The idea of the hedge in the Svensson (1994) model is to create a global portfolio with:

- the bond portfolio to hedge whose price and face value are denoted by *P* and N_P
- four hedging instruments whose prices and face values are denoted by G_i and N_{G_i} for *i = 1,2,3* and *4*.

and to make it neutral to changes of parameters b_0, b_1, b_2 *and* b_3.

We therefore look for the quantities q_0, q_1, q_2 *and* q_3 to invest, respectively, in the four hedging instruments G_0, G_1, G_2 *and* G_3 that satisfy the following linear system

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[6] Of course, \$duration D_3 is only obtained in the Svensson (1994) model.

[7] See also Barrett, Gosnell and Heuson (1995) and Willner (1996).

$$\begin{cases} q_1 N_{G_1}\dfrac{\partial G_1}{\partial b_0} + q_2 N_{G_2}\dfrac{\partial G_2}{\partial b_0} + q_3 N_{G_3}\dfrac{\partial G_3}{\partial b_0} + q_4 N_{G_4}\dfrac{\partial G_4}{\partial b_0} = -N_P D_0 \\ q_1 N_{G_1}\dfrac{\partial G_1}{\partial b_1} + q_2 N_{G_2}\dfrac{\partial G_2}{\partial b_1} + q_3 N_{G_3}\dfrac{\partial G_3}{\partial b_1} + q_4 N_{G_4}\dfrac{\partial G_4}{\partial b_1} = -N_P D_1 \\ q_1 N_{G_1}\dfrac{\partial G_1}{\partial b_2} + q_2 N_{G_2}\dfrac{\partial G_2}{\partial b_2} + q_3 N_{G_3}\dfrac{\partial G_3}{\partial b_2} + q_4 N_{G_4}\dfrac{\partial G_4}{\partial b_2} = -N_P D_2 \\ q_1 N_{G_1}\dfrac{\partial G_1}{\partial b_3} + q_2 N_{G_2}\dfrac{\partial G_2}{\partial b_3} + q_3 N_{G_3}\dfrac{\partial G_3}{\partial b_3} + q_4 N_{G_4}\dfrac{\partial G_4}{\partial b_3} = -N_P D_3 \end{cases} \quad \textbf{(11)}$$

In the Nelson and Siegel (1987) model, we only have three hedging instruments because there are only three parameters. Then $q_4 = 0$, and the last equation of system (11) disappears.

6 Comparative Analysis of Various Hedging Techniques

We now analyze the hedging performance of three methods, the duration hedge, the duration/convexity hedge and the Nelson-Siegel $durations hedge in the context of a specific bond portfolio.
At date *t=0*, the continuously compounded zero-coupon yield curve is described by the following set of parameters of the Nelson and Siegel model[8]

b_0	b_1	b_2	t
8%	*-3%*	*-1%*	*3*

This corresponds to a standard increasing curve. We consider a bond portfolio whose features are summarized in Table 2. The price is expressed in % of the face value which is equal to $100,000,000. We compute the yield to maturity (YTM), the $duration, the $convexity, and the level, slope and curvature $durations of the bond portfolio as given by equation (10).

Table 2 - Characteristics of the Bond Portfolio to Be Hedged

Price	YTM	$Duration	$Convexity	Level D_0	Slope D_1	Curvature D_2
972.376	7.169%	-5709.59	79662.17	-6118.91	-1820.02	-1243.28

To hedge the bond portfolio, we use three plain vanilla 6-month Libor swaps whose features are summarized in Table 3. $duration, $convexity, level, slope and curvature $durations are those of the fixed coupon bond contained in the swap. The principal amount of the swaps is $1,000,000. They all have an initial price of zero.

[8] Note that we can obtain the annualized compounded zero-coupon yield curve by using the following equation

$$R(t,q) = \exp\left[R^C(t,q)\right] - 1$$

where $R^C(t,q)$ is the continuously compounded zero-coupon rate at date t with maturity q, and $R(t,q)$ its annualized compounded equivalent.

Table 3 - Characteristics of the Swap Instruments

Maturity	Swap Rate	$Duration	$Convexity	Level D_0	Slope D_1	Curvature D_2
2 years	5.7451%	-184.00	517.09	-194.55	-142.66	-41.66
7 years	6.6717%	-545.15	3809.39	-579.80	-242.66	-166.22
15 years	7.2309%	-897.66	11002.57	-948.31	-254.58	-206.69

We consider that the bond portfolio and the swap instruments present the same default risk so that we are not concerned with that additional source of uncertainty, and we can use the same yield curve to price them. This curve is the one described above with the Nelson and Siegel parameters.

To measure the performance of the three hedge methods, we assume 10 different movements for the yield curve. These 10 scenarios are obtained by assuming the following changes of the beta parameters in the Nelson and Siegel model:

- small parallel shifts with $b_0 = +0.1\%$ and $b_0 = -0.1\%$.
- large parallel shifts with $b_0 = +1\%$ and $b_0 = -1\%$.
- decrease and increase of the spread short to long-term spread with $b_1 = +1\%$ and $b_1 = -1\%$.
- curvature moves with $b_2 = +0.6\%$ and $b_2 = -0.6\%$.
- flattening and steepening moves of the yield curve with ($b_0 = -0.4\%, b_1 = +1.2\%$) and ($b_0 = +0.4\%, b_1 = -1.2\%$).

The six last scenarios, which represent non-parallel shifts, are displayed in Figures 1, 2 and 3.

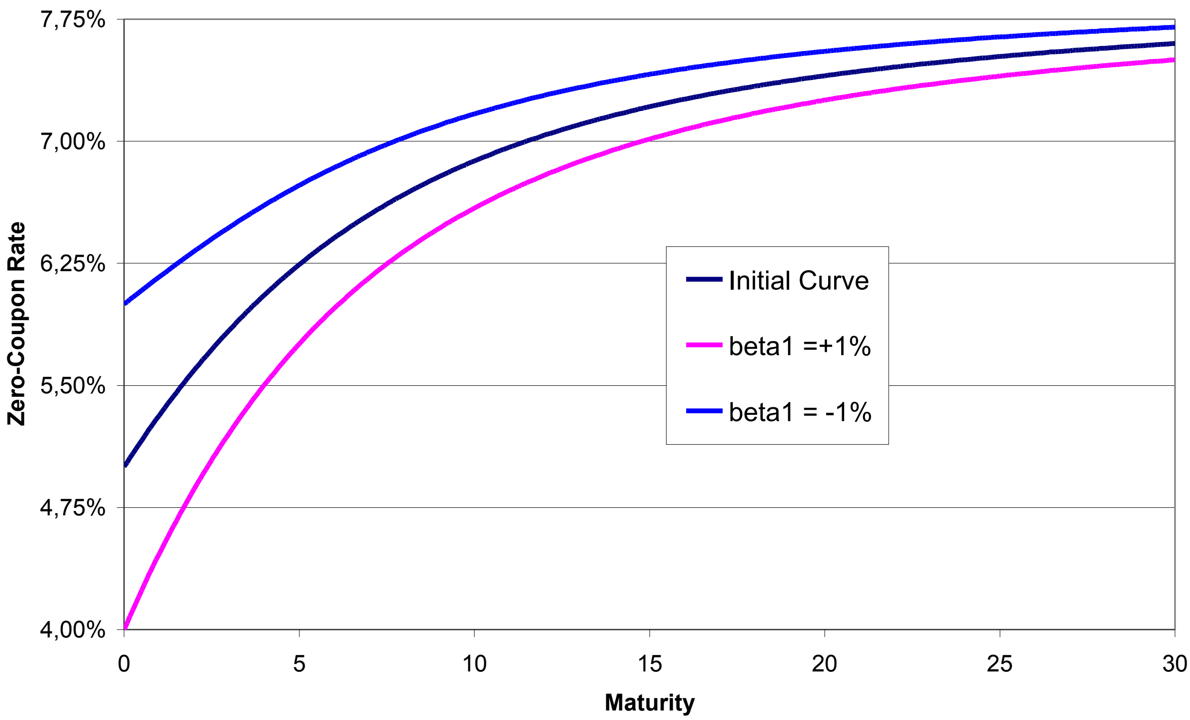

Figure 1 - New Yield Curve after an Increase and a Decrease of the Slope Factor ($b_1 = +1\%$) and ($b_1 = -1\%$)

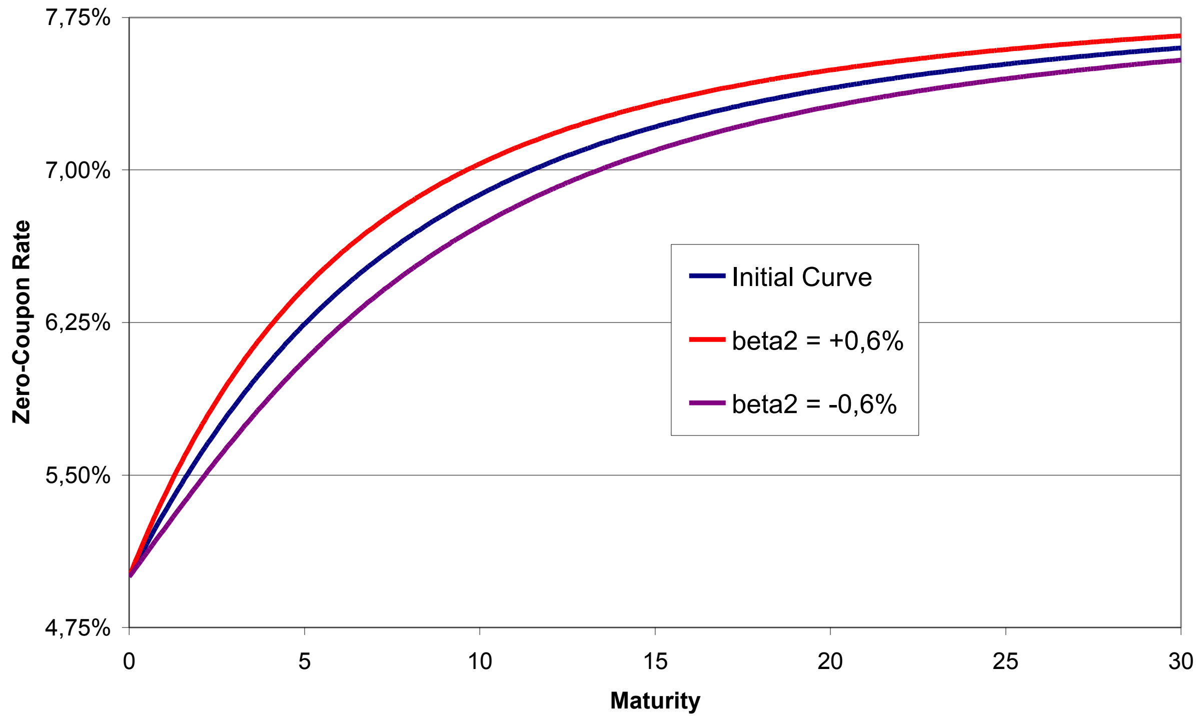

Figure 2 - New Yield Curve after an Increase and a Decrease of the Curvature Factor ($b_2 = +0.6\%$) and ($b_2 = -0.6\%$)

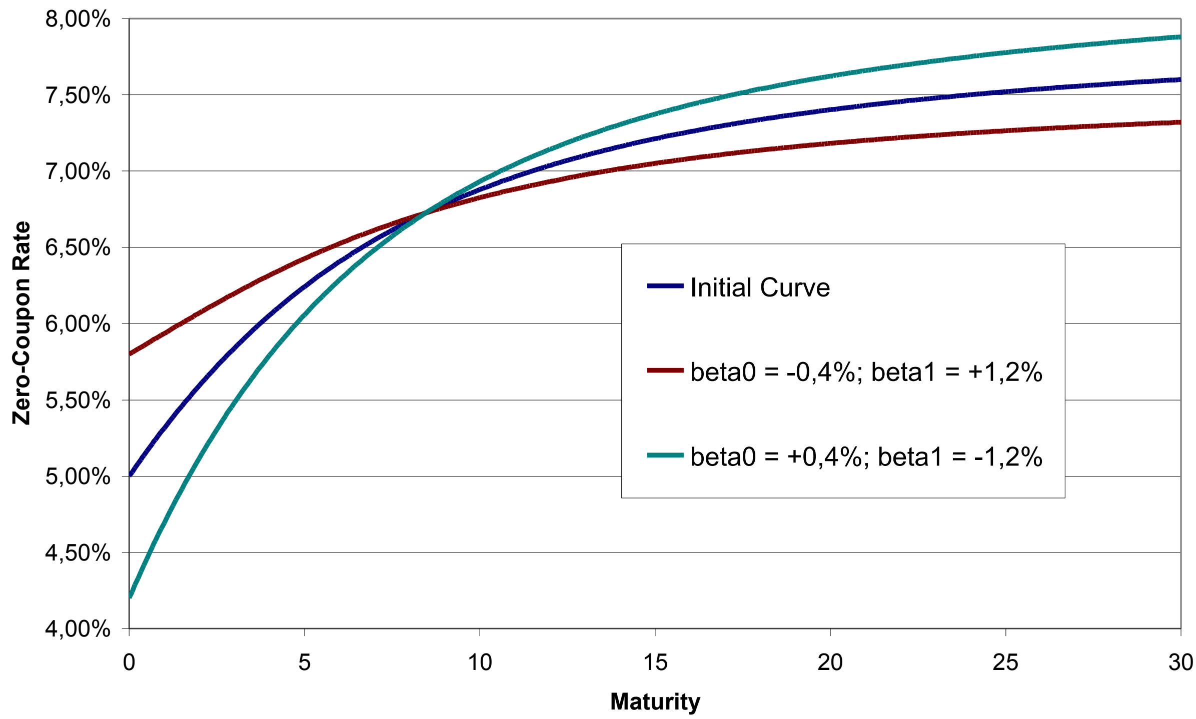

Figure 3 - New Yield Curve after a Flattening Movement ($b_0 = -0.4\%, b_1 = +1.2\%$) and a Steepening Movement ($b_0 = +0.4\%, b_1 = -1.2\%$)

Duration hedging is performed with the 7-year maturity swap using equation (7). We have to sell 1047 swaps. Duration/Convexity is performed with the 7- year and 15-year maturity swaps using equation (9). We have to buy 337 7-year maturity swaps and to sell 841 15-year maturity swaps. Nelson and Siegel $Durations hedge is performed with the three swaps using equation (11). We have to sell 407 2-year maturity swaps, to buy 219 7-year maturity swaps and to sell 696 15-year maturity swaps. Results are given in Table 4, where we display the change in value of the global portfolio (which aggregates the change in value on the bond portfolio and the hedging instruments) assuming that the yield curve scenario occurs instantaneously. This change of value can be regarded as the hedging error for the strategy. It would be exactly zero for a perfect hedge.

Table 4 - Hedging Errors in $ of the Three Different Methods, Duration, Duration/Convexity and Nelson and Siegel $Durations

Yield Curve Scenario	No Hedge	Duration	Duration/Convexity	Nelson-Siegel $Durations
$b_0 = +0.1\%$	-6,076,494	-25,627	-97,826	8,573
$b_0 = -0.1\%$	6,161,872	71,692	97,435	2,013
$b_0 = +1\%$	-57,176,627	1,605,853	-1,050,770	475,891
$b_0 = -1\%$	65,743,922	3,028,609	1,026,043	599,622
$b_1 = +1\%$	-17,982,901	7,103,063	-4,934,261	15,557
$b_1 = -1\%$	18,421,236	-7,311,245	5,001,463	-4,959
$b_2 = +0.6\%$	-7,410,125	2,972,451	-400,339	1,131
$b_2 = -0.6\%$	7,509,714	-2,991,594	381,400	-9,038
$b_0 = +0.4\%, b_1 = -1.2\%$	-2,438,405	-8,665,316	5,661,669	90,991
$b_0 = -0.4\%, b_1 = +1.2\%$	2,839,537	9,024,298	-5,474,877	94,636

The value of the bond portfolio is equal to $972,375,756.[9] With no hedge, we clearly see that the loss in portfolio value can be significant in all adverse scenarios.

As expected, duration hedging appears to be effective only for small parallel shifts of the yield curve. The hedging error is positive for large parallel shifts because of the positive convexity of the portfolio. For non parallel shifts, the loss incurred by the global portfolio can be very significant. For example, the portfolio value drops by $7,311,245 in the pure slope scenario when $b_1 = -1\%$, and $8,665,316 in the steepening scenario.

As expected, duration/convexity hedge is better than duration hedge when large parallel shifts occur. On the other hand, it appears to be ineffective for all other scenarios, even if the hedging errors are still better (smaller) than those obtained with duration hedging.

[9] Opposite results in terms of hedging errors would be obtained if the investor was short the bond portfolio.

Finally, we see that the Nelson and Siegel $durations hedge is a very reliable method for all kinds of yield curve scenario. In all cases, the hedging error appears to be negligible when compared to the value of the bond portfolio to hedge.

8 Conclusion

This paper addresses the problem of hedging a portfolio of fixed-income cash-flows. Because there is ample empirical evidence that changes in the yield curve can be large and multi-dimensional, we argue that simple duration hedging techniques achieve limited efficiency in most market conditions. We explain how to relax the assumptions of small and parallel shifts of the yield curve, implicit in duration hedging, and show that satisfying hedging results can be achieved by using instead a three-factor model for the yield curve. Besides, by implementing semi-hedged strategies, this model enables a portfolio manager to take specific bets on particular changes of the yield curve while being hedged against the others (see Martellini, Priaulet and Priaulet (2002) for such an implementation in the case of the butterfly strategy).

9 References

Barber, J.R., and M.L. Copper, Immunization Using Principal Component Analysis, Journal of Portfolio Management, Fall 1996, p.99-105.

Barrett W., Gosnell Jr. and Heuson A., Yield Curve Shifts and the Selection of Immunization Strategies, Journal of Fixed Income, vol. 5, 2, September 1995, p.53-64.

Bühler A., and H. Zimmermann, A Statistical Analysis of the Term Structure of Interest Rates in Switzerland and Germany, Journal of Fixed Income, vol. 6, 3, December 1996, p.55-67.

Chance D.M., and J.V. Jordan, "Duration, Convexity, and Time as Components of Bond Returns", Journal of Fixed Income, vol. 6, 2, September 1996, p.88-96.

Cox J.C., Ingersoll J.E. and Ross S.A., A Theory of the Term Structure of Interest Rates, Econometrica, vol. 53, 2, March 1985, p.385-407.

D'Ecclesia R.L. and Zenios S.A., Risk Factor Analysis and Portfolio Immunization in the Italian Bond Market, Journal of Fixed Income, vol. 4, 2, September 1994, p.51-58.

Fama, E. F. and K R. French., The Cross-Section of Expected Stock Returns, Journal of Finance, vol. 47, 2, June 1992, p.427-465.

Golub B.W. and L.M. Tilman., Measuring Yield Curve Risk Using Principal Components Analysis, Value at Risk, and Key Rate Durations, Journal of Portfolio Management, Summer 1997, p. 72-84.

Kärki J. and Reyes C., Model Relationship, Risk, vol. 7, 12, December 1994, p.32-35.

Lardic, S., Priaulet, P., and S. Priaulet, PCA of Yield Curve Dynamics: Questions of Methodologies, Working Paper, HSBC-CCF, 2001.

Litterman, R., and J. Scheinkman, Common Factors Affecting Bond Returns, Journal of Fixed Income, vol.1, 1, 1991, p. 54-61.

Martellini, L., and P. Priaulet, Fixed-Income Securities: Dynamic Methods for Interest Rate Risk Pricing and Hedging, John Wiley and Sons, 2000.

Martellini, L., Priaulet, P., and S. Priaulet, Understanding the Butterfly Strategy, Journal of Bond Trading and Management, vol. 1, 1, June 2002, p. 1-11.

Martellini, L., Priaulet, P., and S. Priaulet, Fixed-Income Securities: Alternative and Traditional Investment Strategies, John Wiley and Sons, forthcoming, 2003.

Merton, R.C., An Intertemporal Capital Asset Pricing Model, Econometrica, 41, 1973, p.867-888.
Nelson, C.R., and A.F. Siegel, Parsimonious Modeling of Yield Curves, Journal of Business, 60 (4), 1987, p.473-489.
Ross, S., The Arbitrage Theory of Capital Asset Pricing, Journal of Economic Theory, 13, 1976, p.341-360.
Svensson L., Estimating and Interpreting Forward Interest Rates: Sweden 1992-94, CEPR discussion paper 1051, October 1994.
Vasicek O.A., An Equilibrium Characterisation of the Term Structure, Journal of Financial Economics, vol. 5, 2, November 1977, p.177-188.
Willner R., A New Tool for Portfolio Managers: Level, Slope and Curvature Durations, Journal of Fixed Income, vol. 6, 1, June 1996, p.48-59.