Pricing Simple Interest-Rate Derivatives

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Abstract

We introduce the basic concepts of quantitative pricing for interest-rate derivatives without optionality, focusing on the risk-free discount curve. We show how to compute the non-arbitrage value of simple interest-rate derivatives from the knowledge of the discount factor. The type of interest-rate derivatives that can be computed in this way include inter-bank deposits, interest-rate swaps, and foreign-exchange forward contracts. Finally, we describe in details the bootstrap of the discount curve from a series of quoted deposit and swap rates.

1 Introduction

In the last three decades the use of interest-rate derivatives has spread so much that terms like *swap* and *futures* are often heard even outside the financial community. In particular, the wide-spread use of adjustable-rate mortgages has helped to popularize the quoting of LIBOR indexes even by non-financially-oriented newspapers. Many books and papers have been written on interest-rate derivatives; some of them require a very strong mathematical background on stochastic processes, while other are too simplistic. We take an intermediate approach: all the details of pricing simple interest-rate derivatives are spelled out with the use of a reduced number of mathematical concepts. Most readers, with a minimal mathematical background, should be able to understand the discussion and grasp the formulae written in this paper.

The pricing of interest-rate derivatives is a necessary step to perform any type of risk management on these instruments. Indeed, the first and foremost form of risk management on a financial instrument is the ability to compute its mark-to-market price. In this paper risk management is the underlying motivation to compute an instrument price and its possible future price scenarios. Furthermore, this paper can be read by students interested in fixed-income models.

The goal of this paper is to describe a number of practical tools necessary to price, i.e. compute the value, of many simple interest-rate derivatives starting from a small number of quoted deposit and swap rates. The term *instrument* value denotes the present value of all the future cash flows that the instruments entitle. By the term *simple interest-rate derivative* we define a contract with the following features:

1. the instrument obliges the contract holder to receive, or pay, a finite number of cash flows within a finite maturity;

2. the instrument value is solely dependent on the level of past, present, and future interest rates (i.e. the effects of variations in the stock market, credit market, and the other markets are only indirect);

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3. cash flows are not directly affected by the credit of the parties involved;

4. the cash flows, to be paid or received, will occur at future dates known at the beginning of the contract;

5. the instrument contracts do not have any type of optionality.

Assumption 1 is the definition of interest-rate derivatives, while the remaining assumptions specify the meaning given to the word *simple*. Examples of derivatives satisfying these conditions include forward rate agreements, interest-rate swaps, and foreign-exchange forwards. This definition implies that it is possible to compute the present value of financial instruments as the sum of the present values of their future cash flows. We show how the price of an instrument satisfying this set of assumptions can be computed using arbitrage-free considerations from the knowledge of the risk-free discount curve.

The present work is intended to be more of a practical handbook than a theoretical text on interest-rate modeling. After reading this paper one should be able to write a computer program, or a spreadsheet, to compute the value of the derivatives described here. For example, this has been already accomplished by the QuantLib project (see reference [6]). Because of this focus on the implementation details, the next section, section 2, describes the basic accounting conventions and their use in financial engineering. Section 3 introduces the discount factor and outlines the general approach to instrument pricing. In sections 4 and 5 we apply the general approach to pricing to two different asset classes: the interest-rate swap and the foreign-exchange forward. Finally, in section 6 we describe how to build, or bootstrap, the discount curve from market data.

2   Accounting conventions

In simple terms the pricing of a financial instrument is just the computation of the schedule of its future cash flows and the determination of their current fair value. Clearly, the understanding of fair present value is the most important concept in pricing; however, one should not underestimate the importance of cash-flow dates. Hence, before we look at the pricing-model details, it is necessary to describe the algebra and the metric associated with dates and the time lags between them. As explained in section 2.1, the algebra of dates consists of adding or subtracting days, months, or years to a given date in order to obtain another date. The metric associated with dates is the time lag between two dates; it is usually expressed as a fraction of one calendar year and is described in subsection 2.2.

2.1   Calendars and business-day conventions

Banks and stock exchanges are not open for business on every single calendar day. The set of days for which two legal entities agree to exchange cash flows is usually called a *business calendar*. For example, most banks in the Euro area follow the convention named *Trans-European Automated Real-time Gross settlement Express Transfer*, or TARGET for short, which states (see, e.g. reference [2], for details) that transactions can take place on any date other than,

- Any Saturday

- Any Sunday

- New Year's Day: January 1st

- Good Friday (since 2000)

Argentina	Australia	Brazil	Canada
China	Czech Republic	Denmark	Finland
United States	Hong Kong	Hungary	Iceland
India	Indonesia	Italy	Japan
Mexico	New Zealand	Norway	Poland
Saudi Arabia	Singapore	Slovakia	South Africa
South Korea	Sweden	Switzerland	Taiwan
TARGET	Turkey	Ukraine	United Kingdom

Table 1: List of the most popular calendars available in QuantLib. For more details on the actual business dates for each calendar see reference [6].

- Easter Monday (since 2000)

- Labor Day, May 1st (since 2000)

- Christmas, December 25th

- December 26th (since 2000)

- December 31st (1998, 1999, and 2001 only)

In the financial world there are a number of business calendars and they are usually associated to one or more countries. Table 1 lists the most important business calendars used in finance. Each calendar in turn can have different variations. For example, sometimes the bond market, the stock market, and the commodity market have slightly different holiday schedules.

Business-day conventions It is often necessary to perform algebraic operations on calendar dates. For example consider a bond based on the TARGET calendar paying forty cash flows for the next ten years, one every three months, starting three months from today. Some of these future payment dates will not fall on a TARGET business day and it will be necessary to establish at which date the cash flow will be exchanged. In order to address this type of problem *business-day conventions* were introduced by the International Swaps and Derivatives Association (also known as ISDA) and other authorities. Here we list the most commonly used conventions,

- **Following** (ISDA) Choose the first business day after the given holiday.

- **Modified Following** (ISDA) Choose the first business day after the given holiday, unless it belongs to a different month. If the first business day is in a different month, choose the first business day before the holiday.

- **Preceding** (ISDA) Choose the first business day before the given holiday.

- **Modified Preceding** Choose the first business day before the given holiday, unless it belongs to a different month, in which case choose the first business day after the holiday.

- **Unadjusted** Do not adjust: use the given date even if it is an holiday (usually the year fraction, defined next, is computed on the holiday but the payment is deferred to the next day).

Note that all these conventions allow to unique choice of a business day given an arbitrary date.

Actual/360	Actual/365 Fixed	Actual/Actual
Business/252	One Day	Simple
Thirty/360		

Table 2: List of most popular day-count conventions. For a practical implementation of the rules associated to these day-count convention see reference [6].

2.2   Day-count conventions and year fractions

It is customary to express interest payments as the percentage of a certain notional amount per year. This practice would not be a problem if we were always dealing with a whole number of years, however, it is often necessary to compute the cash amount to be paid for time periods less than a year long. Hence for any two dates a *year fraction* is defined as the possible the number of years between them. More formally, given two dates d_1 and d_2, the year fraction between them is denoted as

$$\tau = \tau_{1,2} = T(d_1, d_2) \,. \tag{1}$$

The different ways in which year fractions are computed between two dates are named *day-count conventions*. For example, if n is the number of calendar days between the start date d_1 and the end date d_2 (d_1 excluded and d_2 included), the year fraction according to the *Actual/360* convention is defined as,

$$T_{\text{actual}/360}(d_1, d_2) \,=\, \frac{n}{360} \,. \tag{2}$$

Other day-count conventions can be more complicated both in the way they compute the number of days between two dates and the way in which the denominator of definition (2) is computed. A list of the popular day-count conventions is given in table 2.

In the present paper we will always make a distinction between dates and time fractions. Dates just represent days on a calendar and the only operations allowed on them is the sum, or subtraction, of days, weeks, months, or years. Therefore, it is not possible to multiply a date by a number. On the other hand, year fractions are just delays between two dates, i.e. real numbers, and it is possible to multiply them by a real number (this is actually done in the computation of interests).

For those readers that are more interested in the formal aspects of financial engineering one can view all the dates of a financial calendar together with a business-day convention as a *set* and the day-counter convention as a *metric*, or distance, on this set.

2.3   Interest-rate compounding conventions

Suppose that an interest payment is due for borrowing some notional amount of money N for one day. When the time period is extended to more than one day it is possible to compound the interest payments over the new period in different ways called *interest compounding*. Here we list the most popular compounding conventions.

Simple compounding convention   The simplest compounding possible is no compounding at all. Consider a notional amount N held between two dates, d_1 and d_2, that yields interest at a certain rate r. Then, at date d_2, the capital N will have earned a *simply compounded* cash coupon C given by,

$$C = N\,r\,T(d_1, d_2) \,, \tag{3}$$

where $T(d_1, d_2)$ is the appropriate year fraction between d_1 and d_2. Simple compounding is usually applied on interests earned in one year or less.

Annual compounding convention When two dates d_1 and d_2 are more than one year apart it is customary to express interest rates using the annual compounding convention. In this case the interest coupon can be computed by,

$$C = N \left[(1+r)^{T(d_1, d_2)} - 1 \right].$$

(4)

Continuously compounded rates When dealing with mathematical models of interest-rate curves it is useful to use a compounding conventions that does not change when the maturity date d_2 is more, or less, than one year apart from d_1. Hence financial engineers often use the continuously compounding interest rate defined as,

$$C = N \left[e^{rT(d_1, d_2)} - 1 \right].$$

(5)

It can be shown that, see for example reference [3], the interest rate r defined by expression (5) is equivalent to the interest paid by compounding the rate r over an infinite number of periods from date d_1 to date d_2. The compounding convention defined by expression (5) will be used in section 6 when trying to model the zero curve.

3 Pricing financial instruments

Most financial institutions periodically need to compute the market value of all assets held in their portfolios, a procedure called *mark to market*. The market value is the expected value at which it is possible to buy or sell a certain product to another party. We will use interchangeably the terms market value, current value, and price of an instrument.

We will define pricing a financial instrument as all the activities involved in determining the current value of that product. It is not in the scope of the present paper to illustrate the theory behind all computations; we will only describe a method and try, when possible, to show its consistency. Furthermore, we will restrict the types of instruments that can be priced to those that have a well determined cash flow schedule hence ruling out, for example, callable bonds where it is not known for certain when the cash flow will be paid. We will also neglect all the issues related to the liquidity of assets or to the credit worthiness of future payments and assume that all the operators involved will always honor their cash payments (see reference [7] for more details on how to relax the hypothesis on credit risk). The detailed list of all the assumptions made is written in section 1.

3.1 Discount factors and the future value of money

The value[1] of a unit of currency is not constant in time for at least two reasons: it is possible to obtain future interests by lending an amount of cash; the total amount of a currency that is available changes with its money supply. Therefore, if pricing is the computation of the current value of an asset, the simplest price that can be computed is that of a unit of currency paid at a future date.

Given a certain currency and a certain future date d we will denote with $D(d)$ the present value of a unit of currency paid at date d. In equation form,

$$D(d) = \mathrm{PV}(1 \text{ paid at } d),$$

(6)

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[1]The term *value* here is used in a loose way as "how many units of currency" and should not be confused with the term *intrinsic value*, i.e., the purchasing power of money.

where the symbol PV denotes the present-value operator. Note that if d is the current date, then it must be $D(d){=}1$. For most modern currencies it is observed that $D(d_1) < D(d_2)$ when d_1 is earlier than d_2.

The first challenge faced by a financial engineer is to understand how to compute the discount factor at different dates. Actually, the main goal of this paper is to show one way in which the discount factor can be computed in accordance with market expectations. In this and in the next few sections we will provide the quantitative tools that will allow us, in section 6, to build an expression for the discount factor that is consistent with market prices. Until section 6 therefore we will assume the risk-free discount factor D to be a known function of future dates for any currency.

Arbitrage and discount factor   Note that, because of no arbitrage assumptions, the current discount curve is a deterministic function of a future date. Otherwise two different risk-free investments over the same time period would yield two different interest payments and one could generate unlimited cash by buying one and selling the other. However, it is not known today what the discount curve will look like tomorrow or at any future date. Given two dates d_1 and d_2, with $d_1 < d_2$, we will denote as $D(d_1, d_2)$ the value of the discount rate at d_2 as it will be seen by the world at d_1.

3.2   Deposit rates and LIBOR fixings

In their daily business, banks sometimes require more liquidity than they have access to, while some other days they have a surplus of cash. For this reason an inter-bank lending market has developed so that it is easier to balance the accounting books. Using this inter-bank market one bank can borrow money for time periods that go from one day to a year by making a deal for a money-market deposit with another bank. In the deposit contract, interests are normally charged at the end of the loan period using a simple compounding rule as described in subsection 2.3. The rate charged by one bank to another in a deposit contract is called the *deposit rate.*

The average deposit rate at which big banks lend money is reported by agencies and published for a dozen maturities. For example, the British Bankers' Association (see, e.g., [1]) publishes the LIBOR rate: the *London Interbank Bank Offer Rate.* The LIBOR rates were originally published for deposits exchanged in British pounds but then were extended to many other currencies as well. The standard money market rates for the Euro currency are called Euribor rates and are published for every business day of the TARGET calendar (see reference [5]).

After the LIBOR rates were introduced they became quickly popular: by indexing the coupon paid out on this average rate, banks were able to match the cash flows due to their creditors with those coming from their debtors.

3.3   Forward interest rates

As a first example of how the discount factor can be used in financial engineering, consider the forecasting of interest rate paid for a loan starting at a future date d_1 and maturing at a latter date d_2 (with $d_1 < d_2$). Suppose that we know the risk-free discount factor $D(d)$. The rate r_{fwd} is unknown but can be estimated noting that one unit of currency between d_1 and d_2 accrues an interest of $\tau\, r_{\text{fwd}}$ with $\tau = T(d_1, d_2)$ (T being the appropriate day-count convention). Therefore, using simple compounding, the value at date d_1 of one unit of currency, i.e. $D(d_1)$, must be the same as the value of $1 + \tau\, r_{\text{fwd}}$ units of currency at date d_2. Hence,

$$D(d_1) \,=\, D(d_2)\,(1 + \tau\, r_{\text{fwd}})\,. \tag{7}$$

From this expression it follows that

$$r_{\text{fwd}} = \frac{1}{\tau} \left[\frac{D(d_1)}{D(d_2)} - 1 \right] . \tag{8}$$

The rate r_{fwd} thus computed is known as the forward interest rate between date d_1 and date d_2 implied by the given discount term structure.

Note that r_{fwd} is a function of both d_1 and d_2. When we want to explicit this dependence we write r_{fwd} as $r_{\text{fwd}}(d_1, d_2)$. In general the actual fixing of the LIBOR rate at d_1 will not match the forecast value; however, it can be shown[2] that $r_{\text{fwd}}(d_1, d_2)$ equals the expected rate of future deposit rates and that this is the only possible arbitrage-free value.

3.4 Present value of an instrument

In subsection 3.1 we defined the discount factor D as the present value of one unit of currency at a future date. We now generalize this expression so that it is possible to compute the present value of a string of cash flows paying at future dates.

Therefore, consider a financial instrument that can be split into a number of cash flows, c_1, c_2, ..., c_n to be paid at future dates, d_1, d_2, ..., d_n. We will not consider past cash flows since they do not affect the current value of the instrument and make the assumption that the dates of payments are known with certainty. Furthermore, we allow only constant cash flows or cash flows that can be estimated by the knowledge of the discount factor (such as cash flows depending on the future LIBOR fixing). Since the total present value of a number of cash flows is given by the sum of the present value of each cash flow, the market value of the instrument is given by,

$$\text{PV} = D(d_1)\, c_1 + D(d_2)\, c_2 + \ldots + D(d_n)\, c_n . \tag{9}$$

This formula, although simple, is very important because it allows the computation of many widely-rated interest-rate derivatives.

3.5 Forward rate agreements

A forward rate agreement, or FRA for short, is a contract that locks in a rate r_{fra} for the interests paid on a notional N between a future date d_1, the termination date, and a later date d_2, the effective date. The net position is settled at the termination date with respect to a certain reference rate (usually a LIBOR index).

Using expression (9) the net present value, or NPV for short, of the forward rate agreement is given by,

$$\text{NPV}_{\text{fra}} = D(d_1)\, c_1 + D(d_2)\, c_2 , \tag{10}$$

where

$$c_1 = N , \tag{11}$$

is the amount received at d_1 and

$$c_2 = -N\, (1 + r_{\text{fra}}\hat{\tau}) , \tag{12}$$

is the amount paid at d_2, with $\hat{\tau} = T(d_1, d_2)$ the day-count fraction agreed for the fixed-rate interest payment.[3]

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[2]See reference [3] for a proof.

[3]Since sometimes this day-count convention is not the same as that used for the reference rate, we will use a different symbol for it (i.e., the symbol $\hat{\tau}$).

The net present value in equation (10) can be expanded as,

$$
\begin{aligned}
\mathrm{NPV_{fra}} &= N\,D(d_1) - N\,(1 + r_{\mathrm{fra}}\hat{\tau})\,D(d_2) \\
&= N\,[D(d_1) - D(d_2) - r_{\mathrm{fra}}\hat{\tau}\,D(d_2)] \\
&= N\left[D(d_2)\frac{D(d_1) - D(d_2)}{D(d_2)} - \hat{\tau}\,r_{\mathrm{fra}}D(d_2)\right] \\
&= N\,[D(d_2)\,\tau\,r_{\mathrm{fwd}}(d_1, d_2) - D(d_2)\,\hat{\tau}\,r_{\mathrm{fra}}]\,,
\end{aligned}
\tag{13}
$$

where, in the last step, we used equation (8). This expression can be simplified to yield,

$$
\mathrm{NPV_{fra}} = N\,D(d_2)\,[\tau\,r_{\mathrm{fwd}}(d_1, d_2) - \hat{\tau}\,r_{\mathrm{fra}}]\,.
\tag{14}
$$

When the FRA contract is struck the fixed rate is usually chosen so that the net present value of the contract is zero, i.e. $\mathrm{NPV_{fra}}=0$. From equation (8) one can compute r_{fra},

$$
r_{\mathrm{fra}} = \frac{\tau}{\hat{\tau}}r_{\mathrm{fwd}}(d_1, d_2) \simeq r_{\mathrm{fwd}}(d_1, d_2)\,.
\tag{15}
$$

Hence the fair FRA rate is very close to the forward rate between the termination date and the effective date.

4   Interest rate swaps

An interest rate swap is a contract made between two parties similar to a set of forward rate agreements struck at consecutive date: on a number of future dates a fixed interest rate will be exchanged for an interest rate based on the LIBOR index. All the payments are based on the same notional N and are usually settled at the end of each period.

The net present value of a swap can be computed using equation (14) for each of the forward rate agreements involved,

$$
\begin{aligned}
\mathrm{NPV_{swap}} &= N\,D(d_2)\left[T(d_1, d_2)\,r_{\mathrm{fwd}}(d_1, d_2) - \hat{T}(d_1, d_2)\,r_{\mathrm{swap}}\right] + \dots \\
&\quad \dots + N\,D(d_n)\left[T(d_{n-1}, d_n)\,r_{\mathrm{fwd}}(d_{n-1}, d_n) - \hat{T}(d_{n-1}, d_n)\,r_{\mathrm{swap}}\right]\,.
\end{aligned}
\tag{16}
$$

Rearranging the terms in equation (16) one can see how an interest rate swap is made by the sum of two legs,

$$
\mathrm{NPV_{swap}} = N\,A_{\mathrm{floating\text{-}leg}} - r_{\mathrm{swap}}\,N\,A_{\mathrm{fixed\text{-}leg}}\,,
\tag{17}
$$

the first one, the floating leg, depends on future LIBOR fixings, while the second one, the fixed leg, is proportional to the swap rate r_{swap}. Notice how in a swap there is no exchange of notional at maturity.

The final maturity date on both legs is always the same, however usually the frequency of payments is different from one leg to the other. For example there might be semi-annual payments for the floating leg and yearly payments for the fixed leg. As shown in the next paragraphs, the swap NPV can still be written as in equation (17), even thought the terms $A_{\mathrm{floating\text{-}leg}}$ and $A_{\mathrm{fixed\text{-}leg}}$ will contain a different numbers of terms.

Floating leg   Explicitly, when all the dates are in the future, the floating-leg NPV can be computed as

$$
A_{\mathrm{floating\text{-}leg}} = D(d_2)\,T(d_2, d_1)\,r_{\mathrm{fwd}}(d_1, d_2) + \dots + D(d_n)\,T(d_{n-1}, d_n)r_{\mathrm{fwd}}(d_{n-1}, d_n)\,.
\tag{18}
$$

This expression can be expanded to obtain,

$$
\begin{aligned}
A_{\text{floating-leg}} &= D(d_2) \frac{D(d_1) - D(d_2)}{D(d_2)} + \ldots + D(d_n) \frac{D(d_{n-1}) - D(d_n)}{D(d_n)} = \\
&= D(d_1) - D(d_2) + \ldots + D(d_{n-1}) - D(d_n) = \\
&= D(d_1) - D(d_n) \,,
\end{aligned}
$$

where we used equation (8) to express $T(d_2, d_1) \, r_{\text{fwd}}(d_1, d_2)$ in terms of the discount factor. In summary the floating leg satisfies,

$$
D(d_1) = A_{\text{floating-leg}} + D(d_n) \,. \tag{19}
$$

When time d_1 is in the near future, we have $D(d_1) \simeq 1$, so that expression (19) can be written as

$$
100\% = 1 \simeq D(d_n) + A_{\text{floating-leg}} \,, \tag{20}
$$

i.e. one unit of currency (the left-hand side) is equal the present value of a risk-less floating-rate note that pays the LIBOR rate (the right-hand side). In other words, the present value of a risk-less floating rate note is always close to par.

Sometimes the payment of an additional spread s is added to the floating payments. In this cases the expression for $A_{\text{floating-leg}}$ should be modified as,

$$
\begin{aligned}
A_{\text{floating-leg}} &= D(d_2) \, T(d_2, d_1) \, [r_{\text{fwd}}(d_1, d_2) + s] + \\
&+ \ldots + D(d_n) \, T(d_{n-1}, d_n) \, [r_{\text{fwd}}(d_{n-1}, d_n) + s] \,. \tag{21}
\end{aligned}
$$

Fixed leg The fixed leg is simply given by the sum of all coupons paid with the same fixed rate r_{swap}. The present value of each payment is proportional to the year fraction between the payment dates multiplied by the discount factor at the coupon-payment date,

$$
A_{\text{fixed-leg}} = D(\hat{d}_2) \, \hat{T}(\hat{d}_1, \hat{d}_2) + \ldots + D(\hat{d}_m) \, \hat{T}(\hat{d}_{m-1}, \hat{d}_m) \,. \tag{22}
$$

Comparing this equation with expression (19) we notice that while the term $A_{\text{floating-leg}}$ depends from the discount factor at dates d_1 and d_n, the term $A_{\text{fixed-leg}}$ depends on the discount factor at all the dates from inception to maturity.

Swap fair rate Similarly to the case of a forward rate agreement, when a swap is struck the fixed rate is chosen so that the net present value of the swap is zero. Therefore, given a swap, its *fair rate* is defined as the special fixed rate $r_{\text{swap-fair}}$ for which the net present value of the swap is zero.

From equation (17) it is straightforward to show that the condition

$$
\text{NPV}_{\text{swap}} = 0 \,,
$$

implies,

$$
r_{\text{swap-fair}} = \frac{A_{\text{floating-leg}}}{A_{\text{fixed-leg}}} \,. \tag{23}
$$

Taking the reciprocal, i.e. computing $1/r_{\text{swap-fair}}$, of this equation and using expression (20) one obtains,

$$
\frac{1}{r_{\text{swap-fair}}} \simeq \frac{D(\hat{d}_2) \, \hat{T}(\hat{d}_1, \hat{d}_2) + \ldots + D(\hat{d}_m) \, \hat{T}(\hat{d}_{m-1}, \hat{d}_m)}{1 - D(d_n)} \,. \tag{24}
$$

Since all the terms $\hat{T}(\hat{d}_{i-i}, \hat{d}_i)$ are approximately equal to $T = \hat{T}(\hat{d}_1, \hat{d}_2)$ we can write,

$$\frac{1}{r_{\text{swap-fair}}T} \simeq \frac{D(\hat{d}_2) + \ldots + D(\hat{d}_m)}{1 - D(d_n)}, \tag{25}$$

i.e. the inverse swap fair rate is directly proportional to the average discount factor over the life of the swap. This rule of thumb is often useful for approximating the swap rate.

5   Foreign-exchange forward contracts

So far we have considered contracts struck in one specific currency. Very often derivatives are used in order to hedge against future changes in foreign exchange rates. Therefore we will extend the approach of the previous sections to contracts that involve two different currencies.

Consider a home currency h, together with its risk-free discount curve $D^h(d)$, and a foreign currency f together with its discount curve $D^f(d)$. Finally, let us assume that X_{f-h} is the current currency-exchange rate among the two currencies so that,

$$1(^h) = \frac{1(^f)}{X_{f-h}}, \tag{26}$$

where $1(^h)$ is one unit money in the currency h and $1(^f)$ is one unit if money in the currency f.

Given a certain notional amount N^h in the home currency and a notional amount N^f in the foreign currency, consider the contract that allows, at a certain future date d, to pay N^h and to receive N^f. The present value of N^h in the home currency is given by,

$$\text{PV}^h = D^h(d)\, N^h, \tag{27}$$

where D^h is the discount-rate function in the home currency. On the other end, the present value of the foreign notional in the foreign currency can be written as,

$$\text{PV}^f = D^f(d)\, N^f, \tag{28}$$

where D^f is the discount-rate function in the foreign currency. These expressions can be combined with definition (26) to obtain,

$$\frac{\text{PV}^f}{X_{f-h}} = D^f(d)\frac{N^f}{X_{f-h}}. \tag{29}$$

Therefore, the net present value of the contract in the home currency is given by,

$$\text{NPV}^h_{\text{fx-fwd}} = D^f(d)\frac{N^f}{X_{f-h}} - D^h(d)\, N^h, \tag{30}$$

the same amount can be expressed in the foreign currency as,

$$\text{NPV}^f_{\text{fx-fwd}} = D^f(d)N^f - X_{f-h}D^h(d)\, N^h. \tag{31}$$

The foreign-exchange forward contract it is usually struck so the its NPV is zero so that, from equation (30) with $\text{NPV}^h_{\text{fx-fwd}} = 0$, we have,

$$N^h = \frac{D^f(d)}{X_{f-h}\, D^h(d)}N^f. \tag{32}$$

Comparing this equation with expression (26), it is natural to define the expected forward exchange rate $X_{f-h}(d)$ at date d as,

$$X_{f-h}(d) = X_{f-h} \frac{D^h(d)}{D^f(d)} \,. \tag{33}$$

For all practical purposes one can use the exchange rate $X_{f-h}(d)$ when computing the fair value of a foreign currency at a future date d. Equation (33) highlights how the expected forward foreign-exchange rate between two currencies is highly dependent on the discount curves in each respective currency. It is not surprising, therefore, that any change in monetary policy has an effect (sometimes delayed) on the exchange rate with all other world currencies.

6   Bootstrapping the interest-rate term structure

So far we have assumed the discount curve to be a given function of the maturity date. In this section we show how the discount factor can be be obtained from the quoted deposit rates and interest-rate swaps. First we obtain an expression for the discount factor at a discrete number of nodes with the help of quoted deposit rates. Then we show how to interpolate these values so that the discount curve can be computed between nodes. Finally, we extend the discount curve using the quoted fair rate of interest-rate swaps.

6.1   Discount factor from deposit rates

An overnight deposit is a contract that allows to lend, or borrow, money from today to tomorrow. Given the current overnight rate r_{fix}^{1d} we can obtain the discount factor between today and tomorrow from equation (8),

$$D(d_1) = \frac{1}{1 + T(d_0, d_1)\, r_{\text{fix}}^{1d}} \,, \tag{34}$$

where d_0 is the current date, d_1 is tomorrow's date, and T depends on the day count convention used to compute interests. For the Euro currency the average overnight deposit rate is quoted every day on the TARGET calendar and is named Eonia, see reference [5].

The tomorrow-next deposit, or simply T/N, is a contract for which it is agreed to borrow a certain notional with a settlement date of tomorrow and is returned to the owner the day after. Given the quoted rate r_{fix}^{2d} for the T/N deposit it is straightforward to obtain the discount factor at its maturity date d_2 (two business days from today). Indeed, the discount factor $D(d_2)$ can be computed from equation (8) as,

$$D(d_2) = \frac{D(d_1)}{1 + T(d_1, d_2)\, r_{\text{fix}}^{2d}} \tag{35}$$

where $D(d_1)$ is computed using equation (34). For the Euro currency the average T/N rate is called Eurepo and is published on the Euribor website (see reference [5]).

Consider now a standard deposit contract, with a settlement date two days from today, that matures at date d_j. If the quoted LIBOR fixing is $r_{\text{fix}}^{d_j}$, then proceeding as before we can compute the discount factor at date d_j,

$$D(d_j) = \frac{D(d_2)}{1 + T(d_2, d_j)\, r_{\text{fix}}^{d_j}} \,. \tag{36}$$

Again, for the Euro currency, the average deposit rate is Euribor index quoted in reference [5].

A list of fixings for the Euribor indexes as observed on the financial markets in early June 2008 is given in table 3. Using these quotes and equations (34), (35), and (36) it is possible to compute the discount factor at all the given maturity dates.

	r_{1d}	r_{2d}	r_{1w}	r_{2w}	r_{1m}	r_{2m}	r_{3m}	r_{6m}	r_{9m}	r_{12m}
r_{d_2}	3.89%	3.96%	4.26%	4.37%	4.47%	4.75%	4.96%	5.12%	5.27%	5.41%

Table 3: Values of quoted Eonia, Eurepo, and Euribor rates published in early June 2008.

For risk-management purposes it is usually sufficient to compute the discount curve once a day. However, in order to have a discount curve updated more than once daily one could use the intra-day values of the deposit rate as long as these are sufficiently liquid.

6.2   Interpolation between nodes

So far we have produced the discount curve at discrete nodes between one day and one year for which deposit rates are available. Occasionally, however, one needs to evaluate the discount curve at a date that is not included in the curve nodes. Unfortunately, dates do not allow mathematical operations other than sum or subtraction of days, weeks, and so on. Therefore, in order to make the interpolation process more mathematically tractable, we introduce a day-counter convention so that the date nodes can be transformed into year fractions, i.e. real numbers. Given a reference date d_0, usually the current date, we map the node dates d_1, ..., d_n into time nodes, t_1, ..., t_n,

$$t_i = T(d_0, d_i). \tag{37}$$

Since we would like to map back and forth between dates and time delays, not all day-count conventions are suitable for this purpose. In order to ensure invertibility it is necessary to require the function $d \to T(d_0, d)$ to be a strictly-increasing function of the date. In mathematical terms for any date d we must have

$$T(d_0, d+1) > T(d_0, d), \tag{38}$$

where $d + 1$ is the day after d.

At this point, given the date nodes, d_1, ..., d_n, one can define the discrete discount curve in terms of the time lags, t_1, ..., t_n,

$$D(t_i) = D(d_i). \tag{39}$$

Now, given a date d between two curve nodes d_i and d_{i+1}, one can define the time delay, t, $t = T(d_0, d)$, interpolate the function $t \to D(t)$, defined in equation (39), using standard mathematical techniques.

Continuously-compounded zero rates   One popular choice of interpolation is based on the linear interpolation of the zero rate here defined. For any date d, with a corresponding time lag t, the continuously-compounded zero rate is defined as

$$z(t) = -\frac{\log [D(t)]}{t}, \tag{40}$$

where the symbol 'log' denotes the natural logarithm. The discount factor can be computed from the zero rate as

$$D(t) = \exp [-z(t)\, t], \tag{41}$$

where symbol 'exp' represents the exponential function.

Linear interpolation of zero rate exponent Given two curve nodes t_i and t_{i+1} and a time lag t between these two nodes, we can obtain the zero rate $z(t)$ by linearly interpolating the discount exponent $t\,z(t)$ between the two exponents $t_i\,z(t_i)$ and $t_{i+1}\,z(t_{i+1})$,

$$t\,z(t) \,=\, t_i\,z(t_i) + \frac{t - t_i}{t_{i+1} - t_i}\,t_{i+1}\,z(t_{i+1})\,. \tag{42}$$

The discount factor at t can now be computed using equation (41).

6.3 Modeling the instantaneous forward rate

We can better understand equation (42) by computing the forward rate between two time lags t and $t + dt$ apart a small interval of time dt. Writing equation (8) in terms of time lags, we obtain

$$dt\,f(t) \,=\, \frac{D(t) - D(t + dt)}{D(t + dt)}\,, \tag{43}$$

where $f(t)$ is defined to be the *instantaneous forward rate* at the time lag t.

Expression (43), with the help of equation (41), can be rewritten in terms of the zero rate as

$$
\begin{aligned}
dt\,f(t) &=& \frac{e^{-z(t)t} - e^{-z(t+dt)(t+dt)}}{e^{-z(t+dt)(t+dt)}} = \frac{e^{-z(t)t}}{e^{-z(t+dt)(t+dt)}} - 1 \\
&=& \exp\left[z(t+dt)(t+dt) - z(t)t\right] - 1 \\
&=& \exp\left[z(t+dt)t + z(t+dt)dt - z(t)t\right] - 1 \\
&=& \exp\left[z(t)t + tz'(t)dt + z(t)dt - z(t)t\right] - 1 \\
&=& 1 + \left[tz'(t)dt + z(t)dt\right] - 1 \\
&=& tz'(t)dt + z(t)dt = d\left[t\,z(t)\right]\,,
\end{aligned}
\tag{44}
$$

where we neglected terms of order dt^2 and higher. Integrating both sides of this expression from a time lag of 0 to a time lag t, we can determine the relationship between the zero rate and the instantaneous forward rate,

$$z(t) \,=\, \frac{1}{t}\int_0^t f(\tau)\,d\tau\,, \tag{45}$$

so that the zero rate can be seen as the expected time-average forward rate from time zero to time t, also,

$$D(t) \,=\, \exp\left[-\int_0^t f(\tau)\,d\tau\right]\,. \tag{46}$$

Notice that the instantaneous forward rate is a local property affected only by the expectation of the rate at a future time t. For example, the expression for the expected forward rate (not instantaneous) between two future dates d_1 and d_2, given by equation (8) in terms of the instantaneous forward rate $f(t)$, can be written as,

$$
\begin{aligned}
\tau\,r_{\text{fwd}} &=& \frac{D(t_1)}{D(t_2)} - 1 = \exp\left[z(t_2)t_2 - z(t_1)t_1\right] - 1 \\
&=& \exp\left[\int_0^{t_2} f(\tau)\,d\tau - \int_0^{t_1} f(\tau)\,d\tau\right] - 1 \\
&=& \exp\left[\int_{t_1}^{t_2} f(\tau)\,d\tau\right] - 1\,.
\end{aligned}
\tag{47}
$$

Notice how only the instantaneous forward rate between t_1 and t_2 affects the value r_{fwd}.

Equations (45) and (47) are useful to express the usual interest-rate variables as a function of the more fundamental quantity $f(t)$. Also, one should not be intimidated by the presence of the integral operator in these expressions since we will avoid it in the next paragraph by modeling $f(t)$ as piecewise-flat function.

Piecewise-flat instantaneous forward rate   Since most financially interesting quantities can be expressed as a function of the instantaneous forward rate $f(t)$, it makes sense to model it in the easiest way possible: a constant. Hence, suppose that the function $f(t)$ is a constant,

$$f(t) \;=\; f_0 \qquad \text{for all } t \,, \tag{48}$$

then since,

$$\int_0^t f(\tau)d\tau \;=\; f_0 t \,, \tag{49}$$

from equation (45) it turns out that $z(t) = f_0$ for all t. A discount curve defined in this way, however, cannot explain the price of more than one instrument as more time dependence is needed.

The simplest time-dependent function that is not constant is a piecewise-constant function, i.e. a function constant on separate intervals. We will assume the instantaneous forward rate to be a piecewise-flat function. Given $n+1$ constant forward rates $f_0, f_1, \ldots f_n$ at the time nodes $t_0, t_1, \ldots t_n$, define the instantaneous forward rate function $f(t)$ as,

$$f(t) \;=\; f_i \quad \text{if} \quad t_{i-1} \le t < t_i \qquad \text{for} \qquad i = 1, \ldots, n \,, \tag{50}$$

A plot of $f(t)$ defined in equation (50) is given in figure 1. Note that for any finite f_0 we always have $D(0) = 1$. Also, the integral in equation (45) can be computed explicitly for a given t, so that,

$$z(t)\,t \;=\; (t_1 - t_0)f_1 + \ldots + (t_i - t_{i-1})f_i + (t - t_i)f_{i+1} \,, \qquad \text{with} \qquad t_i \le t < t_{i+1}, \tag{51}$$

which yields,

$$\begin{aligned} D(t) &= \exp\left[-(t_1 - t_0)f_1 + \ldots - (t_i - t_{i-1})f_i - (t - t_i)f_{i+1}\right] \\ &= D(t_{i-1}) \exp\left[-(t - t_i)f_{i+1}\right] \,. \end{aligned} \tag{52}$$

Applying the natural logarithm on both sides we obtain,

$$\log\left[D(t)\right] = \log\left[D(t_{i-1})\right] - (t - t_i)f_{i+1} \,, \tag{53}$$

from which we deduce that between interpolation nodes the logarithm of the discount factor grows linearly with time. For this reason we speak of log-linear interpolation of the discount factor. Unless otherwise stated in this paper and in all papers of the current series the discount curve is interpolated using equation (53).

6.4   Extending the term structure using swap rates

So far it has been straightforward to determine the discount factor from the quoted deposit rates. However, as discussed in subsection 6.1, for reasons related to the credit worthiness of the deposit issuer, it is not advisable to compute the discount rates from deposits that expire later than one year from the current date (basically zero-coupon bonds[4]). Since in an interest-rate swap there is

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[4]In recent years, i.e. after the financial crisis of 2008, deposit rates should be used only up to maturities of six/nine months.
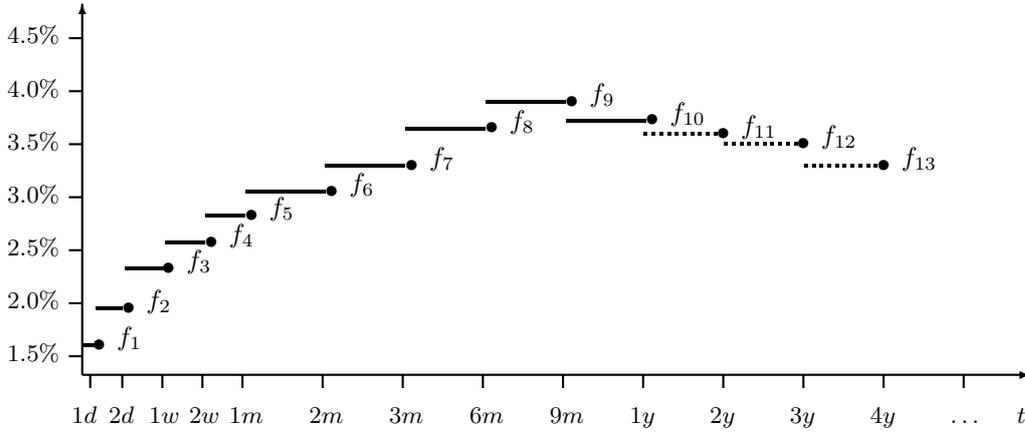
Figure 1: Values of the instantaneous forward rate as function of the time lags t. The nodes corresponding to the deposit and swap maturities are labeled from one day, $1d$, to four years (labeled $4y$). The values of f_i obtained from the deposit rates ($i = 1, \ldots, 10$) are shown as solid lines and the values of f_i obtained from the swap rates ($i = 11, 12, \ldots$) are shown as dotted lines

no exchange of notional, the dependence of the swap NPV from the credit worthiness of its issuer is much reduced. Therefore, we will use the quoted swap rates to imply the discount curve for longer maturities.

Consider a discount curve spanning all dates up to one year, interpolated using the log-linear discount rate as described in subsection 6.3. All the instantaneous forward rates up to one year of maturity are known. In the example of figure 1, the constants f_1, ..., f_{10}, can be computed using the method described in subsection 6.1.

In order to determine the value of constants f_{11}, f_{12}, ... it is necessary to use the swap rates for maturities of two years and longer. Let us first focus our attention on the computation of f_{11}. Hence, define the time node t_{11} as the time lag between the current date and a maturity of two years from now. From the knowledge of the two-year swap rate r_{2y} and from equation (23) it follows that,

$$r_{2y} = \frac{A^{2y}_{\text{float}}}{A^{2y}_{\text{fixed}}}, \tag{54}$$

where the two terms A^{2y}_{float} and A^{2y}_{fixed} are computed in section 4 and can be written as function of the discount factors up to two years. Now, the discount factor up to one year is known at all dates, and, in virtue of equation (52), the discount factor from one year to two years can be computed from the knowledge of f_{11}, i.e.,

$$D(t) = D(t_{10}) \exp\left[-(t - t_{11})f_{11}\right] \qquad \text{for } t_{10} < t \leq t_{11}. \tag{55}$$

Equation (54) can then be explicitly written as,

$$r_{2y} = \frac{A^{2y}_{\text{float}}(f_1, \ldots, f_{10}, f_{11})}{A^{2y}_{\text{fixed}}(f_1, \ldots, f_{10}, f_{11})}, \tag{56}$$

since r_{2y} and $f_1, \ldots, f_{10}$, are known quantities, this equation can be solved for f_{11}.

Let us add another point to the discount curve using the quoted value for the swap rate expiring in three years r_{3y}. Consider the time node t_{12} obtained as the time lag of a date three years from

now. Given a value for f_{12} the discount factor between two and three years can be computed as,

$$D(t) = D(t_{11}) \exp\left[-(t - t_{12})f_{12}\right] \qquad \text{for} \quad t_{11} < t \le t_{12}.$$

(57)

At this point the three-year swap rate can be computed as a function of f_{12},

$$r_{3y} = \frac{A^{3y}_{\text{float}}(f_1, \dots, f_{10}, f_{11}, f_{12})}{A^{3y}_{\text{fixed}}(f_1, \dots, f_{10}, f_{11}, f_{12})},$$

(58)

so that this equation can be solved, as before, to give f_{12}.

We can then proceed in the same way by adding the nodes t_{13}, t_{14}, ..., and compute the values of f_{13}, f_{14}, ..., on from the quoted values of the swap rates r_{4y}, r_{5y}, ...

The procedure just described to obtain the value of the discount factor for all dates given the quotes for the deposit rates and the swap rates is called the *bootstrapping* of the discount curve.

Suppose t_{25} is the last curve node, then it is possible to extrapolate the discount factor beyond t_{25} in the natural way:

$$D(t) = D(t_{25}) \exp\left[-(t - t_{25})f_{25}\right] \qquad \text{for} \quad t \ge t_{25}.$$

(59)

Expression (59) allow us to estimate the fair value of interest-rate derivatives with cash-flow dates that are beyond the maturity date of the longest quoted interest-rate swap.

7   Conclusions

In this paper we have shown how to bootstrap a risk-free discount curve from market data: given a series of deposit and swap rates we can adjust the values of the instantaneous forward rates so that the deposit and swap prices matches those quoted by the market. The bootstrapped risk-free discount curve is then used to price all the simple interest-rate derivatives satisfying the assumptions of section 1.

The evolution of the financial world has brought many instruments for which one or more of the mentioned assumptions are not valid. Specifically, it would be desirable to release the assumption that credit risk is not important to determine the value of an instrument. Releasing this assumption alone would allow us to price a much larger number of instrument types. Pricing in presence of credit risk is a very interesting and much debated subject and is introduced in reference [7].

As mentioned previously, the most important application of the pricing techniques described in this paper is risk management. In order to understand and compute the risk of a product it is necessary to compute its price. Furthermore, in risk management it is customary to simulate what the price would be for changes in the variables that describe financial markets. For example, the historical simulation method, an implementation of which is described in reference [9], allows to compute a distribution of instrument values at a future date. Furthermore one may need to generate stress scenarios in order to simulate what the instrument price would be under certain extreme market movements. As shown in reference [4], it is even possible to combine pricing with credit risk with the historical simulation method in order to compute scenarios that include simulated credit risk.

Finally, even though the analysis presented here can be applied to a limited number of asset classes, the method we described is more general and can be extended to include many other instrument types as shown by reference [8].

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