# Chapter 31

# Cox-Ingersoll-Ross model

In the Hull & White model, $r(t)$ is a Gaussian process. Since, for each $t$, $r(t)$ is normally distributed, there is a positive probability that $r(t) < 0$. The Cox-Ingersoll-Ross model is the simplest one which avoids negative interest rates.

We begin with a $d$-dimensional Brownian motion $(W_1, W_2, \ldots, W_d)$. Let $\beta > 0$ and $\sigma > 0$ be constants. For $j = 1, \ldots, d$, let $X_j(0) \in I\!R$ be given so that

$$X_1^2(0) + X_2^2(0) + \ldots + X_d^2(0) \geq 0,$$

and let $X_j$ be the solution to the stochastic differential equation

$$dX_j(t) = -\tfrac{1}{2}\beta X_j(t)\; dt + \tfrac{1}{2}\sigma\; dW_j(t).$$

$X_j$ is called the *Orstein-Uhlenbeck* process. It always has a drift toward the origin. The solution to this stochastic differential equation is

$$X_j(t) = e^{-\frac{1}{2}\beta t}\left[X_j(0) + \tfrac{1}{2}\sigma \int_0^t e^{\frac{1}{2}\beta u}\; dW_j(u)\right].$$

This solution is a Gaussian process with mean function

$$m_j(t) = e^{-\frac{1}{2}\beta t} X_j(0)$$

and covariance function

$$\rho(s,t) = \frac{1}{4}\sigma^2 e^{-\frac{1}{2}\beta(s+t)} \int_0^{s \wedge t} e^{\beta u}\; du.$$

Define

$$r(t) \stackrel{\triangle}{=} X_1^2(t) + X_2^2(t) + \ldots + X_d^2(t).$$

If $d = 1$, we have $r(t) = X_1^2(t)$ and for each $t$, $I\!P\{r(t) > 0\} = 1$, but (see Fig. 31.1)

$$I\!P\left\{\text{There are infinitely many values of } t > 0 \text{ for which } r(t) = 0\right\} = 1$$

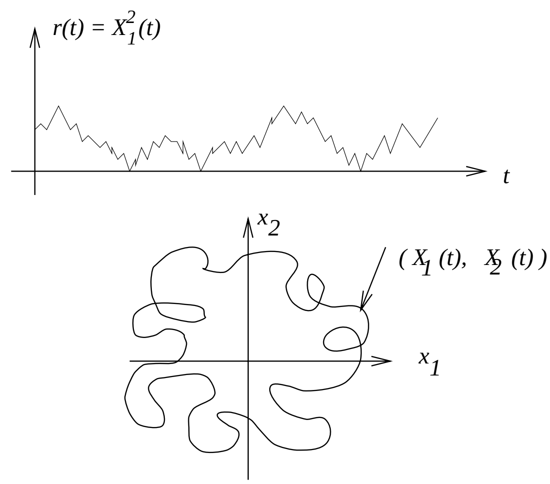


Figure 31.1: $r(t)$ *can be zero.*

If $d \geq 2$, (see Fig. 31.1)

$$\mathbb{P}\{\text{There is at least one value of } t > 0 \text{ for which } r(t) = 0\} \;=\; 0.$$

Let $f(x_1, x_2, \ldots, x_d) = x_1^2 + x_2^2 + \ldots + x_d^2$. Then

$$f_{x_i} = 2x_i, \quad f_{x_i x_j} = \begin{cases} 2 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Itô's formula implies

$$\begin{aligned} dr(t) &= \sum_{i=1}^{d} f_{x_i}\, dX_i + \tfrac{1}{2} \sum_{i=1}^{d} f_{x_i x_i}\, dX_i\, dX_i \\ &= \sum_{i=1}^{d} 2X_i \left(-\tfrac{1}{2}\beta X_i\, dt + \tfrac{1}{2}\sigma\, dW_i(t)\right) + \sum_{i=1}^{d} \frac{1}{4}\sigma^2\, dW_i\, dW_i \\ &= -\beta r(t)\, dt + \sigma \sum_{i=1}^{d} X_i\, dW_i + \frac{d\sigma^2}{4}\, dt \\ &= \left(\frac{d\sigma^2}{4} - \beta r(t)\right) dt + \sigma\sqrt{r(t)} \sum_{i=1}^{d} \frac{X_i(t)}{\sqrt{r(t)}}\, dW_i(t). \end{aligned}$$

Define

$$W(t) = \sum_{i=1}^{d} \int_0^t \frac{X_i(u)}{\sqrt{r(u)}}\, dW_i(u).$$

Then $W$ is a martingale,

$$dW = \sum_{i=1}^{d} \frac{X_i}{\sqrt{r}}\, dW_i,$$
$$dW\ dW = \sum_{i=1}^{d} \frac{X_i^2}{r}\, dt = dt,$$

so $W$ is a Brownian motion. We have

$$dr(t) = \left( \frac{d\sigma^2}{4} - \beta r(t) \right)\, dt + \sigma\sqrt{r(t)}\, dW(t).$$

The *Cox-Ingersoll-Ross (CIR) process* is given by

$$dr(t) = (\alpha - \beta r(t))\, dt + \sigma\sqrt{r(t)}\, dW(t),$$

We define

$$d = \frac{4\alpha}{\sigma^2} > 0.$$

If $d$ happens to be an integer, then we have the representation

$$r(t) = \sum_{i=1}^{d} X_i^2(t),$$

but we do not require $d$ to be an integer. If $d < 2$ (i.e., $\alpha < \frac{1}{2}\sigma^2$), then

$$I\!\!P\{\text{There are infinitely many values of } t > 0 \text{ for which } r(t) = 0\} \ = \ 1.$$

This is not a good parameter choice.

If $d \geq 2$ (i.e., $\alpha \geq \frac{1}{2}\sigma^2$), then

$$I\!\!P\{\text{There is at least one value of } t > 0 \text{ for which } r(t) = 0\} \ = \ 0.$$

With the CIR process, one can derive formulas under the assumption that $d = \frac{4\alpha}{\sigma^2}$ is a positive integer, and they are still correct even when $d$ is not an integer.

For example, here is the distribution of $r(t)$ for fixed $t > 0$. Let $r(0) \geq 0$ be given. Take

$$X_1(0) = 0,\ X_2(0) = 0,\ \ldots,\ X_{d-1}(0) = 0,\ X_d(0) = \sqrt{r(0)}.$$

For $i = 1, 2, \ldots, d-1$, $X_i(t)$ is normal with mean zero and variance

$$\rho(t,t) = \frac{\sigma^2}{4\beta}(1 - e^{-\beta t}).$$

$X_d(t)$ is normal with mean

$$m_d(t) = e^{-\frac{1}{2}\beta t}\sqrt{r(0)}$$

and variance $\rho(t,t)$. Then

$$r(t) = \underbrace{\rho(t,t)\sum_{i=1}^{d-1}\left(\frac{X_i(t)}{\sqrt{\rho(t,t)}}\right)^2}_{\text{Chi-square with } d-1=\frac{4\alpha-\sigma^2}{\sigma^2} \text{ degrees of freedom}} + \underbrace{X_d^2(t)}_{\text{Normal squared and independent of the other term}} \tag{0.1}$$

Thus $r(t)$ has a *non-central chi-square distribution.*

## 31.1 Equilibrium distribution of $r(t)$

As $t\to\infty$, $m_d(t)\to 0$. We have

$$r(t) = \rho(t,t)\sum_{i=1}^{d}\left(\frac{X_i(t)}{\sqrt{\rho(t,t)}}\right)^2.$$

As $t\to\infty$, we have $\rho(t,t) = \frac{\sigma^2}{4\beta}$, and so the limiting distribution of $r(t)$ is $\frac{\sigma^2}{4\beta}$ times a chi-square with $d = \frac{4\alpha}{\sigma^2}$ degrees of freedom. The chi-square density with $\frac{4\alpha}{\sigma^2}$ degrees of freedom is

$$f(y) = \frac{1}{2^{2\alpha/\sigma^2}\Gamma\left(\frac{2\alpha}{\sigma^2}\right)} y^{\frac{2\alpha-\sigma^2}{\sigma^2}} e^{-y/2}.$$

We make the change of variable $r = \frac{\sigma^2}{4\beta}y$. The limiting density for $r(t)$ is

$$\begin{aligned} p(r) &= \frac{4\beta}{\sigma^2}.\frac{1}{2^{2\alpha/\sigma^2}\Gamma\left(\frac{2\alpha}{\sigma^2}\right)}\left(\frac{4\beta}{\sigma^2}r\right)^{\frac{2\alpha-\sigma^2}{\sigma^2}} e^{-\frac{2\beta}{\sigma^2}r} \\ &= \left(\frac{2\beta}{\sigma^2}\right)^{\frac{2\alpha}{\sigma^2}} \frac{1}{\Gamma\left(\frac{2\alpha}{\sigma^2}\right)} r^{\frac{2\alpha-\sigma^2}{\sigma^2}} e^{-\frac{2\beta}{\sigma^2}r}. \end{aligned}$$

We computed the mean and variance of $r(t)$ in Section 15.7.

## 31.2 Kolmogorov forward equation

Consider a Markov process governed by the stochastic differential equation

$$dX(t) = b(X(t))\,dt + \sigma(X(t))\,dW(t).$$

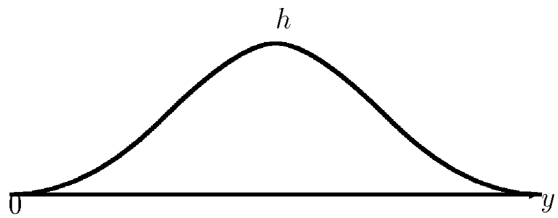


Figure 31.2: *The function* $h(y)$

Because we are going to apply the following analysis to the case $X(t) = r(t)$, we assume that $X(t) \geq 0$ for all $t$.

We start at $X(0) = x \geq 0$ at time 0. Then $X(t)$ is random with density $p(0,t,x,y)$ (in the $y$ variable). Since 0 and $x$ will not change during the following, we omit them and write $p(t,y)$ rather than $p(0,t,x,y)$. We have

$$\mathbb{E}h(X(t)) = \int_0^\infty h(y)p(t,y)\; dy$$

for any function $h$.

The Kolmogorov forward equation (KFE) is a partial differential equation in the "forward" variables $t$ and $y$. We derive it below.

Let $h(y)$ be a smooth function of $y \geq 0$ which vanishes near $y = 0$ and for all large values of $y$ (see Fig. 31.2). Itô's formula implies

$$dh(X(t)) = \left[h'(X(t))b(X(t)) + \tfrac{1}{2}h''(X(t))\sigma^2(X(t))\right]\; dt + h'(X(t))\sigma(X(t))\; dW(t),$$

so

$$\begin{aligned} h(X(t)) &= h(X(0)) + \int_0^t \left[h'(X(s))b(X(s)) + \tfrac{1}{2}h''(X(s))\sigma^2(X(s))\right]\; ds + \\ &\qquad \int_0^t h'(X(s))\sigma(X(s))\; dW(s), \\ \mathbb{E}h(X(t)) &= h(X(0)) + \mathbb{E}\int_0^t \left[h'(X(s))b(X(s))\; dt + \tfrac{1}{2}h''(X(s))\sigma^2(X(s))\right]\; ds, \end{aligned}$$

or equivalently,

$$\int_0^\infty h(y)p(t,y)\ dy = h(X(0)) + \int_0^t \int_0^\infty h'(y)b(y)p(s,y)\ dy\ ds + \\ \tfrac{1}{2}\int_0^t \int_0^\infty h''(y)\sigma^2(y)p(s,y)\ dy\ ds.$$

Differentiate with respect to $t$ to get

$$\int_0^\infty h(y)p_t(t,y)\ dy = \int_0^\infty h'(y)b(y)p(t,y)\ dy + \tfrac{1}{2}\int_0^\infty h''(y)\sigma^2(y)p(t,y)\ dy.$$

Integration by parts yields

$$\int_0^\infty h'(y)b(y)p(t,y)\ dy = \underbrace{h(y)b(y)p(t,y)\Big|_{y=0}^{y=\infty}}_{=0} - \int_0^\infty h(y)\frac{\partial}{\partial y}\left(b(y)p(t,y)\right)\ dy,$$

$$\int_0^\infty h''(y)\sigma^2(y)p(t,y)\ dy = \underbrace{h'(y)\sigma^2(y)p(t,y)\Big|_{y=0}^{y=\infty}}_{=0} - \int_0^\infty h'(y)\frac{\partial}{\partial y}\left(\sigma^2(y)p(t,y)\right)\ dy$$

$$= \underbrace{-h(y)\frac{\partial}{\partial y}\left(\sigma^2(y)p(t,y)\right)\Big|_{y=0}^{y=\infty}}_{=0} + \int_0^\infty h(y)\frac{\partial^2}{\partial y^2}\left(\sigma^2(y)p(t,y)\right)\ dy.$$

Therefore,

$$\int_0^\infty h(y)p_t(t,y)\ dy = -\int_0^\infty h(y)\frac{\partial}{\partial y}\left(b(y)p(t,y)\right)\ dy + \tfrac{1}{2}\int_0^\infty h(y)\frac{\partial^2}{\partial y^2}\left(\sigma^2(y)p(t,y)\right)\ dy,$$

or equivalently,

$$\int_0^\infty h(y)\left[p_t(t,y) + \frac{\partial}{\partial y}\left(b(y)p(t,y)\right) - \tfrac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(y)p(t,y)\right)\right]\ dy = 0.$$

This last equation holds for every function $h$ of the form in Figure 31.2. It implies that

$$p_t(t,y) + \frac{\partial}{\partial y}\left((b(y)p(t,y)\right) - \tfrac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(y)p(t,y)\right) = 0. \qquad \text{(KFE)}$$

If there were a place where (KFE) did not hold, then we could take $h(y) > 0$ at that and nearby points, but take $h$ to be zero elsewhere, and we would obtain

$$\int_0^\infty h\left[p_t + \frac{\partial}{\partial y}(bp) - \tfrac{1}{2}\frac{\partial^2}{\partial y^2}(\sigma^2 p)\right]\ dy \neq 0.$$

If the process $X(t)$ has an equilibrium density, it will be

$$p(y) = \lim_{t \to \infty} p(t, y).$$

In order for this limit to exist, we must have

$$0 = \lim_{t \to \infty} p_t(t, y).$$

Letting $t \to \infty$ in (KFE), we obtain the equilibrium Kolmogorov forward equation

$$\frac{\partial}{\partial y}\left(b(y)p(y)\right) - \tfrac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(y)p(y)\right) = 0.$$

When an equilibrium density exists, it is the unique solution to this equation satisfying

$$p(y) \geq 0 \quad \forall y \geq 0,$$

$$\int_0^\infty p(y)\, dy = 1.$$

## 31.3 Cox-Ingersoll-Ross equilibrium density

We computed this to be

$$p(r) = C r^{\frac{2\alpha - \sigma^2}{\sigma^2}} e^{-\frac{2\beta}{\sigma^2} r},$$

where

$$C = \left(\frac{2\beta}{\sigma^2}\right)^{\frac{2\alpha}{\sigma^2}} \frac{1}{\Gamma\left(\frac{2\alpha}{\sigma^2}\right)}.$$

We compute

$$\begin{aligned}
p'(r) &= \frac{2\alpha - \sigma^2}{\sigma^2} . \frac{p(r)}{r} - \frac{2\beta}{\sigma^2} p(r) \\
&= \frac{2}{\sigma^2 r}\left(\alpha - \tfrac{1}{2}\sigma^2 - \beta r\right) p(r), \\
p''(r) &= -\frac{2}{\sigma^2 r^2}\left(\alpha - \tfrac{1}{2}\sigma^2 - \beta r\right) p(r) + \frac{2}{\sigma^2 r}(-\beta) p(r) + \frac{2}{\sigma^2 r}\left(\alpha - \tfrac{1}{2}\sigma^2 - \beta r\right) p'(r) \\
&= \frac{2}{\sigma^2 r}\left(-\frac{1}{r}(\alpha - \tfrac{1}{2}\sigma^2 - \beta r) - \beta + \frac{2}{\sigma^2 r}(\alpha - \tfrac{1}{2}\sigma^2 - \beta r)^2\right)\ p(r)
\end{aligned}$$

We want to verify the equilibrium Kolmogorov forward equation for the CIR process:

$$\frac{\partial}{\partial r}\left((\alpha - \beta r)p(r)\right) - \tfrac{1}{2}\frac{\partial^2}{\partial r^2}(\sigma^2 r p(r)) = 0. \tag{EKFE}$$

Now

$$
\begin{aligned}
\frac{\partial}{\partial r}\left((\alpha-\beta r)p(r)\right) &= -\beta p(r) + (\alpha-\beta r)p'(r),\\
\frac{\partial^2}{\partial r^2}(\sigma^2 r p(r)) &= \frac{\partial}{\partial r}(\sigma^2 p(r) + \sigma^2 r p'(r))\\
&= 2\sigma^2 p'(r) + \sigma^2 r p''(r).
\end{aligned}
$$

The LHS of (EKFE) becomes

$$
\begin{aligned}
&-\beta p(r) + (\alpha-\beta r)p'(r) - \sigma^2 p'(r) - \tfrac{1}{2}\sigma^2 r p''(r)\\
&= p(r)\Big[-\beta + (\alpha - \beta r - \sigma^2)\frac{2}{\sigma^2 r}(\alpha - \tfrac{1}{2}\sigma^2 - \beta r)\\
&\qquad + \frac{1}{r}(\alpha - \tfrac{1}{2}\sigma^2 - \beta r) + \beta - \frac{2}{\sigma^2 r}(\alpha - \tfrac{1}{2}\sigma^2 - \beta r)^2\Big]\\
&= p(r)\Big[(\alpha - \tfrac{1}{2}\sigma^2 - \beta r)\frac{2}{\sigma^2 r}(\alpha - \tfrac{1}{2}\sigma^2 - \beta r)\\
&\qquad - \tfrac{1}{2}\sigma^2\frac{2}{\sigma^2 r}(\alpha - \tfrac{1}{2}\sigma^2 - \beta r)\\
&\qquad + \frac{1}{r}(\alpha - \tfrac{1}{2}\sigma^2 - \beta r) - \frac{2}{\sigma^2 r}(\alpha - \tfrac{1}{2}\sigma^2 - \beta r)^2\Big]\\
&= 0,
\end{aligned}
$$

as expected.

## 31.4 Bond prices in the CIR model

The interest rate process $r(t)$ is given by

$$dr(t) = (\alpha - \beta r(t))\,dt + \sigma\sqrt{r(t)}\,dW(t),$$

where $r(0)$ is given. The bond price process is

$$B(t,T) = I\!\!E\left[\exp\left\{-\int_t^T r(u)\,du\right\}\Big|\mathcal{F}(t)\right].$$

Because

$$\exp\left\{-\int_0^t r(u)\,du\right\} B(t,T) = I\!\!E\left[\exp\left\{-\int_0^T r(u)\,du\right\}\Big|\mathcal{F}(t)\right],$$

the tower property implies that this is a martingale. The Markov property implies that $B(t,T)$ is random only through a dependence on $r(t)$. Thus, there is a function $B(r,t,T)$ of the three dummy variables $r,t,T$ such that the *process* $B(t,T)$ is the *function* $B(r,t,T)$ evaluated at $r(t),t,T$, i.e.,

$$B(t,T) = B(r(t),t,T).$$

Because $\exp\left\{-\int_0^t r(u)\,du\right\} B(r(t),t,T)$ is a martingale, its differential has no $dt$ term. We compute

$$\begin{aligned} d&\left(\exp\left\{-\int_0^t r(u)\,du\right\} B(r(t),t,T)\right) \\ &= \exp\left\{-\int_0^t r(u)\,du\right\} \Big[-r(t)B(r(t),t,T)\,dt + B_r(r(t),t,T)\,dr(t) + \\ &\qquad \tfrac{1}{2}B_{rr}(r(t),t,T)\,dr(t)\,dr(t) + B_t(r(t),t,T)\,dt\Big]. \end{aligned}$$

The expression in $[\dots]$ equals

$$\begin{aligned} &= -rB\,dt + B_r(\alpha-\beta r)\,dt + B_r\sigma\sqrt{r}\,dW \\ &\qquad + \tfrac{1}{2}B_{rr}\sigma^2 r\,dt + B_t\,dt. \end{aligned}$$

Setting the $dt$ term to zero, we obtain the partial differential equation

$$\begin{aligned} -rB(r,t,T) + B_t(r,t,T) + (\alpha-\beta r)B_r(r,t,T) + \tfrac{1}{2}\sigma^2 rB_{rr}(r,t,T) = 0,& \\ 0 \le t < T,\quad r \ge 0.& \end{aligned} \tag{4.1}$$

The terminal condition is

$$B(r,T,T) = 1, \quad r \ge 0.$$

Surprisingly, this equation has a closed form solution. Using the Hull & White model as a guide, we look for a solution of the form

$$B(r,t,T) = e^{-rC(t,T)-A(t,T)},$$

where $C(T,T)=0,\ A(T,T)=0$. Then we have

$$\begin{aligned} B_t &= (-rC_t - A_t)B, \\ B_r &= -CB, \quad B_{rr} = C^2B, \end{aligned}$$

and the partial differential equation becomes

$$\begin{aligned} 0 &= -rB + (-rC_t - A_t)B - (\alpha-\beta r)CB + \tfrac{1}{2}\sigma^2 rC^2B \\ &= rB(-1 - C_t + \beta C + \tfrac{1}{2}\sigma^2C^2) - B(A_t + \alpha C) \end{aligned}$$

We first solve the ordinary differential equation

$$-1 - C_t(t,T) + \beta C(t,T) + \tfrac{1}{2}\sigma^2C^2(t,T) = 0; \quad C(T,T) = 0,$$

and then set

$$A(t,T) = \alpha\int_t^T C(u,T)\,du,$$

so $A(T,T)=0$ and

$$A_t(t,T) = -\alpha C(t,T).$$

It is tedious but straightforward to check that the solutions are given by

$$C(t,T) = \frac{\sinh(\gamma(T-t))}{\gamma\cosh(\gamma(T-t)) + \frac{1}{2}\beta\sinh(\gamma(T-t))},$$

$$A(t,T) = -\frac{2\alpha}{\sigma^2}\log\left[\frac{\gamma e^{\frac{1}{2}\beta(T-t)}}{\gamma\cosh(\gamma(T-t)) + \frac{1}{2}\beta\sinh(\gamma(T-t))}\right],$$

where

$$\gamma = \tfrac{1}{2}\sqrt{\beta^2 + 2\sigma^2}, \quad \sinh u = \frac{e^u - e^{-u}}{2}, \quad \cosh u = \frac{e^u + e^{-u}}{2}.$$

Thus in the CIR model, we have

$$I\!E\left[\exp\left\{-\int_t^T r(u)\,du\right\}\Big|\mathcal{F}(t)\right] = B(r(t),t,T),$$

where

$$B(r,t,T) = \exp\left\{-rC(t,T) - A(t,T)\right\}, \quad 0 \le t < T, \ \ r \ge 0,$$

and $C(t,T)$ and $A(t,T)$ are given by the formulas above. Because the coefficients in

$$dr(t) = (\alpha - \beta r(t))\,dt + \sigma\sqrt{r(t)}\,dW(t)$$

do not depend on $t$, the function $B(r,t,T)$ depends on $t$ and $T$ only through their difference $\tau = T-t$. Similarly, $C(t,T)$ and $A(t,T)$ are functions of $\tau = T-t$. We write $B(r,\tau)$ instead of $B(r,t,T)$, and we have

$$B(r,\tau) = \exp\left\{-rC(\tau) - A(\tau)\right\}, \quad \tau \ge 0, \ \ r \ge 0,$$

where

$$C(\tau) = \frac{\sinh(\gamma\tau)}{\gamma\cosh(\gamma\tau) + \frac{1}{2}\beta\sinh(\gamma\tau)},$$

$$A(\tau) = -\frac{2\alpha}{\sigma^2}\log\left[\frac{\gamma e^{\frac{1}{2}\beta\tau}}{\gamma\cosh(\gamma\tau) + \frac{1}{2}\beta\sinh(\gamma\tau)}\right],$$

$$\gamma = \tfrac{1}{2}\sqrt{\beta^2 + 2\sigma^2}.$$

We have

$$B(r(0),T) = I\!E\exp\left\{-\int_0^T r(u)\,du\right\}.$$

Now $r(u) > 0$ for each $u$, almost surely, so $B(r(0),T)$ is strictly decreasing in $T$. Moreover,

$$B(r(0),0) = 1,$$

$$\lim_{T\to\infty} B(r(0),T) = I\!\!E \exp\left\{-\int_0^\infty r(u)\,du\right\} = 0.$$

But also,

$$B(r(0),T) = \exp\left\{-r(0)C(T) - A(T)\right\},$$

so

$$r(0)C(0) + A(0) = 0,$$
$$\lim_{T\to\infty}\left[r(0)C(T) + A(T)\right] = \infty,$$

and

$$r(0)C(T) + A(T)$$

is strictly inreasing in $T$.

## 31.5 Option on a bond

The value at time $t$ of an option on a bond in the CIR model is

$$v(t,r(t)) = I\!\!E\left[\exp\left\{-\int_t^{T_1} r(u)\,du\right\}(B(T_1,T_2)-K)^+\Big|\mathcal{F}(t)\right],$$

where $T_1$ is the expiration time of the option, $T_2$ is the maturity time of the bond, and $0 \le t \le T_1 \le T_2$. As usual, $\exp\left\{-\int_0^t r(u)\,du\right\} v(t,r(t))$ is a martingale, and this leads to the partial differential equation

$$-rv + v_t + (\alpha - \beta r)v_r + \tfrac{1}{2}\sigma^2 r v_{rr} = 0, \quad 0 \le t < T_1, \ \ r \ge 0.$$

(where $v = v(t,r)$.) The terminal condition is

$$v(T_1,r) = (B(r,T_1,T_2) - K)^+, \quad r \ge 0.$$

Other European derivative securities on the bond are priced using the same partial differential equation with the terminal condition appropriate for the particular security.

## 31.6 Deterministic time change of CIR model

*Process time scale:* In this time scale, the interest rate $r(t)$ is given by the constant coefficient CIR equation

$$dr(t) = (\alpha - \beta r(t))\,dt + \sigma\sqrt{r(t)}\,dW(t).$$

*Real time scale:* In this time scale, the interest rate $\hat{r}(\hat{t})$ is given by a time-dependent CIR equation

$$d\hat{r}(\hat{t}) = (\hat{\alpha}(\hat{t}) - \hat{\beta}(\hat{t})\hat{r}(\hat{t}))\,d\hat{t} + \hat{\sigma}(\hat{t})\sqrt{\hat{r}(\hat{t})}\,d\hat{W}(\hat{t}).$$

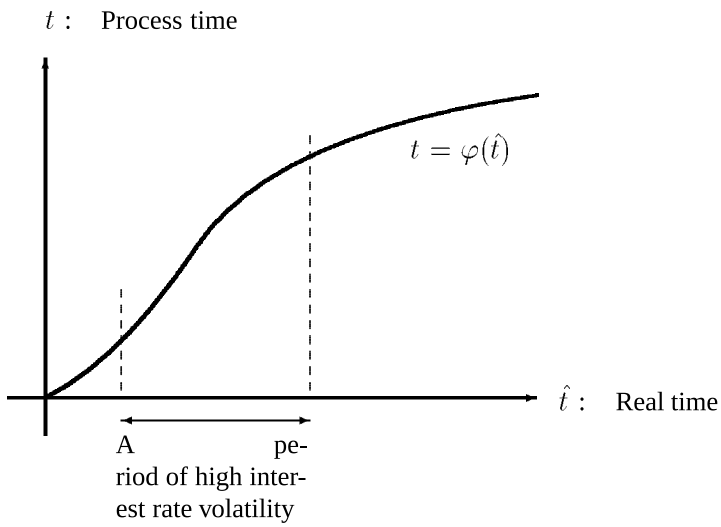


Figure 31.3: *Time change function.*

There is a strictly increasing time change function $t = \varphi(\hat{t})$ which relates the two time scales (See Fig. 31.3).

Let $\hat{B}(\hat{r}, \hat{t}, \hat{T})$ denote the price at real time $\hat{t}$ of a bond with maturity $\hat{T}$ when the interest rate at time $\hat{t}$ is $\hat{r}$. We want to set things up so

$$\hat{B}(\hat{r}, \hat{t}, \hat{T}) = B(r, t, T) = e^{-rC(t,T)-A(t,T)},$$

where $t = \varphi(\hat{t}), T = \varphi(\hat{T})$, and $C(t,T)$ and $A(t,T)$ are as defined previously.

We need to determine the relationship between $\hat{r}$ and $r$. We have

$$\begin{aligned} B(r(0), 0, T) &= I\!\!E \exp\left\{-\int_0^T r(t)\, dt\right\}, \\ B(\hat{r}(0), 0, \hat{T}) &= I\!\!E \exp\left\{-\int_0^{\hat{T}} \hat{r}(\hat{t})\, d\hat{t}\right\}. \end{aligned}$$

With $T = \varphi(\hat{T})$, make the change of variable $t = \varphi(\hat{t})$, $dt = \varphi'(\hat{t})\, d\hat{t}$ in the first integral to get

$$B(r(0), 0, T) = I\!\!E \exp\left\{-\int_0^{\hat{T}} r(\varphi(\hat{t}))\varphi'(\hat{t})\, d\hat{t}\right\},$$

and this will be $B(\hat{r}(0), 0, \hat{T})$ if we set

$$\boxed{\hat{r}(\hat{t}) = r(\varphi(\hat{t}))\, \varphi'(\hat{t}).}$$

## 31.7 Calibration

$$
\begin{aligned}
\hat{B}(\hat{r}(\hat{t}), \hat{t}, \hat{T}) &= B\left(\frac{\hat{r}(\hat{t})}{\varphi'(\hat{t})}, \varphi(\hat{t}), \varphi(\hat{T})\right) \\
&= \exp\left\{-\hat{r}(\hat{t})\frac{C(\varphi(\hat{t}), \varphi(\hat{T}))}{\varphi'(\hat{t})} - A(\varphi(\hat{t}), \varphi(\hat{T}))\right\} \\
&= \exp\left\{-\hat{r}(\hat{t})\hat{C}(\hat{t}, \hat{T}) - \hat{A}(\hat{t}, \hat{T})\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{C}(\hat{t}, \hat{T}) &= \frac{C(\varphi(\hat{t}), \varphi(\hat{T}))}{\varphi'(\hat{t})} \\
\hat{A}(\hat{t}, \hat{T}) &= A(\varphi(\hat{t}), \varphi(\hat{T}))
\end{aligned}
$$

do *not* depend on $\hat{t}$ and $\hat{T}$ only through $\hat{T} - \hat{t}$, since, in the real time scale, the model coefficients are time dependent.

Suppose we know $\hat{r}(0)$ and $\hat{B}(\hat{r}(0), 0, \hat{T})$ for all $\hat{T} \in [0, \hat{T}^*]$. We calibrate by writing the equation

$$
\hat{B}(\hat{r}(0), 0, \hat{T}) = \exp\left\{-\hat{r}(0)\hat{C}(0, \hat{T}) - \hat{A}(0, \hat{T})\right\},
$$

or equivalently,

$$
-\log \hat{B}(\hat{r}(0), 0, \hat{T}) = \frac{\hat{r}(0)}{\varphi'(0)}C(\varphi(0), \varphi(\hat{T})) + A(\varphi(0), \varphi(\hat{T})).
$$

Take $\alpha, \beta$ and $\sigma$ so the equilibrium distribution of $r(t)$ seems reasonable. These values determine the functions $C, A$. Take $\varphi'(0) = 1$ (we justify this in the next section). For each $\hat{T}$, solve the equation for $\varphi(\hat{T})$:

$$
-\log \hat{B}(\hat{r}(0), 0, \hat{T}) = \hat{r}(0)C(0, \varphi(\hat{T})) + A(0, \varphi(\hat{T})). \tag{*}
$$

The right-hand side of this equation is increasing in the $\varphi(\hat{T})$ variable, starting at 0 at time $0$ and having limit $\infty$ at $\infty$, i.e.,

$$
\begin{gathered}
\hat{r}(0)C(0,0) + A(0,0) = 0, \\
\lim_{T \to \infty} [\hat{r}(0)C(0,T) + A(0,T)] = \infty.
\end{gathered}
$$

Since $0 \le -\log \hat{B}(\hat{r}(0), 0, \hat{T}) < \infty$, (*) has a unique solution for each $\hat{T}$. For $\hat{T} = 0$, this solution is $\varphi(0) = 0$. If $\hat{T}_1 < \hat{T}_2$, then

$$
-\log \hat{B}(r(0), 0, \hat{T}_1) < -\log \hat{B}(r(0), 0, \hat{T}_2),
$$

so $\varphi(\hat{T}_1) < \varphi(\hat{T}_2)$. Thus $\varphi$ is a strictly increasing time-change-function with the right properties.

## 31.8 Tracking down $\varphi'(0)$ in the time change of the CIR model

Result for general term structure models:

$$-\frac{\partial}{\partial T}\log B(0,T)\Big|_{T=0} = r(0).$$

Justification:

$$B(0,T) = \mathbb{E}\exp\left\{-\int_0^T r(u)\,du\right\}.$$

$$-\log B(0,T) = -\log \mathbb{E}\exp\left\{-\int_0^T r(u)\,du\right\}$$

$$-\frac{\partial}{\partial T}\log B(0,T) = \frac{\mathbb{E}\left[r(T)e^{-\int_0^T r(u)\,du}\right]}{\mathbb{E}e^{-\int_0^T r(u)\,du}}$$

$$-\frac{\partial}{\partial T}\log B(0,T)\Big|_{T=0} = r(0).$$

In the real time scale associated with the calibration of CIR by time change, we write the bond price as

$$\hat{B}(\hat{r}(0),0,\hat{T}),$$

thereby indicating explicitly the initial interest rate. The above says that

$$-\frac{\partial}{\partial \hat{T}}\log \hat{B}(\hat{r}(0),0,\hat{T})\Big|_{\hat{T}=0} = \hat{r}(0).$$

The calibration of CIR by time change requires that we find a strictly increasing function $\varphi$ with $\varphi(0)=0$ such that

$$-\log \hat{B}(\hat{r}(0),0,\hat{T}) = \frac{1}{\varphi'(0)}\hat{r}(0)C(\varphi(\hat{T})) + A(\varphi(\hat{T})), \quad \hat{T}\geq 0, \tag{cal}$$

where $\hat{B}(\hat{r}(0),0,\hat{T})$, determined by market data, is strictly increasing in $\hat{T}$, starts at 1 when $\hat{T}=0$, and goes to zero as $\hat{T}\to\infty$. Therefore, $-\log \hat{B}(\hat{r}(0),0,\hat{T})$ is as shown in Fig. 31.4.

Consider the function

$$\hat{r}(0)C(T) + A(T),$$

Here $C(T)$ and $A(T)$ are given by

$$C(T) = \frac{\sinh(\gamma T)}{\gamma\cosh(\gamma T)+\frac{1}{2}\beta\sinh(\gamma T)},$$

$$A(T) = -\frac{2\alpha}{\sigma^2}\log\left[\frac{\gamma e^{\frac{1}{2}\beta T}}{\gamma\cosh(\gamma T)+\frac{1}{2}\beta\sinh(\gamma T)}\right],$$

$$\gamma = \tfrac{1}{2}\sqrt{\beta^2+2\sigma^2}.$$

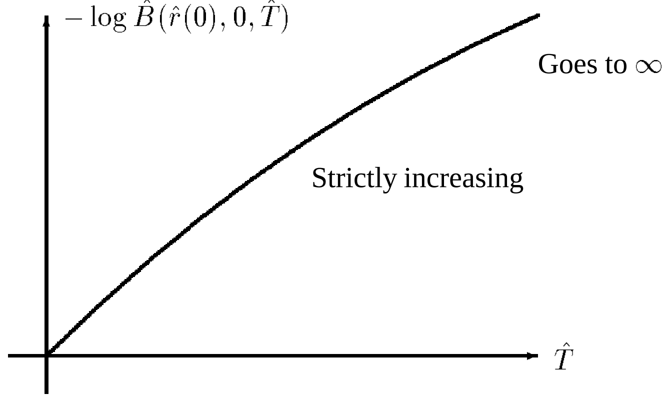


Figure 31.4: *Bond price in CIR model*

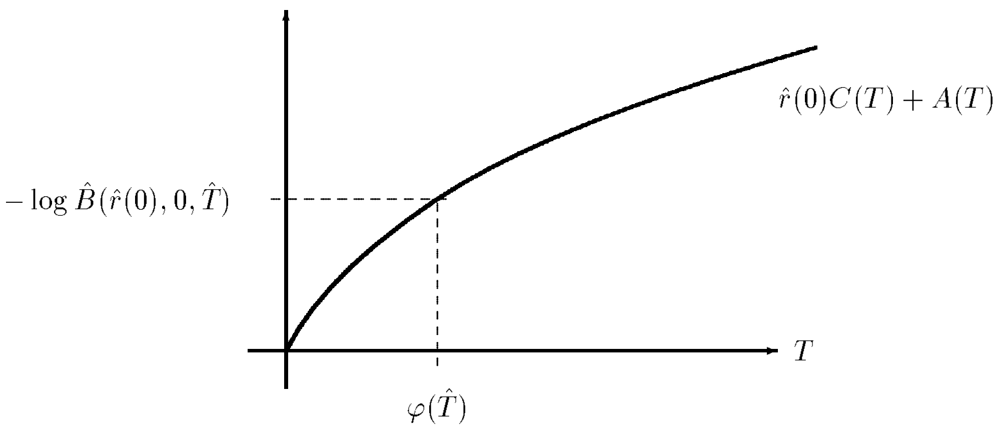


Figure 31.5: *Calibration*

The function $\hat{r}(0)C(T) + A(T)$ is zero at $T = 0$, is strictly increasing in $T$, and goes to $\infty$ as $T \to \infty$. This is because the interest rate is positive in the CIR model (see last paragraph of Section 31.4).

To solve (cal), let us first consider the related equation

$$-\log \hat{B}(\hat{r}(0), 0, \hat{T}) = \hat{r}(0)C(\varphi(\hat{T})) + A(\varphi(\hat{T})). \tag{cal'}$$

Fix $\hat{T}$ and define $\varphi(\hat{T})$ to be the unique $T$ for which (see Fig. 31.5)

$$-\log \hat{B}(\hat{r}(0), 0, \hat{T}) = \hat{r}(0)C(T) + A(T)$$

If $\hat{T} = 0$, then $\varphi(\hat{T}) = 0$. If $\hat{T}_1 < \hat{T}_2$, then $\varphi(\hat{T}_1) < \varphi(\hat{T}_2)$. As $\hat{T} \to \infty$, $\varphi(\hat{T}) \to \infty$. We have thus defined a time-change function $\varphi$ which has all the right properties, except it satisfies (cal') rather than (cal).

We conclude by showing that $\varphi'(0) = 1$ so $\varphi$ also satisfies (cal). From (cal') we compute

$$\begin{aligned}\hat{r}(0) &= -\frac{\partial}{\partial \hat{T}} \log \hat{B}(\hat{r}(0), 0, \hat{T})\Big|_{\hat{T}=0} \\ &= \hat{r}(0)C'(\varphi(0))\varphi'(0) + A'(\varphi(0))\varphi'(0) \\ &= \hat{r}(0)C'(0)\varphi'(0) + A'(0)\varphi'(0).\end{aligned}$$

We show in a moment that $C'(0) = 1$, $A'(0) = 0$, so we have

$$\hat{r}(0) = \hat{r}(0)\varphi'(0).$$

Note that $\hat{r}(0)$ is the initial interest rate, observed in the market, and is striclty positive. Dividing by $\hat{r}(0)$, we obtain

$$\varphi'(0) = 1.$$

Computation of $C'(0)$:

$$\begin{aligned}C'(\tau) &= \frac{1}{\left(\gamma \cosh(\gamma\tau) + \frac{1}{2}\beta \sinh(\gamma\tau)\right)^2}\Big[\gamma \cosh(\gamma\tau)\left(\gamma \cosh(\gamma\tau) + \tfrac{1}{2}\beta \sinh(\gamma\tau)\right) \\ &\qquad - \sinh(\gamma\tau)\left(\gamma^2 \sinh(\gamma\tau) + \tfrac{1}{2}\beta\gamma \cosh(\gamma\tau)\right)\Big] \\ C'(0) &= \frac{1}{\gamma^2}\left[\gamma(\gamma + 0) - 0(0 + \tfrac{1}{2}\beta\gamma)\right] = 1.\end{aligned}$$

Computation of $A'(0)$:

$$\begin{aligned}A'(\tau) &= -\frac{2\alpha}{\sigma^2}\left[\frac{\gamma \cosh(\gamma\tau) + \frac{1}{2}\beta \sinh(\gamma\tau)}{\gamma e^{\beta\tau/2}}\right] \\ &\qquad \times \frac{1}{\left(\gamma \cosh(\gamma\tau) + \frac{1}{2}\beta \sinh(\gamma\tau)\right)^2}\Big[\frac{\beta\gamma}{2}e^{\beta\tau/2}\left(\gamma \cosh(\gamma\tau) + \tfrac{1}{2}\beta \sinh(\gamma\tau)\right) \\ &\qquad - \gamma e^{\beta\tau/2}\left(\gamma^2 \sinh(\gamma\tau) + \tfrac{1}{2}\beta\gamma \cosh(\gamma\tau)\right)\Big], \\ A'(0) &= -\frac{2\alpha}{\sigma^2}\left[\frac{\gamma + 0}{\gamma}\right]\frac{1}{(\gamma + 0)^2}\left[\frac{\beta\gamma}{2}(\gamma + 0) - \gamma(0 + \tfrac{1}{2}\beta\gamma)\right] \\ &= -\frac{2\alpha}{\sigma^2} \cdot \frac{1}{\gamma^2}\left[\frac{\beta\gamma^2}{2} - \tfrac{1}{2}\beta\gamma^2\right] \\ &= 0.\end{aligned}$$