Impact of Different Interest Rate Models on Bond Value Measures

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In the last 10 to 15 years an increasing number of different interest rate models have become available. All are commonly used to price bonds with embedded options. The models have different features, which makes the choice of the correct model confusing for both practitioners and academicians alike. They often provide different values for securities with contingent claim characteristics. Not surprisingly, then, they generate different sensitivity measures such as effective duration (ED), effective convexity (EC), and option-adjusted spreads (OAS).

We document the differences and try to shed some light on their determinants—a phenomenon that has not been documented before. Interest rates are commonly modeled using stochastic differential equations (SDEs). The most common types of models are one-factor and two-factor interest rate models. One-factor models use an SDE to represent the short-term rate, and two factor models use an SDE for both the short-term rate and the long-term rate. The SDEs chosen to model interest rates capture some of the more desirable properties of interest rates such as mean reversion and/or a volatility term that depends on the level of interest rates.

There are two distinct approaches to implement the SDEs in a term structure model: equilibrium and no-arbitrage. Each can be used to value bonds and interest rate derivatives. Each approach starts with similar SDEs, but applies the SDE under a different framework to price securities.

Equilibrium models start with an SDE model and develop pricing mechanisms for bonds under an equilibrium framework. Some of the most common models that use this approach are Vasicek [1977], Brennan and Schwartz [1979, 1982], Cox, Ingersoll, and Ross [1985], Longstaff [1989, 1992], and Longstaff and Schwartz [1992].

No-arbitrage models begin with the same or similar SDE models, but use market prices to generate an interest rate lattice. The lattice produces an array of interest rates that result in prices of bonds that are the same as those observed in the market and follow the behavioral characteristics of the SDEs. Some of the most well-known models that use this approach are Ho and Lee [1986], Black, Derman, and Toy [1990], Hull and White [1990, 1993], Black and Karasinski [1991], and Heath, Jarrow, and Morton [1992].

No-arbitrage models are the preferred framework to value interest rate derivatives. This is because they minimally ensure that the market prices for bonds are exact. Equilibrium models do not price bonds exactly, which can have significant effects on the corresponding contingent claims.

Cheyette [1997] offers an excellent review of how to select the appropriate model; he suggests that one must consider the characteristics of the security to be evaluated in order to select the best model. Cheyette also illustrates empirically that some models may better capture actual interest rate dynamics, but he readily notes that the empirical evidence is far from conclusive.

We examine differences in the effective duration, effective convexity, and the option-adjusted spread resulting from different one-factor no-arbitrage interest rate models. The models considered are: the Ho and Lee (HL) [1986] model; the Kalotay, Williams, and Fabozzi (KWF) [1993] model; the Black, Derman, and Toy (BDT) [1990] model; the Hull and White (HW) [1994] model; and the Black and Karasinski (BK) [1991] model.[1]

I. NO-ARBITRAGE INTEREST RATE MODELS

The interest rate models we examine assume that the short-term interest rate follows a certain process that can be represented by a stochastic differential equation. All the interest rate models are special cases of the general form of changes in the short-term rate:

$$df(r(t)) = (\theta(t) + \rho(t)g(r(t)))dt + \sigma(r(t), t)dz \quad (1)$$

where f and g are suitably chosen functions of the short-term rate and are the same for most models presented here, θ will be shown to be the drift of the short-term rate, and ρ is the mean reversion term to an equilibrium short-term rate. The term σ is the local volatility of the short-term rate, and z is a normally distributed Wiener process that captures the randomness of future changes in the short-term rate.

Equation (1) is a one-factor model that gives only the short-term rate (one factor).[2] Its first component (the dt term) is the expected or average change in the short-term rate over a short period of time. The second component is the risk term, as it includes the random component dz. All the interest rate models we consider are special cases of Equation (1).

The Ho-Lee Model

The Ho-Lee model assumes that changes in the short-term rate can be modeled using Equation (1) by setting $f(r) = r$ and $\rho = 0$, so that the process for the short-term rate is:

$$dr = \theta(t)dt + \sigma(t)dz \quad (2)$$

Since dz is a normally distributed Wiener process, the HL process is a normal process for the short-term rate. As can be seen from Equation (2), the short-term rate may become negative if the random term is large enough to dominate the drift term (dt). This is a serious shortcoming of the HL model, although it is argued that as long as the HL model provides good prices for bonds with embedded options, it does not matter if some of its assumptions are unrealistic. Another possible drawback of the model, however, is that the volatility of the short-term rate does not depend on the level of the rate, and the short-term rate does not mean-revert to a long-term equilibrium rate, as many practitioners believe would hold in reality.

Some of these restrictive assumptions are relaxed in the other models. Note that the distributional properties will have a tendency to bias the values of the embedded contingent claim.

The simplicity of the HL model combined with the fact that it provides reasonable prices under many circumstances makes it a very popular interest rate model.

The Kalotay-Williams-Fabozzi Model

The Kalotay-Williams-Fabozzi model assumes that changes in the short-term rate can be modeled using Equation (1) by setting $f(r) = \ln(r)$ (where ln is the natural logarithm) and $\rho = 0$. Making these adjustments to Equation (1) produces the short-term rate process:

$$d\ln(r) = \theta(t)dt + \sigma(t)dz \quad (3)$$

Comparing Equation (3) to Equation (2), it can be seen that the KWF model is directly analogous to the HL model, except that now the change in the natural logarithm of the short-term rate is modeled instead of the change in the short-term rate itself. Since $\ln(r)$ follows a normal process, r itself follows a lognormal process, and the KWF model is therefore a lognormal interest rate model. Hence, although $\ln(r)$ may become negative if the risk component in Equation (3) dominates the drift component, r itself will never be negative as $r = e^{\ln(r)}$. Therefore, the KWF model eliminates the problem of negative short-term rates that can occur in the HL model.

The actual KWF model does not explicitly incorporate the drift term. As a result, it does not always have a solution for the binomial tree.

While the KWF model is able to avoid negative short-term rates, it still does not capture mean reversion in the short-term rate.

The Black-Derman-Toy Model

One of the main advantages of the Black-Derman-Toy model is that it is a lognormal model that is able to capture a realistic term structure of interest rate volatilities. To accomplish this feature, the short-term rate volatility is allowed to vary over time, and the drift in interest rate movements depends on the level of rates.

While interest rate mean reversion is not modeled explicitly, this property is introduced through the term structure of volatilities. Hence, the extent to which the drift term depends on the level of rates depends on the local volatility process. In other words, the mean reversion is endogenous to the model. Therefore, no additional degree of freedom is required for the mean reversion term, and the BDT SDE can be relatively easily approximated using the binomial tree approach.

The BDT model is obtained from Equation (1) by setting $f(r) = \ln(r)$ and $g(r) = \ln(r)$. Therefore, the short-term rate in the BDT model follows the lognormal process:

$$d\ln(r) = (\theta(t) + \rho(t)\ln(r))dt + \sigma(t)dz \tag{4}$$

The mean reversion term $\rho(t)$ depends on the interest rate local volatility as follows:

$$\rho(t) = \frac{d}{dt}\ln(\sigma(t)) = \frac{\sigma'(t)}{\sigma(t)}$$

which gives

$$d\ln(r) = (\theta(t) + \frac{\sigma'(t)}{\sigma(t)}\ln(r))dt + \sigma(t)dz \tag{5}$$

Comparing Equation (5) with Equation (3), we observe that if the volatility term structure is flat so that $\sigma(t)$ is constant, then $\sigma'(t) = 0$ and $\rho(t) = 0$, so that the BDT model reduces to the KWF model. In this sense, the KWF model is a special case of the BDT for constant local volatility. When the local volatility term structure is decreasing, i.e., if $\sigma'(t) < 0$, the BDT model will exhibit mean reversion. If $\sigma'(t) > 0$, i.e., if the local volatility term structure is increasing, the BDT model will not exhibit mean reversion. Hence, the mean reversion depends entirely on the shape of the local volatility term structure.

While some researchers believe that the mean reversion in the BDT model will be more representative of the market since it is endogenous to the model, others argue that it might be more appropriate to model mean reversion independently of the volatility process. This can be accomplished only in the framework of a binomial model through the use of varying time steps (as in the Hull-White and Black-Karasinski binomial trees), which complicates both the numerical solution and the applicability of the model substantially.

The Hull-White Model

Similar to the Ho-Lee model, the Hull-White model assumes a normal process for the short-term rate. The model can be obtained from Equation (1) by setting $f(r) = g(r) = r$ and $\rho = -\phi$. The process for the short-term rate is thus:

$$dr = (\theta - \phi r)dt + \sigma dz \tag{6}$$

where θ is the long-term equilibrium mean rate, and ϕ is the mean reversion term. Note that if $\phi = 0$, the HW process reduces to the HL process. The HL model is therefore a special case of the HW model when there is no mean reversion.

The HW model explicitly models mean reversion by specifying a central tendency for the short-term rate and by specifying the speed at which the short-term rate reverts to that central tendency. The mean reversion coefficient allows correction for uncontrolled growth or decline in the HW model. The coefficient therefore reduces the probability of negative interest rates, although it does not completely rule out negative interest rates, which makes the HW model subject to the same criticism as the HL model.

Since mean reversion is modeled explicitly, solving the HW SDE numerically using a binomial tree is complicated by the fact that an additional degree of freedom is required. In a binomial framework, this additional degree of freedom can be obtained only by using time steps of varying length, which complicates the analysis. Alternatively, a trinomial lattice may be used for the numerical solution, which is the approach we follow.

The Black-Karasinski Model

In order to obtain the Black-Karasinski short-term rate process we set $f(r) = \ln(r)$, $\rho = -\phi$, and $g(r) = \ln(r)$ in Equation (1), which results in the short-term rate process:

$$d\ln(r) = (\theta - \phi\ln(r))dt + \sigma dz \tag{7}$$

Inspection shows that the BK model is simply the logarithmic analogue to the HW model. In the BK model, ln(r) has the same properties as r in the HW model. As in the KWF model, however, r cannot become negative because $r = e^{\ln(r)}$, which is always positive. This is the advantage of the BK model over the HW model.

Therefore, we see that the BK model is an extension of the KWF process in the same way as the HW process is an extension of the HL process. In fact, as $\phi = 0$, the KWF model is obtained.

The BK model explicitly models mean reversion by specifying a central tendency for the short-term rate and the speed at which the short-term rate reverts to that central tendency. Like the HW model, which also includes a mean reversion term, the BK SDE is solved numerically in the most straightforward way by using a trinomial tree approach.

II. NUMERICAL SOLUTION OF INTEREST RATE MODELS

The models may be solved numerically by either binomial or trinomial methods.

Binomial SDE Approximations

The binomial method models the short-term rate in a geometrically analogous manner as equities in Cox, Ross, and Rubinstein [1979]. That is, the short-term rate for the next period can have only one of two possible values, r_u or r_d where $r_u > r_d$. Continuing in this manner for a number of future time periods results in a binomial tree whose number of nodes increases by two nodes for each time step. As a result, the number of nodes quickly increases over time.

To reduce the number of nodes for computational tractability, a restriction is imposed on the algorithm, namely, the recombination condition, which forces an up move followed by a down move to result in the same future interest rate as a down move followed by an up move. This makes the binomial method more computationally tractable since the number of nodes at each time step increases by only one node.

An up move in the short-term interest rate has a probability q, so the corresponding down move has a probability of $1 - q$. We use $q = 0.5$. It should be noted that setting the up and down probabilities to 0.5 is merely an artificial device to ensure risk-neutrality in solution of the short-term rate SDEs. It by no means implies that the actual probability for an interest rate increase or decrease is equal to 0.5.[3]

Trinomial SDE Approximations

The trinomial method is similar in spirit to the binomial tree, except that there are three possible states instead of two. From each time we call the upward move the up move, the downward move the down move, and the center move the middle move. Again, we have to make sure that the interest rate lattice possesses the recombination property in order for it to be computationally tractable. From any node in the trinomial lattice an up move followed by a down move will get to the same node as two successive middle moves and as a down move followed by an up move. This ensures that the number of nodes in the trinomial lattice increases by only two nodes at each time step.

For the probabilities of an up move, middle move, and down move, we solve both the HW model and the BK model using the Hull and White method (HW version). In each of the two versions the probabilities for an up move, middle move, and down move are given by q_1, q_2, and q_3 with $q_1 + q_2 + q_3 = 1$.

The Hull and White methodology for generating HW and BK trinomial lattices lets the probabilities depend on the mean-reversion term. In order to assure positive probabilities, Hull and White truncate the upper and lower branches of their lattice at a certain maximum and minimum. Below the minimum and above the maximum, they apply a new branching procedure with different probabilities.

III. EFFECTIVE DURATION, EFFECTIVE CONVEXITY, AND OPTION-ADJUSTED SPREAD

Modified duration measures the percentage bond price change from an absolute yield change. It can also be interpreted as the negative of the slope (first derivative) of the price-yield relationship divided by the price. Similarly, convexity is interpreted as the curvature of the price-yield relationship (i.e., the second derivative).

Since modified duration and convexity do not consider that the cash flows for a bond with an embedded option may change due to exercise of the option, they do not provide satisfactory results for bonds with embedded options. Effective duration and convexity do take into consideration how changes in interest rates in the future may alter the cash flows due to exercise of an embedded option.

Effective duration (ED) and effective convexity (EC) are computed as follows:[4]

$$Effective\ Duration(ED) = \frac{P_{down} - P_{up}}{2P_0(\Delta y)} \qquad (8)$$

$$Effective\ Convexity(EC) = \frac{P_{up} + P_{down} - 2P_0}{2P_0(\Delta y)^2} \qquad (9)$$

where Δy is the change in the interest rate used to calculate different estimated bond prices, P_{down} is the estimated bond price if yields decrease by Δy, P_{up} is the estimated bond price if yields increase by Δy, and P_0 is the original bond price.

The option-adjusted spread (OAS) is the constant spread that when added to every rate in an interest rate lattice used to price a bond with an embedded option will make the arbitrage-free value equal to the market price of the security. Hence, if the market price for a bond is below the arbitrage-free value obtained from an interest rate lattice, the OAS will be positive and vice versa (see Fabozzi [1998]).

IV. ED, EC, AND OAS RESULTS

We analyze four types of bonds with embedded options: a callable bond, a putable bond, a callable range note, and a putable range note. Range notes are floating-rate instruments whose coupon is equal to a reference rate as long as the reference rate is within a certain range at the reset date. If the reference rate is outside the range, the coupon rate is set equal to zero for that period.

The characteristics of the bonds are detailed in Exhibit 1. The analysis is performed using five-year maturities.[5] Binomial and trinomial interest rate lattices with six-month time steps are used to price the structures. We use the spot rate and volatility term structure shown in Exhibit 2. Finally, a mean reversion term of 5% is assumed where appropriate.

Effective Duration

Exhibit 3 illustrates an approximation to the price-yield relationship that holds for the four security types. In general, at very low interest rates, we expect the callable bond to have an effective duration of about one year (which is equal to the period from now until the time the callability period starts; *see Exhibit 1*). The effective duration should be low because the issuer has an incentive to call the issue on the first possible call date and refinance its debt at a lower interest rate. Therefore, the slope of the price-yield relationship of the callable bond is slight for low interest rates.

If interest rates are high, the callable bond is much like an option-free bond, since the issuer has no incentive to call the bond issue before maturity. Therefore, the

EXHIBIT 1
Bond Characteristics

	Option Strike	Coupon	Time Option Starts
Callable Bond	$102.50	6.00%	1 year
Putable Bond	$95.00	6.00%	1 year
Callable Range Note	$97.50	floating	1 year
Putable Range Note	$95.00	floating	1 year

The callable and putable bonds are regular bonds that include an embedded option. The range notes pay a floating-rate coupon paid only if the interest rate is within a specified range (in our case between 4% and 8%); otherwise the coupon is equal to zero.

EXHIBIT 2
Spot Rate and Volatility Term Structures Used

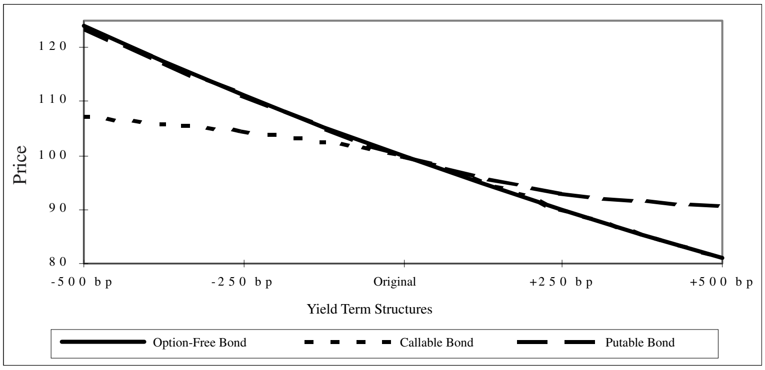
Year	Spot Rates*	Volatility
0.5	6.30	10.00
1.0	6.20	10.00
1.5	6.15	10.00
2.0	6.16	10.00
2.5	6.17	9.00
3.0	6.15	9.00
3.5	6.13	9.00
4.0	6.09	9.00
4.5	6.06	8.00
5.0	6.02	8.00

**All spot rates and volatilities are in percent.*

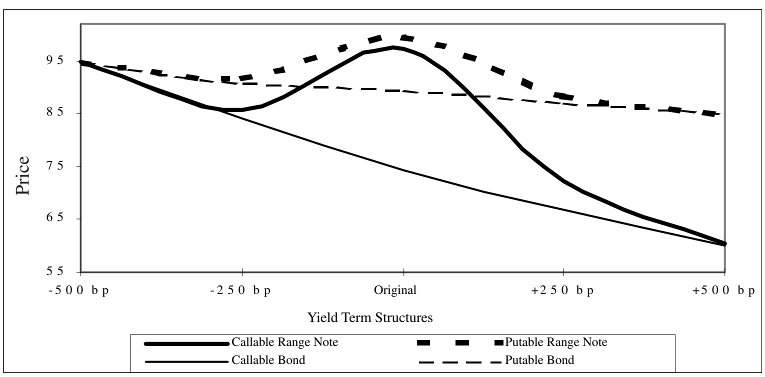
We use this spot rate curve and volatility term structure as a base case for our analysis. In order to compute the effective duration and effective convexity, we shift the spot rate curve in a parallel manner, but we assume that the volatility term structure remains constant. This term structure represents the U.S. Treasury curve in the Spring of 2000.

EXHIBIT 3
Price-Yield Relationships

Regular Callable and Putable Bond



Callable and Putable Range Note



The two graphs illustrate the pricing behavior over a spectrum of yields for the regular callable and putable bond and the callable and putable range note. The purpose of the two graphs is to provide a general intuition for the ED and EC pattern over different yield levels, i.e. for the original term structure provided in Exhibit 2 and the parallel shifts of ±250 bp and ±500 bp. In addition to the regular callable and putable bond, the first graph shows the price-yield relationship for an option-free bond with the same characteristics. The second graph shows the price-yield curve for a callable zero-coupon bond as well as a putable zero-coupon bond, both with the same characteristics as the corresponding range notes. These two securities are shown for reference only, because they provide a lower bound for the callable and the putable range note prices.

effective duration in a high interest rate environment is approximately equal to the effective duration of a corresponding option-free bond.

Exhibit 3 shows that the slopes for the option-free bond and the callable bond are approximately the same for high interest rates and diverge at low interest rates.

Employing similar intuition, the opposite should hold for a putable bond. If interest rates are low, bondholders have no incentive to redeem the bond before maturity. Therefore, the effective duration of the putable bond will be approximately the same as the effective duration of a corresponding option-free bond in a low interest rate environment (the slopes of the price-yield relationships for the two bonds are equal in Exhibit 3 for low interest rates).

At high interest rates, bondholders will redeem the bond early because they are able to get a higher return on an alternative investment. So the effective duration of the putable bond should be low at high interest rates, i.e., about one year in our case (since the bond can be redeemed only after one year from now). As Exhibit 3 illustrates, for high interest rates the putable bond price has a lower bound because of the putability. Therefore, the slope of the price-yield relationship is less steep than for a corresponding option-free bond.

Exhibit 3 shows that the slopes for the option-free bond and the putable bond are approximately the same for low interest rates and diverge at high interest rates.

For the range notes, this relationship is more complicated. The range note graph in Exhibit 3 includes the price-yield curve for the corresponding zero-coupon callable and putable bonds with the same strike prices and the same callability periods. The range notes differ from these zero-coupon bonds only in that they pay a floating-rate coupon whenever the interest rate is between the lower and the upper interest rate limits of 4% and 8%. When the interest rate is outside this range, the coupon is set equal to zero.

As a result, the callable range note must be worth the same or more than the corresponding zero-coupon callable bond, and the putable range note must be worth the same or more than the corresponding zero-coupon putable bond. Consequently, the duration pattern of the range notes at different interest rate levels differs from that for the corresponding zero-coupon bonds only when the interest rate is within the range where a floating-rate coupon is paid. The durations of the range notes are similar to the durations of the corresponding zero-coupon bonds when interest rates are substantially outside this range.

In general, there are two interest rate effects for range notes that work in the same direction at very high interest rates and in the opposite direction at very low interest rates. If interest rates are very high, the range notes' redemption value is discounted at a higher interest rate, and therefore its present value will be low. If interest rates are high enough so that they exceed the upper limit of the interest rate range (8% in our example), then bondholders will receive no coupon. *Both* effects depress the range note value to the bondholder, and there is no incentive to redeem the putable range note. Similarly, the issuer has no incentive to call the callable range note before maturity under such a scenario.

When interest rates are low, however, the two interest rate effects work in the opposite direction. On the one hand, the redemption value is discounted at a lower rate, and therefore its present value is high. On the other hand, the interest rate might fall below the lower limit of the interest rate range (4% in our case) so the range note has a zero coupon. For the issuer, this provides cheap financing, so there is no incentive to call the callable range note as long as the coupon is equal to zero. The range note holder may have a strong incentive to redeem the putable range note before maturity under such a scenario.

These characteristics will have a significant influence on the effective duration of the range notes. At very high interest rates, the effective duration of the callable range note will be high, and the effective duration of the putable range note will be very low. At very low interest rates, the effective duration will be high for the callable range note and low for the putable range note. This is seen in Exhibit 3.

The slope of the price-yield relationship is high in absolute value for the callable range note in high interest rate environments, and it is low in absolute value for the putable range note. The same pattern holds for low interest rates. Here, the slope is relatively high in absolute value for the callable range note and low for the putable range note.

The range notes exhibit this peculiarly non-monotonic price-yield relationship because of the imposed boundary conditions from the interest rate ranges. For example, if interest rates increase from below the lower limit of the range (where the coupon is zero) to a rate that is within the range (so that the coupon jumps from zero to the floating rate level), the value of the range note to its holder increases. Under this scenario, the range note price increases if the interest rate rises, for both the callable and the putable range note.

This pattern is a result of the note becoming a coupon-paying security, and this has more influence over the value of the note than the lower discount factors. Therefore, the coupon effect dominates the present value effect so that the range note value increases and the effective duration will be negative near the lower limit of the interest rate range.

Exhibit 3 illustrates this phenomenon through the positively sloped price-yield relationship, indicating a negative effective duration. The value of the range note then reaches a maximum somewhere between the lower and the upper limit of the interest rate range. Beyond this interest rate level, the value decreases again with increasing interest rates.

The range note price sensitivity to interest rate changes is especially high close to the upper limit of the interest rate range (here 8%), since the coupon jumps from the floating-rate level to zero above the upper limit. The present value effect of higher interest rates further depresses the range note value, so the effective duration is relatively high close to the upper interest rate limit (especially for the callable range note). This can be seen in Exhibit 3. The slope of the price-yield relationship is relatively high in absolute value close to the upper interest rate limit.

Exhibit 4 shows the effective duration results for the HL, the KWF, and the BDT models. We computed the effective duration for the original term structures shown in Exhibit 2 using a yield change Δy of 50 basis points. The results are no different for any yield change of less than 100 bp. We then shift the original term structure up and down in a parallel manner by ± 250 basis points and by ± 500 basis points.

The KWF and the BDT effective duration estimates are very similar, while the HL model sometimes produces substantially different estimates. This is to be expected; the HL model is a normal interest rate model, while the KWF and BDT models are lognormal interest rate models.

While the KWF model does not incorporate interest rate mean-reversion, the BDT model includes an implicit mean reversion term introduced through the term structure of local volatilities. The differences in their effective duration estimates are relatively minor, especially for the more extreme interest rate term structures. Therefore, the impact of the implicit mean-reversion term on the effective duration estimate in the BDT model is minor.

For the HL model, the effective duration does not vary much with the level of interest rates. The HL model produces ED estimates that are not as representative of the price-yield properties as the estimates resulting from the lognormal models. For example, when interest rates are extremely low, as in the first column of Exhibit 4 (–500 bp), the callable bond should have a very short effective duration. In fact, the one-year delayed call should dictate an effective duration close to 1.0. The KWF and BDT models are in line with this estimate, but the HL model

EXHIBIT 4
Effective Duration (Binomial)

	–500 bp	–250 bp	Original	+250 bp	+500 bp
			Ho-Lee		
Callable Bond	2.1843	2.5365	2.4526	2.5544	3.0556
Putable Bond	3.4528	3.2383	2.8964	2.7253	2.4569
Callable Range Note	4.3656	–1.7185	2.2345	4.2559	2.7632
Putable Range Note	4.4676	–0.0342	1.9460	1.9601	0.6527
			Kalotay-Williams-Fabozzi		
Callable Bond	0.9802	0.9680	3.6940	4.1764	4.0875
Putable Bond	4.4452	4.3552	4.1643	1.4121	0.9322
Callable Range Note	4.4769	–7.3821	1.9101	12.1257	6.0632
Putable Range Note	4.2303	–3.8566	0.5662	3.3333	0.9470
			Black-Derman-Toy		
Callable Bond	0.9802	0.9680	3.8620	4.1765	4.0875
Putable Bond	4.4453	4.3552	4.2656	1.2215	0.9322
Callable Range Note	4.4768	–7.5520	1.1107	13.9608	5.7683
Putable Range Note	4.2419	–3.5625	0.5573	3.6847	0.9470

Effective duration estimates obtained using the Ho-Lee, Kalotay-Williams-Fabozzi, and the Black-Derman-Toy interest rate models for the different bond structures. The duration estimates are shown for the original term structure as shown in Exhibit 2 as well as for parallel shifts of this term structure of ±250 and ±500 bp. The effective duration is computed as shown in Equation (8) using an interest rate change (Δy) of 50 bp.

produces an effective duration estimate of more than two years (more than a 100% difference). The same is true for the HL model estimate for the putable bond in an environment of extremely high interest rates.

Since our HL model does not truncate the returns at zero, the results in Exhibit 4 should not be surprising. At the very low interest rates (i.e., –500 bp and –250 bp), the HL model will have a large number of negative rates in the lattice. Combine this with the hyperbolic nature of value with respect to the discount rate, and the higher ED values are easily explained. Small changes in rates (Δy) will produce greater price differentials for the HL model than any of the other models, which results in higher ED values. If we were to truncate the values at zero, this result would be reduced somewhat.[6]

Exhibits 5 and 6 present the effective duration estimates produced by the trinomial versions of the HW and BK models. As we have noted, both the HW and the BK models are able to explicitly incorporate mean reversion. Recall that the HW model is a normal interest rate model and the BK model is a lognormal interest rate model. In this sense, we can make a similar comparison between normal and lognormal models as we do in explaining Exhibit 4.

The HW model, as a normal model, produces effective duration estimates that vary less with interest rate levels than the BK effective duration estimates. As for the HL model, the HW model produces an effective duration estimate for the callable bond that is high at very low interest rates and low at very high interest rates. The reverse is true for the putable bond. The BK effective duration estimates are more variable for different interest rate levels and seem to be more in line with theoretical reasoning.

A comparison of Exhibits 5 and 6 shows that the duration estimates for the two models can be substantially different, especially for more complex securities such as the range notes.

Effective Convexity

Effective convexity (EC) is an approximation to the second derivative (the curvature) of the price-yield relationship while taking the embedded option into account.

EXHIBIT 5
Effective Duration (HW Trinomial)

	–500 bp	–250 bp	Original	+250 bp	+500 bp
			Hull-White (HW Version)		
Callable Bond	2.5483	2.4641	2.4236	2.3411	3.3456
Putable Bond	3.5943	3.3607	3.2409	2.9184	2.3306
Callable Range Note	1.8843	–3.0793	5.4636	3.3510	3.0201
Putable Range Note	2.9997	–0.9756	2.9736	1.7582	1.5931

Effective duration estimates obtained using the Hull-White (HW version) interest rate model for the different bond structures. The duration estimates are shown for the original term structure as shown in Exhibit 2 as well as for parallel shifts of this term structure of ±250 bp and ±500 bp. The effective duration is computed as shown in Equation (8) using an interest rate change (Δy) of 50 bp.

EXHIBIT 6
Effective Duration (BK Trinomial)

	–500 bp	–250 bp	Original	+250 bp	+500 bp
			Black-Karasinski (HW Version)		
Callable Bond	0.9802	0.9681	3.7210	4.1455	4.0584
Putable Bond	4.4809	4.3207	4.1841	1.2907	0.9322
Callable Range Note	4.9290	–8.4850	1.1348	11.0725	5.6202
Putable Range Note	4.2399	–4.5037	0.3010	2.1743	0.9471

Effective duration estimates obtained using the Black-Karasinski (HW version) interest rate model for the different bond structures. The duration estimates are shown for the original term structure as shown in Exhibit 2 as well as for parallel shifts of this term structure of ±250 bp and ± 500 bp. The effective duration is computed as shown in Equation (8) using an interest rate change (Δy) of 50 bp.

As can be seen in Exhibit 3, for option-free bonds, convexity/curvature of the price-yield curve is always positive. This is not always the case for callable bonds. The callability feature effectively places a value cap on the bond price at low interest rates. When interest rates fall, it is optimal for the issuer to exercise the call option. This phenomenon, referred to as price compression, is explained in detail in Buetow and Johnson [2000] and Fabozzi, Buetow, and Johnson [2001].

Therefore, the price-yield relationship is convex for high interest rates, but for interest rates at or near the call yield a callable bond tends to exhibit negative convexity (i.e., concavity). At extremely low interest rates the price-yield relationship becomes linear and the convexity approaches zero. Exhibit 3 illustrates all these characteristics for the callable bond.

The opposite is true for a putable bond. The put option effectively puts a price floor below the bond price that is equal to the exercise price of the put option. Whenever the option-free bond price drops below the exercise price, it is optimal for the bondholder to exercise the put option and redeem the bond at the exercise price. Therefore, during the putability period, the putable bond cannot fall below the exercise price of the put option.

This phenomenon is referred to as price truncation (explained in detail in Buetow and Johnson [2000] and Fabozzi, Buetow, and Johnson [2001]). At extremely high interest rates the price-yield relationship becomes linear and the convexity approaches zero. Exhibit 3 illustrates these characteristics for the putable bond.

For more complex bond structures such as the range notes, the convexity pattern is more complicated. As the price-yield pattern for range notes in Exhibit 3 shows, there is substantial negative convexity/curvature within the interest rate range (between 4% and 8% in our case), because the range note prices reach their maximum within this interest rate region. Since our original term structure is relatively flat at around 6%, as shown in Exhibit 2, we expect the curvature/convexity of the price-yield relationship in Exhibit 3 to be negative around the original term structure.

Exhibit 3 shows that the prices of the callable and putable range notes have a lower bound that is equal to the price of the corresponding zero-coupon callable or putable bond, respectively. As can be seen in the graph, the price-yield curves of the range notes approach their lower bounds asymptotically for interest rates that are substantially below or above their coupon interest rate range. As a result, the concavity within the coupon interest rate range has to be offset by higher (positive) convexity above and below the concavity range in order for the range note price-yield curves to converge to their lower bounds. This pattern is illustrated in the second graph of Exhibit 3.

Exhibit 7 shows the EC estimates resulting from the HL, KWF, and BDT models. Exhibit 8 shows the EC estimates produced by the HW (HW version) model. The EC estimates for the BK (HW version) models are shown in Exhibit 9.

For the callable bond, once again the three lognormal models (KWF, BDT, and BK) show a distinctive EC pattern. The EC tends to be relatively high and positive for high interest rates; it then turns into concavity for medium interest rates and is fairly linear for low interest rates (i.e., the EC is close to zero). Therefore, the possibility that the callable bond might be called if interest rates drop is already anticipated at medium interest rates (for the original term structure), and the EC therefore becomes negative (i.e., concave) at this interest rate level.

The HL model (as a normal interest rate model) shows a somewhat different EC pattern with concavity at high interest rates (+250 bp) and very low interest rates (–500 bp). It suffers from EC estimates that do not match the necessary pricing behavior.

EC estimates for putable bonds are all positive as required by the pricing behavior. Most normal interest rate models tend to produce EC estimates that are relatively high at all interest rate levels. The lognormal models produce EC patterns that are more representative of the pricing behavior. The EC is generally highest for intermediate interest rate levels, but in all cases becomes close to zero (i.e., a linear relationship between yields and bond prices) for very high interest rates, which again is due to the putability at high interest rates.

The EC pattern for range notes is substantially more volatile than for regular callable and putable bonds. This is due to the boundary effects of the interest rate range (4% and 8% in our case) that determine the amount of the coupon and whether a coupon will be paid or not.

At low interest rates, we expect positive convexity (–500 bp and –250 bp). Convexity should be highest around the level of the lower interest rate range limit, which coincides approximately with our –250 bp shift. Our results in Exhibits 7, 8, and 9 confirm this expectation, especially for the callable range note.

It is interesting to observe that in all cases the lognormal models produce a significantly higher convexity at this interest rate level than the normal interest rate models. For the callable range note, the convexity estimate ranges between 311.37 (HL) and 1669.74 (BK-HW version).

The range notes achieve their price maximum within the interest rate range between 4% and 8%. Therefore, we expect the price-yield curve to be concave within most of this region, which is confirmed by most of our results. Here again, this pattern is more pronounced for the lognormal models than for the normal interest rate models. Beyond the maximum range note price, the price-yield curve returns to its lower bound—the corresponding callable or putable zero-coupon bond—for higher interest rates. Therefore, concavity should turn into convexity for higher rate levels.

There are substantial differences among the interest rate models, however. The HL model, for example, still shows a concave price-yield relationship for the +500 bp term structure shift for both the callable and the putable range note, while the BDT and BK (HW) models generate price-yield relationships that quickly revert to convexity for the callable range note and seem to produce results that are most consistent with the pricing behavior. For high interest rates, there is no clearly distinguishable pattern between normal and lognormal interest rate models as was the case for lower interest rate levels.

In general, Exhibits 7, 8, and 9 illustrate that the extent of the convexity estimates is strongly influenced by the interest rate model used. Differences can be significant. In some cases, different interest rate models yield convexity estimates for the same bond that are of large absolute value but opposite sign. This is true not only for more complex structures such as the range notes that we examine but also for the regular bonds with embedded options. The BDT and the BK (HW) models seem to generate estimates that are consistent with the pricing behavior.

Option-Adjusted Spread

The option-adjusted spread (OAS) is the constant spread that when added to every rate in an interest rate lattice used to price a security will make the model price equal

EXHIBIT 7
Effective Convexity (Binomial)

	–500 bp	–250 bp	Original	+250 bp	+500 bp
			Ho-Lee		
Callable Bond	–6.5042	4.9239	4.6533	–9.9469	6.3074
Putable Bond	18.3791	7.4845	9.4005	5.7927	4.9184
Callable Range Note	667.5951	311.3662	6.2264	–209.2758	–665.7050
Putable Range Note	399.6786	226.8234	5.0027	-3.9026	–212.8750
			Kalotay-Williams-Fabozzi		
Callable Bond	0.7283	0.7089	–46.2580	10.5003	10.1265
Putable Bond	11.6832	11.2774	30.1305	25.5506	0.6608
Callable Range Note	11.1333	1303.2938	–258.0827	–277.2039	232.9752
Putable Range Note	162.6136	846.8329	–97.3774	–390.5430	0.6749
			Black-Derman-Toy		
Callable Bond	0.7273	0.7094	–52.0860	10.5224	10.0998
Putable Bond	11.7159	11.2700	10.8943	42.8777	0.6585
Callable Range Note	11.1932	1394.6886	–132.1511	378.1289	125.4250
Putable Range Note	160.3531	845.9795	–37.9984	–307.8412	0.6726

Effective convexity estimates obtained using the Ho-Lee, Kalotay-Williams-Fabozzi, and the Black-Derman-Toy interest rate models for the different bond structures. The convexity estimates are shown for the original term structure as shown in Exhibit 2 as well as for parallel shifts of this term structure of ±250 bp and ±500 bp. The effective convexity is computed as shown in Equation (9) using an interest rate change (Δy) of 50 bp.

EXHIBIT 8
Effective Convexity (HW Trinomial)

	–500 bp	–250 bp	Original	+250 bp	+500 bp
			Hull-White (HW Version)		
Callable Bond	4.9668	4.6948	4.6353	4.3647	7.3021
Putable Bond	19.0310	8.0771	7.6229	20.1459	30.1028
Callable Range Note	115.8107	758.8392	–65.5413	82.9525	111.4673
Putable Range Note	96.6740	570.1957	–46.8147	25.2619	31.9267

Effective convexity estimates obtained using the Hull-White (HW version) interest rate model for the different bond structures. The effective convexity estimates are shown for the original term structure as shown in Exhibit 2 as well as for parallel shifts of this term structure of ±250 bp and ± 500 bp. The effective convexity is computed as shown in Equation (9) using an interest rate change (Δy) of 50 bp.

to the market price of the security. We compare the OAS that results from using different interest rate models for the structures in Exhibit 1. For this analysis, we price the structures using the original interest rate term structure shown in Exhibit 2, and assume a market price for each of these bond structures is 3% below the model price. The estimates we obtain are shown in Exhibit 10.

As with the effective duration and effective convexity estimates, we can observe that the OAS estimates obtained using different interest rate models differ substantially. The estimates differ in some cases by more than 100% (e.g., the OAS estimate for the callable range note obtained using the HL model is 0.7215%, while it is only 0.3050% using the BK-HW model).

In general, the OAS estimates obtained from the normal interest rate models (HL and HW) are higher for almost all the bond types than the estimate obtained from the lognormal interest rate models (KWF, BDT, and BK). This is due to the distributional differences between the types of models. The HL and the HW model allow for very low and even negative interest rates, while the lognormal-based models do not. This results in higher OAS estimates.

EXHIBIT 9
Effective Convexity (BK Trinomial)

	–500 bp	–250 bp	Original	+250 bp	+500 bp
			Black-Karasinski (HW Version)		
Callable Bond	0.7575	0.6920	–48.5904	10.3039	9.9813
Putable Bond	11.4815	11.1116	18.8605	45.3959	0.6522
Callable Range Note	13.3549	1669.7437	–247.1297	10.2581	200.2952
Putable Range Note	151.1884	937.3596	–120.4923	–204.6208	0.6660

Effective convexity estimates obtained using the Black-Karasinski (HW version) interest rate model for the different bond structures. The effective convexity estimates are shown for the original term structure as shown in Exhibit 2 as well as for parallel shifts of this term structure of ±250 bp and ±500 bp. The effective convexity is computed as shown in Equation (9) using an interest rate change (Δy) of 50 bp.

EXHIBIT 10
Option-Adjusted Spread (in percent)

	Callable	Putable	Callable Range Note	Putable Range Note
HL	1.2350	1.0619	0.7215	1.0447
KWF	0.7598	0.7509	0.5077	0.9697
BDT	0.7270	0.7219	0.3610	0.6732
HW (HW)	1.2661	1.0186	0.5219	1.5998
BK (HW)	0.7413	0.7359	0.3050	1.0255

Option-adjusted spread (OAS) estimate obtained using the different interest rate models for each bond structure. The OAS is computed using the original interest rate term structure as shown in Exhibit 2 and an assumed market price for each bond structure that is 3% below the price obtained from the model.

V. CONCLUSION

We have examined the common interest rate risk metrics—effective duration, effective convexity, and option-adjusted spread—produced by different one-factor no-arbitrage interest rate models that are solved numerically using binomial or trinomial lattices. The models are: Ho and Lee; Kalotay, Williams, and Fabozzi; Black, Derman, and Toy; Hull and White; and Black and Karasinski. In the trinomial framework, we use the implementation technique of Hull and White [1990, 1993, 1994].

We show that these interest rate models produce substantially different effective duration, effective convexity, and option-adjusted spread estimates for different bond structures, namely, regular callable and putable bonds and callable and putable range notes. This is true for regular callable and putable bonds, but the differences are even more pronounced for more complex bond structures such as range notes.

Our findings highlight the need for a careful interpretation of these metrics. The differences become more problematic as the structures become more complicated—structured notes, collateralized mortgage obligations, asset-backed securities, and so on. While no model or implementation framework is deemed superior, users must ensure that they thoroughly understand the model they are using before they apply any of the metrics in an investment or risk management context.

ENDNOTES

[1]The Kalotay, Williams, and Fabozzi model we present is a variation of the actual KWF model in that we allow for a non-zero drift rate.

[2]An exposition of these models appears in Buetow and Sochacki [2001].

[3]See Jarrow [1996] for a more thorough explanation of deriving the probabilities in the risk-neutral framework. A more rigorous development can be found in Harrison and

Kreps [1979] and Harrison and Pliska [1981]. A $q = 0.5$ also ensures that the moments of the distribution are matched.

[4]This is illustrated in Buetow and Johnson [2000] and in Fabozzi, Buetow, and Johnson [2001].

[5]Results for the ten-year maturities are available at www.bfrcservices.com.

[6]The effective duration results for the truncated HL model are as follows for the original term structure: Callable bond = 2.2980, putable bond = 1.4166, callable range note = 1.7095, putable range note = 0.4263. In general, the ED estimate for the callable bond is mainly a function of the call option delay (one year in our example). If there is no delay, the callable price cannot exceed the call price at any time, and the ED should be low for all models. As the delay increases, the HL ED estimate will be even higher.

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