A Unique Unconditionally Arbitrage-Free Solution to the Term Structure of Interest Rates

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Abstract

This paper points out that the fundamental partial differential equation (PDE) implies that the drifts of the underlying processes are *irrelevant*, regardless whether the state factors are tradable or not. Since the simple boundary condition for default-free discount bonds can be satisfied by a linear discount function, the variances and covariances of the underlying processes are also irrelevant. This paper proves that the linear solution is unique, which indicates that the bond market has an unconditionally arbitrage-free equilibrium. The unique linear solution, namely, the Exponential Polynomial (EP) model, is equivalent to the Exponential Spline model of Vasicek and Fong (1982) without spline fitting. The empirical results support the model.

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A Unique Unconditionally Arbitrage-Free Solution to the Term Structure of Interest Rates

The term structure of interest rates can be defined as a discount function of a vector of state factors, $P(x,s,t)$, $s \geq t$, which describes the prices of default-free discount bonds of all possible maturities. Under the usual perfect-market assumptions, absence of riskless arbitrage requires the discount function to satisfy a fundamental partial differential equation (PDE), which has been derived by many authors, such as Langetieg (1980), Cox, Ingersoll, and Ross (CIR) (1981) and (1985). The fundamental PDE is unspecified because it does not require identification of the underlying state factors, and the coefficients of the PDE are neither specified nor restricted by the arbitrage argument.

Because the absence of arbitrage simply requires the *existence* of the market prices of risk, the drifts of the underlying processes are *irrelevant*, regardless whether the underlying factors are tradable or not. Bond pricing has a unique advantage over option pricing because the boundary condition, $P(x,s,s) \equiv P(s,s) = 1$, is so simple that it can be satisfied by a linear discount function. Hence, the variances and covariances of the underlying processes are also irrelevant. Moreover, since the boundary condition simply stipulates bond issuer to pay the face value of the bond at maturity, *regardless* of the economic conditions, the underlying factors can be specified after the solution is found. Hence, the bond market should have an unconditionally arbitrage-free equilibrium, which is completely independent of the identity of the underlying factors and their stochastic processes.

This paper proves that the unconditionally arbitrage-free equilibrium is unique, because the linear solution to the unspecified PDE and the boundary condition is unique. The unique solution is a linear combination of some exponential functions, which is referred to as the Exponential Polynomial (EP) model, because it is equivalent to the component function of the Exponential Spline model of Vasicek and Fong (1982) defined on a subinterval of the maturity range. In other words, the EP model is equivalent to the Vasicek-Fong model without spline fitting, i.e., with the subinterval stretched to the entire maturity range $[0, \infty)$.

Intuitively, the EP model represents a term structure space that is linearly spanned by its state factors on an exponential basis that consists of a number of distinct exponential functions. Since the boundary condition does not contractually specify any state factor, the state factors are, in fact, defined relatively to a given exponential basis. The stochastic processes of the state factors can be objectively identified by empirical studies because they are unrestricted by the solution. As long as the basis is time invariant, the term structure shift is guaranteed arbitrage-free within the linear space. Using the monthly US treasury security data in the CRSP Bond File, our empirical investigation supports the existence of a time-invariant basis.

The traditional methodology has misinterpreted the unspecified first-order coefficients of the fundamental PDE. Many authors thought that, since the underlying factors are not tradable, there could be extra freedom to manipulate the components of first-order coefficients, i.e., the drifts and the market prices of risk. They failed to realize that a solution to such a specified, i.e., restricted, PDE would force the bond market equilibrium to be conditional on some imaginary constraints.

Vasicek (1977) first designated the instantaneous interest rate (spot rate) as a state factor. The existing single-factor models, such as Vasicek (1977) and CIR (1985), force the bond market equilibrium to be conditional on the spot rate following their corresponding processes. Since the spot rate is just the limiting value of the derivative of the discount function, it is by definition a function of the underlying factors in a multi-factor setting. Hence, any assumed spot rate process is equivalent to a severe constraint imposed on the underlying factors. As a result, the existing two-factor models are just marginal extensions of the single-factor models, because the second factor is only part of the spot rate process. For example, the two-factor model of Richard (1978) has simply decomposed the nominal spot rate into the real spot rate and inflation rate. For another example, the second factor in Longstaff and Schwartz (1992) is just the volatility of the spot rate. Brennan and Schwartz (1979) wanted to introduce the long rate as the second factor, but could not find a closed form solution.

Since there is no reason why the bond market equilibrium should be conditional on some imaginary spot rate process, it is not surprising that none of the existing solutions fits the observed term structure. As further solutions seem unlikely, many academics have turned to the new arbitrage-pricing methodology proposed by Ho and Lee (1986), and Heath, Jarrow, and Morton (HJM 1992). HJM (1992) criticized the traditional methodology for manipulating the drifts and market prices of risk. They removed the spot rate constraint and allowed multiple factors that do not necessarily have specific economic meanings. However, by "taking the current term structure as given", this methodology has, in fact, presumed that the current bond market equilibrium cannot be represented by any equilibrium model.

The new methodology argues that the directly imposed stochastic structure of forward rates can be consistent with the current term structure because its parameters can be calibrated from the taken-as-given current term structure. However, calibration is always possible regardless of the specification of the stochastic structure. This methodology argues that, under certain conditions, especially if the drift function follows a peculiar form, there exists a unique equivalent martingale probability measure. However, whether a *P*-measure has a unique equivalent *Q*-measure is strictly within the context of the Girsanov Theorem on changing the probability measure, which cannot justify the arbitrariness of the exogenous specification of the stochastic structure, or the *P*-measure. Essentially, HJM (1992) attempted to drop out the market prices of risk by changing the probability measure. However, it is the drifts that are irrelevant.

From a more general perspective, arbitrage pricing in a perfect market is a high standard for asset pricing, because otherwise any pricing scheme may be easily justified as arbitrage-free by certain constraints, whether realistic or imaginary. Since the bond market, at least the US treasury security market, is already reasonably efficient and will be more efficient, we have already *assumed away* all realistic market imperfections, such as transactions costs, taxes, and restrictions on short sales. Hence, there is no reason why the arbitrage equilibrium of the bond market should be forced to be conditional on some imaginary constraints.

The rest of the paper is organized as follows. Section I reviews the derivation of the PDE and discusses the methodological problems in the existing solutions. Section II derives the unique linear EP solution. Section III discusses parameter specification and empirical estimation. Section IV presents empirical results, and Section V concludes.

I. The Fundamental Partial Differential Equation

This section reviews the derivation of Langetieg (1980) because its expression is easy to follow. The CIR (1981) derivation is similar except that it presumes the existence of the instantaneous interest rate.

Suppose the discount function can be described by a state-factor function, $P(x(t), s, t)$, which describes the discount bond prices at time *t* with terminal payoff $1 at their respective maturity dates $s \geq t$. If the state vector, $x(t) = [x_1(t), \cdots, x_n(t)]$, follows a joint Ito process,

$$dx_i = \mu_i(x(t), t)dt + \sigma_i(x(t), t)dz_i, \quad i = 1, \cdots, n, \tag{1}$$

the instantaneous change of the bond price can be expressed by Ito's formula as:

$$\begin{aligned} dP(x(t), s, t) &= \left(\frac{\partial P}{\partial t} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} \sigma_i \sigma_j \frac{\partial^2 P}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \mu_i \frac{\partial P}{\partial x_i} \right) dt + \sum_{i=1}^{n} \sigma_i \frac{\partial P}{\partial x_i} dz_i \\ &= \overline{P}_t dt + \sum_{i=1}^{n} \sigma_i P_{x_i} dz_i. \end{aligned} \tag{2}$$

Notice that both the identity of the underlying factors and the coefficients of their stochastic processes are unspecified.

Let $\mathbf{P}$ be a vector of $n+2$ bond prices of different maturities, the above equation can be expressed in vector form as

$$d\mathbf{P} = \overline{\mathbf{P}}_t dt + \sum_{i=1}^{n} \sigma_i \mathbf{P}_{x_i} dz_i \quad . \tag{3}$$

Assume the bond market is perfect in the usual sense, such as the bonds are perfectly divisible and tradable continuously without transactions costs, taxes, and restrictions on short sales. In such a market, any bond portfolio, represented by vector *w*, that requires zero investment ($w'\mathbf{P} = 0$) and bears zero risk ($w'\mathbf{P}_{x_i} = 0, \ \forall i$), must earn exactly zero return ($w' d\mathbf{P} = w'\overline{\mathbf{P}}_t = 0$). Hence, the $n+2$ vectors, $\mathbf{P}, \overline{\mathbf{P}}_t$, and $\mathbf{P}_{x_i}$, must be linearly dependent. The mathematical definition of linear dependency simply means that there *exist* $n+2$ scalars, not all zero, such that the linear combination of the $n+2$ vectors is a zero vector.

Since there can be at most $n+1$ independent scalars, the linearity can be expressed, without loss of generality, as

$$f_0 \mathbf{P} = \overline{\mathbf{P}}_t + \sum_{i=1}^{n} f_i s_i \mathbf{P}_{x_i} . \tag{4}$$

The $n+1$ scalars, $f_i(x(t),t)$, $i = 0,1,\cdots,n$, can be arbitrary functions of the state factors and the current time *t*, but not the maturity date *s*. Hence, the well-known fundamental PDE for bond pricing with arbitrary maturity is just an arbitrary row of equation (4), which can be re-arranged to

$$\begin{aligned} &\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 P[x(t),t,s]}{\partial x_i \partial x_j} s_{ij}[x(t),t] + \sum_{i=1}^{n}\frac{\partial P[x(t),t.s]}{\partial x_i}[m_i(x(t),t) - f_i(x(t),t)s_i(x(t),t] \\ &+ \frac{\partial P[x(t),t,s]}{\partial t} - f_0(x(t),t)P[x(t),t,s] = 0. \end{aligned} \tag{5}$$

It is clear that the PDE is a logical result of the perfect-market assumptions. The arbitrage argument is just the mathematical definition of linear dependency, which is valid regardless of the specification of the underlying factors and their stochastic processes.

The scalar function $f_0(x(t),t)$ deserves a close examination. Langetieg (p. 80) stated that, "Assuming there is a riskless instantaneous interest rate, denoted by *r*, then $f_0 = r$". In CIR (1981), this scalar is denoted by *r* from the very beginning because the authors have presumed the existence of the instantaneous interest rate. However, there is no reason to *assume* the existence of the instantaneous interest rate, because the term structure discount function has already described the pricing of default-free discount bonds of all maturities. *By definition*, the instantaneous interest rate is nothing more than the limiting value of the default-free discount function:

$$r(x,t) = \left. \frac{\P\, P(x(t),s,t)}{\P\, t} \right|_{s=t} . \qquad (6)$$

It exists because any bond can, at least theoretically, be traded up to the moment of its maturity. Like any bond price of arbitrary maturity is a function of the underlying factors, so is the instantaneous interest rate.

The scalars functions, $f_i(x(t),t),\ \ i=1,\cdots,n$, are well known as the market prices of risk for the state factors. Langetieg (p. 80) stated that, "If an underlying stochastic factor x_i is tradable, then $f_i = (m_i - f_0 x_i)/s_i$. When x_i is not tradable, then f_i must be empirically estimated or theoretically specified". This is a fatal misinterpretation of the PDE, because the only correct interpretation should be that *the solution to the PDE should not be dependent on the drifts of the underlying processes*. Notice that the market price of risk $f_i(x(t),t)$ is part of the first-order coefficient for factor *i*, which can be denoted as a single coefficient function,

$$h_i(x(t),t) = m_i(x(t),t) - f_i(x(t),t)s_i(x(t),t) . \qquad (7)$$

If the factor is tradable, the first-order coefficient is uniquely determined as $h_i = f_0 x_i$, hence, $f_i = (m_i - f_0 x_i)/s_i$ regardless the functional form of the drift. In this case, we say that the drift is irrelevant, because whatever it might be, it will be complemented by the market price of risk to meet the unique first-order coefficient $h_i = f_0 x_i$. When the underlying factor is not tradable, the first-order coefficient $h_i(x(t),t)$ should be determined from solving the unspecified PDE. If a solution is found, the first-order coefficient is obviously determined from the solution process. Whatever the drift might be, the market price of risk would complement the drift to meet the first-order coefficient, exactly like the case where the factor is tradable. Hence, the drifts of the underlying processes are *irrelevant* under any circumstance.

The traditional methodology thought that the nontradable factors of the term structure offered some extra freedom for manipulating the components of the first-order coefficients. For example, Langetieg (1980) arbitrarily assumed the state factors following a jointly elastic random walk, and the corresponding market prices of risk being constant, without even slightest knowledge of the state factors. There is, of course, no hope to find any meaningful solution from such an arbitrarily specified, i.e., restricted, PDE.

Other authors thought that, by exogenously designating some seemingly meaningful economic factors, the drifts and the market prices could still be manipulated. Unfortunately, nothing seems eligible except for the instantaneous interest rate (spot rate). The presumption of the existence of the instantaneous interest rate has led Vasicek (1977) and CIR (1985) to manipulate the spot rate process and the market price of risk, and found two well-known single-factor closed-form solutions. Unfortunately, these celebrated solutions have also marked the beginning of the end of the traditional methodology. Since the spot rate is a function of the underlying factors in a multi-factor case, any assumption on the spot rate process is equivalent to a constraint on the underlying factors. This constraint is apparently so severe that the second factor in the existing two-factor closed-form solutions, such as Richard (1978) and Longstaff and Schwartz (1992), has been nothing but part of the spot rate process, as mentioned early.

In option pricing, since the nonlinear option boundary cannot be satisfied by any linear solution, the volatility of the underlying asset is relevant. However, bond pricing has a unique advantage over option pricing, because its simple boundary condition can be satisfied by a linear discount function. Since the linearity implies that all the variances and covariances of the underlying processes are irrelevant, and the drifts are already irrelevant, the existence of a linear solution indicates the existence of an unconditionally arbitrage-free equilibrium in the bond market. Unlike the case of option pricing where the underlying asset is contractually specified in the option boundary as a state factor, the bond boundary does not specify any state factor. Hence, there is no reason to imagine some stochastic processes of unknown factors in order to force some nonlinear solutions. The next section proves that the linear solution not only exists, but is also unique.

II. The Unique Unconditionally Arbitrage-free Solution

The existing solutions have all been time-homogenous in the sense that

$$P(x(t), s, t) = P(x(t), s-t) = P(x(t), T) \text{ with } T = s - t \geq 0\,, \tag{8}$$

because the dependency of a bond price on the current time can be implicit in its dependency on the current state vector $x(t)$, and its dependency on the maturity date *s* is only implicit in its dependency on its term to maturity *T*. We will simplify all the notations by the time homogeneity and denote the current state vector as x rather than $x(t)$ whenever the context is clear.

Since a linear solution is, by definition,

$$\frac{\partial^2 P(x, s-t)}{\partial x_i \partial x_j} \equiv 0, \text{ for } i, j = 1, \cdots, n, \tag{9}$$

it can be found from solving the linear part of the fundamental PDE,

$$\sum_{i=1}^{n} h_i(x) \frac{\partial P(x,T)}{\partial x_i} - \frac{\partial P(x,T)}{\partial T} - f_0(x) P(x,T) = 0, \text{ s.t. } P(x,0) \equiv 1. \tag{10}$$

Note that $\partial P / \partial t = -\partial P / \partial T$. Because the boundary condition is independent of *x*, a linear discount function that satisfies the boundary must be of the following form:

$$P(x,T) = h_0(T) + \sum_{i=1}^{n} x_i g_i(T), \text{ with } h_0(0) = 1 \text{ and } g_i(0) = 0 \ \ \forall i, \tag{11}$$

where the component functions $h_0(T)$ and $g_i(T)$ must not contain the state factors. The partial derivatives of the discount function can be found as

$$\frac{\partial P}{\partial x_i} = g_i(T), \tag{12}$$

and

$$\frac{\partial P}{\partial t} = -\frac{\partial P}{\partial T} = -h_0'(T) - \sum_{i=1}^{n} x_i g_i'(T), \tag{13}$$

where the "prime" denotes the partial derivatives with respect to *T*. Substituting (11) to (13) into PDE (10) and re-arranging yields

$$\sum_{i=1}^{n} \left\{ [h_i(x) - f_0(x) x_i] g_i(T) - x_i g_i'(T) \right\} - \left\{ f_0(x) h_0(T) + h_0'(T) \right\} = 0, \tag{14}$$

where the notations for the coefficients are simplified by the time-homogeneity of the discount function.

If discount function (11) satisfies (14) regardless the values of the state vector *x*, the braced terms in (14) must be all individually zero. However, if the last braced term is equal to zero,

$$\{f_0(x)h_0(T) + h_0'(T)\} = 0 \ , \tag{15}$$

the solution, $h_0(T) = e^{-f_0(x)T}$, clearly contains the state vector. The coefficient function $f_0(x)$ cannot be constant because it is the instantaneous interest rate.

The only possibility left is for functions $g_i(T)$ to contain separable element $h_0(T)$. This can be expressed, without loss of generality[1], as

$$g_i(T) = h_i(T) - h_0(T), \quad i = 1, \cdots, n \,, \tag{16}$$

where no restrictions are imposed on $h_i(T)$ except that $h_i(0) = 1$. Substituting (16) into (14) and re-arranging yields

$$\begin{aligned} &\sum_{i=1}^{n} \{[h_i(x) - f_0(x)x_i]h_i(T) - x_i h_i'(T)\} \\ &- \left\{ h_0'(T)\left[1 - \sum_{i=1}^{n} x_i\right] + h_0(T)\left[f_0(x) + \sum_{i=1}^{n}[h_i(x) - f_0(x)x_i]\right]\right\} = 0. \end{aligned} \tag{17}$$

By setting the i^{th} term to zero,

$$[h_i(x) - f_0(x)x_i]h_i(T) - x_i h_i'(T) = 0 \,, \tag{18}$$

it can be re-arranged to

$$d \ln h_i(T) / dT = [h_i(x) - f_0(x)x_i] / x_i \,. \tag{19}$$

The solution to (19), subject to $h_i(0) = 1$, can be found uniquely as

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[1] The i[th] component of the linear discount function may be specified more generally as $x_i g_i(T) = x_i[a h_i(T) - b h_0(T)]$. As linearity implies that a and b cannot be functions of *x*, and time-homogeneity implies that a and b cannot be functions of *t*, a and b can only be arbitrary

$$h_i(T) = e^{-l_i T}, \tag{20}$$

where

$$l_i = -[h_i(x) - f_0(x)x_i]/x_i. \tag{21}$$

Hence, l_i is constant if and only if the first-order coefficient is

$$h_i(x) = [f_0(x) - l_i]\,x_i. \tag{22}$$

As a natural result, the first-order coefficient is uniquely determined from solving the unspecified PDE. By equation (7), the market price of risk for factor i has a unique expression:

$$f_i(x) = \left[m_i(x) + l_i x_i - f_0(x)x_i\right]/s_i(x), \tag{23}$$

regardless the specification of the underlying process.

Similarly, $h_0(T)$ can be solved from setting the last braced term to zero:

$$h_0(T) = e^{-l_0 T}, \tag{24}$$

with

$$l_0 = \frac{f_0(x) + \sum_{i=1}^{n}[h_i(x) - f_0(x)x_i]}{1 - \sum_{i=1}^{n} x_i}. \tag{25}$$

By equations (22) and (25), l_0 is constant if and only if the coefficient function $f_0(x)$ is

$$f_0(x) = l_0 + \sum_{i=1}^{n}(l_i - l_0)x_i. \tag{26}$$

By substituting (20) and (24) into (11), the unique unconditionally arbitrage-free solution can be formally expressed as

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constants. Hence, $x_i[ah_i(T) - bh_0(T)] = bx_i[(a/b)h_i(T) - h_0(T)]$. By scaling x_i and $h_i(T)$, this

$$P(x,s-t)=\left(1-\sum_{i=1}^{n}x_i\right)e^{-\lambda_0(s-t)}+\sum_{i=1}^{n}x_i e^{-\lambda_i(s-t)} \quad , \tag{27}$$

which is a linear combination of some exponential functions. We refer this solution as the Exponential Polynomial (EP) discount function, because it can be recognized as the component function of the Exponential Spline model of Vasicek and Fong (1982) defined on a subinterval of the maturity range. In other words, the EP solution is equivalent to the Vasicek-Fong spline model without spline fitting, i.e., with the subinterval stretched to the entire maturity range $T \in [0,\infty)$.

As mentioned early, there is no need to assume the existence of the instantaneous interest rate or to set exogenously the coefficient $f_0 = r$. Instead, the solution should justify that its limiting value is identical to the coefficient function $f_0(x)$. From equation (27), it is straightforward to verify that the instantaneous interest rate is, indeed, equal to $f_0(x)$:

$$r(x) \equiv \left.\frac{\partial P(x,s-t)}{\partial t}\right|_{s=t} = \lambda_0 + \sum_{i=1}^{n}(\lambda_i - \lambda_0)x_i = f_0(x). \tag{28}$$

It is interesting to notice that the *derived* expression of the instantaneous interest rate from the EP solution coincides with the *presumed* spot rate expression in Langetieg (1980):

$$r = w_0 + \sum_{i=1}^{n} w_i x_i \ . \tag{29}$$

The yield function can be found from the following transformation,

$$y(x,s-t) = -\frac{\ln P(x,s-t)}{s-t}, \quad s>t \ , \tag{30}$$

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specification is equivalent equation (16).

with the instantaneous interest rate given by (28). The time-homogeneity of the discount function allows the forward function to be found from any of the following transformations:

$$X \qquad i \qquad . \qquad (31)$$

III. Empirical Estimation of the EP Model

In order for (27) to be a legitimate discount function, all the exponential parameters must be strictly positive. Intuitively, a set of distinct exponential parameters defines an exponential basis, upon which the EP model represents a term structure space that is linearly spanned by its state factors. Hence, the state factors are, in fact, defined relatively to a given basis. If the bond market equilibrium is unique, there should exist a unique basis. However, since the boundary condition does not contractually specify any state factor and parameter, the basis may have to be identified by trial-and-error. For example, it could be first exogenously designed, then empirically tested, then modified and tested. Fortunately, the EP model allows sufficient flexibility for the basis design, because it does not impose any restriction on the stochastic processes of the state factors. The exponential parameters cannot be estimated simultaneously with the state factors because the latter are defined relatively to the former. In order for the state factors to be well defined, the basis should be sufficiently stable, i.e., the exponential parameters should remain as time invariant as possible. Hence, these parameters have to be exogenously specified.

Assume, without loss of generality, that l_0 is the smallest exponential parameter. As $s \to \infty$, since the EP discount function approaches

$$P(x, s-t) \to \left(1 - \sum_{i=1}^{n} x_i\right) e^{-l_0 (s-t)} \text{ as } s \to \infty \quad , \tag{32}$$

l_0 is the asymptotic forward rate, or the long rate. Although not directly observable, the long rate may reflect the general level of interest rates, thus, it may not be kept constant. If the long rate is sufficiently stochastic, the EP model may be inconsistent with its claimed no-arbitrage property, because the long rate risk cannot be hedged unless the bond portfolio has zero duration. Hence, we have to hope that the long rate is smooth enough to be treated as a time-varying parameter. In the existing literature, the long rate has always been designated as either a parameter (e.g.,Vasicek (1977)) or a combination of parameters (e.g., CIR (1985) and Longstaff and Schwartz (1992)).

For simplicity of notation, we denoted the long rate as *R*. To minimize the potential inconsistency of the EP model with its no-arbitrage claim, we specify all the exponential parameters as fixed multiples of the long rate:

$$l_0 = R, \quad l_i = (a_i + 1)R, \quad i = 1, \cdots, n, \tag{33}$$

where a_i, $i = 1, \cdots, n$ are constant parameters. This specification is similar to Vasicek and Fong (1982), in which $a_i = i$. Under this specification, the state factors are measured on a basis that is indexed by the long rate *R*. For example, the instantaneous interest rate expression in (28) becomes

$$r(x(t)) = R(t) + \sum_{i=1}^{n} (l_i - R) x_i(t) = R(t)\left(1 + \sum_{i=1}^{n} a_i x_i(t)\right). \tag{34}$$

This specification allows the EP model to be transformed to

$$P(x,\hat{s},\hat{t}) = \left(1-\sum_{i=1}^{n} x_i\right) e^{-(\hat{s}-\hat{t})} + \sum_{i=1}^{n} x_i e^{-(a_i+1)(\hat{s}-\hat{t})} \quad , \tag{35}$$

by changing variables

$$\hat{t} = Rt, \text{ and } \hat{s} = Rs . \tag{36}$$

It can be easily verified that $P(x,\hat{s},\hat{t})$ satisfies the following equation:

$$\sum_{i=1}^{n} [r-(a_i+1)R] \frac{\partial P}{\partial x_i} + R\frac{\partial P}{\partial \hat{t}} - rP = 0 . \tag{37}$$

If $R(t)$ is sufficiently smooth such that $(d\,\hat{t})^2$ can be ignored, the instantaneous price change can be expressed by the first-order Taylor series,

$$dP = \frac{\partial P}{\partial \hat{t}} d\,\hat{t} + \sum_{i=1}^{n} \frac{\partial P}{\partial x_i} dx_i . \tag{38}$$

By the same arbitrage argument and vector notations as in the previous section, equations (37) and (38) imply that any bond portfolio that requires zero investment ($w'\mathbf{P} = 0$) and bears zero risk ($w'\mathbf{P}_{x_i} = 0, \ \forall i$) would earn exactly zero return ($w'd\mathbf{P} = w'\mathbf{P}_{\hat{t}} = 0$) over the transformed time interval $d\,\hat{t}$.

Notice that (32) implies a restriction to the sum of the state factors as

$$\sum_{i=1}^{n} x_i < 1 \quad , \tag{39}$$

because the discount function must be strictly positive. Theoretically, there is an implicit domain for the state vector *x*, relative to a given basis, such that the EP forward function (31) is nonnegative. Empirically, we have found that (39) is almost sufficient to ensure the forward function nonnegative, if the basis is reasonably designed.

In our empirical investigation with the CRSP Bond File, we have found that the cross-sectional estimation can take up to eight factors without overfitting. The constant parameters in equation (33) are presented in Table 1 by a descending order:

Table 1
Fixed Parameters

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
50	40	30	20	10	5	2	1

In the Vasicek and Fong (1982) model, these parameters, in ascending order, are 1, 2, 3, 4, ..., etc.. We have found that the Vasicek-Fong parameters often cause multicollinearity, because the component exponential functions are not sufficiently distinct among themselves. Hence, we have deliberately chosen these parameters as far apart from each other as possible. We have attempted different selection of the parameters, but could not find any significant difference in the performance of the EP model, so long as the parameters are set sufficiently apart. As mentioned above, there may exist a unique time-invariant basis, but we do not know how to identify it, at least for now. Hence, our objective in the current empirical investigation is limited to verifying the existence of a relatively time-invariant basis. We hope an optimal basis can be identified in future research.

The estimation of the EP model is identical to that of the Vasicek and Fong (1982) model. Let $B_k(t, s_k, c_k)$ denote the observed price of the k^{th} bond at t for maturity at s_k, which has \$1 face value, and an annual coupon rate c_k with semiannual interest payments. Let A_k be its accrued interest. Similarly to equation (7) of Vasicek and Fong (1982), the linearity of the EP discount function allows the relationship between the observed price and the present value of the cash flows of the bond to be expressed as

$$B_k + A_k = \left(1 - \sum_{i=1}^{n} x_i\right) B_k^0 + \sum_{i=1}^{n} x_i B_k^i + e_k \,, \tag{40}$$

where B_k^0 is the present value of the cash flows of the bond discounted by the rate R, and B_k^i is the present value of the bond discounted by the rate l_i, etc., and e_k is the pricing error. For a cross-sectional sample of *m* treasury securities, equation (40) can be arranged to an Ordinary Least Squares (OLS) regression as

$$(B_k + A_k - B_k^0) = \sum_{i=1}^{n} x_i (B_k^i - B_k^0) + e_k \,, \quad k = 1,\ 2,\ \cdots,\ m \,. \tag{41}$$

McCulloch (1971) introduced the spline method because the observed cross-sectional samples of treasury securities usually contain far more short-term securities than the long-term ones. To accommodate the data structure, he suggested dividing the maturity range into subintervals, such that the number of securities in each subinterval is roughly the same. By dividing the maturity, he applied the same spline function, e.g., a cubic function, to all the subintervals. Since each spline function is defined only on a subinterval, every pair of adjacent spline functions have to be carefully connected to ensure continuity and differentiability at the knot. Vasicek and Fong (1982) replaced the polynomial spline function of McCulloch by exponential polynomial spline function, but still followed the spline method. Since the spline discount function is artificially continuous and differentiable at the knots, the resulting forward function often exhibits extreme instability at the knots, as shown in Shea (1984) and (1985). Unfortunately, nobody has ever questioned whether this purely technical design is consistent with no-arbitrage. Now we know that these linear spline discount function models cannot be consistent with no-arbitrage, because the EP model is a unique linear solution.

The EP model takes care the technical concern of McCulloch naturally. Notice that all the component exponential functions are effective at the short end of the maturity range. Since the exponential functions with larger parameters decay rapidly with maturity, there are less and less effective exponential functions for longer-term treasury securities as maturity increases. Hence, the number of effective exponential functions fits naturally with the distribution of the number of treasury securities over the observable maturity range.

IV. Empirical Results

A. Data

From the CRSP (Center for Research in Security Prices) Bond File, we extracted the price data of US Treasury bills, notes and bonds that are non-callable and without special tax status from January 1960 to December 1991. Every monthly sample consists of the bid and ask quotes of the treasury securities at 3:30 p.m. Eastern Time on the last business day of the month. The bid-ask average is used for the term structure estimation. The yields (yield-to-maturity) in the CRSP file are also used for comparison with the estimated yield function.

B. Dimensionality

The dimension of the EP model is primarily determined by the maturity range of the bonds in each sample, with the January 1973 sample as a dividing point, because the 20-year treasury bond first appeared there. Prior to January 1973, since the maturity range is much shorter, most of the samples require only five factors, and some require six. After January 1973, all the samples can take eight factors without overfitting, except for the period between February 1977 and December 1979, during which the number of factors have to be reduced to seven in order to fit the short end of the term structure accurately.

For computational convenience, we have always maintained the eight-factor basis. If significant multicollinearity is encountered in a sample, as indicated by significantly large absolute values of x_i, we know the dimension of the EP model is too high, so we impose $x_8 = 0$, which is equivalent to eliminating the last column, $(B_k^8 - B_k^0)$. If the multicollinearity is still significant, the second last column can be further eliminated, and so on, until multicollinearity disappears. Because the dimensionality is strictly related to the maturity range, we have arranged the exponential parameters in a descending order, as shown in Table 1, in order for the exponential function with the slowest rate of decay to be first eliminated.

C. Stability of the Long Rate

For each cross-sectional sample of bond prices, the long rate *R* is determined by a loop, within which the OLS Regression (41) is invoked for each trial value *R*. Each iteration finds the estimates of x, and the root mean squared errors (RMSE) of the sample. Since the short-term treasury securities are effectively zero-coupon securities, their yields in the CRSP file should coincide with the estimated yield function in absence of pricing errors. Because the pricing errors of the short-term securities contribute very little to the overall sample RMSE, the accuracy of the estimation, especially in the short maturity range, cannot rely solely on the sample RMSE. Towards the end of the looping, the short term

RMSE (SRMSE), i.e., the root mean squared errors of the securities with maturities less than one year, is used to guide the fine-tuning of R. The estimation experience shows that constraint (39) could be violated if R is too low. When R cannot be lowered further, the number of factors has to be adjusted. At the end of the estimation, the forward function up to 50 years maturity is computed. If any part of the forward function is negative, the number of factors and/or the long rate needs adjusted.

The numerical value of R is found sensitive to changing the number of factors. For example, R was 10.2% for the January 1977 sample, which required eight factors. Between February 1977 and December 1979, because the number of factor dropped to seven, R dropped to around 5% level. When the number of factors came back to eight in the sample of January 1980, R jumped back to 12.5%. As a result, the non-observable long rate parameter R can reflect the general level of interest rate only if the basis is constant. Despite the occasional change in the basis, the eight-factor basis has dominated most of the samples since 1973, and has not changed after January 1980.

As mentioned early, the EP model remains consistent with no-arbitrage as long as the long rate is sufficiently smooth. Figure 1 plots the time-series of the long rate and the histogram of the first difference of the long rate against the normal distribution. It shows that the long rate usually changes very little on monthly basis, except for a few sudden shifts that are caused by changing the dimension of the EP model. Hence, it is reasonable to treat the long rate as a time-varying parameter rather than a stochastic variable. Notice that the last sharp decline in December 1985 is not caused by the dimensionality. It appears to reflect some structural shift of the bond market, because the long rate has since remained around that lower level.

D. Goodness of fit

Figure 2 plots the sample RMSE and SRMSE for general assessment of the goodness of fit. The SRMSE's are generally less than 0.2 cent per $1 face value cross all the samples. In contrast, the RMSE's are much larger. The sample of October 29, 1982 has the highest RMSE of 1.17 cents per $1 face value. According to Elton and Green (1998), the pricing errors in the CRSP Bond File are generally attributable to nonsynchronous trading, especially for those illiquid bonds. Using high quality daily data provided by GovPX Inc., they find that the tax/coupon effects are much less significant than found by some previous studies. Since the main objective of this empirical investigation is to check if the basis is time invariant, we have ignored the tax/coupon effects.

It is difficult to compare the goodness of fit of the EP model with other models, because most of the empirical studies do not report the goodness of fit on *individual samples.* So far we have only found two comparable studies that have used the same data and reported results on individual samples. One study by Jordan (1984) reported SERB (standard error of regression before tax adjustment, which is equivalent to RMSE) on selected *individual* samples of the CRSP Bond File between 1970 and 1979 in his Table IV, based on the cubic spline of McCulloch (1975) with various tax adjustments. Table 2 compares Jordan's SERB with our RMSE for these samples, which shows that the EP model is more accurate than the cubic spline model. Another comparable study by Coleman, Fisher, and Ibbotson (CFI 1992) proposed a piecewise linear forward function model with various specifications and tax adjustments. For selected CRSP Bond File samples, they reported a range of RMSE's resulted from different estimation methods in their Exhibit 5A. Table 3 compares my RMSE with their best RMSE. Even without tax adjustment, the EP model is clearly comparable with CFI's results.

Table 2
Comparison of Goodness of Fit with Jordan (1984)

The SERB is the standard error of regression before tax adjustment reported in Table IV of Jordan (1984). The RMSE is my root mean squared error equivalent to the SERB.

Year	Month	SERB	RMSE
1975	1	0.46	0.203
	4	0.69	0.585
	7	0.61	0.584
	10	0.74	0.304
1976	1	0.57	0.452
	4	0.75	0.285
	7	0.56	0.372
	10	0.74	0.368
1977	1	0.91	0.461
	4	0.77	0.367
	7	1.04	0.315
	10	0.93	0.218
1978	1	0.86	0.252
	4	0.72	0.198
	7	1.11	0.131
	10	1.23	0.220
1979	1	1.03	0.179
	4	1.09	0.324
	7	0.76	0.250
	10	1.14	0.251

Table 3
Comparison of goodness of fit with CFI (1992)

Sample	CFI's Best RMSE	RMSE of EP Model
Dec. 31, 1979	0.52	0.386
Dec. 30, 1983	0.43	0.352
Nov. 30, 1984	0.31	0.423

Figures 3 to 6 present some specific examples for more intuitive understanding of the goodness of fit. Figure 3 illustrates the ability of the EP model to fit a complicated yield curve on October 31, 1974. It can be seen that the yield curve fits the yields of the short term securities accurately and smoothly, and naturally reaches to the instantaneous nominal interest rate at the zero maturity. Note that the estimation is to minimize the pricing errors rather than the yield errors. The pricing errors are shown in the lower chart. The values of the state factors are reported at the bottom of the figure. Note that the yield curve in the upper chart is transformed from the estimated discount function according to equation (30). The yields are from the CRSP Bond File. They are depicted for comparison purposes. Although the yield curve should coincide with the yields of the zero-coupon securities in absence of pricing errors, it needs not fit the yields, i.e., yield-to-maturity, of the coupon bonds, especially the long term coupon bonds. This is more apparent in Figures 5 and 6.

Figure 4 illustrates the estimation of the April 30, 1975 sample, which is the same sample estimated by Shea (1984) and (1985), using the cubic spline of McCulloch (1975) and exponential spline of Vasicek and Fong (1982), respectively. In both estimations, Shea showed that the forward rate curve dropped to negative after the 20-year maturity. The EP model has never encountered such instability. This result is consistent with Ferguson and Raymar (1998), who found that the Vasicek-Fong component discount function (without spline) is not only sufficient for the term structure estimation but also far more stable than with spline.

Figure 5 illustrates the most erroneous sample of October 29, 1982 among the 384 samples. Notice that the EP model fits the short end extremely accurately, and the pricing errors for the medium and longer term securities are distributed quite symmetrically. The widely dispersed yields of the treasury securities indicate that the major source of the RMSE is data error, but data error has not impaired the ability of the EP model to find a smooth yield curve. Figure 6 illustrates the sample of October 31, 1989, in which the yields of the short and medium term securities exhibit significant dispersion. Once again, the data error has not impaired the estimation.

V. Conclusion

Arbitrage pricing in an ideally perfect market is a high standard for asset pricing, because otherwise any pricing scheme may be easily justified as arbitrage-free by certain constraints, whether realistic or imaginary. Since all possible realistic constraints have been assumed away by the perfect-market assumptions, there is no reason why the bond market equilibrium should be conditional on some imaginary constraints. By taking advantage of the simple boundary, this paper has proved that the fundamental PDE has a unique unconditionally arbitrage-free solution, which is an exponential polynomial (EP) discount function.

While the functional form of the solution is unique, the exponential basis is still unknown. Since the EP model does not impose any restriction on the state factors and their stochastic processes the basis can be exogenously specified, then empirically tested. In this paper we have designed and tested a basis, which is found sufficient stable over a long period of time. We hope a better basis can be identified with more frequent and better quality data in future research.

The traditional methodology has misunderstood the unspecified first-order coefficients of the fundamental PDE. Many authors have thought that the components of the first-order coefficients, i.e., the drifts and the market prices of risk, can be exogenously and separately manipulated, as long as the underlying state factors are not tradable. In fact, since absence of arbitrage is equivalent to the existence of the market prices of risk, it is the drifts that are always irrelevant, regardless whether the state factors are tradable or not. When the state factors are tradable, the first-order coefficients are known, so the drifts are clearly irrelevant. When the state factors are not tradable, the first-order coefficients should be determined from solving the fundamental PDE, so the drifts are also irrelevant.

By arbitrarily selecting the state factors and manipulating both the drifts and the market prices of risk, the traditional methodology has imposed many unreasonable constraints on the bond market equilibrium. Although a few special solutions have been found in the past, none of them is able to fit the observed term structure. The exogenously specified spot rate process has actually blocked further search for multi-factor solutions, because it has imposed a severe constraint on the underlying factors.

HJM (1992) criticized the traditional methodology for exogenously and separately specifying both the drifts and the market prices of risk, but did not realize the irrelevancy of the drifts. They observed that the market prices of risk can be “dropped out” in the “inversion of the term structure”. Hence they attempted to drop out the market prices of risk by manipulating the drift, which was shooting a wrong target. In order to fit the Girsanov Theorem, HJM (1992) directly imposed a stochastic structure of the forward rates, then relied on certain conditions, especially a peculiar form of the drift function, to justify the existence of a unique equivalent martingale probability measure. However, whether a *P*-measure has a unique equivalent *Q*-measure has nothing to with whether the arbitrary specification of the stochastic structure is correct. Since there is neither restriction nor guidance on the specification of the stochastic structure, there can be infinite alternatives to specify the stochastic structure. For example, even within a framework of one or two factors, Amin and Morton (1994) have specified and tested six alternative specifications. It does not make any economic sense to have so many unique equivalent martingale probability measures.

The new methodology claims itself as consistent with the currently observed term structure, because the parameters of the stochastic structure can be calibrated from the taken-as-given current term structure. However, calibration is always possible regardless of the specification of the stochastic structure. Unless the stochastic structure itself and the Brownian Motion assumption on its associated random factors are both correct, the calibrated stochastic structure is neither stable nor reliable. It is very difficult to imagine that, for an exogenously specified stochastic structure of the forward rates, the associated random factors could happen to be not only pure Brownian Motions, but also independent.

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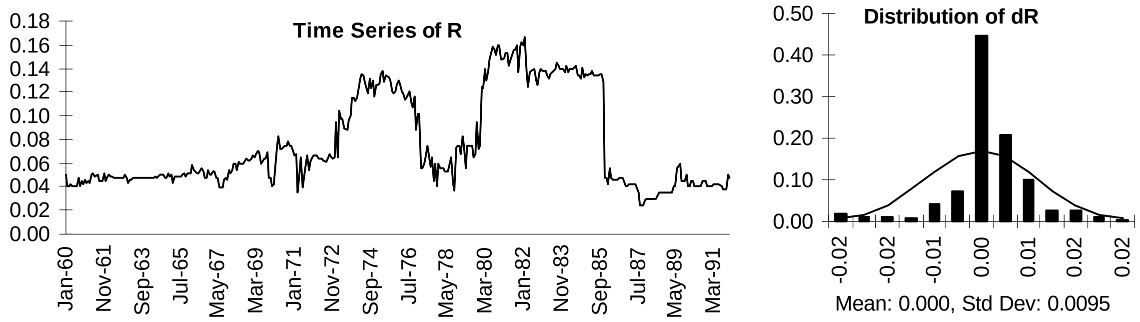


Figure 1. Time series plot of the long rate parameter *R*, and the distribution of the first difference $dR = R_t - R_{t-1}$ against the normal distribution.

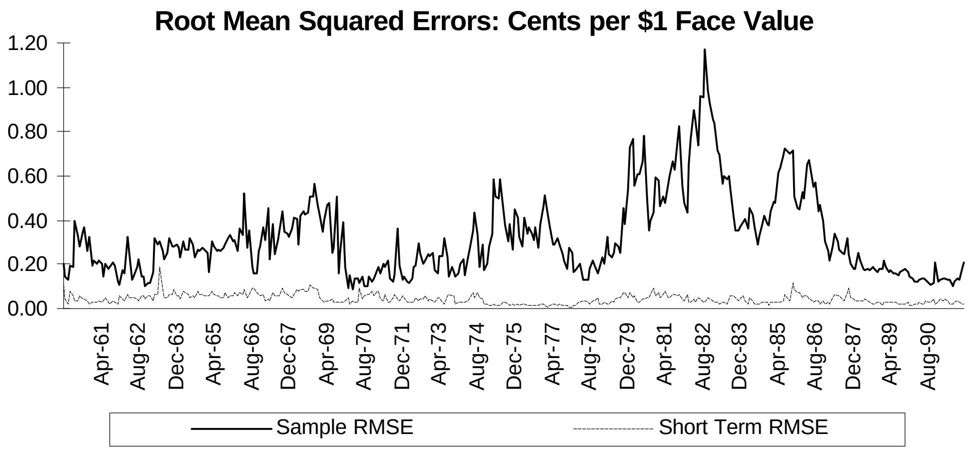
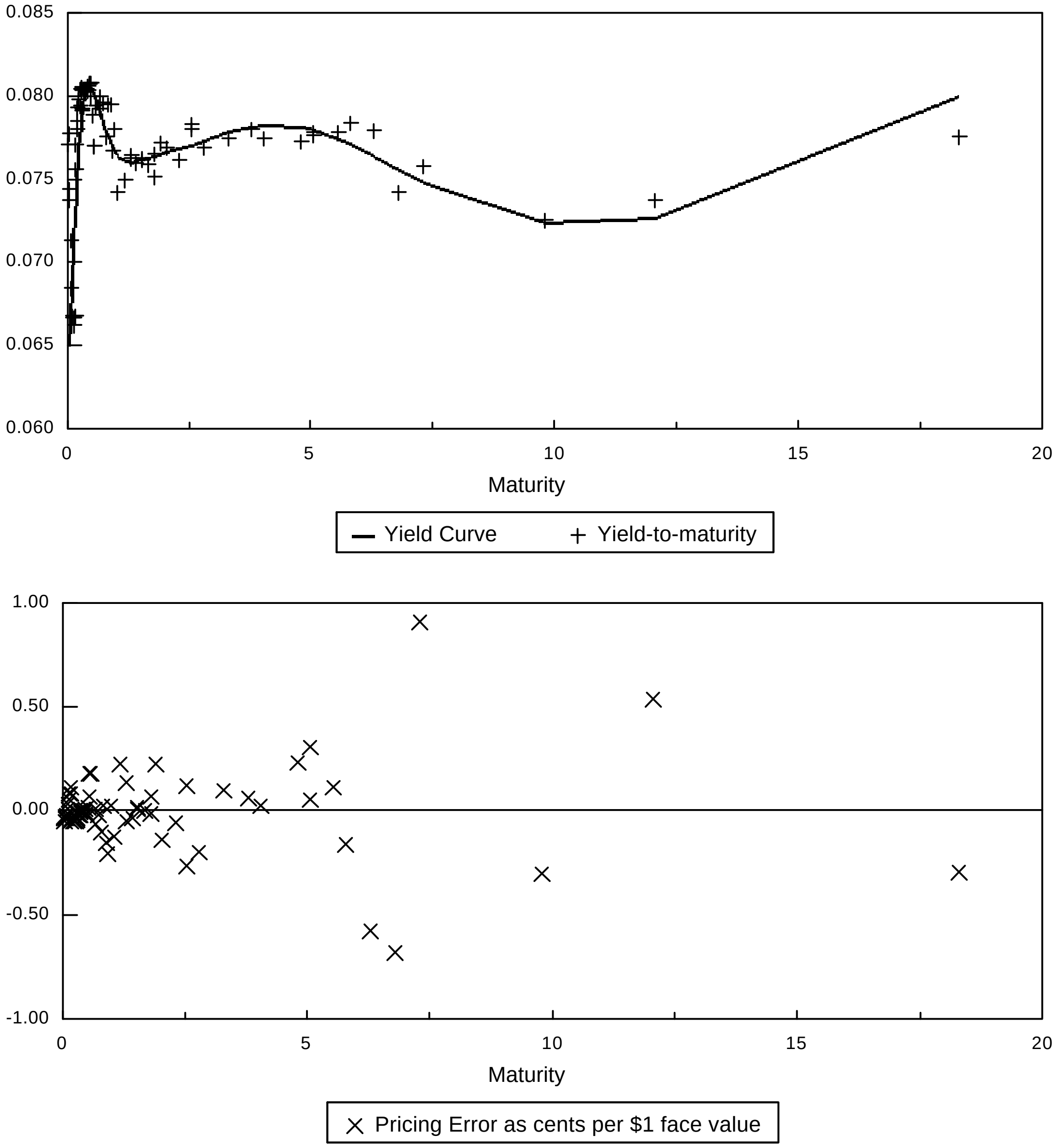
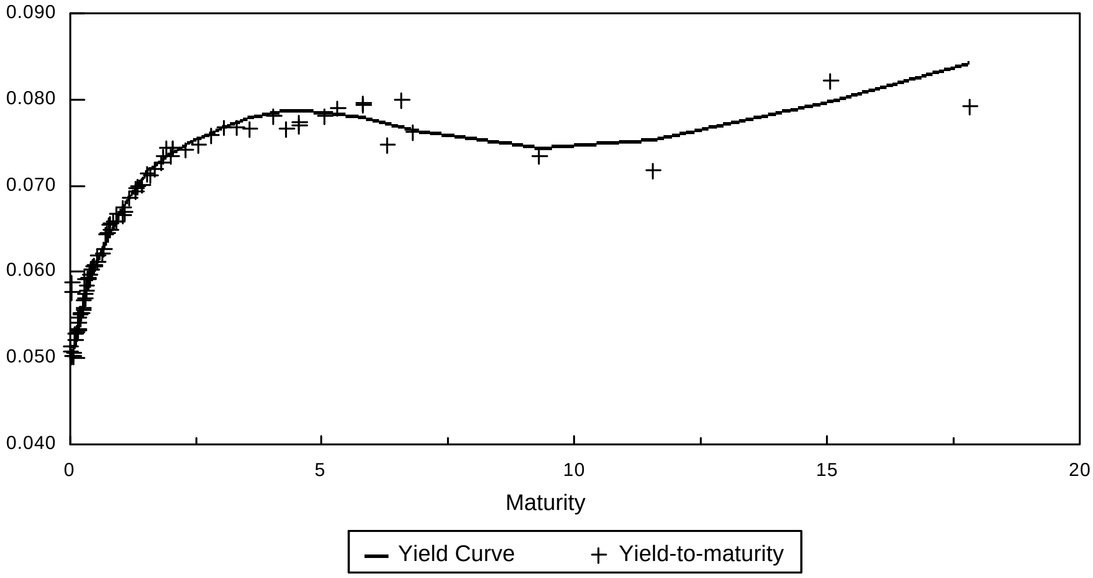


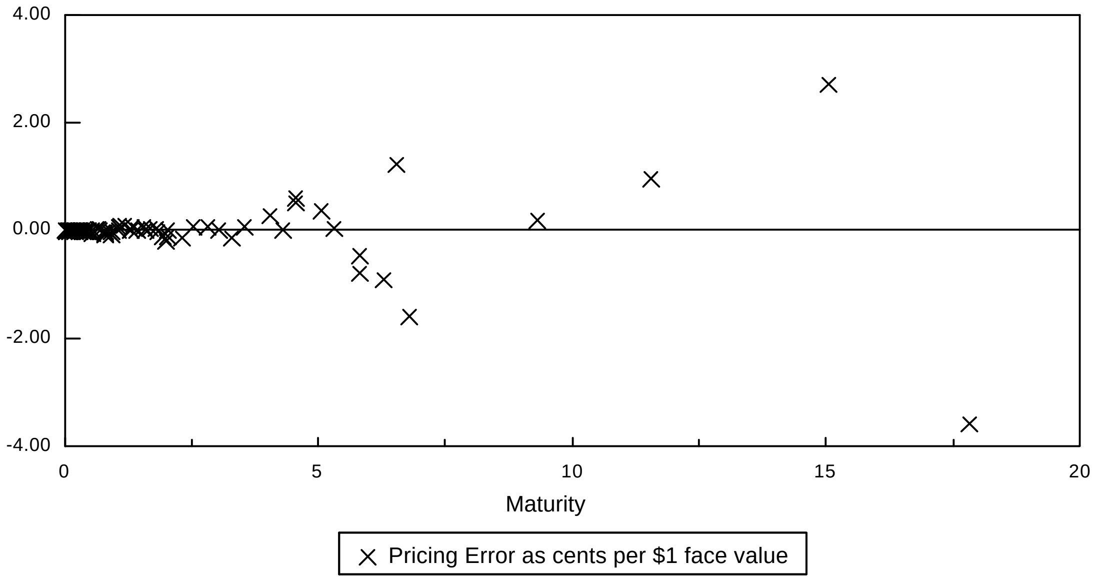
Figure 2. Sample RMSE and short term RMSE. The Short Term RMSE is the root mean squared errors of the treasury securities with maturity less than one year.



Date	Size	R	r	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RMSE
10/31/74	78	0.130	0.064	0.3200	-1.1976	1.7665	-1.4612	1.4983	-2.4979	5.5557	-5.9859	0.1895

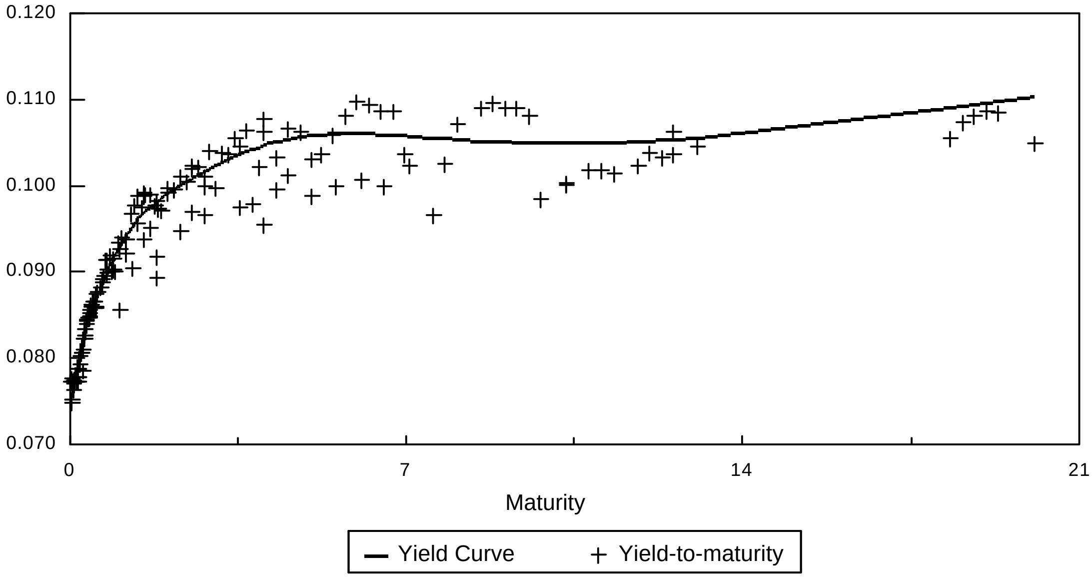
Figure 3. Term structure of October 31, 1974.

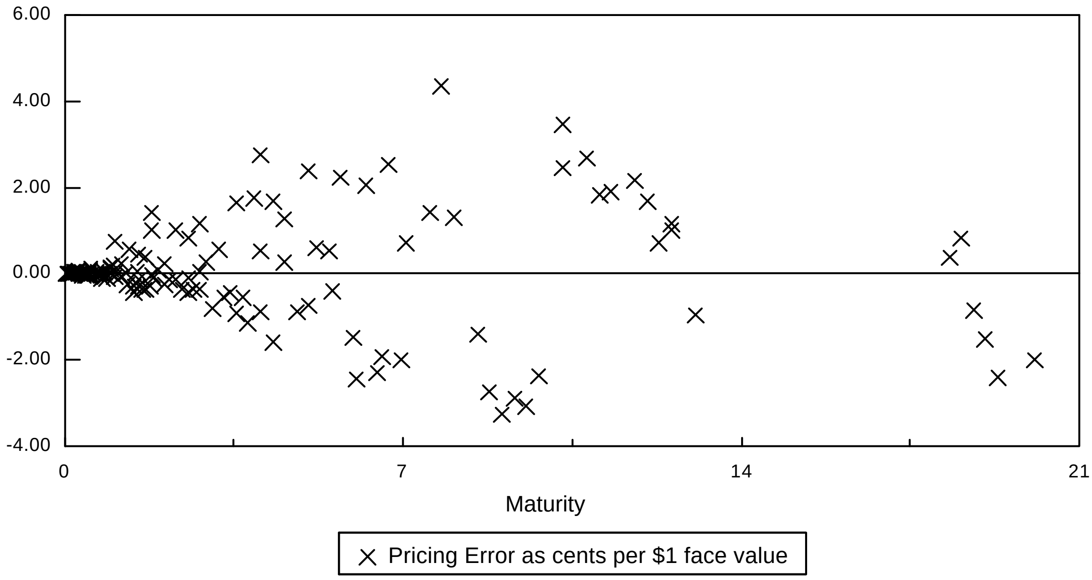




Date	Size	R	r	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RMSE
4/30/75	87	0.138	0.052	0.2284	-0.8692	1.3885	-1.3336	1.5492	-2.6157	5.7593	-6.1877	0.5849

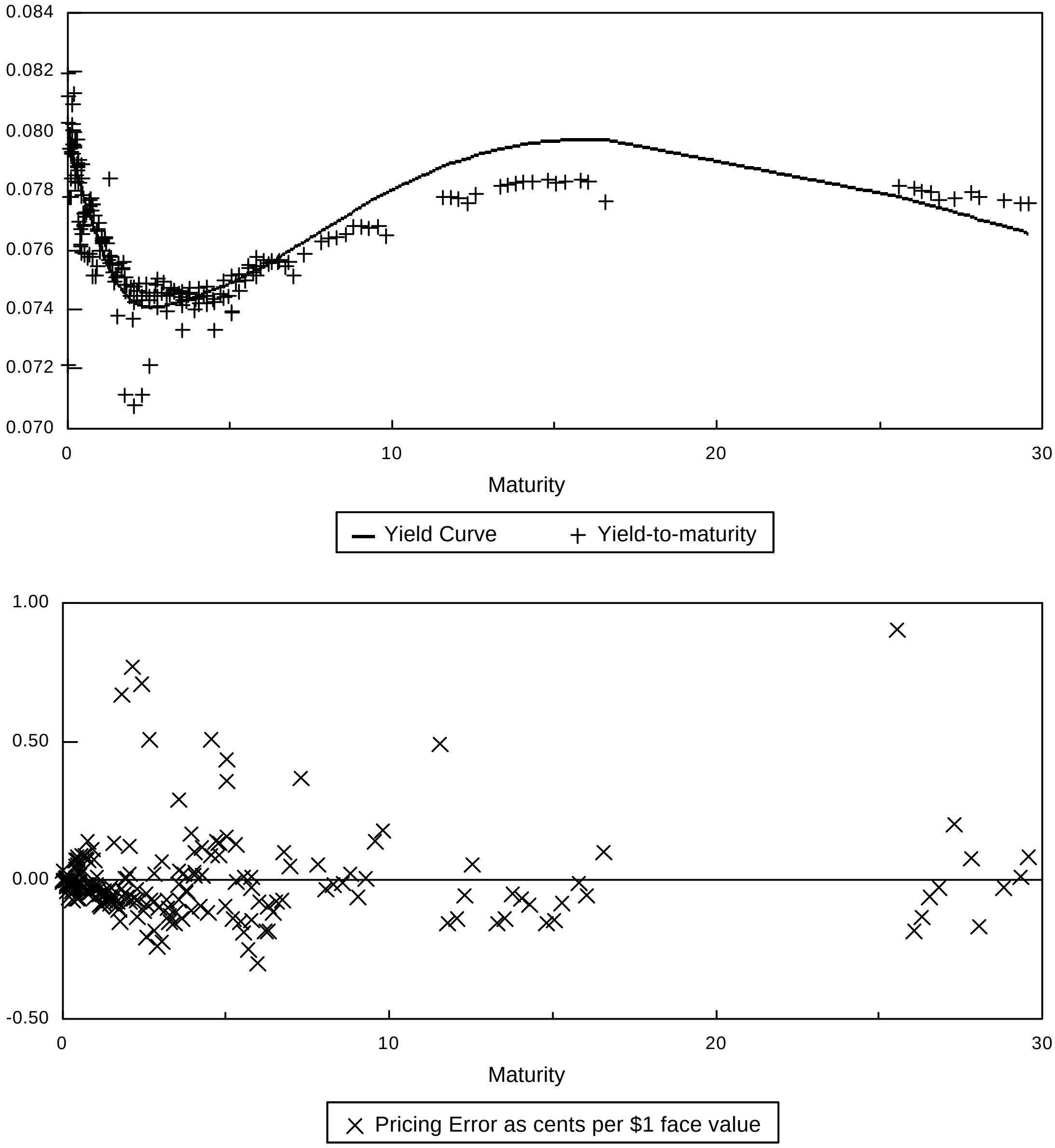
Figure 4. Term structure of April 30, 1975.





Date	Size	R	r	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RMSE
10/29/82	149	0.137	0.075	0.0564	-0.2441	0.4480	-0.5018	0.6510	-1.1118	2.2812	-2.4253	1.1738

Figure 5. Term structure of October 29, 1982.



Date	Size	R	r	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RMSE
10/31/89	207	0.045	0.079	0.0460	-0.2646	0.5684	-0.6426	0.7249	-0.9895	1.3176	-0.0947	0.1769

Figure 6. Term structure of October 31, 1989.