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## Vasicek: The Fixed Income Benchmark

1. Prospectus

2. Models and their uses

3. Spot rates and their properties

4. Fundamental theorem of arbitrage-free pricing

5. The Vasicek model:
   - Solution and properties
   - Parameter values ("calibration")
   - Hedging: Vasicek v. Duration
   - Where are the bodies buried?

6. Summary and final thoughts

## 1. Prospectus

- Regard this course as an experiment: no one has ever taught such advanced material at this level
- What makes this possible?
  - unique approach minimizes technical demands
  - custom software does much of the work
  - steady pace: we refuse to get bogged down in details
  - great teaching!
- Our approach features
  - discrete time (much simpler than stochastic calculus)
  - continuous states (more accurate than "trees")
  - emphasis on numbers (a good model with bad parameters is a bad model)
  - hands-on experience (work in progress, you be the judge)
- Software options:
  - Excel: the old reliable, but awkward for complex problems
  - Matlab: more effort initially, but much more powerful
- Our guarantees:
  - you won't understand everything, but you'll learn a lot
  - an honest attempt gets at least a B

## 2. Uses of Models

- Valuation: What's this option worth?

  Given prices of underlying assets, estimate the value of related derivatives.

- Hedging: How do I offset the risk in these DM swaptions?

  Find combinations of assets that have relatively little risk.

- Replication: How can I synthetically reproduce the cash flows of an American swaption?

  Construct combinations of simple assets to approximate the returns of a more complex asset.

- Risk management: How bad can things get?

  Estimate the statistical distribution of the one-day return of a portfolio of assets.

- Summary and assessment:

  Different objectives demand different models. If we thought we knew everything, the perfect model could be used for all four purposes. Instead, models are designed with strengths that suit their uses. They invariably exhibit weaknesses in other uses. Thus complex statistical models are helpful in quantifying risk, but generally ignore the pricing of risk. Our interest lies, instead, in simpler models in which the pricing of risk is explicit.

## 3. Spot Rates and Other Definitions

- The *discount factor* $b_t^n$ is the price of an $n$ period zero: the value at $t$ of one dollar payable $n$ periods in the future.

- The *spot rate* $y_t^n$ is related to the discount factor $b_t^n$ by

$$
\begin{aligned}
b_t^n &= e^{-n y_t^n} \\
y_t^n &= -n^{-1} \log b_t^n
\end{aligned}
$$

  (This is based on continuous compounding, an arbitrary convention that helps us down the road.)

- The *short rate* is $r_t = y_t^1$.

- The time interval is $h$ years. For periods of one month, $h = 1/12$ years.

- To express spot rates as annual percentage rates:

$$
\text{Annual Percentage Rate} \;=\; y_t^n \times 100/h
$$

  For monthly data, we multiply by 1200.

## 4. Statistics Review

- Data: we have observations $y_t$ for $t = 1, \ldots, T$.

- The sample mean is

$$\bar{y} = \frac{\sum_{t=1}^{T} y_t}{T}$$

- The sample variance is

$$s^2 = \frac{\sum_{t=1}^{T} (y_t - \bar{y})^2}{T}$$

  ($s$ is the standard deviation. Some people divide by $T - 1$, an minor issue we'll ignore.)

- The sample autocorrelation (an indicator of dynamics)

$$Corr(y_t, y_{t-1}) = \frac{\sum_{t=2}^{T} (y_t - \bar{y})(y_{t-1} - \bar{y})}{\sum_{t=1}^{T} (y_t - \bar{y})^2}$$

  (Values close to one indicate that changes tend to last — they're persistent.)

- Higher moments (for future reference only):

$$m_j \equiv \frac{\sum_{t=1}^{T} (y_t - \bar{y})^j}{T} \qquad \text{for } j \geq 2$$

$$\text{skewness} = \frac{m_3}{(m_2)^{3/2}}$$

$$\text{kurtosis} = \frac{m_4}{(m_2)^2} - 3$$

  (Skewness measures asymmetry. Kurtosis summarizes the probability of extreme events. Both are zero for normal random variables.)

## 5. Properties of Spot Rates

US Treasury spot rates, 1970-95, monthly data

| Maturity | Mean | St Dev | Skewness | Kurtosis | Autocorr |
|---|---|---|---|---|---|
| 1 month | 6.683 | 2.699 | 1.073 | 1.256 | 0.959 |
| 3 months | 7.039 | 2.776 | 1.007 | 0.974 | 0.971 |
| 6 months | 7.297 | 2.769 | 0.957 | 0.821 | 0.971 |
| 9 months | 7.441 | 2.721 | 0.924 | 0.733 | 0.970 |
| 12 months | 7.544 | 2.667 | 0.903 | 0.669 | 0.970 |
| 24 months | 7.819 | 2.491 | 0.887 | 0.484 | 0.973 |
| 36 months | 8.008 | 2.370 | 0.913 | 0.378 | 0.976 |
| 48 months | 8.148 | 2.284 | 0.944 | 0.321 | 0.977 |
| 60 months | 8.253 | 2.221 | 0.971 | 0.288 | 0.978 |
| 84 months | 8.398 | 2.138 | 1.008 | 0.251 | 0.979 |
| 120 months | 8.529 | 2.069 | 1.035 | 0.217 | 0.981 |

Note

- Means: increase with maturity (picture coming)
- Standard deviations
- Autocorrelations: close to one (very persistent!)

## 5. Properties of Spot Rates (continued)

Spreads Over Short Rate: $y_t^n - y_t^1$

| Maturity | Mean | St Dev | Skewness | Kurtosis | Autocorr |
|---|---|---|---|---|---|
| 3 months | 0.357 | 0.350 | 1.811 | 5.642 | 0.403 |
| 6 months | 0.614 | 0.507 | 1.325 | 4.538 | 0.527 |
| 9 months | 0.758 | 0.601 | 1.117 | 4.647 | 0.618 |
| 12 months | 0.861 | 0.683 | 0.863 | 4.177 | 0.680 |
| 24 months | 1.136 | 0.929 | –0.017 | 2.197 | 0.792 |
| 36 months | 1.325 | 1.095 | –0.424 | 1.368 | 0.833 |
| 48 months | 1.465 | 1.213 | –0.599 | 0.983 | 0.855 |
| 60 months | 1.570 | 1.300 | –0.677 | 0.765 | 0.868 |
| 84 months | 1.715 | 1.420 | –0.728 | 0.532 | 0.882 |
| 120 months | 1.846 | 1.526 | –0.735 | 0.385 | 0.892 |

(Skip for now)

## 5. Properties of Spot Rates (continued)

One-Month Changes: $y_t^n - y_{t-1}^n$

| Maturity | Mean | St Dev | Skewness | Kurtosis | Autocorr |
|---|---|---|---|---|---|
| 1 month | –0.009 | 0.764 | –1.043 | 9.939 | 0.003 |
| 3 months | –0.010 | 0.662 | –1.747 | 11.158 | 0.146 |
| 6 months | –0.010 | 0.656 | –1.352 | 10.691 | 0.150 |
| 9 months | –0.009 | 0.648 | –1.088 | 10.514 | 0.153 |
| 12 months | –0.009 | 0.634 | –0.926 | 10.251 | 0.154 |
| 24 months | –0.009 | 0.558 | –0.583 | 7.902 | 0.157 |
| 36 months | –0.009 | 0.499 | –0.424 | 5.829 | 0.157 |
| 48 months | –0.009 | 0.461 | –0.345 | 4.598 | 0.152 |
| 60 months | –0.008 | 0.434 | –0.300 | 3.845 | 0.143 |
| 84 months | –0.008 | 0.401 | –0.256 | 3.000 | 0.120 |
| 120 months | –0.007 | 0.374 | –0.220 | 2.337 | 0.094 |

(Skip for now)

## 6. The Fundamental Theorem

- An *arbitrage opportunity* is a combination of long and short positions that costs nothing but has a positive return (a free lunch, in other words).

- We say prices are *arbitrage-free* if there are no arbitrage opportunities. (Statement of faith in markets, good approximation but not perfect.)

- Notation: let $R_{t+1}$ be the gross return between dates $t$ and $t+1$. For example, the return on an $n$-period zero is $R_{t+1} = b_{t+1}^{n-1}/b_t^n$.

- **Theorem.** In any arbitrage-free setting, there is a positive random variable $m$ (a pricing kernel) satisfying

$$1 \;=\; E_t\left(m_{t+1} R_{t+1}^j\right)$$

  for all assets $j$. $E_t$ means the expectation as of time $t$ (when the trade is done). (Ignore the $t$'s for now, they're a distraction.)

- Risk premiums are governed by covariance with $m$:
  - Let $R_{t+1}^1 = 1/b_t^1$ be the riskfree one-period return
  - Excess returns are $x_{t+1}^j = R_{t+1}^j - R_{t+1}^1$.
  - Risk premiums are

$$E_t\left(x_{t+1}^j\right) \;=\; -\frac{Cov_t(m_{t+1}, x_{t+1}^j)}{E_t m_{t+1}}$$

  - Comment: this is a lot like the CAPM, with $m$ playing the role of the market return.

- Needed: a good $m$

## 7. Vasicek: The Model

- Goal: a useful description of the behavior of $m$

- The archetype: the Vasicek model

$$\begin{aligned} -\log m_{t+1} &= \lambda^2/2 + z_t + \lambda \varepsilon_{t+1} \\ z_{t+1} &= (1-\varphi)\theta + \varphi z_t + \sigma \varepsilon_{t+1} \end{aligned}$$

  where $\varepsilon$ is normal with mean zero and variance one and $0 < \varphi < 1$. (The two-part structure is typical.)

- The pricing kernel $m$ controls risk:

  - log guarantees positive $m$ (if $m$ is positive $\log m$ can be anything, and vice versa)

  - If $\lambda = 0$, $m$ discounts at rate $z$:

    $$m_{t+1} = e^{-z_t}$$

  - $\lambda$ controls risk (recall: risk premiums are proportional to covariance with $m$)

  - $\lambda^2/2$ is largely irrelevant (you'll see)

## 7. Vasicek: The Model (continued)

- The "state variable" $z$ will turn out to be the short rate. Its properties include:
  - Over one period, the conditional mean and variance are:
  $$\begin{aligned} E_t z_{t+1} &= (1-\varphi)\theta + \varphi z_t \\ Var_t z_{t+1} &= \sigma^2 \end{aligned}$$
  (Stop if this isn't clear.)
  - Over two periods, $z$ looks like this:
  $$\begin{aligned} z_{t+2} &= (1-\varphi)\theta + \varphi z_{t+1} + \sigma\varepsilon_{t+2} \\ &= (1-\varphi^2)\theta + \varphi^2 z_t + \sigma(\varepsilon_{t+2} + \varphi\varepsilon_{t+1}) \end{aligned}$$
  The conditional mean and variance are:
  $$\begin{aligned} E_t z_{t+2} &= (1-\varphi^2)\theta + \varphi^2 z_t \\ Var_t z_{t+2} &= \sigma^2(1+\varphi^2) \end{aligned}$$
  - Over $n$ periods:
  $$z_{t+n} = (1-\varphi^n)\theta + \varphi^n z_t + \sigma \sum_{j=0}^{n-1} \varphi^j \varepsilon_{t+n-j}$$
  The conditional mean and variance are:
  $$\begin{aligned} E_t z_{t+n} &= (1-\varphi^n)\theta + \varphi^n z_t \\ Var_t z_{t+n} &= \sigma^2(1+\varphi^2+\cdots+\varphi^{2(n-1)}) \end{aligned}$$
  - As $n$ grows, the mean and variance approach
  $$\begin{aligned} Ez &= \theta \\ Var z &= \sigma^2(1+\varphi^2+\varphi^4+\cdots) = \frac{\sigma^2}{1-\varphi^2} \end{aligned}$$
  We refer to these as the *unconditional* mean and variance, the theoretical analogs of the mean and variance we might estimate from a sample time series of $z$.

## 8. Vasicek: The Solution

- Discount factors are log-linear functions of $z$:

$$-\log b_t^n \;=\; A_n + B_n z_t$$

  This relatively simple structure is one of the most useful features of the model. All we need to do is find the coefficients $(A_n, B_n)$.

- The fundamental theorem gives us

$$\begin{aligned} A_{n+1} &= A_n + \lambda^2/2 + B_n(1-\varphi)\theta - (\lambda + B_n\sigma)^2/2 \\ &= A_n + B_n(1-\varphi)\theta - \lambda B_n\sigma - (B_n\sigma)^2/2 \\ B_{n+1} &= 1 + B_n\varphi \end{aligned}$$

- For future reference, note that

$$B_n \;=\; 1 + \varphi + \cdots + \varphi^{n-1} \;=\; (1-\varphi^n)/(1-\varphi)$$

- Starting point: $b_t^0 = 1$ (a dollar today costs a dollar), so

$$\log b_t^0 \;=\; -A_0 - B_0 z_t \;=\; 0 \quad\Rightarrow\quad A_0 = B_0 = 0$$

- These formulas are easily computed in a spreadsheet: starting with $A_0 = B_0 = 0$, we compute $(A_{n+1}, B_{n+1})$ from $(A_n, B_n)$.

- Needed: values for the parameters $(\theta, \sigma, \varphi, \lambda)$ — coming soon.

## 9. Vasicek: How We Got There

Step 1: Fundamental theorem applied to zeros

- Theorem says returns $R$ satisfy

$$1 \;=\; E_t\left(m_{t+1} R_{t+1}\right)$$

- Returns on zeros are

$$R^{n+1}_{t+1} \;=\; b^{n}_{t+1}/b^{n+1}_{t}$$

  (ratio of sale price to purchase price)

- Theorem becomes

$$b^{n+1}_{t} \;=\; E_t\left(m_{t+1} b^{n}_{t+1}\right)$$

  (An $n+1$-period zero is a claim to an $n$-period zero one period in the future.)

## 9. Vasicek: How We Got There (continued)

Step 2: Show that $z$ is the short rate

- Since $b_t^0 = 1$, the short rate satisfies:

$$e^{-y_t^1} = b_t^1 = E_t\left(m_{t+1}\right)$$

  (Stop now if you have any questions about this.)

- Fact: if $\log x$ is normal with mean $a$ and variance $b$, then

$$\begin{aligned} E(x) &= e^{a + b/2} \\ \log E(x) &= a + b/2 \end{aligned}$$

  In words: the log of the mean $(a + b/2)$ isn't quite the mean of the log $(a)$. There's an extra term due to the variance $b$.

- Since $\varepsilon$ is normal, $\log m_{t+1}$ is normal, too:

$$\log m_{t+1} = -\lambda^2/2 - z_t - \lambda\varepsilon_{t+1}$$

  Its conditional mean and variance are

$$\begin{aligned} E_t(\log m_{t+1}) &= -\lambda^2/2 - z_t \\ Var_t(\log m_{t+1}) &= \lambda^2 \end{aligned}$$

- Apply the fact:

$$E_t\left(m_{t+1}\right) = e^{-\lambda^2/2 - z_t + \lambda^2/2} = e^{-z_t},$$

- Result: $z$ is the short rate.

## 9. Vasicek: How We Got There (continued)

Step 3: Solution for an arbitrary maturity

- Guess that bond prices are log-linear functions of $z$:

$$-\log b_t^n = A_n + B_n z_t$$

(We know this works for $n = 1$: $A_1 = 0$ and $B_1 = 1$.)

- Given a solution for $n$, find $n+1$:

$$b_t^{n+1} = E_t\left(m_{t+1} b_{t+1}^n\right)$$

- Work on right-hand-side:

$$\begin{aligned}\log m_{t+1} &= -\lambda^2/2 - z_t - \lambda\varepsilon_{t+1}\\ \log b_{t+1}^n &= -A_n - B_n z_{t+1}\\ &= -A_n - B_n\left[(1-\varphi)\theta + \varphi z_t + \sigma\varepsilon_{t+1}\right]\end{aligned}$$

The sum is

$$\begin{aligned}\log m_{t+1} b_{t+1}^n &= -\lambda^2/2 - A_n - B_n(1-\varphi)\theta - (1+B_n\varphi)z_t\\ &\quad -(\lambda + B_n\sigma)\varepsilon_{t+1}.\end{aligned}$$

The conditional mean and variance are

$$\begin{aligned}E_t\left(\log m_{t+1} b_{t+1}^n\right) &= -\lambda^2/2 - A_n - B_n(1-\varphi)\theta - (1+B_n\varphi)z_t\\ Var_t\left(\log m_{t+1} b_{t+1}^n\right) &= (\lambda + B_n\sigma)^2\end{aligned}$$

- Apply the fact:

$$\begin{aligned}\log b_t^{n+1} &= -\lambda^2/2 - A_n - B_n(1-\varphi)\theta - (1+B_n\varphi)z_t\\ &\quad +(\lambda + B_n\sigma)^2/2\\ &= -A_{n+1} - B_{n+1}z_t.\end{aligned}$$

- Recursions for computing $(A_n, B_n)$ one maturity at a time:

$$\begin{aligned}A_{n+1} &= A_n + \lambda^2/2 + B_n(1-\varphi)\theta - (\lambda + B_n\sigma)^2/2\\ B_{n+1} &= 1 + B_n\varphi\end{aligned}$$

## 10. Vasicek: Choosing Parameters

- Premise: a model with bad parameters is a bad model
- Question: How do we choose them $(\theta, \sigma, \varphi, \lambda)$?
- Answer (for now): reproduce broad featues of spot rates (see table several pages above)
- Mean and variance of spot rates:

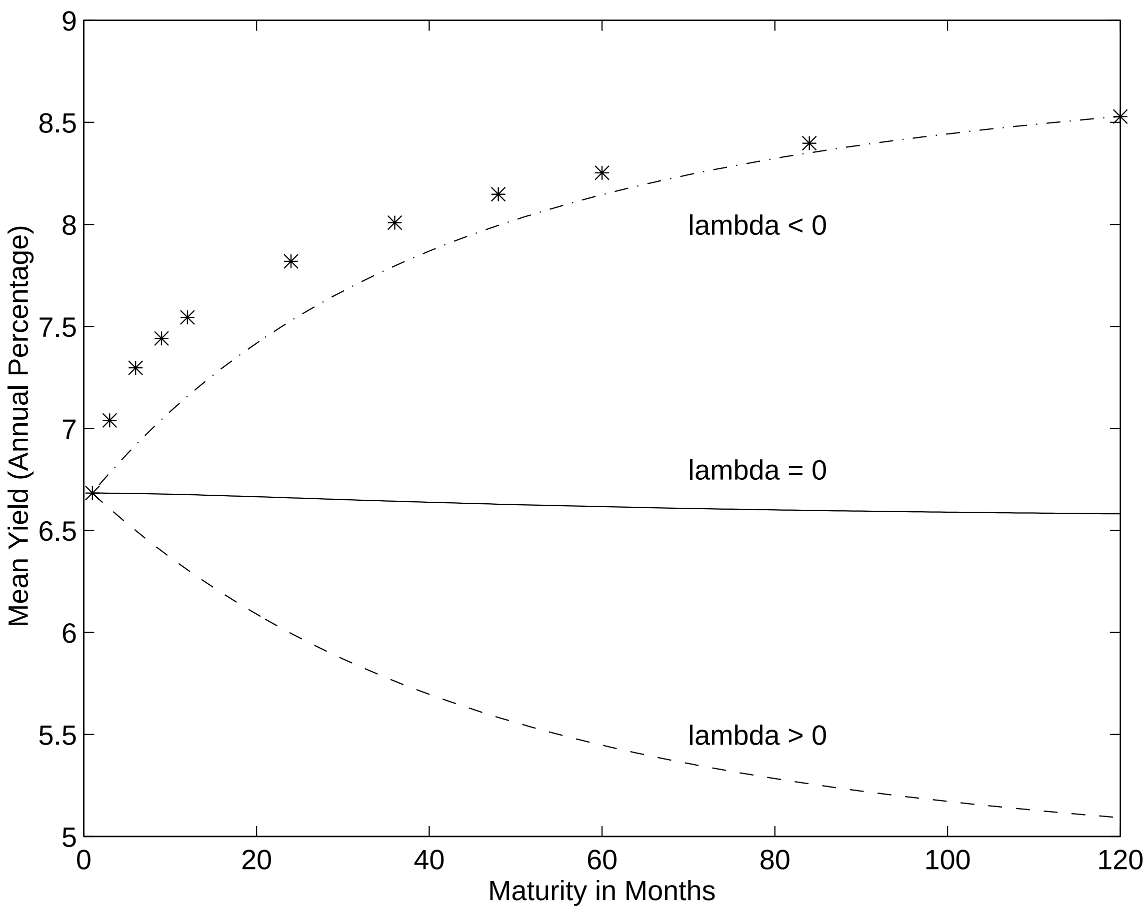
$$
\begin{aligned}
y_t^n &= n^{-1}(A_n + B_n z_t) \\
E(y^n) &= n^{-1}(A_n + B_n \theta) \\
Var(y^n) &= \left(\frac{B_n}{n}\right)^2 Var(z) \;=\; \left(\frac{B_n}{n}\right)^2 \frac{\sigma^2}{1-\varphi^2} \\
Corr(y_t^n, y_{t-1}^n) &= \varphi
\end{aligned}
$$

  (The last follows since linear functions of $z$ inherit $z$'s autocorrelation.)

- Parameter values:
  - Choose $\theta$ to match the mean short rate:
  $$\theta = \frac{6.683}{1200} = 0.005569$$
  - Choose $\varphi$ to match autocorrelation of short rate:
  $$\varphi = 0.959$$
  - Choose $\sigma$ to match variance of short rate:
  $$\frac{\sigma^2}{1-\varphi^2} = \left(\frac{2.699}{1200}\right)^2 \quad \Rightarrow \quad \sigma = 0.0006374$$
  - What about $\lambda$?

## 10. Vasicek: Choosing Parameters (continued)

- $\lambda$ controls risk premiums on long bonds
- Mean spot rates in model and data for different choices of $\lambda$:




- Result: $\lambda = -0.1308$ (matches mean 10-year rate)
- Why? Need negative value to get right sign on risk premium
- Problem: miss the curvature

## 11. Hedging

Duration-based hedging of a 5-year zero

- Sensitivity to yield changes:

$$
\begin{aligned}
b^n &= e^{-ny^n} \\
\Delta b^n &\approx -ne^{-ny^n}\Delta y^n \;=\; -nb^n \Delta y^n
\end{aligned}
$$

  ($n$ is the duration, measured in months.)

- Arrange positions in $n$-month zero that offset risk in 60-month:
  - Portfolio consists of

$$
v \;=\; x^{60}b^{60} + x^n b^n
$$

  - Change in value:

$$
\begin{aligned}
\Delta v &= x^{60}\Delta b^{60} + x^n \Delta b^n \\
&\approx -x^{60}(60b^{60}\Delta y^{60}) - x^n(nb^n \Delta y^n)
\end{aligned}
$$

  - Hedge ratio results from setting $\Delta v = 0$ (the definition of a hedge) and $\Delta y^{60} = \Delta y^n$ (the assumption underlying duration):

$$
\text{Hedge Ratio} \;=\; \frac{x^n b^n}{x^{60}b^{60}} \;=\; -\frac{60}{n}
$$

    The usual: the ratio of the durations.

## 11. Hedging (continued)

Vasicek-based hedging of a 5-year zero

- Sensitivity to $z$ changes:

$$\begin{aligned} b^n &= e^{-A_n - B_n z} \\ \Delta b^n &\approx -B_n b^n \Delta z \end{aligned}$$

- Arrange positions in $n$-month zero that offset risk in 60-month:
  - Portfolio consists (again) of

$$v = x^{60} b^{60} + x^n b^n$$

  - Change in value:

$$\begin{aligned} \Delta v &= x^{60} \Delta b^{60} + x^n \Delta b^n \\ &\approx -x^{60}(B_{60} b^{60} \Delta z) - x^n (B_n b^n \Delta z) \end{aligned}$$

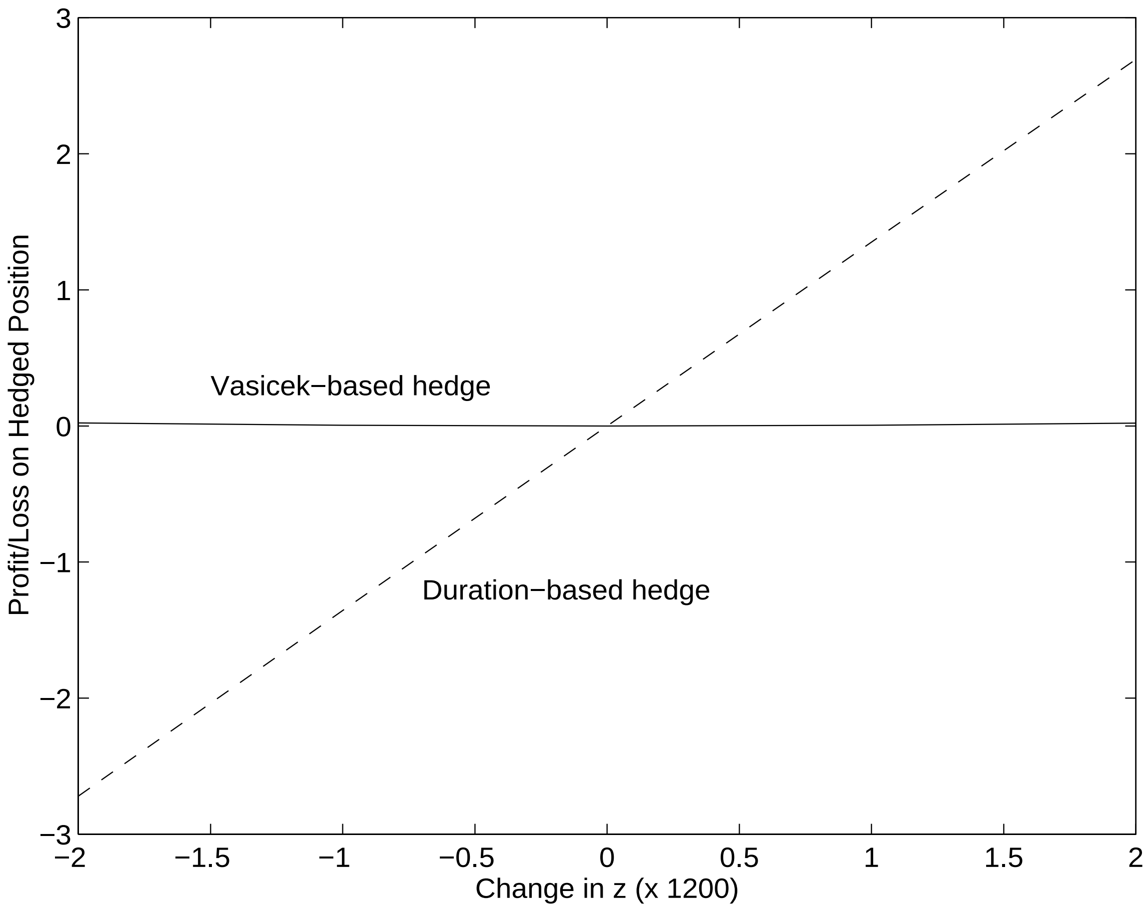
  - Hedge ratio results from setting $\Delta v = 0$ (the definition of a hedge):

$$\text{Hedge Ratio} = \frac{x^n b^n}{x^{60} b^{60}} = -\frac{B_{60}}{B_n}$$

- Comparison:
  - Recall: $B_n = 1 + \varphi + \cdots + \varphi^{n-1}$
  - When $\varphi = 1$, $B_n = n$ and Vasicek- and duration-based hedge ratios are the same
  - When $0 < \varphi < 1$, $B_n < n$: Vasicek attributes relatively less risk to long zeros than duration

## 11. Hedging (continued)

- Thought experiment: If the world obeyed the Vasicek model, how far wrong could you go by using duration to hedge?
  - Buy $100 of 5-year zeros
  - Construct duration and Vasicek hedges with 24-month zeros
  - Compute profit/loss for given changes in $z$



## 11. Hedging (continued)

- Comments on the thought experiment
  - Duration hedging: you're overhedged
  - Vasicek hedging: not exactly zero due to convexity
  - Is this reasonable? Qualitatively yes: less volatility for long rates than duration assumes (see table for monthly changes)

## 12. Where are the Bodies Buried?

- Strengths of the Vasicek model:
  - relatively simple and transparent
  - fits general features of bond prices
  - Black-like option prices (coming soon)
- Weaknesses:
  - insufficient curvature in mean spot rates
  - crude approximation to current spot rates
  - volatility is constant ($\sigma$)
  - persistence same for spot rates and spreads
  - allows negative interest rates

## Summary

1. Modern pricing theory is based on a pricing kernel, which plays a role analogous to the market return in the CAPM

2. Models contain

   - descriptions of interest rate dynamics $(\theta, \sigma, \varphi)$
   - assessment of risk $(\lambda)$

3. The Vasicek model

   - an archetype: many other models have similar structures
   - strengths include simplicity, Black-like option formulas
   - weaknesses include its single factor, constant volatility

4. Coming up:

   - derivatives pricing
   - comparisons with other models

5. Vote on these topics:

   - convexity adjustment for eurocurrency futures
   - Hull and White model
   - Heath-Jarrow-Morton approach
   - volatility smiles for options
   - stochastic volatility
   - two-factor models
   - American options
   - CMT/CMS swaps
   - Currency and commodity derivatives