South African Financial Markets

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I would like to thank

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- Dr Jaco van der Walt for writing much of, and Dr Johan van den Heever of the SARB for contributions to, Chapter 2.

Financial Modelling Agency is the name of the author's consultancy. He is available for training on, consulting on and building of financial models, of both a derivative and a risk measurement nature.

This is a course which attempts to apply the theory of finance to the unique features of South African markets. So, for example, we completely avoid foreign exchange markets, as the practice globally is standardised. The South African market is characterised by subtly different formulae, and sparse data. So we will see that practice can be a long way from theory:

The experimentalist comes running excitedly into the theorist's office, waving a graph taken off his latest experiment. "Hmmm," says the theorist, "That's exactly where you'd expect to see that peak. Here's the reason." A long logical explanation follows. In the middle of it, the experimentalist says "Wait a minute", studies the chart for a second, and says, "Oops, this is upside down." He fixes it. "Hmmm," says the theorist, "you'd expect to see a dip in exactly that position. Here's the reason...".

Nevertheless, one can be precise, and such precision has its rewards in being able to be more than a naïve user of systems:

"The breakdown of the formulae for the above models is not discussed here, as one would have to have a good understanding of advanced mathematics in order to grasp the workings. However, all these calculations are performed on a computer, which would require one simply to input the relevant variables, such as volatility." Alexander [1996]

Contents

2007 Munchkin Calender

January

Sun	Mon	Tue	Wed	Thu	Fri	Sat
	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
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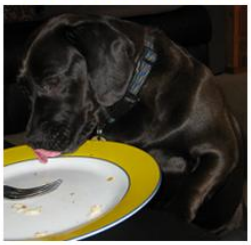
Matlab and Mini-Me's 2008 Munchkin Calender



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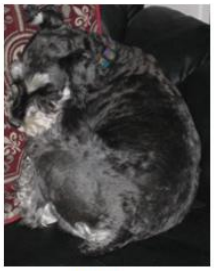

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Chapter 1

The time value of money

1.1 Discount and capitalisation functions

The owner of funds charges a fee, known as interest, for the usage of funds. Thus future values are more than present values; present values are less than future values. We have two functions

$$C(t,T) \tag{1.1}$$

the capitalisation function, denoting the amount that needs to be paid at time T in return for usage of 1 in the time $[t,T]$; and

$$Z(t,T) \tag{1.2}$$

the present value function, denoting the amount that needs to be handed over at time t and used in time $[t,T]$ in return for repayment of 1 at time T.[1]

Note that

$$Z(t,T) = \frac{1}{C(t,T)} \tag{1.3}$$

and $Z(t,t) = 1 = C(t,t)$; $Z(t,T)$ is decreasing, with $Z(t,T) \to 0$ as $T \to \infty$.

A fundamental result in Mathematics of Finance is (intuitively) that the value of any instrument is the present value of it's expected cash flows. So the present value function is important.

Time is measured in years, with the current time typically being 0. So, in this notation, one might rather write $Z(0,T)$ or $C(0,T)$ if it is understood without doubt when now is.

Example 1.1.1. Can pay 103 in 3 months or 113 in 1 year for use of 100 now.
$C(0, \frac{3}{12}) = 1.03$, $C(0,1) = 1.13$.

1.2 Compounding rates

Example 1.2.1. I can receive 12.5% at the end of the year, or 3% at the end of each 3 months for a year. Which option is preferable?

[1] Also denoted $B(t,T)$ (B for bond) or $D(t,T)$ (D for discount) in some sources. Our Z represents zero, for zero coupon bond.

Option 1: receive 112.5% at end of year. So $C(0,1) = 1.125$. We say: we earn 12.5% NACA (Nominal Annual Compounded Annually)
Option 2: receive 3% at end of 3, 6 and 9 months, 103% at end of year. We say: we earn 12% NACQ (Nominal Annual Compounded Quarterly). Then $C(0,1) = 1.03^4 = 1.1255...$, so this option is preferable.

This introduces the idea of compounding frequency. We have NAC* rates (* = A, S, Q, M, W, D) as well as simple rates. Note that

A	annual	standard
S	semi-annual	bonds
Q	quarterly	JIBAR, LIBOR
M	monthly	call rate, credit cards
W	weekly	carry market
D	daily	overnight rates

Example 1.2.2. I earn 10% NACA for 5 years. The money earned by time 5 is $(1 + 10\%)^5 = 1.610510$.

Example 1.2.3. I earn 10% NACQ for 5 years. The money earned by time 5 is $\left(1 + \frac{10\%}{4}\right)^{4\cdot 5} = 1.638616$.

Example 1.2.4. I earn 10% NACD for 5 years. The money earned by time 5 is $\left(1 + \frac{10\%}{365}\right)^{365\cdot 5} = 1.648608$.
What do we do about leap years? We'll come to this later.

There is no such calculation for weeks. There are NOT 52 weeks in a year.
If r is the rate of interest NACn, then at the end of one year we have $\left(1 + \frac{r}{n}\right)^n$, in general

$$C(t,T) = \left(1 + \frac{r}{n}\right)^{n(T-t)}$$

We always assume that the cash is being reinvested as it is earned. Now, the sequence $\left(1 + \frac{r}{n}\right)^n$ (for r fixed) is increasing in n, so if we are to earn a certain numerical rate of interest, we would prefer compounding to occur as often as possible. The continuous rate NACC is actually the limit of discrete rates:

$$e^r = \lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^n \tag{1.4}$$

Thus the best possible option is, for some fixed rate of interest, to earn it NACC.
The assumption will always be that, in determining equivalent rates, repayments are reinvested. This implies that for the purpose of these equivalency calculations,

$$C(t,T) = C(t,t+1)^{T-t} \tag{1.5}$$

Example 1.2.5. The annual rate of interest is 12% NACA. Then the six month capitalisation factor (in the absence of any other information) is $\sqrt{1.12}$.

If $r^{(n)}$ denotes a NACn rate, then the following gives the equation of equivalence:

$$\left(1 + \frac{r^{(n)}}{n}\right)^n = C(t,t+1) = \left(1 + \frac{r^{(m)}}{m}\right)^m \tag{1.6}$$

and for a NACC rate,

$$C(t,T) = e^{r(T-t)} \tag{1.7}$$

where time is measured in exact parts of a year, on an actual/365 basis.

In fact, money market interest rates always take into account the EXACT amount of time to payment, on an actual/365 basis, which means that the time in years between two points is the number of days between the points divided by 365.[2] Thus, whenever we hear of a quarterly rate, we must ask: how much exact time is there in the quarter? It could be $\frac{89}{365}$, $\frac{90}{365}$, or some other such value. It is never $\frac{1}{4}$! Thus, in reality, NACM, NACQ etc don't really exist in the money market. The rates are in fact what is known as simple, and the period of compounding must be specified explicitly. Here

$$C(t,T) = 1 + y(T-t) \tag{1.8}$$

where y is the yield rate, and where in all cases time is measured in years. Simple discount rates are also quoted, although, with changes in some of the market mechanisms in the last few years, this is becoming less frequent. The instrument is called a discount instrument. Bankers Acceptances were like this; currently it needs to be specified if a BA is a discount or a yield instrument. Here

$$Z(t,T) = 1 - d(T-t) \tag{1.9}$$

where d is the discount rate, and where in all cases time is measured in years.

Example 1.2.6. Lend 1 at 20% simple for 2 weeks. Then, after 2 weeks, amount owing is $1+0.20\frac{14}{365}$ ie. $C\left(t, t+\frac{14}{365}\right) = 1 + 0.20\frac{14}{365}$.

Example 1.2.7. Lend 1 at 12% simple for 91 days. The amount owing at the end of 91 days is $1 + 0.12\left(\frac{91}{365}\right)$.

Example 1.2.8. Buy a discount instrument of face value 1,000,000. The discount rate is 10% and the maturity is in 31 days. I pay $1,000,000\left(1 - 0.10\frac{31}{365}\right)$ for the instrument.

Another issue that arises is the 'Modified Following' Rule. This rule answers the question - when, exactly, is n months time from today? To answer that question, we apply the following criteria:

- It has to be in the month which is n months from the current month.
- It should be the first business day on or after the date with the same day number as the current. But if this contradicts the above rule, we find the last business day of the correct month.

For the following examples, refer to a calender with the public holidays marked. Once we are familiar with the concept, we can use a suitable excel function (which will be provided).

Example 1.2.9. What is 10-Feb-08 plus 6 months? Answer: 11-Aug-08. This is $19 + 31 + 30 + 31 + 30 + 31 + 11 = 183$ days, so the term is $\frac{183}{365}$.

[2] Actual/actual also occurs internationally, which is a method for taking into account leap years. 30/360 occurs in the USA: every month has 30 days and (hence) there are 360 days in the year. This is certainly the most convenient system for doing calculations. Actual/360 also occurs in the USA, which seems a bit silly.

Example 1.2.10. What is 9-May-08 plus 3 months? Answer: 11-Aug-08. This is $22 + 30 + 31 + 11 = 94$ days, so the term is $\frac{94}{365}$.

Example 1.2.11. What is 31-Aug-08 plus 2 months? Answer: 31-Oct-08. This is $30 + 31 = 61$ days, so the term is $\frac{61}{365}$.

Example 1.2.12. What is 31-Jan-08 plus 1 month? Answer: 29-Feb-08. This is $0 + 29 = 29$ days, so the term is $\frac{29}{365}$.

Always assume that if a rate of interest is given, and the compounding frequency is not otherwise given, that the quotation is a simple rate, and the day count is actual/365, and that Modified Following must be applied.

We now come to the simplest types of instruments which trade using these factors.

1.3 Yield instruments

Negotiable Certificates of Deposit, Forward Rate Agreements, Swaps. The later two are yield based derivatives. The first is a vanilla yield instrument: it trades at some round amount, that amount plus interest is repaid later. They trade on a simple basis, and generally carry no coupon; there can be exceptions to this if the term is long, but in general the term will be three months. Thus the repayment is calculated using (1.8).

The most common NCDs are 3 month and 1 months. The yield rate that they earn is (or is a function of) the JIBAR rate.

1.4 Discount instruments

Bankers Acceptances, Treasury Bills, Debentures, Commercial Paper, Zero Coupon Bonds. These are a promise to pay a certain round amount, and it trades prior to payment date at a discount, on a simple discount basis. The price a discount instrument trades at is calculated using (1.9). Treasury Bills and Debentures are traded between banks and the Reserve Bank in frequent Dutch auctions.

Promissory notes trade both yield and discount, about equally so at time of writing.

The default is that an instrument/rate is a yield instrument/rate, in particular, JIBAR (Johannesburg Inter Bank Agreed Rate) is a yield rate. This wasn't always the case.

If d is the discount rate of a discount instrument and y is the yield rate of an equivalent yield instrument then

$$(1 + y(T - t))\,(1 - d(T - t)) = 1 \tag{1.10}$$

It follows by straightforward arithmetic manipulation that

$$\frac{1}{d} - \frac{1}{y} = T - t \tag{1.11}$$

1.5 Primary and secondary market

Once sold by the originator (primary market) an instrument can usually be traded again (secondary market).

Example 1.5.1. Suppose a 3 month R1m NCD initially trades at 8.50% on 2-Jul-07. On 30-Jul-07, the owner sells it on at a simple discount rate of 8.80%. What are the cash flows?
The expiry date is 2-Oct-07, so the term in days is 92. Thus, it trades at $1,000,000$ for a redemption of $1,000,000 \cdot \left(1 + 8.50\% \frac{92}{365}\right)$.
On 30-Jul-07, the term remaining is 64 days. So, it trades at $1,000,000 \cdot \left(1 + 8.50\% \frac{92}{365}\right) \cdot \left(1 - 8.80\% \frac{64}{365}\right) = 1,005,663.94$.

Example 1.5.2. What rate has been earned NACC?
The capitalisation factor over 28 days is 1.0056639. Thus $r = \frac{365}{28} \ln 1.0056639 \ = 7.3625\%$.

Example 1.5.3. On 15-Aug-07 a 3 month R1m NCD initially trades at 7.00%. On 3-Oct-07, my rate for the expiry date on my yield curve is 6.22% NACC. What is the MtM of the NCD?
The payoff occurs on 15-Nov-07, so the payoff is $1,000,000 \cdot \left(1 + 7.00\% \frac{92}{365}\right)$. On 3-Oct-07, there are 43 days to expiry. Hence the MtM is $e^{-6.22\% \cdot 43/365} 1,000,000 \cdot \left(1 + 7.00\% \frac{92}{365}\right) = 1,010,214.13$.

1.6 South African Forward Rate Agreements

These are the simplest derivatives: a FRA is an OTC contract to fix the yield interest rate for some period starting in the future. If we can borrow at a known rate at time t to date T_1, and we can borrow from T_1 to T_2 at a rate known and fixed at t, then effectively we can borrow at a known rate at t until T_2. Clearly

$$C(t, T_1) C(t; T_1, T_2) = C(t, T_2) \tag{1.12}$$

is the no arbitrage equation: $C(t; T_1, T_2)$ is the forward capitalisation factor for the period from T_1 to T_2 - it has to be this value at time t with the information available at that time, to ensure no arbitrage.
In a FRA the buyer or borrower (the long party) agrees to pay a fixed yield rate over the forward period and to receive a floating yield rate, namely the 3 month JIBAR rate. At the beginning of the forward period, the product is net settled by discounting the cash flow that should occur at the end of the forward period to the beginning of the forward period at the (then current) JIBAR rate. This feature - which is typical internationally - does not have any effect on the pricing.
In South Africa FRA rates are always quoted for 3 month forward periods eg 3v6, 6v9, ..., 21v24 or even further. The quoted rates are simple rates, so if f is the rate, then

$$1 + f(T_2 - T_1) = C(t; T_1, T_2) = \frac{C(t, T_2)}{C(t, T_1)} \tag{1.13}$$

Thus

$$f = \frac{1}{T_2 - T_1} \left(\frac{C(t, T_2)}{C(t, T_1)} - 1 \right) \tag{1.14}$$

Example 1.6.1. On 4-Feb-08 the JIBAR rate is 11.175% and the 5-Aug-08 rate is 11.4290% NACC (as read from my yield curve, for example). What is the 3v6 forward rate?
$t =$ 4-Feb-08, $T_1 =$ 5-May-08, $T_2 =$ 5-Aug-08.

First we have

$$(1 + 11.175\% T_1)\, C(t; T_1, T_2) = e^{11.4290\%\, T_2}$$

where $T_1 = \frac{91}{365}$ and $T_2 = \frac{183}{365}$. So $C(t; T_1, T_2) = 1.030271$ and $f = 12.010\%$.

Now we can formally calculate the value of a FRA.

Suppose a FRA for the period $[T_1, T_2]$ is entered into at time t with a fixed rate of r_K. Suppose we are now at time s, with $t < s < T_1$. How do we value the FRA?

Consider Figure 1.1.

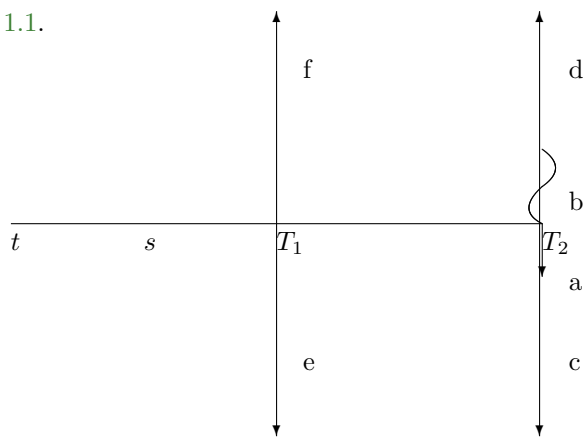


Figure 1.1: Valuing an off the run FRA

The value of the long position in the FRA is the portfolio given by a and b. a is a notional of $r_K(T_2 - T_1)$, b is the floating payment on the notional of 1. But I add in c, d, e, and f, and the value of this is 0, because there is cancellation. All four of these deals have a face value of 1.

However, the value of the portfolio given by b, d and e is also 0: it is a position in a floating rate note. Hence, the value of the FRA is given by the value of a, c and f, which is given by

$$V(s) \quad = \quad Z(s, T_1) - Z(s, T_2)[1 + r_K(T_2 - T_1)] \tag{1.15}$$

One has to be careful to note the period to which a FRA applies. In an $n \times m$ FRA the forward period starts in n months modified following from the deal date. The forward period ends $m - n$ months modified following from that forward date, and NOT m months modified following from the deal date. Here, typically $m - n = 3$. We will call this strip of dates the FRA schedule. In a swap, the ith payment is $3i$ months modified following from the deal date. We will call this strip of dates the swap schedule. Confusion is easy.

Example 1.6.2. On 4-May-07 the JIBAR rate is 9.208%, the 3v6 FRA rate is 9.270%, and the 6v9 FRA rate is 9.200%. What are the 3, 6 and 9 month rates NACC?

Note that

$$C(\text{4-May-07}, \text{6-Aug-07}) = \left(1 + 9.208\% \frac{94}{365}\right) = 1.023714$$

$$C(\text{4-May-07;6-Aug-07}, \text{6-Nov-07}) = \left(1 + 9.270\% \frac{92}{365}\right) = 1.023365$$

$$C(\text{4-May-07}; \text{5-Nov-07}, \text{5-Feb-08}) = \left(1 + 9.200\% \frac{92}{365}\right) = 1.023189$$

Straight away we have $r(\text{6-Aug-07}) = 9.101\%$.
Note that we would really like to have $C(\text{4-May-07; 6-Aug-07}, \text{5-Nov-07})$, but we don't. Also, we would like the sequence to end on 4-Feb-08, but it doesn't. So already, some modelling is required. We have

$$C(\text{4-May-07}, \text{6-Nov-07}) = 1.023714 \cdot 1.023365 = 1.047633$$

We now want to find $C(\text{4-May-07}, \text{5-Nov-07})$.
To model this we interpolate between $C(\text{4-May-07}, \text{6-Aug-07})$ and $C(\text{4-May-07}, \text{6-Nov-07})$. For reasons that are discussed elsewhere, the most desirable simple interpolation scheme is to perform linear interpolation on the logarithm of capitalisation factors - so called raw interpolation. See Hagan and West [2008].
Thus we model that $C(\text{4-May-07}, \text{5-Nov-07}) = 1.047370$.
Thus $r(\text{5-Nov-07}) = 9.131\%$.
Now

$$C(\text{4-May-07}, \text{5-Feb-08}) = 1.047370 \cdot 1.023189 = 1.071658$$

Now interpolate between $C(\text{4-May-07}, \text{6-Nov-07})$ and $C(\text{4-May-07}, \text{5-Feb-08})$ to get

$$C(\text{4-May-07}, \text{4-Feb-08}) = 1.071391$$

Thus $r(\text{4-Feb-08}) = 9.119\%$.
Even within this model, the solution has not been unique. For example, we could have used $C(\text{4-May-07}, \text{6-Aug-07})$ and $C(\text{4-May-07;6-Aug-07}, \text{6-Nov-07})$ unchanged, and interpolated within $C(\text{4-May-07}; \text{5-Nov-07}, \text{5-Feb-08})$ to obtain $C(\text{4-May-07}; \text{6-Nov-07}, \text{4-Feb-08})$. However, this is quite an ad hoc approach. It makes sense to decree that one strictly works from left to right.

1.7 Exercises

1. You have a choice of 2 investments:

   (i) R100 000 invested at 12.60% NACS for 1 year

   (ii) R100 000 invested at 12.50% NACQ for 1 year.

   Which one do you take and why? (Explain Fully)

2. On 17-Jun-08 you sell a 1 Million, 3 month JIBAR instrument. At maturity you receive R1,030,246.58. What was the JIBAR rate?

3. (a) If I have an x% NACA rate, what is the general formula for y, the equivalent NACD rate in i.t.o x?

   (b) Now generalise this to convert from **any** given rate, r_1 (compounding frequency d_1) to another rate r_2 (compounding frequency d_2).

   (c) Now construct an efficient algorithm that converts from or to simple, NAC* or NACC. (Hint: given the input rate, find what the capitalisation factor is (for a year or for the period specified), if the rate is simple. Then, using this capitalisation factor, find the output rate.)

4. On 17-Jun-08 you are given that the 3-month JIBAR rate is 12.00% and the 3x6-FRA rate is 12.10%. What is the 6 month JIBAR rate?

5. If the 4 year rate is 9.00% NACM and the 6 year rate is 9.20% NACA, what is the 2 year forward rate (NACA) for 4 years time?

6. Suppose on 22-Jan-08 the 3 month JIBAR rate is 10.20% and the 6 month JIBAR rate is 10.25%. Show that the fair 3x6 FRA rate is 10.0446%.

7. Suppose on 3-Jan-08 a 6x9 FRA was traded at a rate of 12.00%. It is now 3-Apr-08. The NACC yield curve has the following functional form: $r(t,T) = 11.50\% + 0.01(T-t) - 0.0001(T-t)^2$ where time as usual is measured in years. Find the MtM of the receive fixed, pay floating position in the FRA.

   Note that the 6x9 FRA has now become a 3x6 FRA. However, the rate 12.00% is not the rate that the market would agree on for a new 3x6 FRA. Using the yield curve, what would be the fair rate for a newly agreed 3x6 FRA? (There are two ways of doing this: from scratch, or by using the information already calculated.)

Chapter 2

The short term rates in the market

The yield curve defines the relationship between term to maturity and interest rates, for a specific quality of credit. Typically, a yield curve starts with the overnight point and extends to the 30 year point. In this chapter we look at the overnight and other short term points of the yield curve.

2.1 The repo rate of the SARB

The repurchase (repo) rate of the SARB is the mother of all overnight rates. This rate is set by the South African Reserve Bank (SARB), which is the central bank of South Africa, and it directly affects all other overnight rates that are traded in the market (and it indirectly affects all the rates along the rest of the yield curve).

While the repo rate affects all interest rates in the economy, it is only used for transactions between banks and the SARB. To understand how this works we have to look at the function of the SARB.

The SARB conducts the monetary policy of the country. The primary aim of its monetary policy is "to protect the value of the currency in order to obtain balanced and sustainable economic growth in the country" South African Reserve Bank [6 April 2000]. To achieve this, the SARB has two jobs: (1) to achieve price stability and (2) to maintain stable conditions in the financial sector. To achieve these goals, the SARB operates in the domestic money market. These money market operations are carried out by the Financial Markets Department at the SARB and involves the control over the supply of funds in the interbank market.

By changing the supply of funds, the SARB can affect short term interest rates (which are determined by the demand and supply of funds), which in turn affects the entire yield curve. This in turn affects the overall level of economic activity in the country and hence the general level of prices.

How does the SARB achieve this? Banks need to hold balances with the SARB. These balances are held for two reasons. Firstly, banks require sufficient funds to square off their positions with each other. The second reason is a statutory requirement which compels banks to hold cash reserve deposits with the SARB.

Why do banks require sufficient funds to square off their positions with each other? Every day millions of transactions take place in the economy. All these transactions have to be settled in cash through the banking system. Banks settle these transactions with one another by using their

balances at the SARB. In this way the SARB acts as the banker to all the banks.

If there was no inflow or outflow of cash into the banking system and there was a perfect interbank market, the pool of funds would remain the same, banks would never have to borrow money from the SARB and the repo rate would have no teeth. But in reality, there are several transactions that reduce the balances of banks at the SARB. Such transactions increase the liquidity requirement of banks and force them to borrow money from the SARB. Examples of these transactions include an increase in the notes and coins in circulation outside the SARB, an increase in government deposits with the SARB and the selling of gold and other foreign reserves by the SARB to banks.

While these transactions can reduce the balances of banks at the SARB, similar transactions in the opposite direction can also increase their balances. To ensure that banks are always forced to borrow from the SARB so that the repo rate remains effective, the SARB prefers to keep the market "short". They therefore intervene in the market to create a shortage.

The SARB uses several tools of intervention to create such a shortage. The primary tool goes back to the statutory requirement that we already mentioned. Banks are required by law to hold a percentage (2.5% at present) of their total liabilities (think of it as the public's deposits with banks) as a cash reserve deposit at the SARB. This permanently drains liquidity from the money market and forces the banks to borrow money from the SARB.

A second tool involves the purchase and sale of SARB debentures to banks. This falls under the broader heading of open market operations. The purchase (sale) of debentures by banks drains (injects) liquidity and forces them to borrow more (less) from the SARB.

Other tools include longer term reverse repos (see Chapter 6) and foreign exchange swaps.

Once the SARB has drained sufficient liquidity, the banks have insufficient balances with the SARB and are forced to borrow the shortfall from the SARB.

The SARB is in the process of changing the way in which it conducts its money market operations. A consultative paper appeared in December 2004. Here we briefly outline the current system.

The SARB currently uses five methods of refinancing:

1. Refinancing repo auctions. This is the main source of refinancing. These repurchase agreements are a form of securitised lending. (Please note that although this is a form of securitised lending, it differs from the buy-sell back transactions (carries) that are discussed in Chapter 7.)

   Banks can offer certain financial instruments to the SARB and borrow money against these securities. At maturity the banks pay the principal plus interest to the SARB in exchange for their security. The instruments that qualify as security include central government bonds, Treasury bills, Land bank bills and SARB debentures (proposals are under consideration to include government guaranteed bonds). Note that these instruments are held by banks in their normal course of business as liquid assets. The value of the security has to exceed the amount of the loan to insulate the SARB from adverse movements in the market. These "haircuts" are calculated on the market value of the instruments and vary according to their maturity.

   The refinancing repo auctions currently take place on Wednesdays and the banks borrow the money from the SARB for a week. The simple interest rate convention is used to calculate the size of the interest payments, on an actual/365 basis.

2. Supplementary repo auctions. From time to time there may be small deviations between the

liquidity requirement that the SARB estimates and the actual liquidity requirement. Under these conditions the SARB may announce a supplementary auction. The only difference is that these repos have maturities of one day, and not one week as with the main auctions. The rest of the mechanics is the same: The same instruments qualify as security, the interest calculation convention is simple and the interest rate is equal to the repo rate. It is worth noting that these can be repos or reverse repos, depending if the market is short or long of liquidity.

3. Final-clearing repo auctions. If the difference between the actual and estimated liquidity requirement is small and if the SARB is not responsible for the estimation error, the banks may have to resort to the final-clearing repo auctions. The mechanics are once again the same, but the interest rate is different in this case. The SARB may impose a penalty (currently 150 bps) on these auctions, where funds are borrowed 150 bps above the repo rate (short liquidity) or invested 150 bps below the repo rate (long liquidity).

4. Marginal lending facility. If banks are not squared off and fail to make use of the final-clearing repos, their accounts are automatically squared at the marginal lending rate. This takes place at a very prohibitive rate, where money is currently borrowed at 500 bps above the repo rate and invested at 0%. This facility however rarely gets used.

5. Access to statutory cash reserve deposits. Recall that banks are required by law to hold a percentage (2.5% at present) of their total liabilities as a cash reserve deposit at the SARB. As long as banks comply with the minimum amount of cash reserves on an average basis during the maintenance period (from the 15th working day of every month to the 14th working day of the following month), they can deposit and withdraw from their cash reserve deposit. The cash reserve deposit does not earn any interest at the SARB.

The SARB is currently proposing a change to the refinancing facilities South African Reserve Bank [December 2004]. In short, the SARB wants to use the main refinancing repo auctions, thereafter access to the cash reserve deposits of the banks and lastly the final-clearing repo auctions as the main tools for refinancing. This will essentially create a corridor (floor and ceiling) around the repo rate for interbank rates. The width of the corridor will depend on the penalty associated with the final-clearing repo. It is proposed to narrow this corridor to 50 bps above and below the repo rate.

How do changes in the repo rate affect inflation? One of the goals of monetary policy is to maintain price stability. The SARB uses an inflation targeting approach to achieve this. The present target is to keep the year-on-year increase in the CPIX (consumer price index excluding mortgage interest cost for metropolitan and other urban areas) within the range of 3% to 6%.

The SARB uses the repo rate to achieve this inflation target. The Monetary Policy Committee (MPC), which is chaired by the Governor of the SARB, meets at least every two months to decide if there should be any changes to the repo rate.

Since the refinancing repo auctions take place at the repo rate, the cost of overnight funds of banks is tied to the repo rate. This ensures that the SARB has control over the overnight rates in the interbank market which in turn can affect the entire yield curve. By changing the repo rate, the SARB therefore affects all interest rates in the economy, which in turn affects the overall level of economic activity in the country and hence the general level of prices.

Finally, you may have noticed that the main refinancing repo auction is a one-week transaction whereas overnight transactions are, well, overnight. This assumes a flat yield curve between one day and one week.

2.2 Call rates

Practitioners use the words call rate and overnight rate as synonyms. There are two types of call transactions, the first takes place between banks (interbank call) and the second takes place between banks and their customers. Banks may have different call rates depending on the type of customer. The calculations for interbank call and wholesale call are the same.

2.3 Prime rate

Although the prime rate is not strictly speaking an overnight rate, we discuss it here since banks may link some of their overnight transactions to the prime rate. The prime rate is an announced reference rate at which some low risk clients borrow from banks. Distinction is made between the prime mortgage rate which is used for home loans and the prime overdraft rate which is used for credit card transactions, for example. It is quoted as a simple rate. Interest accrues to the account on the last day of every month. The prime rate generally changes when the repo rate changes. Furthermore, a significant relationship appears to exist between the prime rate and any of the JIBAR rates.

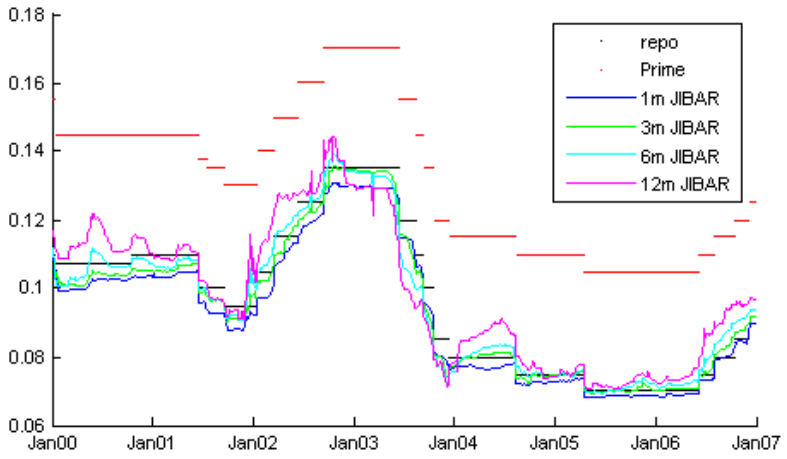


Figure 2.1: repo, JIBAR and Prime Rates

We can model the prime rate as a function of the one month JIBAR rate, and hence as a function of the yield curve, using a co-integration technique discussed in Alexander [2001], and for this particular case in West [2008]. Such models have a very high R^2 regression coefficient (98% under current conditions) and strongly satisfy the requisite hypothesis tests for stationarity.

We see how the model replicates the prime rate in Figure 2.2. One should notice how the cointegration smoothes out the jumps to a great extent, as occurs, for example, throughout 2002. This is due to the fact that the market is pricing into the JIBAR rates anticipated jumps to the prime rate before they occur. Only in cases where the quantum of the jump in prime is unexpected in equilibrium - as in August 2004 - does a jump in JIBAR, and consequentially in the replicating function, occur.

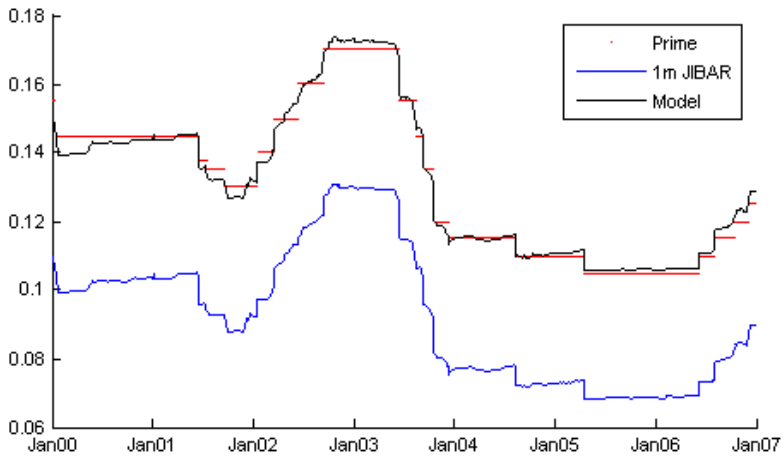


Figure 2.2: The replicating function

2.4 Other overnight rates

In addition to interbank and wholesale call transactions, there are also other transactions that have a maturity of one day. These include buy-sell back transactions (carries) and foreign exchange forwards. Carries are discussed in Chapter 7.

2.5 Indices of overnight rates

To trade in any market, you need to know what the bid and offer rates are. In many markets there is an exchange that facilitates such price discovery, think of the JSE. When trading the overnight area of the yield curve there is limited transparency.

To establish the true overnight rate is unfortunately not so simple. Firstly, there are several different overnight transactions, such as wholesale call, interbank call, carries and forex forwards and there is potentially a different rate for each of these. You therefore need to be very clear why you are building the yield curve to ensure that you reference the correct type of overnight rate.

Secondly, all of these transactions are over-the-counter transactions (OTC) and therefore do not trade on an exchange. To get a handle on the "market" you need to survey all the banks for their prices. Although banks post their rates on their trading screens, the most reliable way to survey the

market is by phone since the screen rates are not always up to date or may only apply to limited volumes. Even though overnight rates don't change much (if at all) during a day, you can imagine what a rigmarole price discovery can be in this market.

There are two indices of overnight rates available in the market. Neither of them is perfect.

2.5.1 SAONIA rate

To facilitate price discovery in the overnight market the SARB compiles the South African Overnight Interbank Average (SAONIA) rate. As its name indicates, this is an index of interbank transactions only.

The SAONIA rate is the weighted average rate paid on unsecured, interbank, overnight funding. This excludes funding that is raised at the prevailing repo rate. (When the latter is included it constitutes the SAONIA plus rate).

Since the SAONIA rate is based on call rates, the interest calculation is identical to that of call rates.

Although the SAONIA rate was launched with the best intentions, it has not achieved everything that it was designed for. Efforts are currently underway to strengthen the interbank market which should in turn make the SAONIA rate more useful South African Reserve Bank [December 2004].

2.5.2 Safex Overnight rate

During their normal course of business, members of Safex place margin with the exchange to ensure that their contracts are honoured. We will deal with this in Chapter 10. These funds are placed overnight at call rates with F1 and A1 banks.

Safex publishes the Safex Overnight rate which is the average rate that it receives on its deposits with the banks, weighted by the size of the investments placed at each bank. Safex only publishes the average rate and not its composition.

Since the Safex overnight rate is based on call rates, the interest calculation is identical to that of call rates.

It should be noted that Safex tries to earn the highest interest for its clients. In addition, Safex deposits represent a very small portion of the overnight funding in South Africa. The Safex overnight rate may therefore not be a good reflection of the weighted average call rates paid on rand deposits by all banks.

Finally, in addition to guiding the rate that Safex pays to their clients (after deducting 25 bps), the Safex overnight is also the reference rate that is used to reset rand overnight deposit swaps (RODS).

Chapter 3

Swaps

3.1 Introduction

These are identical to the swaps in [Hull, 2005, Chapter 7], so refer there for any clarification needed.
A plain vanilla swap is a swap of domestic cash flows, which are related to interest rates. We have a fixed payer and a floating payer; the fixed payer receives the floating payments and the floating payer receives the fixed payments.
In South Africa, the arrangements for a swap are as follows: the fixed payments are a fixed interest rate paid on an actual/365 basis on some unit notional, which we will make 1. These interest rates are made in arrears (at the end of fixed periods), and these periods are always three monthly periods. The ith payment is on date $\text{ModFol}(t, 3i)$ where t is the initiation date of the product, and the period to which the actual/365 day count applies is the period from $\text{ModFol}(t, 3(i-1))$ to $\text{ModFol}(t, 3i)$. Thus, a swap is not merely a strip of FRAs: not only are payments in arrears and not in advance, the day count schedule is slightly different. The floating payments are on a floating interest rate, paid on the same day count basis, and at the same time, on the same notional. The floating rate will be the 3 month JIBAR rate. As we do not know what the JIBAR rate will be in the future, we do not know what these floating payments will be, except for the next one (because that is already set - remember that the rate is set at the beginning of the 3 month period and the interest is paid at the end of the period). The date where the rate is set is known as the reset date.
In other countries, variations are possible. Of course the day count rule could be different. More importantly, it is common for the fixed payments to be made every six months, the floating payments to be made every three months. In these notes, we will in general stick to the situation where the time schedule for the fixed and floating legs are the same.
The notionals are never exchanged, although for some calculation purposes we often imagine that they are, as we will see later. In some more exotic swaps, such as currency swaps, the notional is exchanged at the beginning and at the termination of the product. Of course the fixed and floating payments, occurring on the same day, are net settled. We can, if we like, imagine that notionals ARE exchanged, because they would cancel off net as well. This trick is very useful.
The diagram is that of the long party to the swap - fixed payer, floating receiver. He is making fixed payments, hence they are in the negative direction. He is receiving floating payments, so these are

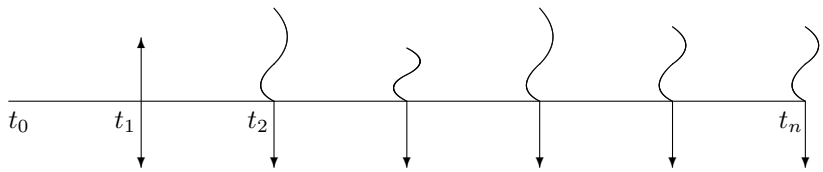


Figure 3.1: Fixed payments (straight lines) and floating payments (wavy lines)

in the positive direction. We don't know what the floating payment is going to be until the fixing date (3 months prior to payment).

3.2 Rationale for entering into swaps

Why would somebody wish to enter into a swap? This is dealt with in great detail in [Hull, 2005, Chapter 7]. The fundamental reason is to transform assets or liabilities from the one type into the other. If a company has assets of the one type and liabilities of the other, they are severely exposed to possible changes in the yield curve. Another reason dealt with there is the Comparative Advantage argument, but this is quite a theoretical concept.

Example 3.2.1. A service provider who charges a monthly premium (for example, DSTV, newspaper delivery, etc.) has undertaken with their clients not to increase the premium for the next year. Thus, their revenues are more or less fixed. However, the payments they make (salaries, interest on loans, purchase of equipment) are floating and/or related to the rate of inflation, which is cointegrated[1] with the floating interest rate. Thus, they would like to enter a swap where they pay fixed and receive floating. They ask their merchant bank to take the other side of the swap. This removes the risk of mismatches in their income and expenditure.

Example 3.2.2. A company that leases out cars on a long term basis receives income that is linked to the prime interest rate, again, this is cointegrated with the JIBAR rate. In order to raise capital for a significant purchase, adding to their fleet, they have issued a fixed coupon bond in the bond market. Thus, they would like to enter a swap where they pay floating and receive fixed. They ask their merchant bank to take the other side of the swap. This removes the risk of mismatches in their income and expenditure.

Of course, if the service provider and the car lease company know about each other's needs, they could arrange the transaction between themselves directly. Instead, they each go to their merchant bank, because

- they don't know about each other: they leave their finance arrangements to specialists.
- they don't wish to take on the credit risk of an unrelated counterparty, rather, a bank, where credit riskiness is supposedly fairly transparent.

[1] A statistical term, for two time series, meaning that they more or less move together over time. It is a different concept to correlation.

- they wish to specify the volume and tenor, the merchant bank will accommodate this; the other counterparty will have the wrong volume and tenor.
- they don't have the resources, sophistication or administration to price or manage the deal.

Sometimes the above arguments are not too convincing. Thus in some instances major service organisations form their own banks or treasuries - for example, Imperial Bank, Eskom Treasury, SAA Treasury, etc.

3.3 Valuation of a just started swap

How do we value such a swap? Assume that we know the capitalisation function (equivalently, the discount function) for every expiry date T, $t < T < \infty$. The fixed payments are clearly worth

$$V_{\text{fix}} = R \sum_{i=1}^{n} \alpha_i Z(t, t_i) \tag{3.1}$$

where R is the agreed fixed rate (known as the swap rate), n is the number of payments outstanding, and α_i is the length of the ith 3 month period on an actual/365 basis. This valuation formula holds whether or not today t is a reset date.

The valuation of the floating payments is more tricky. First we value the whole deal, that will give us certain ideas which will enable us to value each floating payment individually.

Imagine that the notional of 1 is indeed paid at the end. Thus, we have floating interest payments on a notional of 1, paid in arrears every 3 months (as they should be - JIBAR is a yield rate), a final floating interest payment, and repayment of 1 at the end. This deal has a value of 1. Thus, the actual deal (without the payment of notional at the end) has a value of 1 less the present value of that notional, i.e.

$$V_{\text{float}} = 1 - Z(t, t_n) \tag{3.2}$$

We can actually make a more complicated argument (to get to the same answer), this alternative argument will have the virtue that we can find the present value of each of the floating payments.

We would like to know the value of a single floating in arrears payment. Consider the single payment, which is the payment of interest in arrears of the period $[t_{j-1}, t_j]$ for the interest, on an outstanding loan of 1, which has accrued in that period. How do we value this payment? Consider the following diagram.

The payment we wish to value is the wavy line (its size is not known up front). Again I add in b, c, d, and e, and the value of this is 0, because there is cancelation. All four of these deals have a face value of 1. And as before, a, c and e represent a floating rate note, also of value 0. Thus

$$V_a = V_a + V_b + V_c + V_d + V_e = V_d + V_b = Z(t, t_{j-1}) - Z(t, t_j) \tag{3.3}$$

Thus for a general fixed for floating swap,

$$\begin{aligned} V_{\text{float}} &= \sum_{j=1}^{n} Z(t, t_{j-1}) - Z(t, t_j) \\ &= Z(t, t_0) - Z(t, t_n) \end{aligned} \tag{3.4}$$

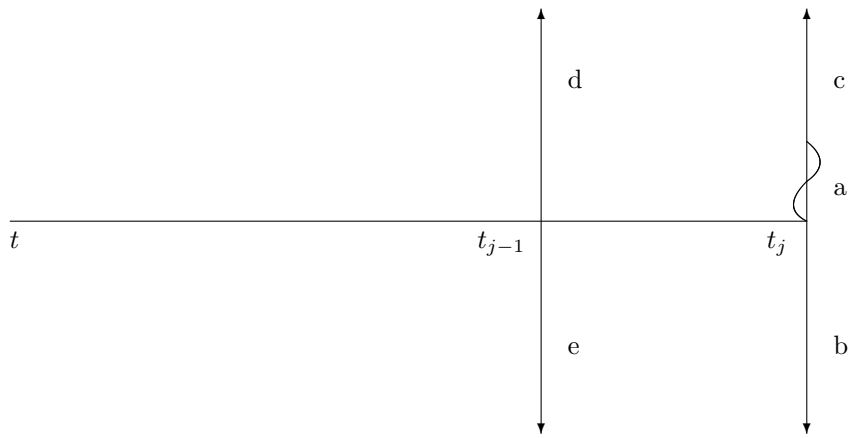


Figure 3.2: A complicated way of getting a single payment!

as a telescoping series.

In the typical situation, $t_0 = t$, and so $Z(t, t_0) = 1$. But the above argument is also valid for a swap which starts at some date t_0 in the future. To 'start' here means that the first observation of the unknown floating rate is made at that date and the first fixed for floating exchange is made at the observation date that follows that date (for example, 3 months later).

On initiation, a swap is usually struck so that the value of the fixed and floating payments are equal. Given the Z/C function, it is fairly trivial to solve for the rate R_n that achieves this:

$$R_n = \frac{1 - Z(t, t_n)}{\sum_{i=1}^{n} \alpha_i Z(t, t_i)} \tag{3.5}$$

R_n is called the swap rate. Of course, when a bank does such a deal with a corporate client, they will weight the actual R_n traded in their favour.

This view has R_n as a function of the Z function. In fact the functional status is exactly the other way round when we do bootstrapping.

3.4 Valuation of off the run swaps

Now suppose that t is not a reset date. Thus, we reset some time previously, and we are in the 'middle' of a 3 month period. To save on introducing more notation, we suppose that the upcoming payment has i index $i = 1$.

In this case, the formula for the value of the fixed payments is unchanged.

r^1_{fix} has already been fixed, at the previous reset date.

Again imagine that the notional is indeed paid at the end. When we reach the next reset date, after the floating payment (which we already know to be $r^1_{\text{fix}}\alpha_1$) is made, the floating leg will again be worth 1. Thus, the full deal is currently worth

$$Z(t, t_1)[r^1_{\text{fix}}\alpha_1 + 1]$$

However, the notional will not be paid on termination, so we need to remove it. Thus

$$V_{\text{float}} \quad = \quad Z(t, t_1)[r^1_{\text{fix}}\alpha_1 + 1] - Z(t, t_n)$$

Alternatively, we can realise that the floating leg is now effectively a fixed payment at t_1 and floating payments at $t_2,\ t_3,\ \ldots,\ t_n$. Thus

$$\begin{aligned} V_{\text{float}} &= Z(t,t_1)r^1_{\text{fix}}\alpha_1 + \sum_{j=2}^{n} Z(t,t_{i-1}) - Z(t,t_i) \\ &= Z(t,t_1)r^1_{\text{fix}}\alpha_1 + Z(t,t_1) - Z(t,t_n) \\ &= Z(t,t_1)[r^1_{\text{fix}}\alpha_1 + 1] - Z(t,t_n) \end{aligned}$$

Chapter 4

BESA Bond Pricing

The issuing of bonds has been the main mechanism for government and high credit worthy institutions borrowing from the public. Government issues (bonds prefixed by an r) comprise about 80% of the market. There is about R382b in outstanding debt in the market. This is the primary market: the issuing of bonds. In the secondary market, there are about 2000 trades per day at the Bond Exchange of South Africa (BESA), invoving a nominal of about R42b. The turnover velocity of bonds at BESA is about 28 ie. on average a bond is bought and sold 28 times per year. Therefore we have a very liquid market.

Bonds trade on yield, not on price. Only the South African, Swedish and Australian markets have this feature. This feature is clearly superior, and is possible because the South African bond market opened in 1973 whereas others opened much earlier eg. the UK in the 1600's. The yield is the IRR as previously, now called the ytm. It is a NACS rate; coupons are semiannual. There is an inverse relationship between yield and price.

Because the market trades on yield but settles on price (the cash price) we need precise rules to take care of the issues that arise. See Bond Exchange of South Africa [2005].

These specifications suffer from the following defects

- the standard unit is a R100 bond. It would be more natural for the standard unit to be R1 (unitless) or R1m, which is the unit a bond does actually trade in. This amount is called the notional, capital, or bullet. We will price everything as a R1 or unitless bond.
- Some of the derivatives are found in a very complicated manner, and can be simplified both for esthetic and computational reasons.

Pricing rules need to take care of

- all timing issues,
- coupons are not only in whole multiples of 6 months away,
- that there are books close dates,
- that the market actually trades $t + 3$.

4.1 The Bond Pricing Formula

The pricing mechanism is as follows. Suppose the settlement date is s.

Every bond has two Coupon Dates and two Books Close Dates per annum, notated in the form mmdd. Thus, for example, the CDs of the r153 are 831 and 228, and the BCDs are 731 and 131.

Note that the coupon will flow according to the regular following business day rule: this is the day in question if it is a business day, and the next business day if not, but here Saturdays are treated as business days (until they are in their own right public holidays). However, the pricing formula will work as if coupons are paid exactly on the coupon dates.

We will use the following notation:

Abbreviation	Notation	Meaning	Explanation
LCD	$\mathcal{L}$	Last coupon date	The CD before or on s
NCD	$\mathcal{N}$	Next coupon date	The CD after s
BCD	$\mathcal{B}$	Book's close date	The BCD of the NCD

$\mathcal{B}$ was typically one month before $\mathcal{N}$. However, in 2004, a number of BESA bonds changed their BCD rule, in order to come into line with international conventions. The bonds that changed were essentially government bonds: r and z bonds, Bophuthatswana bonds, Transkei bonds, Umgeni bonds, WS (TransCaledonian) bonds.[1] The rule is as follows: if

- the bond is one of these bonds, and
- the LCD is after 31 Dec 2003, and
- the NCD is after 29 Feb 2004

then the NCD is unchanged, while the BCD is 10 calendar days before the NCD. It is important to be able to determine whether or not we are before or after the change, otherwise historical calculations will be invalid. Thus, one does not simply hard-code the new BCD.

It is possible to develop very compact and efficient logic for calculating these dates.

1. Determine $\mathcal{L}$, $\mathcal{N}$ such that $\mathcal{L} \leq s < \mathcal{N}$; $\mathcal{B}$ is that one corresponding to $\mathcal{N}$.[2]
2. $e = \begin{cases} 1 & \text{if} \quad s < \mathcal{B} \\ 0 & \text{if} \quad s \geq \mathcal{B} \end{cases}$ is called the 'cumex' flag.

   The cumex flag is an indicator of the ownership of the coupon. $\mathcal{B}$ is the date (up to but NOT including) when whoever the owner was before $\mathcal{B}$ is the person who receives the coupon on $\mathcal{N}$. If you sell the bond on or after $\mathcal{B}$ you will still receive the coupon. Thus e indicates whether or not the bond is being sold with the next coupon.

[1] When using excel functionality, is is important to ensure that the input 'r153' is that string and not 153 South African Rands! Excel will display R 153 (note the space) but the cell is actually populated by the number 153 (an example of what you see is what you typed, but what you get isn't). This occurs if the currency of the computer has been set to ZAR (of course completely typical). The solution is simple: set the format of the cell where the bond code is to be input to be text: format - cells - number - text.

[2] As an example: for the r153, suppose s is 27 Feb 2004. Then $\mathcal{L}$ is 31 Aug 2003, $\mathcal{N}$ is 28 Feb 2004, and $\mathcal{B}$ is 31 January 2004. As another example, suppose s is 31 Aug 2004. Then $\mathcal{L}$ is 31 Aug 2004, $\mathcal{N}$ is 28 Feb 2005, and $\mathcal{B}$ is 18 Feb 2005. What about 29 Feb 2004?

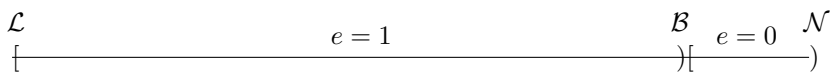


Figure 4.1: The LCD, BCD and NCD

3. Price all cash flows as if we were standing at $\mathcal{N}$.

   $d = (1 + \frac{y}{2})^{-1}$ is the semi-annual discount factor.

   $n = \left[\frac{\text{Maturity}-\mathcal{N}}{182.625}\right]$ is the number of coupons from the $\mathcal{N}$ until maturity, not including the one at $\mathcal{N}$.

   $$V_{\mathcal{N}} = \frac{c}{2}(e + d + d^2 + \cdots + d^n) + d^n = \frac{c}{2}\left(e + d\frac{1-d^n}{1-d}\right) + d^n$$

   where c is the annual coupon percentage.

4. Calculate the discount factor that takes us from $\mathcal{N}$ back to s.

   $$b = \begin{cases} \frac{\mathcal{N}-s}{\mathcal{N}-\mathcal{L}} & \text{if} \quad \mathcal{N} \neq \text{Maturity} \\ \frac{\mathcal{N}-s}{182.5} & \text{if} \quad \mathcal{N} = \text{Maturity} \end{cases}$$

   $$g = \begin{cases} d^b & \text{if} \quad \mathcal{N} \neq \text{Maturity} \\ \frac{d}{d+b(1-d)} & \text{if} \quad \mathcal{N} = \text{Maturity} \end{cases}$$

   are called the broken period and broken period factor respectively.

   Note that $\frac{d}{d+b(1-d)} = \left(1 + b\frac{y}{2}\right)^{-1}$.

   The reason we make distinction is in last period before maturity the bond becomes a money market instrument, and 365 days makes the pricing formula consistent with money market day count conventions.

5. The all-in price is

   $$\mathbb{A} = g \cdot V_{\mathcal{N}} \tag{4.1}$$

In reality, the market trades $\mathbb{A}$ at $t + 3$.[3] For example, we trade the r153on the 15-Feb-07. This is the valuation date. The settlement date is 20-Feb-07. Thus the bond trades ex-coupon ie. $e = 0$. $t + 3$ is known as the standard settlement date of t and is denoted $ssd(t)$.

4.2 Rounding

Traders often think in terms of clean price rather than all in prices. Roughly, the clean price is the price without the next coupon having an impact. How is this calculated?

1. The unrounded all-in Price $\mathbb{A}$ is calculated as above.

[3] In other words, 3 business days after the actual trade date. Thus one must take care of weekends and public holidays. Physical transfer of cash and bonds occur at BESA at this date $t + 3$.

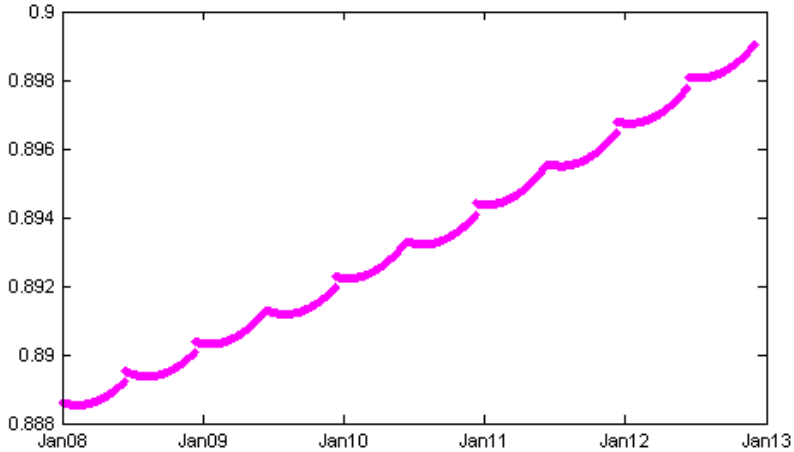


Figure 4.2: There are plenty of anomalies to the clean price formula

2. We calculate how many days $\mathcal{D}$ of accrued interest is included in the bond.

$$\mathcal{D} = \begin{cases} s - \mathcal{L} & \text{if} \quad e = 1 \\ s - \mathcal{N} & \text{if} \quad e = 0 \end{cases} \tag{4.2}$$

   Then

$$\text{Accrued Interest} = c\frac{\mathcal{D}}{365} \tag{4.3}$$

   Of course, there is an anomaly here - compare with the very precise day count methodology seen earlier.

3. The Clean Price is the All in Price minus the Accrued Interest.

4. Round the Clean Price and the Accrued Interest.

5. The Rounded All in Price is the sum of the Rounded Clean Price and the Rounded Accrued Interest.

This is often different to just rounding the All in Price! The rounded all in price will be denoted by $[\mathbb{A}]$.
Not only is the day count anomalous, but furthermore the accrual of interest is simple (linear). It would be better if interest accrued by compounding, as typically interest does.
See Etheredge and West [1999].

4.3 Risk Measures Associated with Bonds

There is an inverse relationship between yield and price.

Recall that by Taylor series

$$f(y+\Delta y)-f(y)=f'(y)\Delta y+\frac{1}{2}f''(y)(\Delta y)^2+\frac{1}{6}f'''(y)(\Delta y)^3+\cdots$$

and so

$$\Delta\mathbb{A}=\frac{\partial\mathbb{A}}{\partial y}\Delta y+\tfrac{1}{2}\frac{\partial^2\mathbb{A}}{\partial y^2}(\Delta y)^2+\frac{1}{6}\frac{\partial^3\mathbb{A}}{\partial y^3}(\Delta y)^3+\cdots \tag{4.4}$$

We have the following terms

$$\frac{\partial\mathbb{A}}{\partial y} \quad \text{called} \quad \text{Delta} \tag{4.5}$$

$$\frac{\partial^2\mathbb{A}}{\partial y^2} \quad \text{called} \quad \text{Gamma} \tag{4.6}$$

$$\frac{\partial^3\mathbb{A}}{\partial y^3} \quad \text{called} \quad \text{Speed} \tag{4.7}$$

$$-\frac{\frac{\partial\mathbb{A}}{\partial y}}{\mathbb{A}} \quad \text{called} \quad \text{Modified duration, in years} \tag{4.8}$$

$$-(1+\frac{y}{2})\frac{\frac{\partial\mathbb{A}}{\partial y}}{\mathbb{A}} \quad \text{called} \quad \text{Duration, in years} \tag{4.9}$$

$$\frac{\frac{\partial^2\mathbb{A}}{\partial y^2}}{\mathbb{A}} \quad \text{called} \quad \text{Convexity} \tag{4.10}$$

$\frac{\partial\mathbb{A}}{\partial y}$ is negative because there is an inverse relationship between yield and price. For a R1 bond, $\frac{\partial\mathbb{A}}{\partial y}$ is -3 say. If I have a R1 bond, and y increases by 1 basis point (1% of 1%), how much money do I make? $\triangle\mathbb{A}\approx\frac{\partial\mathbb{A}}{\partial y}\triangle y=-3\cdot 0.0001=-0.0003$ ie. I lose R0.0003.

By ratio and proportion, if I have R1,000,000 of the same bond and yields go up by 1 basis point, I lose R$0.0003\cdot 1,000,000=$ R300. This number is called the Rand per point of the bond. Thus the Rand per point is the profit you make when the yield goes down 1 basis point on a bond of R1m.

Now let us perform the required calculations:

Derivatives of g with respect to d, the semi-annual discount factor:

$$\frac{\partial g}{\partial d} = \begin{cases} bd^{b-1} & \text{if} \quad \mathcal{N}\neq m_B \\ \frac{b}{[d+b(1-d)]^2} & \text{if} \quad \mathcal{N}=m_B \end{cases} \tag{4.11}$$

$$\frac{\partial^2 g}{\partial d^2} = \begin{cases} b(b-1)d^{b-2} & \text{if} \quad \mathcal{N}\neq m_B \\ \frac{-2b(1-b)}{[d+b(1-d)]^3} & \text{if} \quad \mathcal{N}=m_B \end{cases} \tag{4.12}$$

$$\frac{\partial^3 g}{\partial d^3} = \begin{cases} b(b-1)(b-2)d^{b-3} & \text{if} \quad \mathcal{N}\neq m_B \\ \frac{6b(1-b)^2}{[d+b(1-d)]^4} & \text{if} \quad \mathcal{N}=m_B \end{cases} \tag{4.13}$$

Derivatives of $V_{\mathcal{N}}$, the total coupon amount, with respect to d, the semi-annual discount factor:

$$\frac{\partial V_{\mathcal{N}}}{\partial d} = \frac{c}{2}\,\frac{nd^{n+1} - (n+1)d^n + 1}{(d-1)^2} + nd^{n-1} \tag{4.14}$$

$$\frac{\partial^2 V_{\mathcal{N}}}{\partial d^2} = \frac{c}{2}\,\frac{n(n-1)d^{n+1} - 2(n^2-1)d^n + (n+1)nd^{n-1} - 2}{(d-1)^3} + n(n-1)d^{n-2} \tag{4.15}$$

$$\frac{\partial^3 V_{\mathcal{N}}}{\partial d^3} = \frac{c}{2}\left[\frac{-3}{(d-1)^4}[n(n-1)d^{n+1} - 2(n^2-1)d^n + (n+1)nd^{n-1} - 2] + \frac{(n+1)n(n-1)}{(d-1)^3}[d^n - 2d^{n-1} + d^{n-2}]\right] + n(n-1)(n-2)d^{n-3} \tag{4.16}$$

The formulae derived above are then used to calculate the derivatives of $\mathbb{A}$, the all-in-price.

$$\frac{\partial \mathbb{A}}{\partial d} = \frac{\partial g}{\partial d}V_{\mathcal{N}} + g\frac{\partial V_{\mathcal{N}}}{\partial d} \tag{4.17}$$

$$\frac{\partial^2 \mathbb{A}}{\partial d^2} = \frac{\partial^2 g}{\partial d^2}V_{\mathcal{N}} + 2\,\frac{\partial g}{\partial d}\frac{\partial V_{\mathcal{N}}}{\partial d} + g\frac{\partial^2 V_{\mathcal{N}}}{\partial d^2} \tag{4.18}$$

$$\frac{\partial^3 \mathbb{A}}{\partial d^3} = 3\,\frac{\partial g}{\partial d}\frac{\partial^2 V_{\mathcal{N}}}{\partial d^2} + 3\,\frac{\partial^2 g}{\partial d^2}\frac{\partial V_{\mathcal{N}}}{\partial d} + \frac{\partial^3 g}{\partial d^3}V_{\mathcal{N}} + g\frac{\partial^3 V_{\mathcal{N}}}{\partial d^3} \tag{4.19}$$

Note that:

$$\frac{\partial d}{\partial y} = \frac{-1}{(1+y/2)^2}\cdot\frac{1}{2} = -\frac{d^2}{2} \tag{4.20}$$

The derivatives of the bond all-in-price with respect to yield, y, give the bond delta, gamma and speed:

$$\Delta = \frac{\partial \mathbb{A}}{\partial y} = -\frac{d^2}{2}\frac{\partial \mathbb{A}}{\partial d} \tag{4.21}$$

$$\Gamma = \frac{\partial^2 \mathbb{A}}{\partial y^2} = \frac{d^3}{2}\frac{\partial \mathbb{A}}{\partial d} + \frac{d^4}{4}\frac{\partial^2 \mathbb{A}}{\partial d^2} \tag{4.22}$$

$$\text{Speed} = \frac{\partial^3 \mathbb{A}}{\partial y^3} = -\frac{3d^4}{4}\frac{\partial \mathbb{A}}{\partial d} - \frac{3d^5}{4}\frac{\partial^2 \mathbb{A}}{\partial d^2} - \frac{d^6}{8}\frac{\partial^3 \mathbb{A}}{\partial d^3} \tag{4.23}$$

4.4 Exercises

1. Relevant Bond data for a selection of SA bonds is provided. Given Inputs:

   - Bond Name
   - Settlement Date
   - YTM

   build a Bond Calculator to:

   (a) Bring in (given the above inputs) all the relevant static data for that particular bond - such as maturity, coupon dates etc. (This will be done by means of an excel vlookup table.)

   (b) Calculate:

- The LCD, BCD and NCD as efficiently as possible. A good trick is to break every date you consider into a cell containing the year, a cell containing the month, and a cell containing the day. The logic can then revolve around manipulation with these numbers: manipulation with numbers is so much easier than manipulation with dates. To recover the corresponding date, you use the DATE function.
- The Bond Allin Price, both rounded and unrounded.
- The Clean Price, both rounded and unrounded.
- The Accrued interest amount, both rounded and unrounded.
- the delta and rand per point;
- the duration and modified duration;
- the gamma and convexity.

This must be done entirely in excel. No macros, no vba.

2. Suppose V is the value of a set of cash flows (for example, a bond) which has payments $c_1,\ c_2,\ \ldots,\ c_n$ at times $t_1,\ t_2,\ \ldots,\ t_n$. Note that $V = \sum_i c_i Z(0,t_i) := \sum_i V_i$ where $Z(\cdot)$ is for example calculated off of a yield curve. If the bond has a NACS yield to maturity y, then we take $Z(0,t_i) = (1+\frac{y}{2})^{-2t_i}$.

   Let $w_i = \frac{V_i}{V}$ be the proportion that the ith component makes up of the total value of the bond. We define (Macauley) duration and modified duration as

$$D = \sum_i t_i w_i = \frac{\sum_i t_i V_i}{\sum_i V_i}$$
$$D_m = \frac{D}{1+\frac{y}{2}}$$

   Note that the units of Macauley duration and modified duration are years.

   Show that these quantities can in fact be calculated as in the notes, as functions of the delta of the bond.

Chapter 5

Bootstrap of a South African yield curve

There is a need to value all instruments consistently within a single valuation framework. For this we need a risk free yield curve which will be a NACC zero curve (because this is the standard format, for all option pricing formulae).

Thus, a yield curve is a function $r = r(\tau)$, where a single payment investment for time t will earn a rate $r = r(\tau)$. We create the curve using bootstrapping. We do this using raw data and certain rules.

Such curves have been easily observed in the USA and Europe (in the USA via prices of T-bills and T-bonds which have traded out to a 30 year maturity). With the erosion of liquidity of even these instruments the use of swaps is becoming more common.

5.1 The overnight capitalisation factor

How does the call rate work? In many international markets there exists the possibility for a depositor to place cash with a bank, and obtain the capital back with interest the next day. We easily get the overnight capitalisation factor, and hence we could calculate the overnight NACC rate on our yield curve.

There is in fact no true overnight (call) rate in the South African market. In most cases, the interest is calculated every day, but the interest is only paid at the end of every month. (It is however possible to request payment of the interest when funds are withdrawn during the month. But this is a once off event.)

To create a yield curve, it is however necessary to include a one-day rate (for example, for pricing of short dated equity derivatives). A number of approaches are possible; all of them are decrees rather than fact.

What occurs in reality is different to a true overnight deposit. An amount of 1 is deposited, 1 is returned the following business day WITHOUT interest, and the interest that accrues at the so-called overnight rate is only paid on the last calendar day of the month. In particular, there is no

compounding of interest. Usually, the depositor leaves its money on deposit for the whole month, receiving their capital back at the end and the arithmetic sum (no compounding) of the interest payments.

It is clear that (for a given overnight rate) the actual capitalisation that the depositor is earning improves during the month, as the payment of interest is occurring relatively earlier. However, as pointed out in Bond Exchange of South Africa/Actuarial Society of South Africa [2003], there is no evidence that this factor is taken into consideration.

One approach to calculating the overnight capitalisation factor is:

$$C(0, 1d) = \left(1 + \frac{r}{12}\right)^{12/365} \tag{5.1}$$

Of course, this is equivalent to

$$d = Z(0, 1d) = \left(1 + \frac{r}{12}\right)^{-12/365} \tag{5.2}$$

Another approach might be

$$C(0, 1d) = 1 + r\frac{1}{365} \tag{5.3}$$

that is, the overnight rate is modelled as a true overnight rate.

Of course these approaches are plausible art, not science - there is no arbitrage reason why this is the overnight compounding factor. There are other solutions to this problem. Some institutions, for example, calculate the one month rate, and then simply extrapolate horizontally from there back to the 1 day term node. A solution is required, but there is no point in claiming that any given solution is 'correct': the overnight compounding factor does not actually exist.

5.2 Bootstrap the money market part of the curve

We will consider one bootstrap of a NACC yield curve via an example. Suppose on 17-Jun-08, we have the inputs in Table 5.1.

Bond Curve Inputs			
Data	Type	Rate type	Input rate
1 day	on	simple yield	12.000%
1 Month	JIBAR	simple yield	12.021%
3 Month	JIBAR	simple yield	12.229%
3x6	fra	simple yield	12.670%
6x9	fra	simple yield	12.760%
9x12	fra	simple yield	12.730%
12x15	fra	simple yield	12.700%
15x18	fra	simple yield	12.590%

Table 5.1: Money market curve data for 17-Jun-08

The ON rate is a one day rate, regardless of whether or not the next day is a business day. For the JIBAR and FRA rates we take into account the Modified Following rule, thus, the ON, JIBAR and FRA terms are (in days) 1, 30, 92, 183, 273, 365, 457, and 548.

- We will use (5.3) to calculate the overnight rate. $C(0,1d) = 1 + \frac{12.000\%}{365} = 1.0003288$, so $r_{1d} = 365 \ \ln C(0,1d) = 11.9980\%$.
- $C(0,30d) = \left(1 + 12.021\% \frac{30}{365}\right) = 1.0098803$, so $r_{30d} = \frac{365}{30} \ \ln C(0,30d) = 11.9620\%$
- $C(0,92d) = \left(1 + 12.229\% \frac{92}{365}\right) = 1.0308238$, so $r_{92d} = \frac{365}{92} \ \ln C(0,92d) = 12.0443\%$

From the above listed observation dates, the fra periods are of length 91, 90, 92, 92, 91 days. We now use (1.12) over and over. Thus

- $C(0,183d) = C(0,92d) \cdot \left[1 + 12.670\% \frac{91}{365}\right]$ and so $r_{183d} = 12.2580\%$
- $C(0,273d) = C(0,183d) \cdot \left[1 + 12.760\% \frac{90}{365}\right]$ and so $r_{273d} = 12.3587\%$
- $C(0,365d) = C(0,273d) \cdot \left[1 + 12.730\% \frac{92}{365}\right]$ and so $r_{365d} = 12.4019\%$
- $C(0,457d) = C(0,365d) \cdot \left[1 + 12.700\% \frac{92}{365}\right]$ and so $r_{457d} = 12.4218\%$
- $C(0,548d) = C(0,457d) \cdot \left[1 + 12.590\% \frac{91}{365}\right]$ and so $r_{548d} = 12.4176\%$

5.3 Bootstrap using default-free bonds

In South Africa the traditional approach has been to use JIBAR's in the short part of the curve, FRA's in the medium part of the curve and r bonds in the long part of the curve. What is the major difference between JIBAR's and Bonds in terms of credit worthiness? In the JIBAR market, if you lend money you may not get it back, but in the bond market you will get it back. Thus different levels of credit worthiness are used to construct the curve, although this is forced by the lack of liquid and perfect credits in the short and medium term.

Thus we are looking at a curve which is not default free (and does not fit in with typical theoretical assumptions in options pricing) because there are not government bonds on the left (short) side of the curve. This will not occur in the US - you can construct an entire yield curve just using treasury bills. In SA there are no liquid and short-dated government bonds.

Suppose we also have the input data in Table 5.2. Now for the hard step. We only extend the curve

Bond Curve Inputs			
Data	Type	Rate type	Input rate
r153	Bond	ytm	11.590%
r201	Bond	ytm	10.530%
r157	Bond	ytm	10.460%
r203	Bond	ytm	10.450%
r204	Bond	ytm	10.420%
r186	Bond	ytm	10.270%

Table 5.2: Bond curve data for 17-Jun-08

to 31-Aug-10, which is the maturity date of the r153 (at which point we will stop). For the r153,

$[\mathbb{A}] = 1.0659167$ for settlement 20-Jun-08. We will consider two different ways of realising this value:

$$[\mathbb{A}] \exp[-r_\alpha \alpha] = \sum_{i=0}^{N} p_i \exp[-r_{\tau_i} \tau_i] \tag{5.4}$$

where

$$\begin{aligned}
\alpha &= \text{3 business day term, measured in years} \\
p_0 &= e\frac{c}{2} \\
p_1,\ p_2,\ \ldots,\ p_{N-1} &= \frac{c}{2} \\
p_N &= 1 + \frac{c}{2}\text{[1]} \\
N &= \text{The } N \text{ in the bond pricing formula} \\
\tau_i &= \text{term in years until the coupon flow } i\text{[2]}
\end{aligned}$$

In this example, $\alpha = 3d$. Note that this is the critical equation of value: whether we realise the value in the bond market, or we strip the cash flows one at a time, we must (in principle) get the same value.

We will use linear interpolation to find the rate at any point which is between known rates[3]. In other words, we assume that such rates are known.

Using this interpolation, the 3d rate is 11.9955%. Thus, the left hand side of (5.4) for the r153 is 1.06486629. The cash flows on the right hand side are as in Table 5.3.

date	cash flow	rate	pv
01-Sep-08	0.0650000	12.0231%	0.063393
28-Feb-09	0.0650000	12.3396%	0.059611
31-Aug-09	0.0650000	12.4181%	0.055963
17-Dec-09	0.0000000	12.4176%	0.000000
01-Mar-10	0.0650000	??	??
31-Aug-10	1.0650000	??	??

Table 5.3: Cash flows for r153, iteration 0

The one equation has two unknowns. What we now do is impose a model - we make r_N the subject of the equation. Hence

$$r_{\tau_N} = -\frac{1}{\tau_N}\left[\ln\left(\mathbb{A}\exp[-r_\alpha\alpha] - \sum_{i=0}^{N-1} p_i \exp[-r_{\tau_i}\tau_i]\right) - \ln p_N\right] \tag{5.5}$$

[2]In the case that $N = 0$ we have $p_N = 1 + e\frac{c}{2}$ and the equation that will follow solves immediately, without iteration.

[2]Note that this is found using the regular following date of the the designated coupon date.

[3] If $a < x < b$, and we know $f(a)$, $f(b)$, then $f(x) = \frac{x-a}{b-a}f(b) + \frac{b-x}{b-a}f(a)$. In real applications of this approach, a much more sophisticated interpolation scheme would be used.

Now, this equation is somewhat self-referential - it is an under-specified system - because the r_{τ_i} are previously estimated by interpolation, which involves the given estimate of r_{τ_N}. This observation, however, sets up the iterative procedure, which may coincide with the simultaneous bootstrap method suggested in Smit and van Niekerk [1997].

We guess a rate for 31-Aug-10 and then apply linear interpolation on the interval [17-Dec-09, 31-Aug-10] using the known 17-Dec-09 rate and the guessed 31-Aug-10 rate at the endpoints. The seed for the 31-Aug-10 rate will for example be the NACC equivalent of the r153's ytm, a NACS rate, namely $2\ln\left(1+\frac{y}{2}\right) = 11.2666\%$.

We can then value all the cash flows using the interpolation method chosen, as in Table 5.4.

date	cash flow	rate	pv
01-Sep-08	0.0650000	12.0231%	0.063393
28-Feb-09	0.0650000	12.3396%	0.059611
31-Aug-09	0.0650000	12.4181%	0.055963
17-Dec-09	0.0000000	12.4176%	0.000000
01-Mar-10	0.0650000	12.0862%	0.052901
31-Aug-10	1.0650000	11.2666%	0.830682

Table 5.4: Cash flows for r153, iteration 1

Using (5.5) we get a new estimate of the 31-Aug-10 rate, namely 11.1404%. We then apply the same process:

date	cash flow	rate	pv
01-Sep-08	0.0650000	12.0231%	0.063393
28-Feb-09	0.0650000	12.3396%	0.059611
31-Aug-09	0.0650000	12.4181%	0.055963
17-Dec-09	0.0000000	12.4176%	0.000000
01-Mar-10	0.0650000	12.0498%	0.052934
31-Aug-10	1.0650000	11.1404%	0.832998

Table 5.5: Cash flows for r153, iteration 2

The new estimate of the 31-Aug-10 rate is 11.1422%. We iterate to convergence. Convergence to 4 decimal places - in this case 11.1421% - occurs after about four iterations.

5.4 Bootstrap using swap rates

Suppose we also have the input data in Table 5.6.

At inception, a swap is dealt at a rate $R = R_n$ which makes the value of the swap above 0. R_n is then called the fair swap rate. Then

$$R_n \sum_{i=1}^{n} \alpha_i Z(t, t_i) \quad = \quad 1 - Z(t, t_n) \tag{5.6}$$

Data	Type	Input rate
swap	2y	11.930%
swap	3y	11.480%
swap	4y	11.140%
swap	5y	10.870%
swap	6y	10.660%
swap	7y	10.490%
swap	8y	10.350%
swap	9y	10.230%
swap	10y	10.140%
swap	15y	9.780%
swap	20y	9.540%
swap	25y	9.340%

Table 5.6: Swap curve data for 17-Jun-08

The rates quoted in the market are, by virtue of the forces of supply and demand, deemed to be the fair rates for the following purposes: we can inductively suppose that $Z(t, t_j)$ is known for $j = 1,\ 2,\ \ldots,\ n-1$, and R_n is known, to get

$$Z(t, t_n) = \frac{1 - R_n \sum_{j=1}^{n-1} \alpha_j Z(t, t_j)}{1 + R_n \alpha_n} \tag{5.7}$$

Now $Z(t, t_j)$ is known for small j from the money market. Swap rates are quoted up to 30 years. The above formula can be used to bootstrap the curve out for that term.

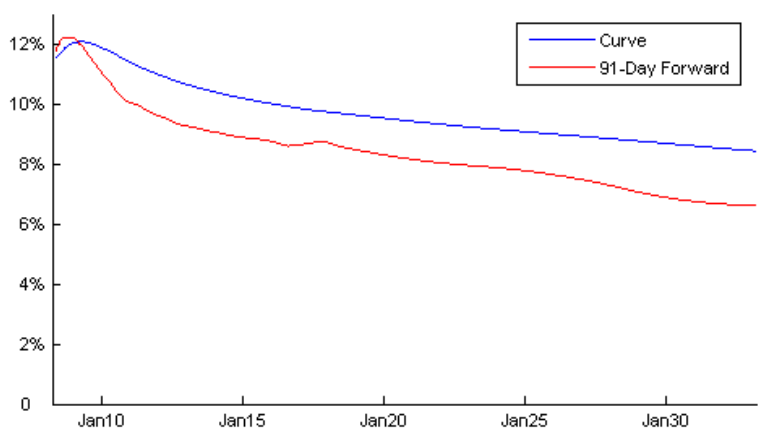


Figure 5.1: The NACC yield curve derived from swap data on 14-May-08

5.5 What to do about holes in the term structure?

What happens if it is not the case that swaps for all expiries are quoted? This is what happens in reality, certainly in the middle and longer part of the curve, where typically only annual expiries are quoted, but even in the shorter part of the curve as well. We can rewrite (5.7) as

$$r_n = \frac{-1}{\tau_n} \ln \left[\frac{1 - R_n \sum_{j=1}^{n-1} \alpha_j Z(t, t_j)}{1 + R_n \alpha_n} \right] \tag{5.8}$$

and this gives us a very useful iterative formula (the same sort of thing happens with bond curves): we guess initial rates r_n for each of the quoted expiries, perform interpolation using our chosen method of the yield curve itself to determine any missing r_j, and hence any $Z(t, t_j)$, and use this formula to extract new estimates of the r_n's. The initial guess might for example be the continuous equivalent of the input swap rate, but in reality, any guess will suffice. We then iterate; convergence is fast over the entire yield curve.

This algorithm was introduced in Hagan and West [2006], Hagan and West [2008].

5.6 Exercises

1. Suppose on 3-Mar-08 I have the JIBAR and FRA data given in the table.

jibar	fra	fra	fra	fra	fra	fra	fra
3m	3v6	6v9	9v12	12v15	15v18	18v21	21v24
11.33%	11.43%	11.41%	11.29%	11.04%	10.74%	10.74%	10.74%

   (a) Bootstrap a the rates for every 3 month interval out to 24 months.

   (b) What is the fair swap rate for a 24 month swap?

   (c) Suppose we decide to use raw interpolation to find rates at dates which are not are not node points on my yield curve. Write a vba function to find the NACC rate at any non-node point.

   (d) Suppose on 12-Feb-08 the 3m JIBAR rate was 11.20% and the 6v9 FRA rate was 11.40%. I entered into a pay fixed 6v9 FRA. What is the MtM of the FRA now (on 3-Mar-08)?

   (e) Suppose on 12-Feb-08 I entered into a 1 year swap, paying a fixed rate of 11.30%. What is the MtM of the swap now (on 3-Mar-08)?

2. Suppose on 3-Apr-08 my yield curve is 12.00% NACC for a term of zero, and increases by one-tenth of a basis point every calendar day into the future (relevant for this question).

   - Find the fair rate for vanilla swaps with expiry at 6m, 9m, 12m, etc. to 20y.
   - What can you observe about the trend in the fair swap rates? What is the reason for this?
   - If I am paying fixed, receiving floating, in what periods do I expect to be receiving payments and in what periods do I expect to make payments?

3. (UCT exam 2004) Suppose we are given the inputs to the swap curve as follows:

Date	31-Mar-05	
Swap Curve Inputs		
Data	Type	Rate
1 day	on	7.6400%
1 month	jibar	7.2210%
3 month	jibar	7.4040%
3x6	fra	7.2500%
6x9	fra	7.2900%
9x12	fra	7.4000%
12x15	fra	7.8500%
15x18	fra	8.1900%
2y	swap	7.8300%
3y	swap	8.3200%
4y	swap	8.6200%
5y	swap	8.8700%

Bootstrap the yield curve out to 5 years using the method discussed in class. Report the rates for every 3 month point following the valuation date out for 5 years.

To determine these three month points, use the modfol function provided. Use raw interpolation on the NACC rates, for which it is recommended that you write your own linear interpolation function in vba.

4. (UCT exam 2008) Suppose a $T_1 \times T_2$ FRA is traded today date t at rate F.

Suppose I have a complete yield curve, that is, I can borrow or lend zero coupon bonds for any maturity. The mark to market borrowing price for date T is denoted $Z(t,T)$. There are bid and offer curves: $Z^b(t,T)$ and $Z^o(t,T)$. For avoidance of doubt:

- $Z^b(t,T)$ is the bid curve: there are participants in the market willing to pay $Z^b(t,T)$ at date t and receive 1 at date T.
- $Z(t,T)$ is the MtM price on date t of a payment of 1 on date T.
- $Z^o(t,T)$ is the offer curve: there are participants in the market willing to receive $Z^o(t,T)$ at date t and pay 1 at date T.

These quantities exist for every date T.

(a) What is a FRA?

(b) Write down a formula for today's MtM value V of the pay fixed, receive floating side of the above FRA.

(c) What is the no-arbitrage range for the FRA rate F? (It will be a function of $Z^b(t,T_1)$, $Z^o(t,T_1)$, $Z^b(t,T_2)$ and $Z^o(t,T_2)$.) Prove your assertions.

You may assume that FRAs are settled in arrears, and that at time T_1 one can either pay or receive the then-ruling JIBAR rate J (that is, there is no spread on J).

Chapter 6

The Johannesburg Stock Exchange equities market

The JSE has four electronic trading and settlement platforms:

- equities, which we consider in this chapter, along with some general equity issues.
- financial futures and agricultural products, run by SAFEX, which is a wholly owned subsidiary of the JSE. We look at SAFEX in Chapter 10.
- Yield-X, an interest rate futures market.

The JSE is the world's 15th largest exchange (on a value traded basis).

6.1 Some commonly occurring acronyms

- STRATE: Share Transactions Totally Electronic. An electronic settlement system for the South African equities market.
- MST: Marketable Securities Tax, a tax payable on the purchase of shares in the market in terms of the Income Tax Act, 1992, currently 0.25% of market price.
- SETS: the trading system of the partnership of the JSE with the London Stock Exchange. SETS may offer a number of potential strategic benefits, including remote membership and primary dual listing opportunities, the possibility of which should decrease the exodus of blue chip South African companies to foreign exchanges. The LSE and the JSE have also put in place a separate business agreement covering dual listing for issuers, remote access for JSE and Exchange member firms, and the marketing and sale of each other's market information.

  The government shut the door after five of the JSE's largest companies moved to London as a prerequisite for inclusion in the FTSE 100 index. Dual primary listings could reopen this door without exposing the country to capital flight.

- SENS: The JSE Securities Exchange South Africa News Service was established with the aim of facilitating early, equal and wide dissemination of relevant company information, and improving communication between companies and the market. It is a real time news service for the dissemination of company announcements and price sensitive information. The company must submit all relevant company and price sensitive information to SENS as soon as possible after authorisation. An announcement must be sent to SENS before it's published in the press.

6.2 Indices on the JSE

Indices are needed as an indication of the level of the market. The JSE All-Share index comprises 100% of JSE shares, and has existed since 1995. Prior to that the indices comprised the top 80% by market capitalisation and it was these indices that the futures market was based. This index comprised about 140 of the 620 shares listed on the JSE.

In 1995 there was the realisation that indices for futures need to comprise a small number of shares which have high market capitalisation and are highly traded. This permits hedging and arbitrage operations through trades in baskets of shares. Nevertheless, arbitrage trades will rarely include all the relevant shares and so there will be some residual correlation risk (tracking error).

The JSE, in collaboration with the SA Actuarial Society, determine a number of indices which are used as barometers of the market (or sectors thereof). The most popular of these are:

- ALSI (All share index). This consists of all shares on the JSE bar about 100, these being, for example, pyramids or debentures.
- TOPI (Top 40 listed companies index). Until June 2002 called the ALSI40.
- INDI25 (Top 25 listed industrial companies index)
- FINI15 (Top 15 listed financial companies index)
- RESI20 (Top 20 listed resources companies index)

The primary derivatives indices are the TOPI, INDI25. There was/is also the GLDI10, RESI20, FINI15, FINDI30. There are no futures contracts based on the ALSI.

The indices are revised quarterly to synchronise with the futures closeouts, being on the third Thursday of March, June, September and December. If this is a public holiday we go to the previous business day. The criteria for inclusion/exclusion are market capitalisation (measured in rands) and the daily average value traded (measured in rands). These two criteria are equally important. Shares are listed according to their dual rank, which is the larger of the two ranks, and then selected in order. Ties are broken by higher market capitalisation.

How is the index constructed? The index has some nominated but arbitrary starting date and value I_0. For example, the ALSI40 started on 19 June 1995 at 2000, the INDI25 on the same date at 3000. There is an initial k-factor given by

$$k_0 = I_0 \frac{\sum_{i=1}^{n} W_i}{\sum_{i=1}^{n} P_i W_i} \tag{6.1}$$

where the index consists of the n shares, the prices are $P_1,\ P_2,\ \ldots,\ P_n$ and the number of eligible shares[1] are $W_1,\ W_2,\ \ldots,\ W_n$. Now

$$I_t = k_t \frac{\sum_{i=1}^{n} P_{t,i} W_{t,i}}{\sum_{i=1}^{n} W_{t,i}} \tag{6.2}$$

The ratio $\frac{\sum_{i=1}^{n} P_{t,i} W_{t,i}}{\sum_{i=1}^{n} W_{t,i}}$ is the average price of the shares making up the index. We adjust by the k-factor to make the index level manageable.

Until now, k_t for $t > 0$ is undefined. However, such a factor cannot be omitted. Occasionally certain changes occur to certain shares which should not affect the index. For example, when there is a share split, constituent reselection, or if a share is delisted or suspended the index should evolve continuously. Essentially,

$$k_{t^-} \frac{\sum_{i \in I_{t^-}} P_{t^-,i} W_{t^-,i}}{\sum_{i \in I_{t^-}} W_{t^-,i}} = k_{t^+} \frac{\sum_{i \in I_{t^+}} P_{t^+,i} W_{t^+,i}}{\sum_{i \in I_{t^+}} W_{t^+,i}}. \tag{6.3}$$

This ensures that at the instant of the event, the level of the index is unchanged.

[1] At the June 2002 closeout, this number of eligible shares changed from shares in issue, to free float shares in issue. These are the shares which trade freely in the market, and are not held in family ownership, etc. And that closeout was made a day late, on the Friday.

	Day 1		Day 2		Between Days 2 and 3		Day 3 closing		Between Days 3 and 4		Day 4 closing	
	k value	51.9286834	k value	51.9286834	k value	57.3144941	k value	57.3144941	k value	51.7378892	k value	51.7378892
	Price	ffSISS	Price	ffSISS	Price	ffSISS	Price	ffSISS	Price	ffSISS	Price	ffSISS
AAA	23.00	4,000,000	24.00	4,000,000	24.00	4,000,000	24.00	4,000,000	24.00	4,000,000	24.00	4,000,000
BBB	1.20	3,542,564	1.20	3,542,564	1.20	3,542,564	1.20	3,542,564	1.20	3,542,564	1.20	3,542,564
CCC	0.55	10,000,500	0.55	10,000,500	0.55	10,000,500	0.55	10,000,500	0.55	10,000,500	0.55	10,000,500
DDD	300.00	555,666	310.00	555,666	31.00	5,556,660	34.00	5,556,660	34.00	5,556,660	34.00	5,556,660
EEE	120.00	4,040,400	120.00	4,040,400	120.00	4,040,400	118.00	4,040,400	118.00	4,040,400	118.00	4,040,400
FFF	12.55	10,000,255	13.00	10,000,255	13.00	10,000,255	13.00	10,000,255	13.00	10,000,255	13.00	10,000,255
GGG	4.40	1,078,989	4.50	1,078,989	4.50	1,078,989	4.50	1,078,989	4.50	1,078,989	4.50	1,078,989
HHH	3.00	15,000,000	2.50	15,000,000	2.50	15,000,000	2.50	15,000,000				
III									4.10	10,000,000	4.20	10,000,000
den		48,218,374		48,218,374		53,219,368		53,219,368		48,219,368		48,219,368
num		**928,549,904**		**935,214,577**		**935,214,577**		**943,803,757**		**947,303,757**		**948,303,757**
Avg.	19.26		19.40		17.57		17.73		19.65		19.67	
Index	1000		1007.18		1007.18		1016.43		1016.43		1017.50	

6.3 Performance measures

Definition 6.3.1. The simple return on a financial instrument $P_.$ is $R_t = \frac{P_t - P_{t-1}}{P_{t-1}}$.

This definition has a number of caveats:

- The time from $t-1$ to t is one business day. Thus it is the daily return. We could also be interested in monthly returns, annual returns, etc.
- P_t is a price. Sometimes a conversion needs to be made to the raw data in order to achieve this. For example, if we start with bond yields y_t, it doesn't make much sense to focus on the return as formulated above. Why?
- We need to worry about other income sources, such as dividends, coupons, etc.
- The continuous return is $\ln \frac{P_t}{P_{t-1}}$. This has better mathematical and modelling properties than the simple return above. For example, it is what occurs in all financial modelling and (hence) is what we use for calibrating a historical volatility calculator.

Let

$$w_{t,i} = \frac{P_{t,i} W_{t,i}}{\sum_{i \in I} P_{t,i} W_{t,i}} \tag{6.4}$$

be the proportion of the wealth of the market at time t that is comprised by stock i.

For any one stock, this proportion will usually be small. A very big share is about 10 - 15%. (In Finland, Nokia makes up 50 - 55% of the index which makes it very volatile). Fund managers have to track indices, in order to do this, they need to own the shares in the correct proportions (which is why, for example, the demand for a South African share goes up before listing on a foreign exchange such as the FTSE - because FTSE index trackers need the share in order to track the index).

In order to calculate performance measures of the index as a function of the performance measures of the stocks, $w_.$ becomes critical. Let us suppose (as is typical for most days) that between $t-1$ and t there were no corporate actions or other index rebalancing. Then

$$\begin{aligned}
R_{t,I} &= \frac{k_t \frac{\sum_{i=1}^n W_i P_{t,i}}{\sum_{i=1}^n W_i} - k_{t-1} \frac{\sum_{i=1}^n W_i P_{t-1,i}}{\sum_{i=1}^n W_i}}{k_{t-1} \frac{\sum_{i=1}^n W_i P_{t-1,i}}{\sum_{i=1}^n W_i}} \\
&= \frac{\sum_{i=1}^n W_i P_{t,i} - \sum_{i=1}^n W_i P_{t-1,i}}{\sum_{j=1}^n W_j P_{t-1,j}} \\
&= \frac{\sum_{i=1}^n W_i (P_{t,i} - P_{t-1,i})}{\sum_{j=1}^n W_j P_{t-1,j}} \\
&= \frac{\sum_{i=1}^n W_i P_{t-1,i} \frac{P_{t,i} - P_{t-1,i}}{P_{t-1,i}}}{\sum_{j=1}^n W_j P_{t-1,j}} \\
&= \frac{\sum_{i=1}^n W_i P_{t-1,i} R_{t,i}}{\sum_{j=1}^n W_j P_{t-1,j}} \\
&= \sum_{i=1}^n \left[\frac{W_i P_{t-1,i}}{\sum_{j=1}^n W_j P_{t-1,j}} \right] R_{t,i} \\
&= \sum_{i=1}^n w_{t-1,i} R_{t,i}
\end{aligned}$$

These are simple daily returns. A similar calculation involving continuous returns would not work because the log of a sum is not the sum of the logs. Nevertheless, volatility calculations are usually made using this assumption, which is valid to first order. See [J.P.Morgan and Reuters, December 18, 1996, TD4ePt_2.pdf, §4.1] for additional information.

6.4 Forwards, forwards and futures

A forward is a legally binding agreement for the long party to buy an equity from the short party at a certain subsequent date at a price (the delivery price) that is agreed upon today.

Theorem 6.4.1. Suppose today is date t and the future date is date T. Suppose that a forward on an equity, which is not going to receive any dividends, is struck with a strike of K. Then, in the absence of arbitrage,

$$K = S_t C(t, T) \tag{6.5}$$

where time is measured in years.

Proof: suppose $K < S_t C(t, T)$. Then we go long the forward, borrow the stock and sell it, investing the proceeds. At time T we close the bank account, for an inflow of $S_t C(t, T)$, use K to buy the stock, and use this stock to close out the stock borrowing. There is an arbitrage profit of $S_t C(t, T) - K$.
Suppose $K > S_t C(t, T)$. Then we go short the forward, borrow S_t cash and buy the stock. At time T we deliver the stock for K and use $S_t C(t, T)$ to close the borrowing. There is an arbitrage profit of $K - S_t C(t, T)$. $\diamond$
This delivery value is known as the fair forward level. We will denote it $f_{t,T}$, or just f, if the dates are clear.
If we deal such a forward, it might have value 0 at inception, but the chances of it having value 0 at any time during its life again are small.

Theorem 6.4.2. Suppose today is date t and the future date is date T. Suppose that a forward on an equity, which is not going to receive any dividends, is struck (has already previously been dealt) with a strike of K. Then the fair valuation of the forward is

$$V = S_t - Z(t, T)K \tag{6.6}$$

where time is measured in years.

Proof: I am long the forward. I go short (possibly with another counterparty) a fair forward with a strike of $S_t C(t, T)$, from the above theorem, this has no cost. Thus, the flow of equity will cancel, but I will pay K cash in the first deal and recieve $S_t C(t, T)$ in the second deal. Hence the excess value of this is

$$Z(t, T)[S_t C(t, T) - K]$$

and we are done. $\diamond$
The arbitrage formula for an index is the same. How do you sell an index? You sell each share in proportion. That is why an index subject to arbitrage operations must comprise of a few liquid shares.

A forward is a deal struck between two legal entities in the open market, known as an Over The Counter deal (OTC). A future is something completely different. A future is a type of bet that is traded on an exchange, in South Africa on the South African Futures Exchange (SAFEX), which is now a branch of the JSE. Futures will be denoted $F_{t,T}$ or just F, if the dates are clear. $F_{t,T}$ is in some sense the market consensus or equilibrium view at time t of what the value of the stock price will be at time T i.e. the value $S(T)$ (of course unknown at time t). If a speculator believes that the value of $S(T)$ will in fact be higher than $F_{t,T}$ (in other words, he believes that the market is underestimating the forthcoming performance of the stock) then he will go long the future. If a speculator believes the value will be lower (in other words, the market is overestimating the forthcoming performance of the stock) then he will go short the future. This is because as the consensus value F goes up the long party makes money and the short party loses money, and as it goes down the short party makes money and the long party loses money. The consensus value is driven by the forces of supply and demand; going long or short is mechanistically exactly a buy/sell, so F truely is a consensus or equilibrium value.

In a sense which is not mathematically very solid, both f and F are predictions of where the market should be. Under certain assumptions, for example if interest rates are non-stochastic, $f = F$. This was first proved in Cox et al. [1981], see also [Hull, 2002, Appendix 3A]. Of course, the assumption that interest rates are non-stochastic is not practicable, but nevertheless, the not unreasonable assumption is usually made that $f \approx F$. A common strategy is the 'spot-futures arbitrage': to be long/short futures and short/long spot according to the above theorem, as if futures and forwards were the same. This strategy is not an arbitrage, it is a value play! It can be shown that this strategy can lead to a loss.

6.5 Dividends and dividend yields

Dividends are income paid by the company to shareholders. Companies pay dividends because they have to share profits with shareholders, otherwise the shares would not have long term value. However, some companies offer value to shareholders by promising growth in the share price (historically MicroSoft, and more recently, most .com enterprises); nevertheless the value of the company must eventually be realised by dividends or other income. In fact, one very ancient theory is that the value of the stock is equal to the sum of the present value of the dividends and other income, which is called the Rule of Present Worth, and was first formulated in Williams [1938].

If there are dividends whose value at time t is $Q[t,T)$,[2] then

$$f_{t,T} = (S_t - Q[t,T))C(t,T) \tag{6.7}$$

To prove this, we borrow against the value of the dividends. If there is a continuous dividend yield $q = q[t,T)$, then

$$f_{t,T} = S_t e^{-q(T-t)} C(t,T) \tag{6.8}$$

which is known as the Merton model for forwards. To prove this, arrange that dividends are instantly reinvested in the stock as they are paid. See [Hull, 2002, Chapter 3]. Last, but not least

[2] This notation is appropriate, as shares go ex-dividend at close of business. Thus, the dividends include those at time t and exclude those at time T.

Theorem 6.5.1. Suppose today is date t and the future date is date T. Suppose that a forward on an equity, which is going to receive dividends, is struck (has already previously been dealt) with a strike of K. Suppose the present value of all the dividends to be received in the period is $Q[t,T)$. Then the fair valuation of the forward is

$$V = S_t - Q[t,T) - Z(t,T)K \tag{6.9}$$

where time is measured in years.

Proof: Experience (that is, an exam) has shown that the understanding of how to construct proofs of results like this is quite miserable. So, we give a proof in *every detail*.

Suppose the forward is trading at a price of $P < S_t - Q[t,T) - Z(t,T)K$. We construct an arbitrage. The forward is cheap, so I go long the forward for a cost of P. I borrow stock and sell it for an income of S_t. Suppose there are dividends D_i at various times $t_i \in [t,T)$. I deposit $D_i Z(t,t_i)$ for maturity date t_i; so in total I deposit $Q[t,T) = \sum_i D_i Z(t,t_i)$. Finally, I deposit $Z(t,T)K$ for maturity date T.

Income has been S_t and costs have been $P + Q[t,T) + Z(t,T)K$, so I am in net profit of $S_t - P - Q[t,T) - Z(t,T)K > 0$.

As each dividend becomes due, my deposits $D_i Z(t,t_i)$ mature with value D_i and I use these to make good the dividends with the party that I borrowed the stock from. At time T my deposit $Z(t,T)K$ matures with value K; I use this with the long forward to buy the stock. This stock I return to the lender.

Thus, besides the initial profit, there have been no net flows. This is an arbitrage profit.

Suppose the forward is trading at a price of $P > S_t - Q[t,T) - Z(t,T)K$. We construct an arbitrage. Now the forward is expensive, so I sell it for an income of P. I borrow $D_i Z(t,t_i)$ for maturity date t_i; so in total I borrow $Q[t,T) = \sum_i D_i Z(t,t_i)$. I borrow $Z(t,T)K$ for maturity date T. Finally, I buy the stock for S_t.

Income has been $P + Q[t,T) + Z(t,T)K$ and cost S_t, so I am in net profit of $P + Q[t,T) + Z(t,T)K - S_t > 0$.

As each dividend is received, my borrowing of $D_i Z(t,t_i)$ mature with value D_i and I use the dividend to eliminate what I have borrowed. At time T my borrowing of $Z(t,T)K$ matures with value K; I eliminate this by selling the stock for K in the short forward.

Thus, besides the initial profit, there have been no net flows. This is an arbitrage profit.

Assumptions made are that the dividend sizes are known, even though they may not yet have been declared. Also we ignore the value lag between the LDT date and the actual dividend receipt date. $\diamond$

There may be a desire to formulate results in terms of a continuous dividend yield. This is most appropriate because option pricing may be formulated in this manner, and one can determine sensitivities w.r.t. q. Since $(S_t - Q[t,T))C(t,T) = S_t e^{-q(T-t)} C(t,T)$, one has that

$$Q[t,T) = S_t[1 - e^{-q(T-t)}] \tag{6.10}$$

$$q[t,T) = \frac{-1}{T-t} \ln \frac{S_t - Q[t,T)}{S_t} \tag{6.11}$$

The value of q so obtained will be called the expected dividend yield and denoted $\mathbb{E}[q]$. Most importantly, if there is no dividend in the period under consideration then $\mathbb{E}[q] = 0$. In South

Africa there are typically two dividends a year, the interim dividend consisting of about 30% of the dividend and the final dividend constituting the remainder. Special dividend payments may also be made.

However, unlike in this nice textbook theory, the share price does not drop at the very instant the dividend is paid. The share price drops the business day after the LDT (Last Day to Trade). This is the last day to trade the stock cum the dividend. The LDT is usually a Friday. Some time later the dividend will be paid.

The LDR (Last Day To Register) is the date by which securities must be lodged with the company's office to qualify for dividends rights or other corporate actions.

How much does the stock price drop? Intuitively, it should be about the present value of the dividend. We have seen it claimed that this relationship is exact, to avoid arbitrage: if t is an LDT date for a dividend of size D with pay date T then $S(t^-) = S(t^+) + Z(t,T)D$. Of course, this statement is completely absurd: it assumes pre-knowledge of the stock price $S(t^+)$ at the time t^- - an instant before. The stock price is not previsible, but forward prices are. The following would be correct:

Theorem 6.5.2. Suppose today is date t. Suppose two forwards for a date T are available for trade in the market (and we ignore market frictions such as bid-offer spreads). A dividend of size D has already been declared with a ldt of date T and a payment date of T_p. The one forward, with a strike of K_1 is for the stock cum-dividend, while the other with a strike of K_2 is for the stock ex-dividend. Prove that

$$K_1 - K_2 = DZ(t;T,T_p)$$

where $Z(t;T,T_p)$ is the forward discount factor as seen now for the period from T to T_p.

Proof: First suppose $K_1 - K_2 < DZ(t;T,T_p)$. Then K_1 is 'too low' relative to K_2, so I

(1) go long the forward with strike K_1.

(2) go short the forward with a strike of K_2.

(3) lend $Z(t,T)K_1$ for maturity T.

(4) borrow $Z(t,T)K_2$ for maturity T.

(5) borrow $Z(t,T_p)D$ for maturity T_p.

At time t there is a net flow of

$$\begin{aligned}
-Z(t,T)K_1 + Z(t,T)K_2 + Z(t,T_p)D &= Z(t,T)\left[K_2 - K_1 + \frac{Z(t,T_p)}{Z(t,T)}D\right] \\
&= Z(t,T)\left[K_2 - K_1 + Z(t;T,T_p)D\right] \\
&> 0
\end{aligned}$$

At time T I receive in (3) K_1 which is use to pay for the stock in (1). I then deliver the stock and receive K_2 in (2). This money I use to repay (4). I now have the rights to the dividend D which I receive at T_p. This I use to repay (5).

Now suppose $K_1 - K_2 > DZ(t;T,T_p)$. I do the opposite, so

(1) go short the forward with strike K_1.

(2) go long the forward with a strike of K_2.

(3) borrow $Z(t,T)K_1$ for maturity T.

(4) lend $Z(t,T)K_2$ for maturity T.

(5) lend $Z(t,T_p)D$ for maturity T_p.

At time t there is a net flow of

$$\begin{aligned} Z(t,T)K_1 - Z(t,T)K_2 - Z(t,T_p)D &= Z(t,T)\left[K_1 - K_2 - \frac{Z(t,T_p)}{Z(t,T)}D\right] \\ &= Z(t,T)\left[K_1 - K_2 - Z(t;T,T_p)D\right] \\ &> 0 \end{aligned}$$

At time T I receive K_2 in (4). I use this to pay for the stock in (2). I then deliver the stock in (1) and receive K_1. This I use to repay (3). At time T_p I receive D in (5). This I use to manufacture a dividend payment for the long party in (1). $\diamond$
As a corollary, we have

Corollary 6.5.1. Suppose t is an LDT date for a dividend of size D with pay date T. Let $f(t^-,t^+)$ be the strike of a forward which is dealt at t^- and expires at time t^+ (effectively, a forward for immediate delivery, but of an ex stock). Then

$$S(t^-) = f(t^-,t^+) + Z(t,T)D$$

Proof: The forward for immediate delivery is just a purchase of stock, so has strike $S(t^-)$. $\diamond$

6.6 The term structure of forward prices

We are going to make extensive use of (6.7). In order to do so, we reformulate that equation so that we have a simple recursive procedure for determining the forward price for any date.
Let the dividend dates be t_i. Let $f(t_i^-)$ be the forward for that date but prior to the stock going ex, and let $f(t_i^+)$ be the forward post going ex. Let the current date be t_0. Then

$$\begin{aligned} f(t_0) &= S(t_0) \\ f(t_i^-) &= f(t_{i-1}^+)C(t_0;t_{i-1},t_i) \\ f(t_i^+) &= f(t_i^-) - D_i Z(t_0;t_i,t_i^{\text{pay}}) \end{aligned}$$

for $i = 1,\ 2,\ \ldots$.

Note that we can also perform a forward calculation into the past (a 'backward'?) It is a function of today's stock price, and the historical yield curves, but not the historical prices.

$$
\begin{aligned}
f(t_0) &= S(t_0) \\
f(t_i^-) &= f(t_i^+) + D_i \\
f(t_{i-1}^+) &= f(t_i^-) Z(t_{i-1}, t_i)
\end{aligned}
$$

for $i = -1,\ -2,\ \ldots$.

6.7 A simple model for long term dividends

We would like to construct a model that, given a valuation date t, and an expiry date T, will output a continuous dividend yield $q[t, T)$. Remember that continuous dividend yields are purely useful mathematical fictions - it is a number that can be used in comparing stocks very much in the same way that volatility can, and it is the q that will be used in the Black-Scholes formula.

Our intention is to model the evolution of the stock price and so the ex-dividend decrease in stock price is of higher concern than the actual valuation of dividend payments. Special dividends will be included in valuations and the calculation of dividend yields, but will not be used for prediction, so, of course, it will not be predicted that special dividends will reoccur.

Our model will take into account the period from one year BEFORE the valuation date until the expiry date.

Let us establish some notation:

- $t \ \sim$ valuation date.
- $T \ \sim$ expiry date.
- $\tau = \frac{T-t}{365}$
- $t_i \ \sim$ the ith LDT for the stock.
- $\tau_i = \frac{t_i - t}{365}$ - this could be positive or negative: negative for dividends occuring in the past.
- $vD_i \ \sim$ the value on its LDT date of the discrete dividend paid corresponding to LDT date t_i.
- $PV_i \ \sim$ the present value of the discrete dividend paid corresponding to LDT date t_i.
- $Q[t, T) = \sum_i PV_i$ where $t_i \in [t, T)$.
- $S_t \ \sim$ the share price at valuation date for the stock.
- $q_i \ \sim$ the simple dividend yield for the dividend at time t_i.
- $r_i \ \sim$ the riskfree rate for the period $[t, t_i]$ or the period $[t_i, t]$
- $q \ \sim$ the continuous dividend yield for the period $[t, T)$

Some dividends are known as cash amounts, for example, the known past, the short term known future (where the dividend amount has been declared by the company), or broker forecast future (where brokers have predicted the cash amount of the dividend). For example, there could be two historic dividends, a special dividend, and two forecast dividends. Typically, of course, we will have dividends much further into the future, and these dividends that will require some creativity.

It is possible to have a model for dividend yields which is a 'mixed' function i.e. a function of both the cash dividends and the dividend yields. However, it is easier to convert the cash dividends into what are called simple dividend yields, and have a more specific formula for dividend yields. Thus, each dividend paid will have an associated simple dividend yield q_i; the continuous dividend yield will be a function of all the q_i's.

We first define the simple dividend yield rates q_i as follows:

$$q_i = \frac{vD_i}{f(t_i^-)} \tag{6.12}$$

for either forward or backward dividends. This formulation is consistent with the observed occurrence that companies attempt to pay dividends which are consistent in Rand value, rather than consistent as a proportion of share price.

For dividends which are further into the future, the fundamental modelling assumption will be that the LDT dates of regular (non-special) dividends will reoccur in the years in the future and that the simple dividend yields of those dividends will also reoccur. Forecast dividends will override the corresponding historic dividends from the previous year.

In principle this model requires several yield curves: not only the valuation date curve, but also some historical curves. Empirically it turns out that the risk free rates are not a major factor in this model, and it usually suffices to take a flat risk free rate for all dates and all expiries.

That is our model. Everything that follows is model independent. It now can be shown that

$$PV_n = q_n \left(S_t - \sum_{i=1}^{n-1} PV_i \right) \tag{6.13}$$

$$PV_n = q_n \prod_{i=1}^{n-1} (1 - q_i) S_t \tag{6.14}$$

$$\sum_{i=1}^{n} PV_i = S_t \left(1 - \prod_{i=1}^{n} (1 - q_i) \right) \tag{6.15}$$

The first statement follows directly from a simple manipulation of (6.12) using Theorem 6.5.1. The second statement then follows by a careful induction argument. The third statement will again be induction; a trivial application of the second statement. (See tutorial.) This is consistent with the Rule of Present Worth.

Using the sum of the present values of the forecasted dividends for the stock that fall between today

t, and the expiry date T we can calculate the NACC dividend yield applicable over the period.

$$\begin{aligned} S_t \exp(-q\tau) &= S_t - Q[t,T) \\ &= S_t \prod_{i=1}^{n}(1-q_i) \\ \exp(-q\tau) &= \prod_{i=1}^{n}(1-q_i) \qquad (6.16) \\ q &= -\frac{1}{\tau}\sum_{i=1}^{n}\ln(1-q_i) \qquad (6.17) \end{aligned}$$

(6.17) is the required dividend yield for the option, to be input into the appropriate pricing formula. (6.16) is the fundamental dividend yield equation, and is model independent i.e. it is still valid under other models of determining the q_i's. Either side represents the proportional loss in the value of the stock if the owner of the stock forego's the dividends in the period $[t,T)$.

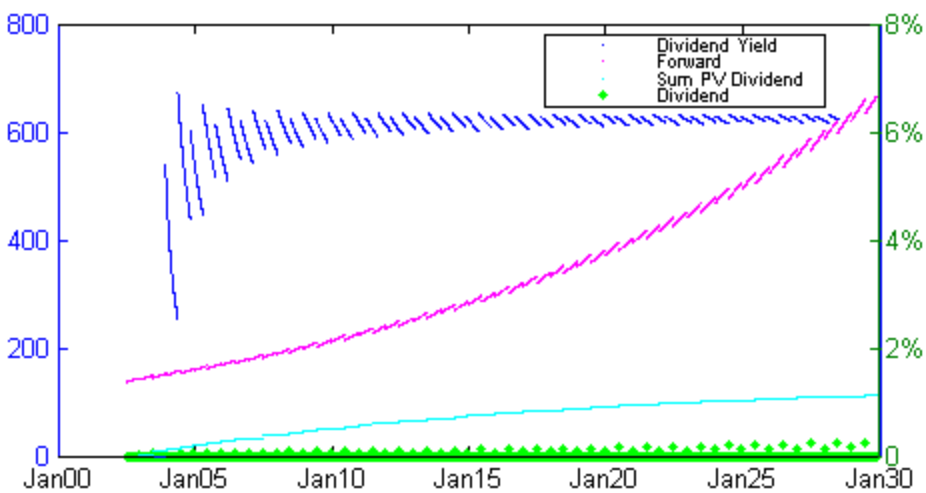


Figure 6.1: Dividends, values of dividends, forward values (left axis) and dividend yields (right axis)

The case of indices involves the same calculations in principle, taking into account the index constituents and their weights. In forward looking models the assumption will be that the constituents and their weights will be unchanged into the future.

These expected dividend yields need to be contrasted with the historic dividend yield, which is quoted by the JSE and all the data vendors, but is a completely useless piece of information. The historic dividend yield is the cash dividends in the last 12 months as a proportion of the most recent market value ie. the sum of all dividends in the past 12 months divided by the share price. The dividends are calculated relative to payment dates rather than ex-dividend dates.

The expected dividend yield and historic dividend yield are very different.

- $\mathbb{E}[q]$ is very often equal to 0 for short periods, and this, not the historical dividend yield needs to be used for option pricing. It can also be very large for short dated options over a LDR.

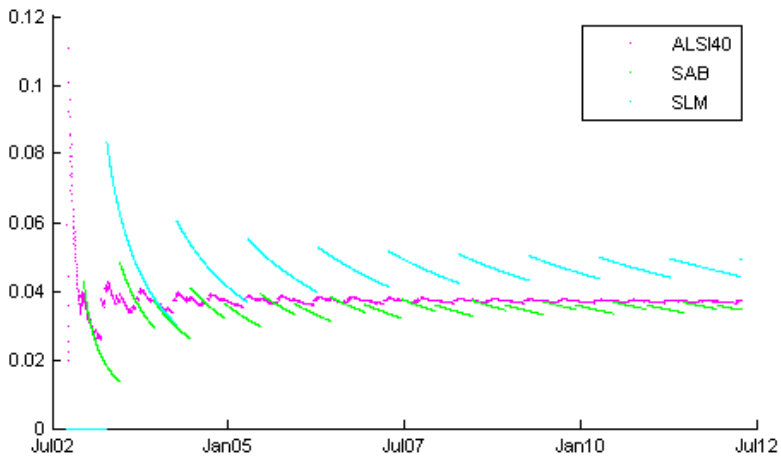


Figure 6.2: Dividend yields using the above model

- Using the historical dividend yield can lead to severe mispricing. Shares pay 0, 1 or 2 dividends a year. Most shares have year end at December or June so the dividends occur at March and September. In other words we have a clustering of dividends at specific times. So the error of using the historical dividend yield does not wash out when we move from a single equity to an index.

6.8 Exercises

1. Suppose that an index consists of the following shares:

Share	Price	ffSISS
AMS	22.00	6 722 223
BOC	4.52	40 000 000
MMD	3.20	22 331 700
ZZZ	0.45	17 010 000
ABC	7.20	10 000 555

The index level at close of business today is 8005.22. Two events now occur:

(a) AMS has a 5 for 1 share split.

(b) ZZZ has deteriorated in terms of market capitalisation and is removed from the index. It is replaced by PAR which has a closing price of 12.02 and a ffSISS of 10 000 270.

Calculate the old and the new basing constant (k Factor).

Answer: 1602.7576 and 1566.8818.

2. Consider the recent history of dividends for the share, ABC:

LDT Date	Dividend Amount
18-Jun-07	4.50
18-Dec-07	3.75

Using an interest rate of 12% NACC throughout, and the dividend model discussed in class, calculate for 26 February 2008,

(a) the JSE quoted dividend yield

(b) the expected dividend yield for a 3 month period

(c) the expected dividend yield for a 6 month period

(d) the expected dividend yield for a 5 year period

The share price for ABC on 26 February 2008 was 122.00 and there were no corporate actions in the last year.

3. (UCT exam 2007) Suppose share ABC has the dividend information as follows:

LDR	Pay	Classification	Amount
08-Sep-06	15-Sep-06	Occurred	2.00
09-Mar-07	16-Mar-07	Occurred	1.50
06-Jul-07	13-Jul-07	Special, declared	1.00
07-Sep-07	14-Sep-07	Forecast	2.50

Today is 3 June 2007 and the stock price is R150. Using a risk free rate of 8% throughout, use the model discussed in class to calculate the dividend yield for an option which expires on 31 Dec 2011.

4. (UCT Exam 2008) Dividend forecasts on a stock are provided. Today is 31-Aug-08 and the stock price is 100.00.

LDR	Pay	Classification	Amount
30-Nov-07	07-Dec-07	Occurred	2.00
11-Apr-08	18-Apr-08	Occurred	1.50
03-Oct-08	10-Oct-08	Special, declared	0.50
28-Nov-08	05-Dec-08	Forecast	2.50

Using a risk free rate of 11% throughout, use the model discussed in class to calculate the dividend yield for an option which expires on 31 Dec 2012.

5. Suppose that we assume that, at least for moderate moves in stock price, the present value of dividends in the short term is unchanged. Derive a formula that shows, under these assumptions, how the dividend yield changes, as a function of the old dividend yield, the old stock price, and the new stock price.

   Why is the phrase 'in the short term' important?

6. Suppose a stock pays percentage dividends q_i at times t_i. What is the value today of the dividend payment at time t_2? How would you replicate this?

7. Prove equations (6.14) and (6.15) by induction.

8. Suppose a stock pays percentage dividends q_i at times t_i. What is the value today of the dividend payment at time t_2? How would you replicate this (no arbitrage replication)?

Chapter 7

The carry market

7.1 Internal rates of return

Let us briefly return to the NACM, NACS, etc. situation.

Example 7.1.1. Suppose an investment offers R6 at the end of every 6 months until the 24th month and 106 at the end of the 30th month. This investment is traded at 14% NACS. What is the charge?
Let $d = 1.07^{-1}$, the 6 month discount factor. The charge is $6d + 6d^2 + 6d^3 + 6d^4 + 106d^5 = 95.8998$.

Conversely, suppose we are charged 95.8998 for this investment. What interest rate is charged? Answer: 14% NACS.
Suppose we are charged 99. What is the rate? Such a rate is known as the internal rate of return.
x is required where $d = \left(1 + \frac{x}{2}\right)^{-1}$ and

$$6d + 6d^2 + 6d^3 + 6d^4 + 106d^5 = 99$$

We want the positive root of the polynomial $p(d) = 6d + 6d^2 + 6d^3 + 6d^4 + 106d^5 - 99$. Taking the derivative, we see that this is an increasing function on $(0, \infty)$. We now solve for d using Newton-Rhapson, and then $x = 2\left(\frac{1}{d} - 1\right)$ is the NACS IRR.

7.2 Yield given price

In pricing bonds, we saw that given

- the bond name, or bond info,
- the settlement date,
- the yield to maturity

one could use the BPF to calculate the all in price $\mathbb{A}$. We will on occasion, have

- the bond name, or bond info,

- the settlement date,
- the all in price

and need to know which yield to maturity coincides with this $\mathbb{A}$.
Rounding when required will be to 7dp, and denoted by $[\cdot]$ as before. There is still no guarantee that a given all in price will have an exact corresponding yield.
In order to avoid confusion, we will denote the all in price by $\mathbb{A}(y)$ or $\mathbb{A}(y, s)$ whenever it is necessary to avoid confusion about the input yield y and/or the input settlement date s.
We have an analytic method for $y \to \mathbb{A}$, and both first and second derivatives. To solve $\mathbb{A} \to y$ we use an extension of Newton's method which in [Bond Exchange of South Africa, 2005, §7] is called the Newton-Bailey method - see also McCalla [1967]. In general, this is

$$x_{m+1} = x_m - \frac{f(x_m)}{f'(x_m) - \frac{f(x_m)f''(x_m)}{2f'(x_m)}}$$

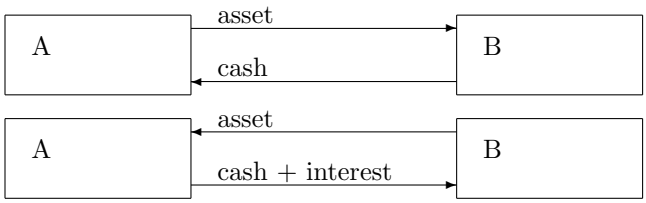
and so in this application

$$y_{n+1} = y_n - \frac{\mathbb{A}(y_n) - \mathbb{A}}{\Delta(y_n) - \frac{(\mathbb{A}(y_n)-\mathbb{A})\Gamma(y_n)}{2\Delta(y_n)}} \tag{7.1}$$

This is iterated until the successive values coincide in the seventh decimal place.
See [Bond Exchange of South Africa, 2005, §7].

7.3 The carry market

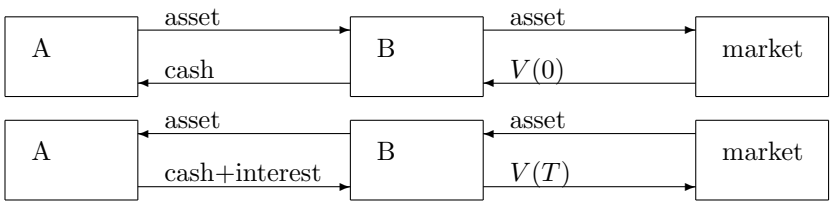
A repo is the following flows:



at time $t = 0$,

at time $t = T$.

Viewed simply like this, the repo is borrowing by A where their placement of the asset with B provides security for the loan. A is said to perform a repo and B a reverse repo.
A might need bridging finance i.e. they need immediate money, and use their assets to synthetically obtain this money. On the other hand, B might find the interest earned here more attractive that placing the money in a call account.
However, B typically has another motive for performing this trade, and that is to sell the asset short in the market, either for speculative or hedging purposes.



at $t = 0$,

at $t = T$.

A repo typically can take on one of the following forms: see Steiner [1997]

- classic repo: here any cash flows (such as coupons or dividends) which B receives, are paid over to A, as those cash flows are received. This can be administratively quite onerous.
- securities lending: similar to classic repo, except it is B that needs the asset, A does not need cash but instead takes another asset as collateral. In the second leg A takes a fee.
- buy sell back, or carry: the transfer of ownership is 'true', so any cash flows received by B are kept by B. A is compensated for the loss of these flows in a discount to the price they pay when repurchaisng the instrument. Administrative burdens with these transactions are much lower.

The repos with bonds seen in the South African market are almost always buy sell backs, and almost never classic repos. They are referred to as carries, BSBs, or repos. A buy sell back can easily be recorded, as trades at BESA can be booked for any settlement date at any time.
The price that A pays B for return of the bond will be the forward price of the bond, on an all-in basis. The way we calculate this forward all-in price will in principle be the same as for the calculation of forward prices of equities. The difference will be that we know the cash flows that are going to occur, rather than having to predict them; and we will be very precise about cum-ex issues, which we were not when we did equities.

Example 7.3.1. Suppose A borrows 10,300,000.00 on 3-Mar-08 at 11.00% NACC from party B, paying back 10,371,642.53 on 26-Mar-08.
Suppose A sells to B notional 10,000,000.00 of r153 at 11.64205% for settlement 3-Mar-08, and they book a trade at the same time that A buys the bond back from B at 11.64711% on the 26-Mar-08. Thus B pays A 10,300,000.00 on 3-Mar-08 and A pays B 10,371,643.00 on 26-Mar-08.

These scenarios are equivalent, by executing the trade at the bond exchange, B has security for the loan. If A should go bankrupt between 3-Mar-08 and 26-Mar-08 then B can take legal actions to keep the bond. Any remaining loss that B suffers (because the bond has deteriorated in value relative to the loan) can in principle be made up by money kept in a fund by the bond exchange for this purpose, and because they are first in line amongst the creditors when it comes to liquidation. Some market participants have expressed doubts to me that this principle would hold true in practice.
Terminology and notes:

- Trades at the bond exchange can be booked for any settlement date. In South Africa standard bonds trade $t+3$ but not so here i.e. can trade $t+2$, for example. Unless otherwise specified, we will assume a carry trades $t+2$, which is the standard. Thus, a carry can be booked from $t+2$ to $t+3$.
- One does not necessarily get the same bond back that you sold. They are all the same kind of thing, like a cash note, for example.
- A buys the bond back at the second date at yield agreed at the start, which is not at all related to the market price at the second date. Consequently, this market allows you to speculate on bonds without having to actually own them.
- The first trade is known as the first leg, the buy leg or the reference leg.

- The second trade is known as the second leg or the sell back.
- Typically carries are one week, so, from $t+2$ to $t+2+7d = t+2+5 = t+7$ (assuming no public holidays).

Because there is legal transfer of ownership, we have to worry about coupons. This is what distinguishes the carry from a classic repo transaction. Coupons are paid by the bond issuer on the NCD according to the party that was legal owner immediately prior to the BCD. Thus, if a carry is from settlement date s to forward settlement date $\mathbb{f}_s$ then we transfer all coupons satisfying

$$s < \text{BCD} \leq \mathbb{f}_s \tag{7.2}$$

Example 7.3.2. Carry the r153 on the date 6-Feb-07 to 22-Feb-07 at 8.70% NACC with a buy leg yield of 8.50%.[1]
$[\mathbb{A}(8.50\%,\ \text{8-Feb-07})] = 1.1933066$.
This grows to $1.1933066e^{8.70\% \cdot 14/365} = 1.19729530$. All other things being equal, this would be the amount owing on 22-Feb-07.
However, there is a coupon of 0.065 which will transfer. It will be paid on the 28-Feb-07, so its value on 22-Feb-07 is $0.065e^{-8.70\% \cdot 6/365} = 0.06490711$. So the amount owing on 22-Feb-07 is 1.13238819. Hence the second leg yield is 8.49554% and the rounded second leg price is 1.1323883.

Example 7.3.3. Suppose the r201 is trading at 8.00% on the 17-Apr-07. Carry the bond to 14-Jun-07 at 8.60% NACC. Calculate the forward all in price, and hence determine the forward yield.
$[\mathbb{A}(8.00\%,\ \text{19-Apr-07})] = 1.0708071$.
This grows to $[\mathbb{A}]e^{8.60\%\ 56/365}$. There is a coupon of 0.0438 on 21-Jun-07 which will transfer. Its value on 14-Jun-07 is $0.0438e^{-8.60\%\ 7/365}$.
Hence the forward all in price is $[\mathbb{A}]e^{8.60\%\ 56/365} - 0.0438e^{-8.60\%\ 7/365} = 1.04135161$. So the forward yield is 7.97853% and the rounded forward price is 1.0413518.

Example 7.3.4. For a 56 day period, the simple equivalent of 8.60% NACC is 8.65699%. Suppose the r201 is trading at 8.00% on the 17-Apr-07. Carry the bond to 14-Jun-07 at 8.65699% simple. Calculate the forward all in price, and hence determine the forward yield.
Again, $[\mathbb{A}(8.00\%,\ \text{19-Apr-07})] = 1.0708071$. This grows to $[\mathbb{A}](1 + 8.65699\%\ 56/365)$. There is a coupon of 0.0438 on 21-Jun-07 which will transfer. Its value on 14-Jun-07 is $0.0438\,(1 + 8.65699\%\ 7/365)^{-1}$.
Hence the forward all in price is

$$[\mathbb{A}](1 + 8.65699\%\ 56/365) - 0.0438\,(1 + 8.65699\%\ 7/365)^{-1} = 1.04135202$$

Hence the forward yield is 7.97853% and the rounded forward price is 1.0413518.
The two answers are different. See the tutorial.

We will denote the carried all in price by $\mathbb{f}_{\mathbb{A}}$ (forward all-in-price) and the carried yield by $\mathbb{f}_y$ (forward yield)

[1] The r153 traded at 8.50% on 6-Feb-07. Even though the first leg can be completed at any yield, it is convenient to use 8.50%. It is convention to use this ytm even though as a market rate it applies to the bond at 9-Feb-07.

Thus the (unrounded) forward bond price is calculated as follows Bond Exchange of South Africa [2006]:

$$\mathbb{f}_{\mathbb{A}}(y, s, \mathbb{f}_s) = [\mathbb{A}(y, s)]\, C(s, \mathbb{f}_s) - \frac{c}{2} \sum_{s < \mathrm{BCD}_i \leq \mathbb{f}_s} C(\mathrm{CD}_i, \mathbb{f}_s) \tag{7.3}$$

Please note that it is of course the semi-annual coupons that need to be adjusted for, not the annual coupon amount. This trivial division by 2 is often carelessly overlooked.

Thus, we grow the entire $[\mathbb{A}]$, and adjust for the value of coupons that are not received, taking into account when they are received. Note that we often have $\mathrm{CD}_i > \mathbb{f}_s$, in which case the capitalisation is really discounting.

Example 7.3.5. The first leg ytm is fairly arbitrary: suppose we carry the r153 on 13-Mar-07 to 17-May-07 to earn 8.50% NACC, starting with a ytm of:

1. 8.87%. $[\mathbb{A}(8.87\%, \text{15-Mar-07})] = 1.1259504$

   So $\mathbb{f}_{\mathbb{A}} = [\mathbb{A}]e^{8.50\% \cdot 63/365} = 1.14259125$, so $\mathbb{f}_y = 8.87710\%$ and $[\mathbb{f}_{\mathbb{A}}] = 1.1425913$.

2. 8.88%. $[\mathbb{A}(8.88\%,\ \text{15-Mar-07})] = 1.1256361$

   So $\mathbb{f}_{\mathbb{A}} = [\mathbb{A}]e^{8.50\% \cdot 63/365} = 1.14227231$, so $\mathbb{f}_y = 8.88773\%$ and $[\mathbb{f}_{\mathbb{A}}] = 1.1422722$.

The only difference between the two deals is the amount we borrow. The important thing is that we are borrowing money at 8.50% NACC for both deals. Traders worry about the difference in y and $\mathbb{f}_y$, and negotiate to "carry someone at such-and-such basis points". We talk about the difference in ytm between the first and second leg and call this the carry points. In the above example, the carry is at about 1basis points.

The pricing of the second leg of a forward deal is essentially a matter of finding the present or future values of cash flows. The present practice is to use simple interest calculations for this. This suffices for short periods (up to six months, say), but even then can still lead to disagreements. For longer periods (for example, long forwards; or for the forward pricing which is implicit in option pricing) compound interest should be used.

Surrendered coupons are valued as if they are received on the coupon date - even if this is a public holiday. This is market convention, but can be modified. The deal is logged through the two transactions at BESA, in which the rate of interest or indeed its compounding frequency is not recorded. This part is the responsibility of the trader.

See also Etheredge and West [1999] and Bond Exchange of South Africa [2006].

7.4 Marking a Carry to Market

A bond carry will involve a buy of a bond at date s_1 for a consideration of C_1 and a sell back of the bond at date s_2 for a consideration of C_2. There are two inputs involved in the valuation of the transaction at the valuation date t:

- the present value of the excess or shortfall of the value of receiving the bond at s_1 (if $t < s_1$) for the consideration of C_1 rather than the price at which it can be obtained in the market, denoted V_1;

- the present value of the excess or shortfall of the value of selling the bond at s_2 for the consideration of C_2 rather than the (expectation of the) price at which it can be sold at in the market, denoted V_2.

The value of the carry after close of business is

$$V_1 = \begin{cases} 0 & \text{if} \quad t \geq s_1 \\ Z(t, s_1)[\mathbb{f}_{\mathbb{A}}(y(t), ssd(t), s_1) - C_1] & \text{if} \quad t < s_1 \end{cases} \tag{7.4}$$

$$V_2 = Z(t, s_2)[C_2 - \mathbb{f}A(y(t), ssd(t), s_2)] \tag{7.5}$$

$$V = V_1 + V_2 \tag{7.6}$$

Note that there are some odd cases in disguise here. For example, if t is the initiation date of the carry, then $s_1 = t + 2$ is typical, so we have a carry from $t + 3$ to $t + 2$ i.e. a carry backwards in time. A similar occurrence will take place as t approaches s_2. There is nothing in principle that prevents us from calculating backward carries. Note that some such argument needs to be made to avoid problems with possible mismatch of cum-ex flags.

It can also be argued that rather than using $ssd(t)$ above we should use $t + 2$, as if we were to replicate the trade it would probably be done in the carry market itself rather than in the vanilla bond market.

7.5 Exercises

1. Suppose we expect an investment, that costs P at time t_0 to deliver cashflows c_i at times t_i, for $i = 1, 2, \ldots, n$. Write down the algorithm that uses Newton Rhapson to calculate the IRR (NACC) of the investment.

2. Build a loop in your bond spreadsheet that, given the all-in-price, gives the next guess at the ytm given a previous guess, using the Newton-Bailey method. (This method converges within 2 or 3 iterations, so by simply cutting and pasting the VALUE of the output cell into the input cell, and then recalculating, one can iterate. You can even write a macro that will do this copy-paste values.)

3. A R1,000,000 nominal r153 carry is booked with BESA. The first leg settles on 5-May-08 at 12.00% and the second leg settles on 12-May-08. The carry is executed at 11.50% simple. What is the consideration and the yield to maturity of the second leg?

4. A loan is entered into from 3-Mar-08 maturing 17-Mar-08. The loan is for approximately R100,000,000 and interest to be charged is 11.50% simple. The deal is to securitised by a carry transaction in the bond market with a carry of r153 bonds. The ytm on the first leg is 12.00%.

   Describe in detail the deals that have to be entered into at the bond exchange, specifying the relevant settlement dates, the nominal of the carry, and ytm of both the buy and sell-back leg. (The nominal will be a whole lot ie. a whole number of R1,000,000's.)

5. This example illustrates as discussed in class how one can speculate on bonds without ever owning any. Suppose you are bearish on bonds, that is, you believe that they are going to lose value (yields are going to increase). The following story describes what you could do.

   On 21-May-08 you sell R1 million nominal of the r153 in the ordinary bond market. When settlement is due you have already dealt to carry the position for one week - you carry at 8.40% simple yield. When settlement is again due, you again have already dealt to carry again for another two weeks at 8.30% simple interest.

   In order to close out the position you do an ordinary trade in the bond market at the appropriate time.

   Set out a schedule of settlements, show the timing and quantum of all cash and scrip flows. You must take into consideration all weekends and public holidays in your timing considerations. The bids and offers of the r153 on all the business days in the period of interest are shown below:

	Bid	Offer
21-May-08	10.64%	10.62%
22-May-08	10.74%	10.72%
23-May-08	10.76%	10.74%
26-May-08	10.76%	10.74%
27-May-08	10.91%	10.89%
28-May-08	11.06%	11.04%
29-May-08	11.38%	11.36%
30-May-08	11.54%	11.52%
02-Jun-08	11.60%	11.58%
03-Jun-08	11.55%	11.53%
04-Jun-08	11.57%	11.55%
05-Jun-08	11.54%	11.52%
06-Jun-08	11.57%	11.55%
09-Jun-08	11.69%	11.67%
10-Jun-08	11.73%	11.71%
11-Jun-08	11.74%	11.72%
12-Jun-08	11.53%	11.51%
13-Jun-08	11.63%	11.61%
17-Jun-08	11.60%	11.58%
18-Jun-08	11.67%	11.65%
19-Jun-08	11.60%	11.58%
20-Jun-08	11.58%	11.56%

   As we can see, yields have indeed gone up. Have you made money? How much?

6. A r153 carry is entered into on 29-Feb-08 at a trade ytm of 12.00%, for expiry 16-Oct-08. The carry rate is a simple rate of 11.50%. Calculate the reference price of the carry, the second consideration and the second yield to maturity.

Now convert the quoted carry rate to a NACC rate and repeat the example.

For the big prize, explain carefully why the answers are different. Which is correct?

For a smaller prize, or to get yourself started, answer the following question: if I earn a certain simple rate for a period and then the same simple rate for the next period, have I earned that simple rate for the entire period?

7. Describe how relative value plays can be implemented if there is a significant disparity between the bond curve and the swap curve. (Phrase your answer as follows: 'If the bond curve is much lower/higher than the swap curve, then I will go long/short a bond and pay fixed/floating in a swap. During the swap, I ..... From this I make a profit, because.....') Numerical calculations or examples are not required!

Chapter 8

Inflation linked products

8.1 The Consumer Price Index

The value of the consumer price index (CPI) is published monthly by Statistics South Africa.[1] Here however we conform with the methodology of BESA which calculates a CPI value on a daily basis for the purpose of daily MtM of inflation linked products.

On any date t, CPI(t) is calculated by interpolation as follows:

$$\mathrm{CPI}(t) = \frac{D-d+1}{D}\mathrm{CPI}_{\text{M-4}} + \frac{d-1}{D}\mathrm{CPI}_{\text{M-3}} \tag{8.1}$$

where

$$\begin{aligned} d &\sim \text{calendar day corresponding to } t; \\ M &\sim \text{calendar month corresponding to } t; \\ D &\sim \text{number of days in the calendar month } M; \\ \mathrm{CPI}_{\mathrm{M}} &\sim \text{published CPI for the calendar month } M \\ \mathrm{CPI}(t) &\sim \text{CPI for date } t \text{ for use in bond pricing.} \end{aligned}$$

Please note carefully the distinction between $\mathrm{CPI}_{\mathrm{M}}$ (which is a number published by StatsSA) and CPI(t) (which is a calculated, unpublished, number).

This formula is easy to construct from first principles; one simply remembers that for t being the first day of the month, $\mathrm{CPI}(t) = \mathrm{CPI}_{\text{M-4}}$. So, for any date, find CPI(t) for the first day of that month and the first day of the following month, and then interpolate linearly between those two dates and values.

This 3-4 month delay in the use of the figures ensures that the calculation is always possible, as due to data collection criteria, the CPI index is always published retrospectively. The number for month M is typically published on or about the last Wednesday of month $M+1$, although there is some variation in the exact timing.

[1] The document http://www.statssa.gov.za/keyindicators/CPI/CPIHistory.pdf provides a history of this value, up to and including the most recent value. Publication dates are at http://www.statssa.gov.za/Publications/AdvanceCalendar.asp.

8.2 Inflation Index Linked Bonds

Inflation index linked bonds are such that the coupon payments and maturity value of the bond adjust with the change in the CPI ie. they increase at the same rate as the CPI. Thus payments at every coupon date will be made of the amount

$$\frac{\text{CPI}(CD_i)}{\text{CPI}(ssd(\text{issue}))}\, p_i$$

where CD_1, CD_2, $\ldots$, CD_n are the coupon dates to follow $s = ssd(t)$, $p_1 = e\frac{c}{2}$, $p_i = \frac{c}{2}$ for $1 < i < n$, and $p_n = 1 + \frac{c}{2}$, and 'issue' is the issue date of the bond. CPI(ssd(issue)) is called the Base CPI.

8.2.1 Pricing using a real yield curve

Let us imagine that there is a term structure of real rates of interest $R(\tau)$, a term structure of rates of inflation $X(\tau)$, and a term structure of nominal rates of interest $r(\tau)$, so that $r(\tau) = R(\tau) + X(\tau)$. This relationship $r(\tau) = R(\tau) + X(\tau)$ can be called the Fisher equation, although one has to be careful about saying then what $X(\tau)$ means: is it simply expectations of inflation or does a risk premium need to be included? We simply define the problem away: we will call the difference between nominal rate and real rates the break-even inflation rate. If retrospectively the inflation rate turned out to be equal to the break-even rate, then investors in the nominal market and the real market have earned the same amount of money. Henceforth, we just refer to break-even inflation as inflation.

All rates are NACC. Let the period from settlement date s to CD_i be τ_i. The value is

$$\begin{aligned}
\mathbb{A}_{\text{CPI}} &= \mathbb{E}_t\left[\sum_{i=1}^{n} e^{-r(\tau_i)\tau_i}\frac{\text{CPI}(\text{CD}_i)}{\text{Base CPI}}p_i\right] \\
&= \frac{\text{CPI}(s)}{\text{Base CPI}}\sum_{i=1}^{n}\mathbb{E}_t\left[e^{-r(\tau_i)\tau_i}\frac{\text{CPI}(\text{CD}_i)}{\text{CPI}(s)}p_i\right] \\
&= \frac{\text{CPI}(s)}{\text{Base CPI}}\sum_{i=1}^{n} e^{-r(\tau_i)\tau_i}p_i\mathbb{E}_t\left[\frac{\text{CPI}(\text{CD}_i)}{\text{CPI}(s)}\right] \\
&= \frac{\text{CPI}(s)}{\text{Base CPI}}\sum_{i=1}^{n} e^{-r(\tau_i)\tau_i}p_i e^{X(\tau_i)\tau_i} \\
&= \frac{\text{CPI}(s)}{\text{Base CPI}}\sum_{i=1}^{n} e^{-R(\tau_i)\tau_i}p_i \qquad (8.2)
\end{aligned}$$

In this way, any determined (non-optional) set of cash flows can be valued.

Allowing the pass from a term structure of interest rates to flat interest rates, and the pass from NACC to NACS, this is a heuristic for the formal pricing mechanism for inflation linked BESA bonds, which follows.

8.2.2 Pricing using BESA specifications

Pricing of this bond is done according to BESA specifications with an additional factor adjusting for the effect of inflation, as specified in Raffaelli [2006].

The BESA bonds which are CPI linked are the r189, the r197, the ws05, the r202, and the r210. Previously there was the r198 which matured on 31-Mar-08. The r209 was only recently issued, so it currently does not have much liquidity. Due to differing credit the ws05 trades away from the r bonds, so it will not be used in curve building processes.

All-in-price

Let $[\mathbb{A}(y,s)]$ be the rounded all-in-price for the settlement date s of an ordinary bond as specified by BESA, then the unrounded all-in-price of the inflation bond, $\mathbb{A}_{\text{CPI}}$ is defined as:

$$\mathbb{A}_{\text{CPI}}(y,s) = \frac{\text{CPI}(s)}{\text{Base CPI}}[\mathbb{A}(y,s)]$$

where the input yield-to-maturity is the quoted real yield of the CPI bond.
Rounding takes place as follows: we simply round the above quantity arithmetically to 7 decimal places.

Accrued Interest

The accrued interest in obtained by adjusting the accrued interest of a vanilla bond for the inflation index. The rounded accrued interest obtained from the BESA pricing formula is multiplied by the unrounded index ratio to give the unrounded accrued interest on the inflation linked bond, which is then rounded arithmetically.

Clean Price

The rounded clean price of the inflation linked bond is the difference between the rounded all-in-price and rounded accrued interest of the inflation linked bond.

8.3 Inflation linked swaps

8.3.1 Zero coupon swaps

To start this off, we first consider the case of a zero coupon inflation linked bond. Suppose such a bond trades at a NACS yield of R. This means that for a deposit of 1 now (time t), I will receive $\frac{\text{CPI}(T)}{\text{CPI}(t)}\left(1+\frac{R}{2}\right)^{2(T-t)}$ at time T.
Let $\tau = T - t$.
For now we ignore any credit risk in this deal; the way credit risk is mitigated will become clear as we proceed. The question is: what is the fair value of R? We show that R is the NACS equivalent of the NACC rate $R(\tau)$, where $R(\cdot)$ is the zero coupon real curve.
In equilibrium the cost of the deal is the present value (under nominal discounting!) of the expected

cash flows. Since $r(\tau) = R(\tau) + X(\tau)$, we have

$$\begin{aligned}
1 &= e^{-r(\tau)\tau}\mathbb{E}_t\left[\frac{\mathrm{CPI}(T)}{\mathrm{CPI}(t)}\left(1+\tfrac{R}{2}\right)^{2\tau}\right] \\
&= e^{-r(\tau)\tau}\mathbb{E}_t\left[\frac{\mathrm{CPI}(T)}{\mathrm{CPI}(t)}\right]\left(1+\tfrac{R}{2}\right)^{2\tau} \\
&= e^{-r(\tau)\tau}e^{X(\tau)\tau}\left(1+\tfrac{R}{2}\right)^{2\tau} \\
&= e^{-R(\tau)\tau}\left(1+\tfrac{R}{2}\right)^{2\tau}
\end{aligned}$$

and so R is the semi-annual equivalent of $R(\tau)$.

The above argument is fine as long as we ignore credit risk. The way credit risk is dealt with is by making the instrument into a swap instead of a bond. Instead of paying 1 at time t, I pay at time T the amount that 1 would have grown to by continual reinvestment in the JIBAR market. That is, I deposit 1 for 3 months, after 3 months I deposit the proceeds for another 3 months, etc. What I receive at time T is the same as before. From a pure pricing perspective nothing has changed; however, because there is only one net flow at termination date, the credit risk is mitigated. Thus we have:

In an inflation linked swap, the fixed and floating legs are as follows:[2]

- Fixed: $\frac{\mathrm{CPI}(T)}{\mathrm{CPI}(t_0)}\left(1+\frac{R}{2}\right)^{2(T-t_0)}$. R is the contractually agreed real rate.
- Floating: $\prod 1 + J_{i-1,i}\alpha_i$, where $J_{i-1,i}$ refers to the JIBAR rate for the ith period, observed at time t_{i-1}, and $\alpha_i = t_i - t_{i-1}$ its day count proportion. This is then the amount that 1 has capitalised to in the JIBAR market during the life of the swap.

Then R is the NACS equivalent of the NACC rate $R(\tau)$.

8.3.2 Coupon swaps

In these there will be a fixed for floating swap on a net basis every three months: at time t_i the payment will be:

- Floating: the floating payment $J_{i-1,i}\alpha_i$ as above, on 1 in the nominal market.
- Fixed: $\frac{\mathrm{CPI}(t_i)}{\mathrm{CPI}(t)}R\alpha_i$ where R is the fixed real rate.

Notice that even though we refer to the latter as fixed payments, we don't know what they are in advance because the CPI values are not known in advance.

We haven't finished: the floating payments in the nominal market are those for a floating rate note, so there is a principle of 1 at the end. The fixed payments are in the real market - they are fixed

[2] This is contrary to what occurs internationally. There, in an inflation linked swap, the fixed and floating legs are as follows:

- Fixed: $\left(1+\frac{X}{2}\right)^{2(T-t_0)} - 1$. X is the contractually agreed inflation rate.
- Floating: $\frac{\mathrm{CPI}(T)}{\mathrm{CPI}(t_0)} - 1$.

See [Kerkhof, 2005, §5].

These products are subtly different. The South African swap fixes the real rate, the international swap fixes the inflation rate. Thus, a lot of the international theory would not apply in South Africa - so, no change there.

payments at a fair rate of R, but this means that the principle at the end is 1 in the real market, which is $\frac{\text{CPI}(T)}{\text{CPI}(t)}$ in the nominal market. They aren't the same, so they need to be exchanged: the floating receiver will also receive 1 at termination and the fixed receiver will also receive $\frac{\text{CPI}(T)}{\text{CPI}(t)}$ at termination (both of these amounts as measured in the nominal market).

Note that because of this exchange of principle there is a material credit risk in these instruments.

8.4 The real yield curve

What we have said so far assumes that a real yield curve $R(\tau)$ exists. Of course, such a curve needs to be bootstrapped. Typically, the inputs to such a curve might be

- Known or modelled CPI values in the short term (in other words, $X(\tau)$ is known for small τ, and of course $r(\tau)$ is known, so we get $R(\tau) = r(\tau) - X(\tau)$).
- Government issued inflation linked bonds.
- Inflation linked zero coupon swaps.

Inflation linked coupon swaps would probably not be included as with the exchange of notional at the end these instruments have a materially different credit risk to the instruments listed above.

8.5 Exercises

1. Find the rounded all in price of the r189 for the trade date of the 30-May-08 with a ytm of 2.90%. The basing constant (base CPI) of the r189 is 95.683870988.[3]

[3].... and relevant CPI data can be found at `www.statssa.gov.za`. You don't need to subscribe: the requisite data is on the front page.

Chapter 9

Statistical issues and option pricing

9.1 The cumulative normal function

Our concern here is the function $N(\cdot)$. The Cumulative Standard Normal Integral is the function:

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{X^2}{2}} dX \tag{9.1}$$

A closed form solution does not exist for this integral, so a numerical approximation needs to be implemented. Most common is an approximation which involves an exponential and a fifth degree polynomial, given in Abramowitz and Stegun [1974], and repeated in [Hull, 2002, §12.9] and [Haug, 1998, Appendix A], for example. The fifth degree function is used by most option exchanges for futures option pricing and margining, and hence may be preferred to better methods, in order to maintain consistency with the results from the exchange.

However, another option is one that first appears in Hart [1968]. This algorithm uses high degree rational functions to obtain the approximation. This function is accurate to double precision throughout the real line.

9.2 Pricing formulae

We will consider vanilla option pricing formulae. The inputs to these formulae will be (some of) spot S, future F, risk free rate r, dividend yield q, strike K, volatilility σ, valuation date t and expiry date T. So far, all of these symbols have been discussed, and we know how to derive them, with the exception of σ, which for the moment is just an input.

Black-Scholes

$$\mathrm{V_{EC}} = Se^{-q\tau}N(d_1) - Ke^{-r\tau}N(d_2) \tag{9.2}$$

$$\mathrm{V_{EP}} = Ke^{-r\tau}N(-d_2) - Se^{-q\tau}N(-d_1) \tag{9.3}$$

where

$$d_{1,2} = \frac{\ln(S/K) + (r - q \pm \sigma^2/2)\tau}{\sigma\sqrt{\tau}} \tag{9.4}$$

This formula is used for a European option on stock.

Forward form of Black-Scholes

$$\begin{aligned} \mathrm{V_{EC}} &= e^{-r\tau}[fN(d_1) - KN(d_2)] && (9.5) \\ \mathrm{V_{EP}} &= e^{-r\tau}[KN(-d_2) - fN(-d_1)] && (9.6) \end{aligned}$$

where

$$d_{1,2} = \frac{\ln(f/K) \pm (\sigma^2/2)\tau}{\sigma\sqrt{\tau}} \tag{9.7}$$

This is still the Black-Scholes formula, using the fact that $f = Se^{(r-q)\tau}$.

Standard Black

$$\begin{aligned} \mathrm{V_{EC}} &= e^{-r\tau}[FN(d_1) - KN(d_2)] && (9.8) \\ \mathrm{V_{EP}} &= e^{-r\tau}[KN(-d_2) - FN(-d_1)] && (9.9) \end{aligned}$$

where

$$d_{1,2} = \frac{\ln(F/K) \pm (\sigma^2/2)\tau}{\sigma\sqrt{\tau}} \tag{9.10}$$

This formula is used internationally for a European option on a future.

SAFEX Black

$$\begin{aligned} \mathrm{V_{AC}} &= FN(d_1) - KN(d_2) && (9.11) \\ \mathrm{V_{AP}} &= KN(-d_2) - FN(-d_1) && (9.12) \end{aligned}$$

where

$$d_{1,2} = \frac{\ln(F/K) \pm (\sigma^2/2)\tau}{\sigma\sqrt{\tau}} \tag{9.13}$$

This formula is used in South Africa (and Australia) for an American option on a future.

Forward version of the SAFEX Black

If we assume $F = f$ ie. $F = Se^{(r-q)\tau}$ then we get

$$\begin{aligned} \mathrm{V_{AC}} &= Se^{(r-q)\tau}N(d_1) - KN(d_2) && (9.14) \\ \mathrm{V_{AP}} &= KN(-d_2) - Se^{(r-q)\tau}N(-d_1) && (9.15) \end{aligned}$$

where

$$d_{1,2} = \frac{\ln(S/K) + (r - q \pm \sigma^2/2)\tau}{\sigma\sqrt{\tau}} \tag{9.16}$$

The unifying formula

Note that in all cases

$$V = \xi\eta[\mathbb{f}N(\eta d_+) - KN(\eta d_-)] \tag{9.17}$$

$$d_\pm = \frac{\ln(\mathbb{f}/K) \pm \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}} \tag{9.18}$$

where

- ξ is $e^{-r\tau}$ for an European Equity Option and for Standard Black, and 1 for SAFEX Black Futures Options and SAFEX Black Forward Options.
- $\eta = 1$ for a call and $\eta = -1$ for a put,
- $\mathbb{f} = f = Se^{(r-q)\tau}$ is the forward value for an European Equity Option and for SAFEX Black Forward Options, and $\mathbb{f} = F$ is the futures value for Standard Black and SAFEX Black Futures Options.

In doing any calculations (greeks, for example) the following equations are key:

$$d_+ = d_- + \sigma\sqrt{\tau} \tag{9.19}$$

$$d_+^2 = d_-^2 + 2\ln\frac{\mathbb{f}}{K} \tag{9.20}$$

$$N'(x) = N'(\eta x) \tag{9.21}$$

$$N'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \tag{9.22}$$

$$\mathbb{f}\, N'(d_+) = K\, N'(d_-) \tag{9.23}$$

9.3 What is Volatility?

Volatility is a measure of the jumpiness of the time series (price, rate or yield) of data we have. There are two very different notions of volatility:

- implied volatility: the volatility which can be backed out of an appropriate pricing formula. Such a volatility may be explicitly traded if there is agreement on the appropriate pricing formula, or one may imply it by taking the price and determining the appropriate volatility using an algorithm such as Newton's method.

  The trader charges for the option what he thinks the market will bear. This notion is forward looking, based on investor sentiment.

  Implied volatility is a function of the term $\tau = T - t$ and the strike K.

- realised or historic volatility. This is backward looking. It is a measure of the 'jumpiness' that has occured.

Implied volatility includes market sentiment and nervousness, not to mention fees, so will usually be higher than historical volatility.

9.4 Calculating historic volatility

For more information, including a full mathematical explanation, see [Hull, 2005, Chapter 19].
In the theory of the Geometric Brownian motion model we have

$$dx = \mu x\, dt + \sigma x\, dZ$$

where the time is measured in years, and so by Ito's lemma

$$\ln\left(\frac{x_T}{x_t}\right) \sim \phi\left(\left(\mu - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right).$$

and

$$p_{t,T} := \frac{1}{\sqrt{T-t}} \ln\left(\frac{x_T}{x_t}\right) \sim \phi\left(\left(\mu - \frac{\sigma^2}{2}\right)\sqrt{T-t}, \sigma^2\right). \tag{9.24}$$

In particular,

$$p_t := \ln\left(\frac{x_t}{x_{t-1d}}\right) \sim \phi\left(\left(\mu - \frac{\sigma^2}{2}\right)\frac{1}{250}, \frac{\sigma^2}{250}\right). \tag{9.25}$$

where the data $x_.$ we are taking are daily closes of prices/rates/yields. This follows from the fact that there are 250 trading days in the SA market in the year, and we are assuming they are equally spaced. The number 252 is typical in international texts such as Hull [2005].[1]
Hence:

- σ is estimated to be $\sqrt{250}$ times the standard deviation of $\{p_t\}$. This is the 'unweighted volatility' method. The problem here is that we consider the entire sample, much of which can be old, and stale.

- The second and only slightly less naive method is the N-window moving average method, where statistics are based on the last N observations. N can take any value, typically 30, 60, 90, 180 etc. Thus, on day $t-1$, we calculate s_{t-1} using p_i for $i = t-N, \ldots, t-1$, and on date t we drop p_{t-N} and include the new observation p_t, thus calculating s_t using p_i for $i = t-N+1, \ldots, t$. This allows for the dropping off of old data, because the input data is never older than N observations. However, this drop off is very discrete, and at the time of dropping off, can cause an otherwise unjustified spike down in the volatility estimate.

  We have $N = 30$, $N = 90$, $N = 250$. The volatility graphs described above are labeled unweighted window volatility graphs. The reason for this is that all N observations play an equally important role in the calculation - they are not weighted by perceived importance. This is certainly a problem. Notice that at the October 1997 crash, the volatility shoots up, and then N business days later, it falls off. This is because the October observation is in the window until N business days after it occurred, and then the next day it suddenly disappears. The implicit assumption in this calculation is that an observation from N business days ago is equally relevant as yesterday's, but an observation from $N+1$ business days ago is completely irrelevant. It is the size of the crash observation that is having such an enormous effect on the data, drowning out the informational content of the other $N-1$ observations.

[1] If we were to take monthly rates, we would replace 250/252 with 12, for example, because a month is $\frac{1}{12}$ of a year.

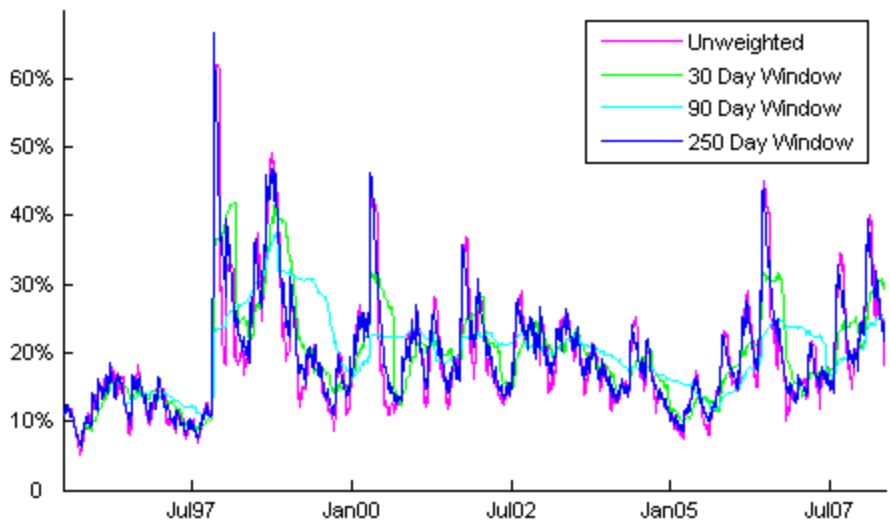


Figure 9.1: Unweighted rolling window volatilities for the ALSI40/TOPI

- A common feature of a variance estimate would be that $\Sigma_t^2 = \sum_{i \le t} \alpha_i p_i^2$ where $\sum_i \alpha_i = 1$.[2] The sum in general will be a finite one - namely, the entire (but finite) history, or the last N observations. Now let us make the sum infinite, at least in principal, so

$$\begin{aligned} \sigma(t)^2 &= 250 \sum_{i=0}^{\infty} \alpha_{t-i} p_{t-i}^2 \\ \sum_{i=0}^{\infty} \alpha_{t-i} &= 1 \end{aligned}$$

We have used the fact that $\sigma^2 = 250\Sigma^2$, ie the square of (annualised) volatility is 250 times the variance of the daily LPRs.

What property would we like? A nice property would be that the importance of an observation is only λ times the importance of the observation which is one day more recent than it. Here λ is called the weight, typically $0.9 < \lambda$, and certainly $\lambda < 1$. But if we put $\alpha_{t-i} = \lambda^i$, we would have $\sum_{i=0}^{\infty} \alpha_{t-i} = \sum_{i=0}^{\infty} \lambda^i = \frac{1}{1-\lambda}$. So instead, let us put $\alpha_{t-i} = (1-\lambda)\lambda^i$. Thus

$$\begin{aligned} \sigma^2(t) &= 250(1-\lambda)\left(p_t^2 + \lambda p_{t-1}^2 + \lambda^2 p_{t-2}^2 + \lambda^3 p_{t-3}^2 + \cdots\right) \\ &= 250(1-\lambda)p_t^2 + \lambda\, 250(1-\lambda)\left(p_{t-1}^2 + \lambda p_{t-2}^2 + \lambda^2 p_{t-3}^2 + \cdots\right) \\ &= 250(1-\lambda)p_t^2 + \lambda\sigma^2(t-1) \end{aligned} \tag{9.26}$$

which is the fundamental updating equation for the EWMA method: Exponentially Weighted Moving Average. If we were to want daily volatilities instead (i.e. not to annualise) we

[2] To understand this, the standard formula would be $\Sigma^2 = \frac{1}{n-1}\sum_{i=1}^{n}(p_i - \overline{p})^2$. First, accept that we will assume that the population mean is zero, so we gain a degree of freedom, and $\Sigma^2 = \frac{1}{n}\sum_{i=1}^{n} p_i^2$. Now we have the above formulation, where $\alpha_i = \frac{1}{n}$ for all i.

could divide the final answer by 250. Easier, we simply drop the 250 from all of the above calculations.

The smaller λ is, the more quickly it forgets past data and the more jumpy it becomes. Large λ forgets past more slowly; in the limit the graph becomes straight. How one chooses λ is a major problem.

However, some comments are in order. It follows immediately that in order to attempt to mimic short term volatility, one should use a lower value of lambda: probably never lower than 0.9, and typically in the regions of 0.95 or so. For longer dated volatility, one will use a higher value of lambda, say 0.99. One can compare these historical measures of volatility with implied volatility; as already discussed, the one is NOT a model for the other, and if historical volatility is to be used as a surrogate for implied volatility, one must take into account additional factors (that implied volatility has a risk price built in). Roughly, however, historical and implied volatility should be cointegrated.

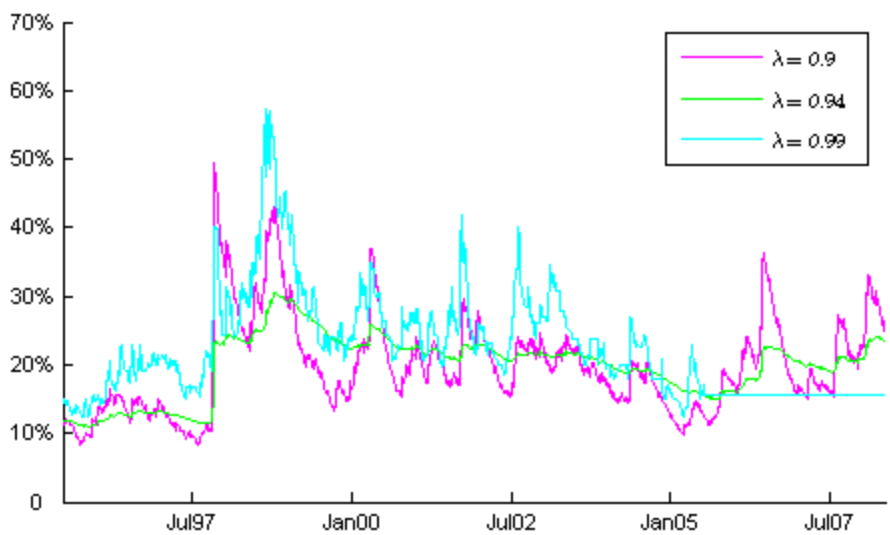


Figure 9.2: Various EWMA volatility measures of ALSI40/TOPI

How do we start the inductive process for determining these EWMA volatilities? The theory developed requires an infinite history, in reality, we do not have this history. And we should not start our estimation at 0.[3] A usable workaround is to take the average of the first few squared returns for the estimate of σ_0^2. We have taken the average of the first 25 observations. Hence the rolling calculator for volatility is

[3]This problem is completely ignored in the texts.

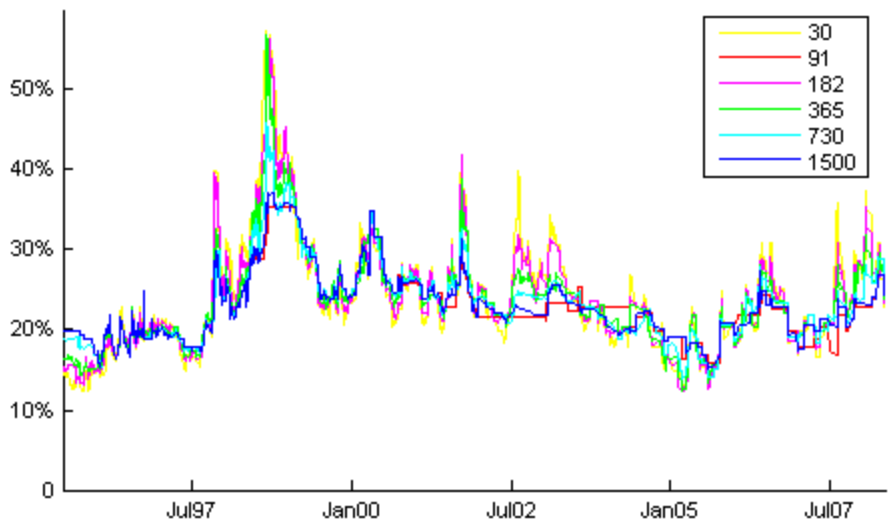


Figure 9.3: Various implied ATM volatilities for ALSI40/TOPI. The key is the number of days to expiry, interpolation between quoted expiries is applied.

The data available is $x_0,\ x_1,\ \ldots,\ x_t$:

$$p_i = \ln \frac{x_i}{x_{i-1}} \quad (1 \leq i \leq t) \tag{9.27}$$

$$\sigma(0) = \sqrt{10 \sum_{i=1}^{25} p_i^2} \tag{9.28}$$

$$\sigma(i) = \sqrt{\lambda \sigma^2(i-1) + (1-\lambda) p_i^2 250} \quad (1 \leq i \leq t) \tag{9.29}$$

- Finally, there are the Generalised Autoregressive Conditional Heteroskedasticity (GARCH) methods. These are logical estensions to EWMA methods (in fact EWMA is a so called baby-GARCH method) but where the process selects its own exponential weighting parameter, on a daily basis, using Maximum Likelihood Estimation techniques. An advantage of these methods is that they have built in mean reversion properties; the EWMA method does not display mean reversion.

  It is known that great care has to be taken to ensure that a GARCH process is sufficiently stable to be meaningful, and even to always converge.

9.5 Other statistical measures

We just consider the EWMA case.

The rolling calculator for covolatility (i.e. annualised covariance) - see [Hull, 2002, §17.7] - is

$$\begin{aligned} \text{covol}_0(x,y) &= \left(\sum_{i=1}^{25} p_i(x)p_i(y)\right) 10 && (9.30) \\ \text{covol}_i(x,y) &= \lambda \text{covol}_{i-1}(x,y) + (1-\lambda)p_i(x)p_i(y)250 \quad (1 \leq i \leq t) && (9.31) \end{aligned}$$

Following on from this, the derived calculators are

$$\begin{aligned} \rho_i(x,y) &= \frac{\text{covol}_i(x,y)}{\sigma_i(x)\sigma_i(y)} && (9.32) \\ \beta_i(x,y) &= \frac{\text{covol}_i(x,y)}{\sigma_i(x)^2} && (9.33) \end{aligned}$$

the latter since the CAP-M β is the linear coefficient in the regression equation in which y is the dependent variable and x is the independent variable. The CAP-M intercept coefficient α has to be found via rolling calculators. Thus

$$\begin{aligned} \overline{p_1(x)} &= 10\sum_{i=1}^{25} p_i(x) && (9.34) \\ \overline{p_i(x)} &= \lambda\overline{p_{i-1}(x)} + (1-\lambda)p_i(x)250 \quad (1 \leq i \leq t) && (9.35) \end{aligned}$$

and likewise for $p(y)$. Then

$$\alpha_i(x,y) = \overline{p_i(y)} - \beta_i(x,y)\overline{p_i(x)} \qquad (9.36)$$

9.6 What option-type products trade on the JSE?

9.6.1 Options issued by the underlying

Examples currently include DMRO - Diamond Core Resorces Ltd. These are options issued by the underlying and which trade like shares on the JSE. The pricing of these instruments is affected by the dilution effect eg. if a call is exercised, the underlying can simply issue more shares (and thereby dilute the price) to honour the call. In textbooks such as Hull [2005] such instruments are known as warrants.

Until 2001 only calls were possible but now puts are also legal. Companies could only issue calls since it was illegal for a company to buy its own shares because of fears of market manipulation. This has now been legalised because of better market surveillance. However, the company cannot actively trade the options - their issue must be strategic.

9.6.2 Warrants

On a stock exchange options issued by a third party are known as warrants.

The third party is always a merchant bank, such as SCMB, Gensec, Investec and Deutsche Bank. The issuer makes a market in these vanilla options ie. will buy/sell the option at the bid/offer. Pricing is a simple matter for the issuer via standard pricing formulae. This price will not necessarily lie between the bid and the offer, but the issuer intends to influence and hedge the position with the theoretical model.

Warrants typically have conversion ratios eg. need 10 warrants to buy one share. Then the value is just 1/10 of the value given by a formula.

It is not allowed to short warrants because the gearing makes this too credit-risky.

Compound warrants (a warrant which entitles you to another warrant on exercise) have been issued by Gensec Bank. Down and out barrier warrants (a warrant which disappears if the stock price goes below a certain level) have been issued by Standard Bank, and out puts and calls were issued by Deutsche Bank on 13 September 2002 (they call them WAVEs).

9.6.3 Debentures

These will be part of the debt structure of the underlying. There are straight debentures (a type of bond) and convertibles.

9.6.4 Nil Paid Letters

A security which is temporarily listed on the stock exchange and which represents the right to take up the shares of a certain company at a certain price and on a certain date, in other words, a call option. Nil paid letters are the result of a rights issue to the existing shareholders (or debenture holders) of a company. A rights issue is one way of raising additional capital by offering existing shareholders the opportunity to take up more shares in the company - usually at a price well below the market price of the shares. These rights are represented by the "nil paid letter" and are renounceable - this means that they may be bought and sold on the stock exchange.

9.7 Program trading?

With the advent of electronic trading, program trading becomes possible. These would be coded modules that facilitate the automation of rule-based execution into the JSE's trading engine.

However, program trading is fraught with dangers, and is typically blamed as the principal culprit of the Wall Street crash of 1987. Major portfolio holders typically hold large portfolios on the stock exchange and portfolio insurance positions on the derivatives exchange. When the portfolio insurance policy comprises a protective put position, no adjustment is required once the strategy is in place. However, when insurance is effected through equivalent dynamic hedging in index futures and risk free bills, it destabilises markets by supporting downward trends.

Dynamic hedging is the synthetic creation of a protective put at a given strike. Recall that the Black-Scholes formula for a put is (9.3). To create this sythetically (as in the derivation of the formula) we have $Xe^{-r\tau}N(-d_2)$ in the money market account and we are short $N(-d_1)$ many of the stock.[4] Now, as the market moves, we rebalance these amounts on an ongoing basis (in the derivation of the formula, this rebalancing occurs continuously; in practice of course transaction costs mean that the rebalancing must occur discretely, possibly according to some trigger rule). And if the market falls, the value of $N(-d_1)$ goes up.[5]

[4] Assume there is no dividend.

[5] When the market falls, the volatility will typically go up, and this causes $N(-d_1)$ to decrease, but this effect is not as dramatic as the effect of the change in stock price.

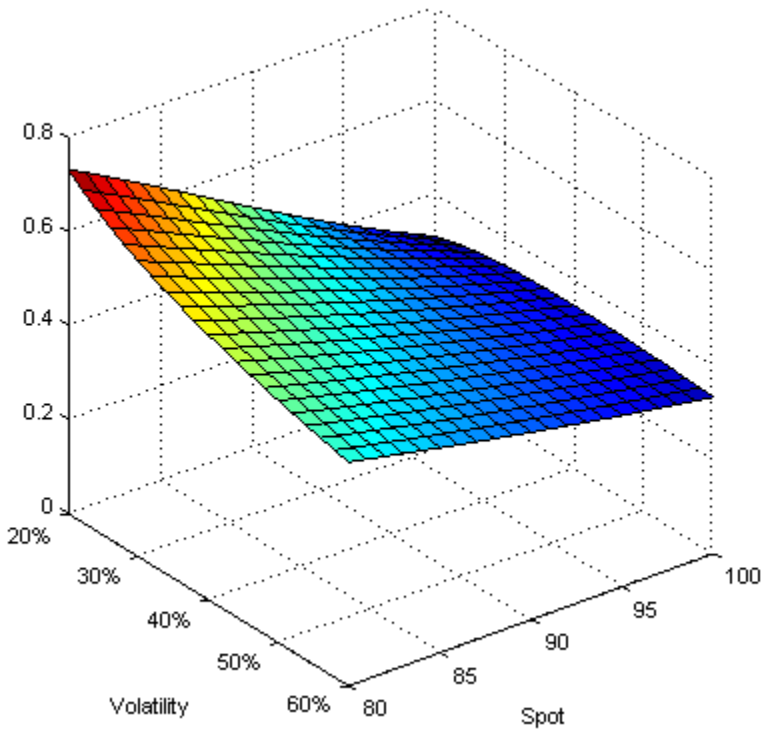


Figure 9.4: The value of $N(-d_1)$

Thus, we cause further downward momentum in the stock price. Alternatively, if we change our delta in the futures market, the prices of index futures will fall below their cost-of-carry value. Then index arbitrageurs step in to close the gap between the futures and the underlying stock market by buying futures and selling stocks through a sell program trade. Thus, either way, the sale of stocks gathers momentum.

For a very well written piece on program trading, see Furbush [2002].

9.8 Exercises

1. As discussed in class, build a spreadsheet to calculate and graph the volatility of a time series of financial prices/yields/rates. Use:
   (a) Unweighted volatility calculation
   (b) 'unweighted window' volatility with N=90
   (c) EWMA volatility with $\lambda = 0.99$
2. Write vba code to price all varieties of the vanilla option pricing formulae. Use the symbols and the general approach from the notes.

Chapter 10

The South African Futures Exchange

Organised exchanges are a method of bringing market participants together. Participants in futures exchanges are hedgers, arbitrageurs, speculators, and curve traders. The exchange takes margin as a measure to ensure the contracts are honoured ie. mitigate against credit risk. As an entity the exchange makes money through the use of this margin on deposit. The exchange does not take positions itself.

See SAFEX [2009].

SAFEX was first organised by RMB in April 1987 and formalised in September 1988.

In order to open positions at SAFEX, one has to pay an initial margin. The market takes this margin as a measure to ensure contracts are honoured (mitigate credit risk). If contracts are not honoured, the contract is closed out by the exchange and this initial margin is used to make good any shortfall.

In order for this strategy to be effective, initial margin is obviously quite a significant amount. Thus it has to earn interest. This is a good interest rate, but not quite as good as the rate earned by SAFEX when they deposit all of the various margin deposits. Thus, as an entity SAFEX makes money through the use of margin on deposit.

If the total margin paid in falls below the maintenance margin then the account has to be topped up to the initial margin.

This is all completely standard, see [Hull, 2002, Chapter 2].

Futures are always fully margined on any exchange. However, what is unusual is that SAFEX futures options are fully margined. No premium is paid up front!! All that is paid up front is the initial margin. However, on a daily basis one pays/receives the change in valuation (MtM). They are called fully margined futures options.

SAFEX trades futures and futures options on indices and individual stocks; on JIBAR rate, bonds etc. We will concentrate on futures and futures options on TOP40, INDI25.

Futures are screen traded via a bid and offer, and are very liquid.

Futures options are slightly less liquid, as they are typically negotiated via brokers and then traded

on the screen, so the screen can be stale. Furthermore, these trades can be for packages ie. a price (implied volatility) is set for a set of options, without determining what the implied volatility for each option is.

Input Data		term (days)	735
Value Date	15-Mar-07	term (years)	2.014
Future	TOPI	d_1	1.0254
Futures Spot	25902	d_2	0.7310
Strike	20000	$N(d_1)$	0.8474
Option Expiry	19-Mar-09	$N(d_2)$	0.7676
Volatility	20.75%	$N(-d_1)$	0.1526
		$N(-d_2)$	0.2324
Prices		Rand Price	
Future	25902.00	R 259,020.00	
Call Price	6597.80	R 65,978.00	
Put Price	695.80	R 6,958.00	
rpp	10.00		

Table 10.1: MtM of SAFEX option

Rounding:

- For indices: futures levels and strikes are always whole point numbers. Unrounded option prices are calculated. The result (for futures or futures options) is multiplied by 10, and then rounded to the nearest rand.
- For equities: futures levels and strikes are in rands and cents. Unrounded option prices are calculated. The result (for futures or futures options) is multiplied by 100, and then rounded to the nearest rand.

Contracts vary by strike and expiry.

- Strike: for indices, discrete strikes allowed, in units of 50 points minimum. The strike levels need to have a reasonable dispersion for liquidity, so that one can find the opposite side for a trade. Too many strikes means low liquidity at each strike.
- Expiry: as far out as there is interest (currently 2012) expiring 3rd Thursday of March, June, Sept, Dec. If this is a holiday then scroll backwards. The March following (currently March 2010) is always the most liquid, for tax reasons.

The volatility we have used is implied, not realised. At the end of each day, SAFEX uses the last implied traded volatility for an at or near the money option for margining purposes. All options for the ALSI40 contracts for that expiry are then margined at a skew, which is dependent on that at the money volatility. All other contracts are insufficiently liquid to have a skew built for them. We will study the skew later, but in the absence of any specific information, we will assume there is no skew.

JSE FINANCIAL DERIVATIVES CLOSING PRICES — DATE: 09-Sep-2008

CONTRACT	SPOT	BID	OFFER	M-T-M	VOL. CHANGES	PREV. VOLS.
FTSE/JSE TOP 40 INDEX (ALSI)						
18-Sep-2008	23052	23035	23050	23043		30.00
18-Dec-2008	23052	23580	23605	23593	31.00	29.00
19-Mar-2009	23052	24027	24167	24097	30.00	28.50
18-Jun-2009	23052	24580	24740	24660		28.50
17-Sep-2009	23052	24926	25106	25016		27.50
17-Dec-2009	23052	25454	25654	25554		27.50
18-Mar-2010	23052	25834	26054	25944		27.50
17-Jun-2010	23052	26312	26552	26432		27.50
16-Sep-2010	23052	26718	26968	26843		27.50
15-Dec-2010	23052	27174	27434	27304		27.50
15-Dec-2011	23052	29106	29386	29246		27.50

10.1 Are fully margined options free?

No. A legendary selling point for paid-up options is that the downside of the investment is at most the premium paid. This is seen as being preferable for risk adverse players to futures, for example, where the gearing is very high and the potential downside is enormous. One can think initially that fully margined options are free, because there is no cost to entering such options[1]. Alternatively, one can think that these options could become unreasonably expensive - in terms of the margin requirements - if, for example, volatility goes up.

Neither viewpoint is correct. Suppose an option is dealt at $V(t)$ and has an initial MtM of $M(t)$ - these will probably not coincide - we buy intraday, we are a price maker or taker, and the SAFEX margin skew is not the same as the traded skew (indeed, for some contracts, SAFEX do not have a skew). Suppose the expiry date MtM is $M(T) = V(T)$. If the options were fully paid up then there would be a flow of $-V(t)$ at date t and a flow of $+V(T)$ on date T.

Now suppose we have the SAFEX situation, where options are fully margined. Suppose the business days in $[t, T]$ are $t = t_0,\ t_1,\ \ldots,\ t_N = T$. Let the MtM on date t_i be denoted $M(t_i)$. The margin flow in the morning of date t_1 is $M(t_0) - V(t_0)$, and the margin flow in the morning of date t_{i+1} is $M(t_i) - M(t_{i-1})$ for $i \geq 1$. Thus, the total margin flow is

$$M(t_0) - V(t_0) + \sum_{i=1}^{N} M(t_i) - M(t_{i-1}) = V(T) - V(t)$$

as a telescoping series. Hence, the net effect is the same: it is merely the timing of cash flows that will be different.

Thus, the downside of the investment is still at most the initial 'cost'. However, there may be funding requirements in the interim which are not necessarily commensurate with that initial cost.

10.2 Deriving the option pricing formula

1. Let F be the futures price, μ the drift, and σ the volatility. F is subject to Geometric Brownian Motion:

$$dF = \mu F dt + \sigma F dz$$

[1] We ignore the initial margin requirements because, whatever they are, they earn a competitive rate of interest at SAFEX.

Suppose $g = g(F, \tau)$ is a derivative. By Ito's lemma,

$$dg = \left(\frac{\partial g}{\partial F} \mu F + \frac{\partial g}{\partial t} + \frac{1}{2} \sigma^2 F^2 \frac{\partial^2 g}{\partial F^2} \right) dt + \frac{\partial g}{\partial F} \sigma F \, dz. \tag{10.1}$$

Let π be the portfolio which is long g and short $\frac{\partial g}{\partial F}$ of F.

$$\begin{aligned} d\pi &= dg - \frac{\partial g}{\partial F} dF \\ &= \left(\frac{\partial g}{\partial t} + \frac{1}{2} \sigma^2 F^2 \frac{\partial^2 g}{\partial F^2} \right) dt \end{aligned}$$

To write $\pi = g - \frac{\partial g}{\partial F} F$ or $\pi = g$ would be highly misleading. In fact, $\pi = 0$ is the initial MONETARY investment into the portfolio. So $d\pi = 0$ and so

$$\frac{\partial g}{\partial t} + \frac{1}{2} \sigma^2 F^2 \frac{\partial^2 g}{\partial F^2} = 0 \tag{10.2}$$

Even now one can show that g is a martingale. r has disappeared because there is no (hedged and riskless) up front investment requiring a return of r.[2] Compare to the original B/S DE (for options on equity)

$$\frac{\partial g}{\partial t} + rS \frac{\partial g}{\partial S} + \tfrac{1}{2} \sigma^2 S^2 \frac{\partial^2 g}{\partial S^2} = rg$$

and Standard Black DE (for paid up European options on futures)

$$\frac{\partial g}{\partial t} + \tfrac{1}{2} \sigma^2 F^2 \frac{\partial^2 g}{\partial F^2} = rg.$$

2. We solve the DE with the usual European boundary conditions:

   - Call: $V(F, t) \to \infty$ as $F(t) \to \infty$, $V(0, t) = 0$, $V(F(T), T) = \max\{F(T) - K, 0\}$.
   - Put: $V(F, t) \to 0$ as $F(t) \to \infty$, $V(0, t) = K$, $V(F(T), T) = \max\{K - F(T), 0\}$.

   For European vanilla options, get

$$V_c = FN(d_1) - KN(d_2) \tag{10.3}$$

$$V_p = KN(-d_2) - FN(-d_1) \tag{10.4}$$

$$d_{1,2} = \frac{\ln \frac{F}{K} \pm \frac{1}{2} \sigma^2 \tau}{\sigma \sqrt{\tau}} \tag{10.5}$$

$$\tau = T - t \tag{10.6}$$

3. However, options are American. If not exercised early, and in the money, then there is automatic exercise on the expiry date.

   To get the American condition, we need to worry about the free boundary condition $V_c \geq \max\{F(t) - K, 0\}$ for a call, $V_p \geq \max\{K - F(t), 0\}$ for a put.

[2] That is quite different to saying that $r = 0$: it isn't. After all, F is of the order of $e^{r\tau}$. However, as a mathematical DE problem, we have eliminated r. This is a subtle point, but rather obvious once pointed out. Nevertheless, there are some otherwise quite sophisticated players in the market who don't understand this point.

To see that $V > 0$, show $V_c(0) = 0$ and show $\frac{\partial V_c}{\partial F} > 0$; $V_p(\infty) = 0$ and $\frac{\partial V_p}{\partial F} < 0$.[3]

For a call,

$$\begin{aligned} V_c - (F - K) &= FN(d_1) - KN(d_2) - F + K \\ &= K[1 - N(d_2)] - F[1 - N(d_1)] \\ &= KN(-d_2) - FN(-d_1) \\ &= V_p \\ &> 0. \end{aligned}$$

In particular, we have put-call parity:

$$V_c + K = V_p + F \tag{10.7}$$

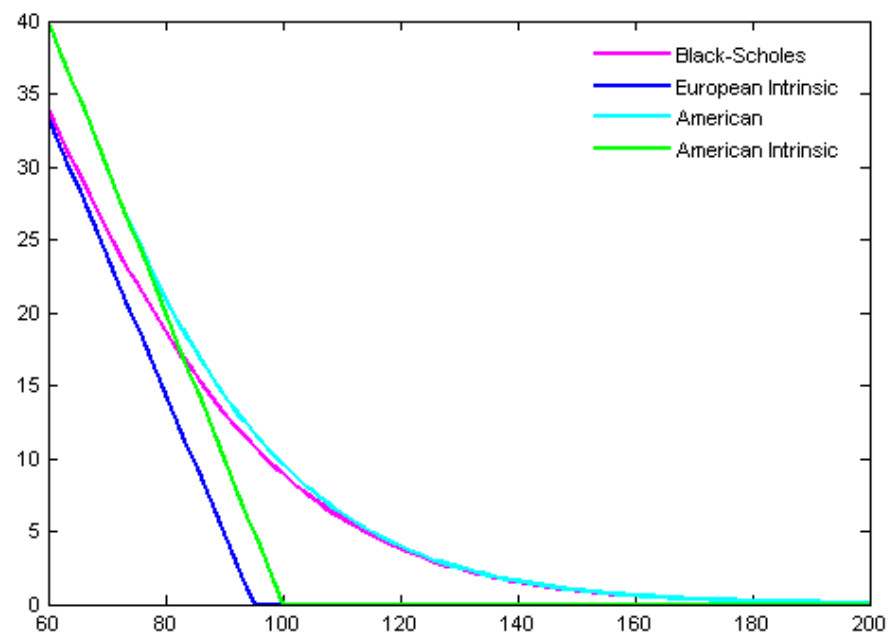


Figure 10.1: The European premium for an equity option violates the American free boundary condition.

Similarly for the put. So the free boundary is never touched. The American pricing formula is the same as the European one, and it is never optimal to exercise early. If someone chooses to

[3]There are some tricks here. Firstly, the differentiation is not completely trivial, as there are some disguised dependences: d_i are functions of F! Use (9.23). Secondly, the proof that $V_p(\infty) = 0$ needs l'Hopital's rule, and (9.23) again. One may not use 'known' properties of option prices when what we are trying to do is establish that we have a valid option pricing formula.

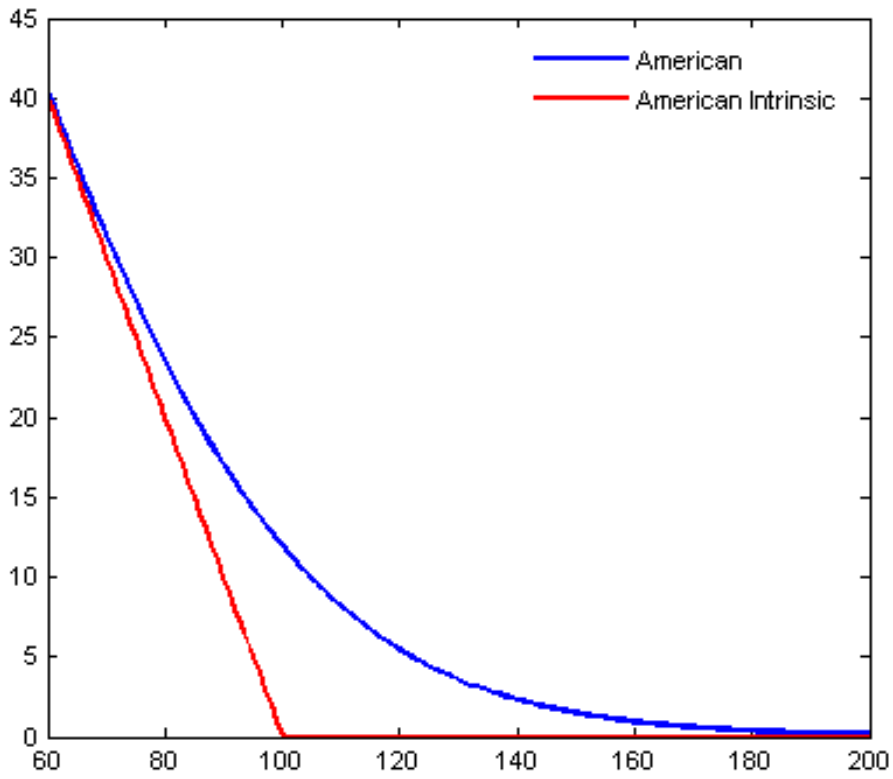


Figure 10.2: The SAFEX Black premium (European formula!) does not violate the free boundary, so is valid for American options.

exercise, SAFEX have a random drawing of individuals who are to be exercised against - lucky people because they are happy because they make money out of it (ie. the person exercising should not have).

See also Etheredge and West [1999].

10.3 The skew

The volatility in the SAFEX Black pricing formula is an implied volatility. This volatility can vary with term $\tau = T - t$ and with strike K.

Until 2001 SAFEX margined on the atm volatility. This caused distortions in the margining of away from the money options on the ALSI40. In 2001 they phased in margining and then MtM for the skew. This was a trigger event for the bankruptcy of RAD - see §10.4. However, the skew published by SAFEX is in general quite noisy, is only updated once a month, and cannot be used for analysis. Market participants will have their own skew which they will keep current.

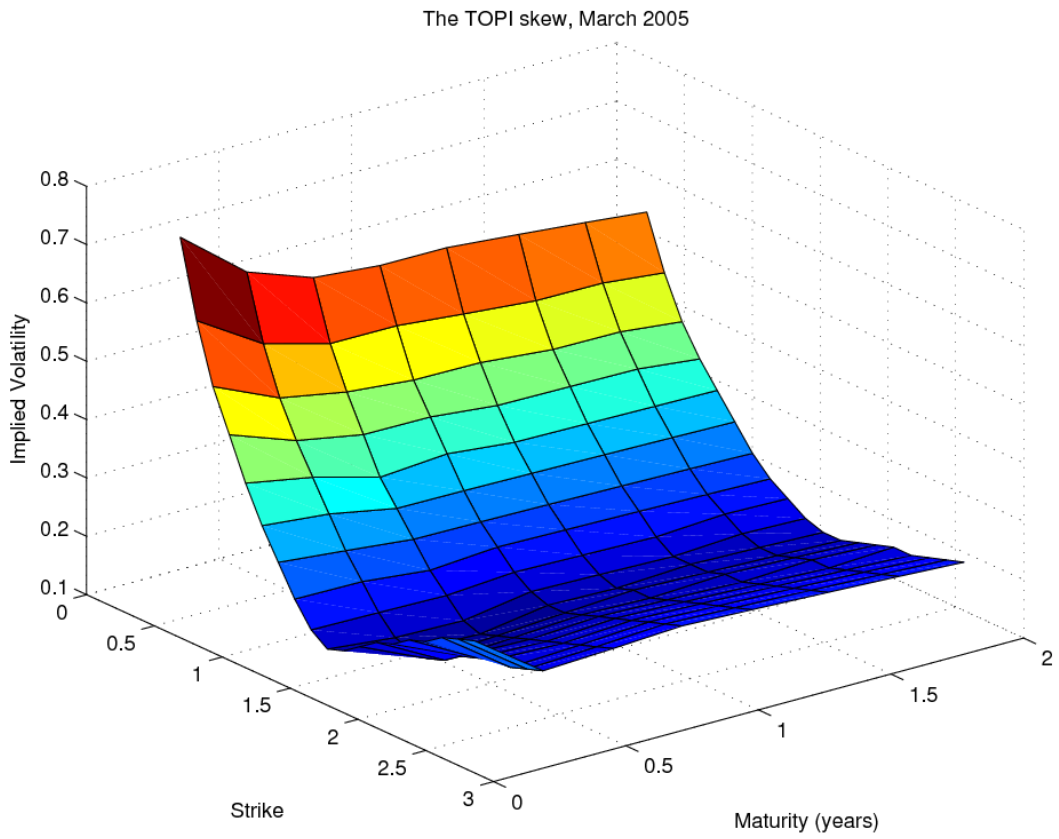


Figure 10.3: The TOP40 skew, by term and strike, March 2005

Why is there a skew? Volatility is not constant. For example, when the stock market falls, index option implied volatility tends to move up, and vice-versa. The reason is as follows: as the stock price decreases, the fear that the price may decrease even further translates into a higher volatility. This is known as the leverage effect as was first noted in **?**. Yet, for option prices to follow the classical form of the Black family of models, volatility must be a constant - the leverage effect shouldn't exist. Thus we have a contradiction, and this explains why there has to be a mechanism for modifying the output of the Black models.

A portfolio hedger will be concerned about how a market fall might happen. Hypothetically, if the fall was guaranteed to be fairly steady, then they can hedge continuously by selling some stock on the way down. In reality, the fall might instead be a sharp jump that cannot be hedged by trading shares. Because of this possibility, they are willing to pay more for those puts. The sellers must charge more for the puts because they too will endeavor - and fail under the above scenario - to hedge the puts, with futures options, for example. When the market goes up, it usually does so in an orderly fashion, and hedging is relatively easy. This is why options struck above the money might be relatively cheap, because there is not much demand for them. Thus the market expects negative skewness of the distribution of returns.

Furthermore, there is the so called 'fat tail' effect - the market expects the tail events to occur with

a far greater probability than is implied by normally distributed returns: the distribution of returns has excess kurtosis. Thus options which pay off in these circumstances need to be priced higher. The expected smooth hedging if the market goes up seems to be a more significant factor than the fat tail there. The modifications are achieved by modifying the volatility input to the option pricing calculator. Vanilla OTC European options or European futures options are priced and often hedged using respectively the Black-Scholes or Black model. In these models there is a one-to-one relation between the price of the option and the volatility parameter σ, and option prices are often quoted by stating the implied volatility σ_{imp}, the unique value of the volatility which yields the option price when used in the formula. In the classical Black-Scholes-Merton world, volatility is a constant. But in reality, options with different strikes require different volatilities to match their market prices. This is the market skew or smile.

Typically, although not always, the word skew is reserved for the slope of the volatility/strike function, and smile for its curvature.

Thus, when a Black formula is used with a certain input volatility, the user is not stating that they believe the underlying is subject to Geometric Brownian Motion at that level of volatility - all that is happening is that the user believes the output premium is the correct one. Nevertheless, the volatility input is useful as a reference, for comparison with other prices and indeed other models, and for the Greek outputs, which are fundamental for hedging.

Because of the leverage effect, the graph of implied volatility looks like a smile or skew. Furthermore, as many index options approach expiration, their smile charts become very steep. In other words, out-of-the-money puts trade at very high implied volatilities relative to the at-the-moneys. These skews are often explained by the demand of portfolio hedgers for protection against losses: they pay more because if the market falls, then volatility will probably rise, and so the puts have become valuable in two different ways - they have gone in-the-money and volatility is higher. To compensate, sellers have to charge more. One way to consider these issues is to consider models with a (negative) correlation between stock prices changes and volatility changes. A negative correlation means that volatility tends to move up when stock price moves down and vice-versa. The models of **?** and **?** are the original benchmark models in this class. Recently the model in **?** has become the standard model for quoting many options, if not for actual trading purposes.

In summary: if there was no skew then the market believes that the underlying is lognormally distributed. In reality the market believes the distribution is leptokurtotic (excessively peaked and fat-tailed). This modification is brought in via the skew. There is a precise mathematical relationship between any distribution function and any skew - **?**, **?**.

To determine this relationship, we proceed as follows: for a SAFEX Black contract, the call option price is given by

$$C = \int_K^\infty (F - K) f(F)\, dF \tag{10.8}$$

where K is the strike and $f(F)$ is the TRUE risk neutral probability density function for the expiry of the option. If $f(F)$ was log-normal, it is a straightforward calculus exercise to derive the Black formula for option pricing. However, f is the actual distribution - with fat tails and so on. Differentiating with respect to K, we get

$$\frac{\partial C}{\partial K} = -\int_K^\infty f(F)\, dF \quad ^{4} \tag{10.9}$$

and again, we get

$$\frac{\partial^2 C}{\partial K^2} = f(K) \tag{10.10}$$

Thus, the risk neutral probability density function as a function of strike is given by $\frac{\partial^2 C}{\partial K^2}$. This of course, is a theoretical result, in that it assumes that a continuum of quoted strikes are available. In reality, only a discrete set of strikes is available. To obtain the continuum, one needs to interpolate. It is known that the most amenable interpolation is natural cubic interpolation on the call prices.

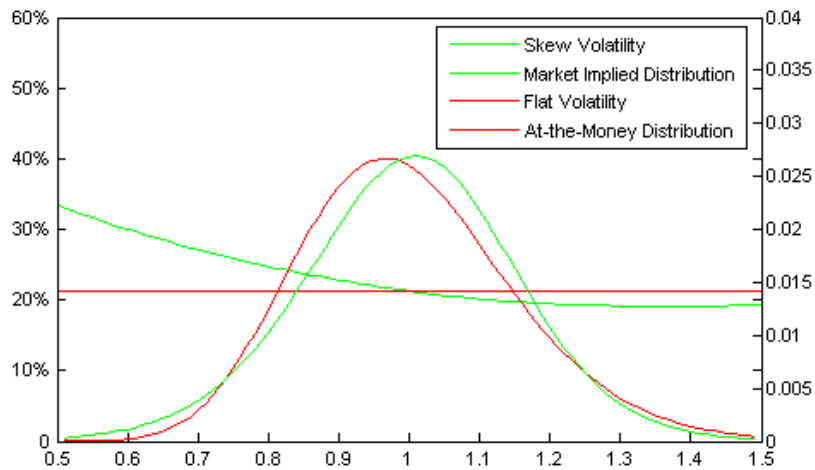


Figure 10.4: The skew and at-the-money volatilities, and the skew and lognormal distributions corresponding to these volatilities. We have a skew, so a thick left tail and a thin right tail.

10.4 Real Africa Durolink

RAD failed early April 2001.

They had a margining call of about R200m when the new SAFEX margining system began using a skew for the margining of SAFEX futures options. Until then the SAFEX margining system and the RAD MtM method had been on a flat volatility. In fact, RAD had paid large bonuses to dealers based on this flawed MtM method.

Furthermore, their exposure was huge. "Even for a big bank, RAD's exposure to equity derivatives would have been excessive. For a small bank, it is simply unthinkable. At the very least there should have been limits on the size of the open position the traders could take." (Anonymous source, Business Day, April 12 2001.) I have been told that the executive at RAD instructed that positions be cut, but this was ignored, and the situation persisted.

[4]Here we are using the Leibnitz rule, for differentiation of a definite integral with respect to a parameter Abramowitz and Stegun [1974]:

$$\frac{d}{d\xi}\int_{\psi(\xi)}^{\phi(\xi)} f(\psi,\xi)\,d\psi = f(\phi(\xi),\xi)\frac{d\phi(\xi)}{d\xi} - f(\psi(\xi),\xi)\frac{d\psi(\xi)}{d\xi} + \int_{\psi(\xi)}^{\phi(\xi)} \frac{d}{d\xi} f(\psi,\xi)\,d\psi$$

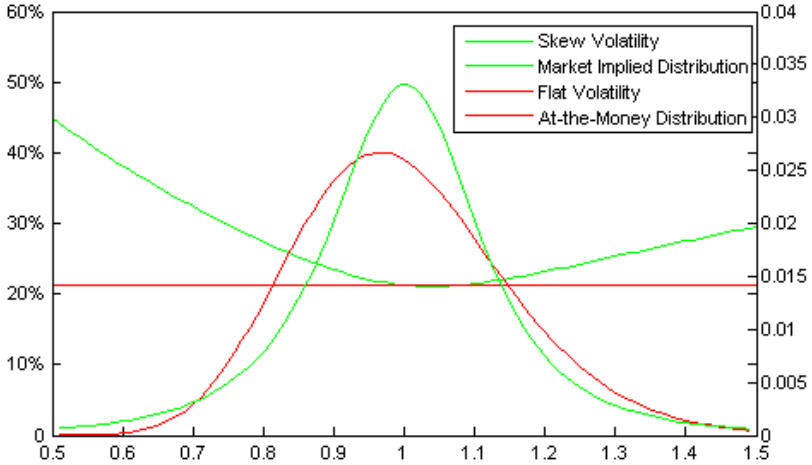


Figure 10.5: Another set of skew and at-the-money volatilities, and the skew and lognormal distributions corresponding to these volatilities. We have a smile, so both tails are thick.

One of the reasons for the failure of RAD was a lack of skills and understanding at the bank. Probably the only person at the bank who knew of the existence of a skew was the dealer James McIntyre (who had no reason to share his knowledge, because this would expose the flaw in his strategy). RAD failed to replace skilled Risk Managers who had left the bank earlier in the year. The book was bought by PSG bank. This deal seemed to typify the PSG approach of buying assets cheaply: PSG acted as the undertaker of the South African banking sector.

10.5 Exercises

1. Find the rand premium for a Dec08 futures put option with the following criteria:
   - Valuation Date: 1-Jun-08
   - Underlying: abt
   - Strike: 100.00
   - Volatility: 42.00%
   - Futures price: 105.00
2. You are managing a portfolio of futures and futures options. If you are short 255 Mar09 ALSI40 futures call options with a strike of 33000.00 and long 150 Mar09 ALSI40 futures what is the mark-to-market of your portfolio (in Rands) on 30-Jun-08? The futures price is 29000.00 and the volatility is 33.00%.
3. (Wits exam 2005) Write a vba function to price SAFEX futures options. For the cumulative normal function, use the cumnorm6 function provided, calling this function from within vba.

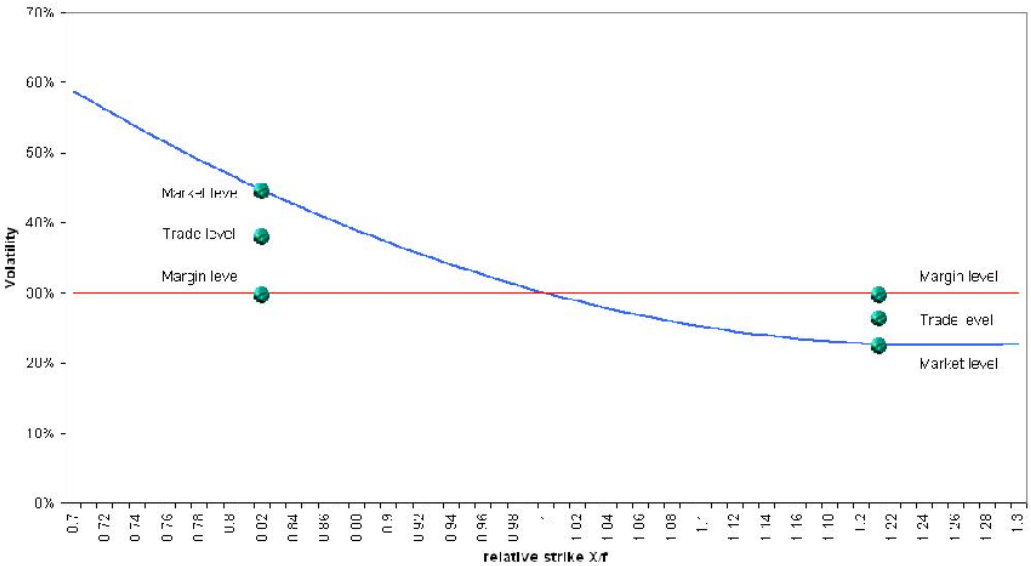


Figure 10.6: The problem with the RAD trading method

Suppose I am long 10000 TOP40 futures call options for expiry March 2006 on 26 and 28 April 05, strike 12050 and short 5000 futures for the same expiry. By using the information on the SAFEX MtM sheets provided, calculate the MtM of my portfolio on 26 and 28 April 05, and hence determine the change in portfolio value between those two days.

Answer the following questions:

(a) What is the delta of the call option on the first day?

(b) Hence, say why the futures position is a delta hedge for the call position.

(c) Nevertheless, if your working is correct, you will find that the p&l between the two days is nearly R7m. Why has that happened? Is it because the hedge is only approximate, because the gains are over two days rather than one, or what?

4. (Wits exam 2005) Explain heuristically how volatility skews and smiles translate to fat or thin tails of the pdf of prices, when we move from an analysis of implied volatility skew to an analysis of the pdf.

   Breeden and Litzenberger showed how this relationship could be made completely precise, by calculating the exact mathematical relationship between the volatility skew, the SAFEX option prices at those volatilities, and the probability density function for the futures level on the expiry date. Carefully state and prove their main result.

   Suppose I have an opinion on what the expiry date pdf should look like. Explain how this could be translated into a skew for trading, and how I would exploit differences between my skew and the skew trading in the market.

5. (UCT exam 2008) Suppose on 1 Jul 2008 the futures spot for expiry 19 March 2009 is 30190. A derivatives dealer provides a collar to a client (thus, the dealer is long the call and short the put). The dealers MtM details are as follows:

	position1	position2
style	call	put
size	100	-100
strike	34000	28000
skew volatility	25%	29%

(a) Build a vba function that will price a SAFEX futures option. The inputs to the function will be: futures spot, strike, valuation date, expiry date, skew volatility, style (± 1), and an optional required as string. The required could be 'p' (for premium) or 'd' (for delta).

(b) Decide on the appropriate futures hedge for the dealer (so that their aggregate delta will be approximately 0). Of course, you can only trade whole numbers of futures.

(c) The dealer does the futures hedge. What is the MtM of his portfolio? (Check: about -20.6m.)

Chapter 11

More on interest rate derivatives

11.1 Options on the bond curve

What remains to consider is bond options.

There is a long history of an OTC market in South Africa in bond options. The market was very liquid, with numerous underlyings, until the market crash of 1998. Then many participants withdrew from the market on the back of large losses.[1] The market is now making something of a recovery, but options are almost exclusively on the r153.

There are bond futures and bond futures options traded at SAFEX, but they are not very liquid.

Options are European or American. Options are struck on yield, because bonds trade on yield. This only causes a complication in the case of American options.[2] At different times during the American option the same ytm gives different prices (and we settle on price) but for a European option there is only one time, so whether the strike is yield or price is simply a matter of convention.

ssd rules apply at both the beginning and end of the option. This causes complications eg. exercise date is T but the cash flows and the deal is booked at BESA for ssd(T).

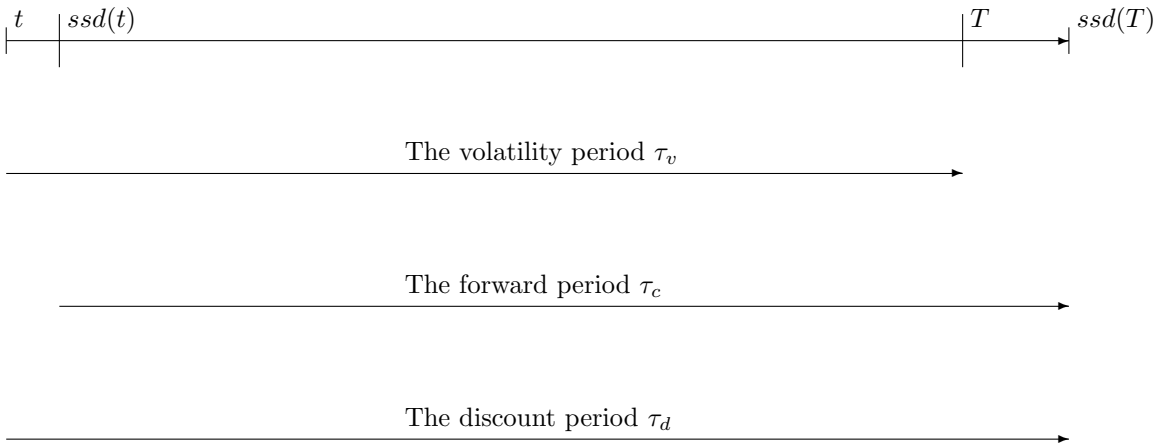
Models in use are typically very primitive. Equity option type models are used - there is an implicit assumption that the underlying is independent of the risk free rate, in equity that assumption is not too unreasonable, whereas for bonds it is not very sensible.

11.1.1 OTC European options

The carry rate - determined from the carry market - may very well be different from the discount rate, which may for example be the risk free rate derived from the yield curve. As a rule of thumb, the carry rate (very often a simple rate) will be used to determine forward prices using our carry model. The discount rate (from our curve, so it is typically NACC) will be used for calculating present values.

[1] In a market crash, with the rates of the position moving so quickly, it becomes impossible to hedge option positions. This causes losses.

[2] If I have a European option, and say it is struck on yield at a yield of $\mathbb{K}_y$, or say it is struck on price at an all in price $\mathbb{K}_{\mathbb{A}}$ which corresponds to $\mathbb{K}_y$ via the BESA formula - it is the same option.



The different time periods in the European bond option formula

The all in price option calculator

$$
\begin{aligned}
V_c &= Z(t, ssd(T))\left[\mathbb{f}_{\mathbb{A}} N(d_1) - \mathbb{K}_{\mathbb{A}} N(d_2)\right] && (11.1)\\
V_p &= Z(t, ssd(T))\left[\mathbb{K}_{\mathbb{A}} N(-d_2) - \mathbb{f}_{\mathbb{A}} N(-d_1)\right] && (11.2)\\
d_{1,2} &= \frac{\ln \frac{\mathbb{f}_{\mathbb{A}}}{\mathbb{K}_{\mathbb{A}}} \pm \frac{1}{2}\sigma^2 \tau_v}{\sigma\sqrt{\tau_v}} && (11.3)\\
\mathbb{f}_{\mathbb{A}} &= \text{carry of underlying over } \tau_c\\
\mathbb{K}_y &= \text{strike yield}\\
\mathbb{K}_{\mathbb{A}} &= \mathbb{A}(\mathbb{K}_y, ssd(T))\\
\sigma &= \text{implied AIP vol at } t \text{ for expiry } T.
\end{aligned}
$$

The clean price option calculator

$$
\begin{aligned}
V_c &= Z(t, ssd(T))\left[\mathbb{f}_{\mathbb{C}} N(d_1) - \mathbb{K}_{\mathbb{C}} N(d_2)\right] && (11.4)\\
V_p &= Z(t, ssd(T))\left[\mathbb{K}_{\mathbb{C}} N(-d_2) - \mathbb{f}_{\mathbb{C}} N(-d_1)\right] && (11.5)\\
d_{1,2} &= \frac{\ln \frac{\mathbb{f}_{\mathbb{C}}}{\mathbb{K}_{\mathbb{C}}} \pm \frac{1}{2}\sigma^2 \tau_v}{\sigma\sqrt{\tau_v}} && (11.6)\\
\mathbb{f}_{\mathbb{C}} &= \text{clean carry of underlying over } \tau_c\text{[3]}\\
\mathbb{K}_{\mathbb{C}} &= \mathbb{C}(\mathbb{K}_y, ssd(T))\\
\sigma &= \text{implied clean vol at } t \text{ for expiry } T.
\end{aligned}
$$

As clean implied volatilities are not quoted, the last formula above is rather moot. Moreover, almost all traded bond options in South Africa are American.

[3] This is by definition the usual all in carry $\mathbb{f}_{\mathbb{A}}$ minus the accrued interest on $ssd(T)$.

All-in Bond Option					days	years
Valuation (t)	30-May-07	ssd(t)	04-Jun-07	τ_d	1259	3.449
Bond	r186			τ_c	1254	3.436
YTM	8.07%			τ_v	1254	3.436
All in price	1.2844831					
Strike yield	8.50%	$\mathbb{K}_{\mathbb{A}}$	1.2140293			
Expiry (T)	04-Nov-10	ssd(T)	09-Nov-10			$N(\cdot)$
Vol	8.00%			d_1	0.0447	0.5178
Carry convention	simple	$\mathbb{F}_{\mathbb{A}}$	1.2517569	d_2	-0.1036	0.4588
Carry rate	9.000%			$-d_1$	-0.0447	0.4822
Discount convention	nacc			$-d_2$	0.1036	0.5412
Discount rate	9.250%					
Discount factor	0.726830743					
	European Call		European Put			
Premium	0.067794		0.040373			

Table 11.1: OTC European options: pricing on all in forward and strike prices

Clean Bond Option					days	years
Valuation (t)	30-May-07	ssd(t)	04-Jun-07	τ_d	1259	3.449
Bond	r186			τ_c	1254	3.436
YTM	8.07%			τ_v	1254	3.436
Strike yield	8.50%	$\mathbb{K}_{\mathbb{C}}$	1.1734677			
Expiry (T)	04-Nov-10	ssd(T)	09-Nov-10			$N(\cdot)$
Vol	8.27%			d_1	0.2831	0.6115
Carry convention	simple	$\mathbb{F}_{\mathbb{C}}$	1.2111953	d_2	0.1299	0.5517
Carry rate	9.00%			$-d_1$	-0.2831	0.3885
Discount convention	nacc			$-d_2$	-0.1299	0.4483
Discount rate	9.25%					
Discount factor	0.726830743					
	European Call		European Put			
Premium	0.067764		0.040343			

Table 11.2: OTC European options: pricing on clean forward and strike prices

11.1.2 OTC American options

For American options, need tree models (trees of prices, not yields cf. BDT model) or more advanced models eg. HW, HJM etc. However, for the latter, the required driving data might not be available, and in the South African market these models might not converge.

Suppose we are using a tree model. Note that the strike is a yield, so there is a different price for each date on the tree. The nodes of the trees must take into account the exact price that would

correspond, under standard settlement rules, to the node. One approach is to build a tree of yields. However, the no arbitrage condition that the up and down probabilities at any node must be positive can be violated.

Another approach is to build a tree of squeaky clean price. Thus one needs to convert the input volatility (be it all-in or yield) to a clean volatility using Ito's lemma. The tree can then be built. A well built tree will take into account the fact that volatility has a term structure, and the pull to par effect.

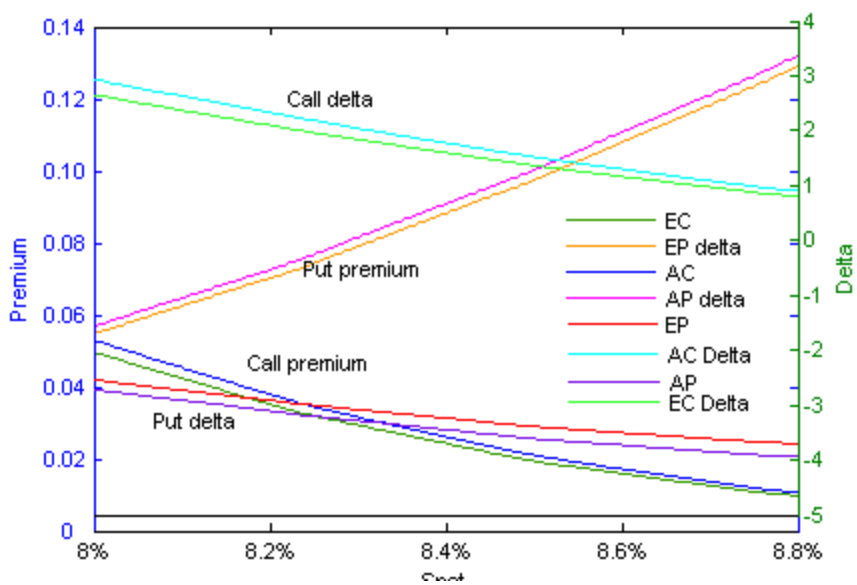


Figure 11.1: Premiums and deltas over a 200bp strike range (spot at centre)

In most models that are built for American options (all proprietary) there can be early exercise opportunities for both calls and puts, such as the model built for RiskWorX by the author - West [2004].

For more commentary on this issue, see Etheredge and West [1999].

11.1.3 Bond Volatility

Two different types of bond volatility can be defined: yield volatilty and price volatility. Price volatility divides into clean price volatility and all-in-price volatility, although, on a rolling (historical) basis, the latter is not meaningful since it is influenced by sudden changes in accrued interest. However, for a fixed expiry date, all three volatility measures for that date are equally meaningful. Define the following variables:

- $\sigma_{\mathbb{A}}$ = the all-in-price volatility of the bond,
- $\sigma_{\mathbb{C}}$ = the clean price volatility of the bond,
- σ_y = the yield volatility of the bond,

- $\sigma_{\mathbb{f}_{\mathbb{A}}}$ = the forward all-in-price volatility of the bond,
- $\sigma_{\mathbb{f}_{\mathbb{C}}}$ = the forward clean price volatility of the bond,
- $\sigma_{\mathbb{f}_y}$ = the forward yield volatility of the bond,

The relationship between the various volatilities of the bond is given via Ito's lemma as

$$\sigma_{\mathbb{A}} = -\frac{\sigma_y y \Delta}{\mathbb{A}} \tag{11.7}$$

$$\sigma_y = -\frac{\sigma_{\mathbb{A}} \mathbb{A}}{y \Delta} \tag{11.8}$$

$$\sigma_{\mathbb{A}} = \frac{\mathbb{C}}{\mathbb{A}} \sigma_{\mathbb{C}} \tag{11.9}$$

The formula's for the forward volatilities are forward versions of the above equations:

$$\sigma_{\mathbb{f}_{\mathbb{A}}} = -\frac{\sigma_{\mathbb{f}_y} \mathbb{f}_y \mathbb{f}_\Delta}{\mathbb{f}_{\mathbb{A}}} \tag{11.10}$$

$$\sigma_{\mathbb{f}_y} = -\frac{\sigma_{\mathbb{f}_{\mathbb{A}}} \mathbb{f}_{\mathbb{A}}}{\mathbb{f}_y \mathbb{f}_\Delta} \tag{11.11}$$

$$\sigma_{\mathbb{f}_{\mathbb{A}}} = \frac{\mathbb{f}_{\mathbb{C}}}{\mathbb{f}_{\mathbb{A}}} \sigma_{\mathbb{f}_{\mathbb{C}}} \tag{11.12}$$

11.2 Options on the swap curve

With the evolution of the swap curve, various options on the swap curve instruments trade. The liquidity here is good. Principle vanilla instruments are swaptions, caplets and floorlets, and caps and floors.

These instruments are in every way like those that trade internationally, so any textbook will provide adequate information. See also West [2009a].

11.3 Futures market

The JSE launched Yield-X, an interest rate futures exchange, early in 2005.

The following products (all either interest bearing securities or derivatives of interest bearing securities) are being listed and traded on Yield-X:

- j-Carries: Buy-sell back transactions
- j-Rods: RODI swaps against 1; 3; 6; 9 or 12 month traded rate
- j-Swaps: Bond look-alike swaps
- j-Notes: futures on notional swaps
- j-FRA's: Forward Rate Agreements
- j-Options: Options on futures

- j-Futures: Futures on bonds
- j-Bonds: Spot and forward bonds (secondary listing of spot bonds on which the JSE currently has futures)

The same principles as at SAFEX apply: futures and futures options are fully margined, so in the case of options, the SAFEX Black formula will apply.

Mark to market and margining are all driven by the yield curve generated by BEASSA. As this is a perfect fit curve, it is guaranteed to properly produce settlement numbers. The construction method of the yield curve is relevant for margining calculations and for 'between reset' valuations.

The idea of launching the market was to introduce an electronic interest rate market which overcomes credit risk issues which are always present in an OTC market. However, many interest rate derivatives, which were previously listed by SAFEX were never traded, and the launch of Yield-X has not necessarily made them more attractive. This market has remained grossly illiquid since inception. As far as we are aware, no banks have been able or inclined to replicate in in-house systems the margining calculations required at Yield-X.

As an example of pricing, bond futures options will proceed as follows:

$$V_C = \mathbb{F}_{\mathbb{A}} N(d_1) - \mathbb{K}_{\mathbb{A}} N(d_2) \tag{11.13}$$

$$V_P = \mathbb{K}_{\mathbb{A}} N(-d_2) - \mathbb{F}_{\mathbb{A}} N(-d_1) \tag{11.14}$$

$$d_{1,2} = \frac{\ln \frac{\mathbb{F}_{\mathbb{A}}}{\mathbb{K}_{\mathbb{A}}} \pm \frac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}} \tag{11.15}$$

$$\begin{aligned}
t &= \text{valuation date} \\
T &= \text{option expiry date} \\
\tau &= T - t, \text{ the volatility period of the option} \\
\mathbb{F}_y &= \text{mark to market future yield of the bond quoted by SAFEX} \\
\mathbb{F}_{\mathbb{A}} &= \mathbb{A}(\mathbb{F}_y, ssd(T)) \\
\mathbb{K}_y &= \text{strike yield of the option} \\
\mathbb{K}_{\mathbb{A}} &= \mathbb{A}(\mathbb{K}_y, ssd(T)) \\
\sigma &= \text{bond future all-in-price volatility quoted by SAFEX}
\end{aligned}$$

Bond futures and futures options closeouts are on the first Thursday of February, May, August and November.

11.4 Exercises

1. What is the price of a European OTC call option on a 1,000,000 nominal r153 Bond with the following details:
   - Deal Date: 1-Jul-08
   - Carry rate: 12.00%

- Discount rate: 11.60%
- Spot Yield: 11.50%
- Strike Yield: 11.25%
- Price Volatility: 12.00%
- Option Expiry: Nov09

Chapter 12

The Pricing of BEE Share Purchase Schemes

A version of this chapter appears as West and West [2009].

12.1 Introduction and history

Black Economic Empowerment (BEE) transactions are very topical, with companies that have to date failed to put in place meaningful BEE schemes under serious pressure from various interested parties. What is involved is the sale of a stake in the business to suitably qualified partners.
The Broad-Based Black Economic Empowerment Act (2003) was introduced in order to assist in ensuring a more equitable distribution of wealth amongst the people of South Africa. This act defines 'black people' and introduces legal mechanisms for their economic upliftment. The core feature of these transactions is that the seller (vendor company) exchanges company value for BEE credentials. The buyer (BEE partner) provides

- the legal requirement
- avenues to new business, in particular statal and parastatal businesses
- promotion of the vendor company, in particular as a BEE compliant company
- assistance for the vendor company in staffing, affirmative action and its social responsibility programmes.

Under most circumstances the designated partner does not have the resources to pay cash for their stake. Thus, structures need to be put in place to facilitate the purchase of the designated stake. In the earliest stages of BEE these transfers were achieved somewhat cynically, by means of fronting. More or less any transfer of equity would be as a gift, in return for the privilege that the vendor would then enjoy of having a black name on letterheads. The next stage is the vendor financing stage. Typically the structure consists of the following scheme: shares are transferred to the BEE partners by the vendor company at or at about market value. To pay for this, typically a small

cash payment (usually 1% to 5%) is made, but the great majority of the payment is set up as debt issued by the BEE partners. The vendor company is the party that buys this debt.

During the debt period the transfer of shares is a legal transfer of ownership; in particular the BEE partner has voting rights.

Over time, the debt rolls up with interest, and rolls down with dividend flows that are received from the shares. At termination, if the share value is higher than the outstanding debt, the BEE partners keep their shares and pay off the outstanding debt, or surrender sufficient shares to pay off the debt. If lower, the BEE partners also surrender their shares, but walk away from the debt.

Thus, what the BEE partner owns is a European call option on the shares with the strike being the level of debt. This option is somewhat exotic because the strike of the otherwise vanilla call option is not known in advance. Reading statements made by participants in these schemes can make for mildly amusing reading given this understanding. When the vendor company wishes to trumpet the successful creation of a BEE structure, they announce that such-and-such a percentage of the company is now held by black partners. On the other hand, if they are being pressured about a grant which some parties - such as the financial press, for example - are viewing as too generous, then they will brazenly retort that the partner owns precisely nothing until such time as the debt has been paid down. As we now see, neither statement is correct: the truth lies somewhere in between. In the next stage, institutional financing comes into play. In these cases, real money is needed to facilitate the transaction - for example, minorities may need to be bought out. In this case equity is purchased using financing from banks and other financial institutions. The financial institution structures their asset into a cascade, with senior, mezzanine and equity components. The senior and/or mezzanine tranches receive a spread above typical interest rates in the market, and will enjoy covenants on the equity. The financial institution might buy the senior tranche in its entirety, and will participate in equity upside.

Currently most common is a mix of the vendor and institutional financing models. We are now starting to see a few straight purchases: BEE companies will have enjoyed income from previous transactions that they can use to enter into new trades. This cuts out the institutions who are acting as quite pricey middle men here. We believe that this type of transaction will become more and more common.

12.2 The BEE Transaction is a Call Option

The vendor company transfers shares to the BEE partner. The BEE partner pays a small deposit, but is unable to pay the full amount. The BEE partner then issues a bond and uses the proceeds to pay the remainder. The vendor company (vendor financing) or a financial institution(s) (institutional financing) or some combination subscribes to this bond.

To fix ideas let us focus for starters on the vendor financing model.

Over time (a period of about 10 years, say) this debt

- rolls up at a fixed rate or is linked to some floating rate such as Prime or JIBAR or the inflation rate
- is paid down by dividends.

The trade sits in an SPV, ring-fenced from any other transactions the BEE company makes. At maturity of the transaction, if the SPV's value of equity holding is

- greater than the debt level, the BEE partner uses the equity to pay down the debt and keep the remainder
- less than the debt level, the BEE partner (technically, the SPV) walks away from the deal.

The only asset of the SPV is these shares, and the only liability this debt. Some vendor companies operate under the erroneous impression that at termination they will have legal recourse to ensure that the BEE party will honour the full debt amount that has accrued. Even in the case where a BEE company owns several assets, one will find that each asset is held by a separate SPV.

Thus, what the BEE partners own (technically, what the SPV own) is a European call option on the shares with the strike being the level of debt. (Sometimes this debt is paid down to zero before the termination date of the deal, in which case there is early termination.) However, as we have already noted, we cannot apply a classic option pricing formula because the strike is not known in advance.

As the vendor company has given the BEE partners the option, it needs to be expensed at inception in accordance with IFRS2 IASCF [2006]. Even in the case where, for example, majority shareholders of the vendor company facilitate the BEE deal on behalf of the company, the company itself will have to recognise the expense so as to be compliant with 'push down' accounting regulations. The expense will be the fair mark-to-market value of the option. The only income they enjoy is the small physical cash payment that was made by the BEE partners. The bond issued by the BEE partners can obviously not be recognised as income.

As we will argue here, to provide appropriate valuation of this option we need to use a Monte Carlo valuation technique. Monte Carlo will pose no implementation problems as the option itself is European.

Roughly speaking, in each Monte Carlo sample, at maturity we test the value of the proportion of the company granted to the BEE partners versus that of the accrued loan; the terminal value of the deal to the BEE partners is the difference if positive. See Figure 12.1 and Figure 12.2.

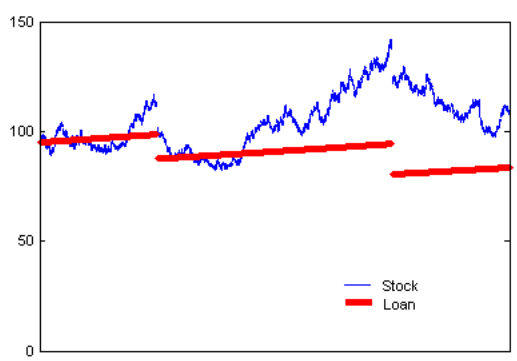


Figure 12.1: A path ending in the money

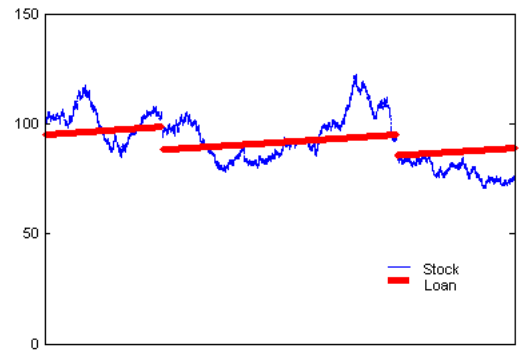


Figure 12.2: A path ending out of the money

12.3 Naïve use of Black-Scholes

Many participants in this market price this option using the Black-Scholes approach, with the strike of the option being set as the forward level of the debt. However, as we shall see, this approach is invalid, due to the path dependency of that level of debt, and can lead to severe mispricing of the option.

When using the Black-Scholes formula for valuation of these options, the strike is calculated as the forward level of the debt. Here the terminal strike price has to be determined "outside the model" (this terminology appears in van der Merwe [2008]) and entered as a fixed input. This treatment ignores the fact that the true final debt amount will depend on the dividends the share pays over the life of the transaction, thus, on the share price evolution: the option is path dependent. The final level of the debt may not be equal to the forward level of the debt.

Suppose as an example a tenor of 10 years, a spot price of 1, the initial debt of 0.95 (thus, the deposit was 5%), a volatility of 30%, a flat risk free rate of 10% (applying both to equity growth and to the debt), and dividends as in Table 12.1.

Term (years)	Amount	Type
0.5	0.02	cash
0.9	0.01	cash
1.5	0.02	cash
1.9, 2.9, ...	1%	yield
2.5, 3.5, ...	2%	yield

Table 12.1: Dividend Schedule

We find the forward level of debt to be 2.08.

The incorrect Black-Scholes approach using the "out of model" strike of 2.08 gives the value of the option of 0.277. With a finite difference approach with the "out of model" strike we get a valuation of 0.266.

The correct approach using Monte Carlo gives a value of 0.331.

A common variation is to declare a large first dividend, to give the debt paydown 'a helping hand'. Of course, this reduces the valuation of the option being granted (which from the vendor point of view will be a good thing). For example, if in this example the first dividend is instead 0.07 cash, with all other dividends unchanged, then the "out of model" strike is 1.97, the Black-Scholes valuation is 0.267, the finite difference valuation is 0.254, and the Monte Carlo valuation is 0.317.

If we suppose the same details except that all dividends are yield dividends, with those dividends being broadly proportional to the cash dividends where provided, the results are similar.

Nevertheless, there are two instances where using the Black-Scholes formula is valid.

Suppose all the dividends are known, or can confidently be forecast as cash amounts. (This is quite unrealistic except when a deal nears maturity.) Then the 'out-of-model' calculation of the strike is valid, and so Black-Scholes can be used - although a different argument will say that it shouldn't, as conversion of cash dividends to a dividend yield for use in the formula is problematic for almost the same reason - see Frishling [2002], Bos and Vandermark [2002].

The only practical occasion in which the Black-Scholes approach is valid is when dividends are paid using the stock, and that stock is paid into the basket of assets, with the debt never being paid down. In this case we use the Black-Scholes with a zero dividend yield.

In preparation for the Nedbank and Mutual and Federal BEE transactions, those companies introduced the option for all shareholders to receive dividends in cash or as stock. (A shareholder's broker will contact them each dividend LDR to determine their election.) However, the BEE partner contractually elected to receive dividends as stock.

The Nedbank deal was restructured in May 2008, as it was felt that too much dilution of the stock was occurring. The modification is that the SPV will receive cash dividends, and will be obligated to buy shares in the open market with that cash, for inclusion in the basket of assets. Thus the valuation approach is unchanged, but the dilution problem is avoided.

12.4 Naïve use of Binomial Trees

Let us now consider pricing such an option in a two step binomial tree. We will explicitly see a numerical example of what can go wrong. Because of the path dependency, the debt tree will not recombine, even if we can ensure the stock price tree recombines.

At time $t = 0$, let $S = 20$, $r = 9\%$, $\sigma = 30\%$, simple dividend after one year of $q = 6\%$ of the stock price at that time, initial debt level $D = 17$, and initial deposit $= 3$.

We consider a two year option and build a two step tree (steps of one year each).

From Cox et al. [1979] we find the up factor $u = 1.350$, the down factor $d = 0.741$, and the risk neutral probability of an up move $\pi = 0.580$.

We calculate the expected dividend

$$q\mathbb{E}\left[S(1)\right] = qSe^{r}.$$

The forward level of debt is calculated as

$$\left(De^{r} - qSe^{r}\right)e^{r} = \left(D - qS\right)e^{2r} = 18.916.$$

However, notice how in Figure 12.3 the debt is path dependent. If the stock goes up and then down, the debt is 18.58; if it goes down and then up, the debt is 19.38.

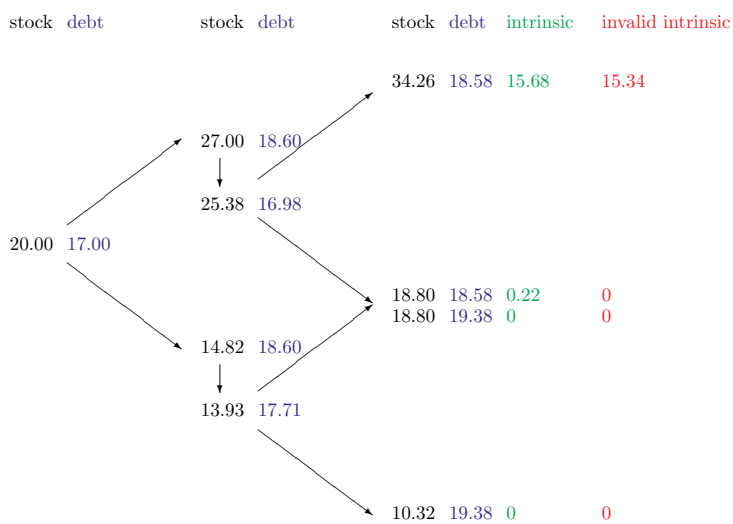


Figure 12.3: The two step binomial tree

- We can price incorrectly, using a call strike of 18.916 throughout, and discounting through the tree in the usual way. Then value then is $e^{-2r}[\pi^2 15.34] = 4.313$. All other paths end out of the money.

- We can price incorrectly, using a call strike of 18.916 throughout, and applying the Black-Scholes formula. We first need to find the dividend yield continuous per annum: this is the value y in $1 - 0.06 = e^{-2y}$, giving $y = 3.094\%$. Then using the Black-Scholes formula we get an option value of 4.643.

  Remember the Black-Scholes price is the limit, as we increase the number of time steps, of the price found using a binomial tree.

- Pricing correctly (within this binomial world) we discount along every path one at a time, gives a value of $e^{-2r}[\pi^2 15.68 + \pi(1-\pi)0.22] = 4.452$.

- Pricing correctly using a Monte Carlo approach gives a value of approximately 4.744.

It may be possible to use a many-stepped tree in the spirit of this example with a path dependent extra factor (being the level of debt) as in [Hull, 2005, §24.4]. However, this approach only works while the underlying tree of stock prices is a recombining tree. Usually this doesn't occur: it only occurred even in our simple two step example, because there was a simple dividend yield.
The problem is if we want to model that the stock pays some discrete dividends, and not only percentage dividends. This will almost always be the case, as broker forecasts will be of cash amounts for the next two or three years, say. Furthermore there will often be other complicating features, which make Monte Carlo pricing almost a forced approach.

Our preferred approach is as follows: use broker forecasts in the short and medium term, and then forecast percentage dividends in the long term using the model of [West, 2009b, §6.6].

12.5 The institutional financing style of BEE transaction

It has become more common for institutions to be involved in the financing of BEE transactions. In this case, the institution buys the bond with real money which is passed to the vendor. In the simplest such cases, the vendor receives full and immediate value for their sale, so they are not required to pass an expense as in §12.2.

Broadly speaking, the institution has bought a bond which will roll up with a contractually specified interest rate, and will roll down with dividends that they receive. At the termination date any residual value of the bond will be extinguished with equity, and the residual equity will accrue to the BEE partner.

Of course, reality will be far more complicated. The equity of the BEE partner is serving as the guarantee for the performance of the bond. Thus the bond will typically have covenants attached to it. If certain asset coverage ratios (ACRs) are not achieved then there might be contractual requirements to pay down the debt by liquidating part of the equity holding. In extreme cases, the deal might be unwound in its entirety. We have seen this occurrence in the market turbulence of early 2008.

Furthermore, it is typical for the bond to be structured as a typical securitised vehicle, with senior and mezzanine debt. In this case the covenants of the senior tranche take precedence over those of the mezzanine tranche. The portion belonging to the BEE partner will be the equity of the vehicle. Typically, the institution will share in this equity as well (the so called 'kicker').

12.6 Our approach

We have focused on the simplest issues that arise. There are much more complicated BEE transactions; some of the features that arise are

- trickle dividends (dividends that are received and are spent, leaving the system)
- the proportion of the dividends paid as trickle vs. those used to pay down debt is a function of some price or earnings target
- American/Bermudan features
- the need to have models of the forward prime curve or forward inflation curve. The debt rollup is often associated with one of these curves. We then have separate curves for the equity growth and for the debt growth.
- the covenants in the institutional financing case can be quite complicated.
- possibly the need to model the evolution of the asset rather than the equity.

These features can be handled by a careful use of Monte Carlo.

We take the usual approach in valuing equity derivatives: we assume that the various interest rates (risk free curve, prime curve, inflation curve) will evolve according to their forward levels. Only the stock price (and possibly its volatility) will evolve randomly.

We bootstrap our own yield curves using market rates and the methods in Hagan and West [2006], Hagan and West [2008].

For major stocks we use implied volatilities provided by a prominent market participant and dividend forecasts from a major broker. For smaller stocks we use historical volatility estimates and note the dividends that have occurred recently. In either case we then forecast percentage dividends further out in time using the simple 'repeating-yield' model of [West, 2009b, §6.6].

In these deals there are several important dates namely

- the commencement date, the effective date, the maturity date
- dividend declare, LDR and payment dates
- debt rollup and paydown dates
- trickle dividend dates.

There might be as many as 150 dates here, although 10-40 is most common. As already discussed, because of the path dependency, we are using the Monte Carlo method to price the transaction. Thus we need low discrepancy sequences in high dimensions. Sobol' sequences are most suitable here, see [Glasserman, 2004, §5.2.3].

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