Chapter 31

Cox-Ingersoll-Ross model

In the Hull & White model, $r(t)$ is a Gaussian process. Since, for each $t$, $r(t)$ is normally distributed, there is a positive probability that $r(t) < 0$. The Cox-Ingersoll-Ross model is the simplest one which avoids negative interest rates.

We begin with a $d$-dimensional Brownian motion $(W_1, W_2, \ldots, W_d)$. Let $\beta > 0$ and $\sigma > 0$ be constants. For $j = 1, \ldots, d$, let $X_j(0) \in \mathbb{R}$ be given so that

$$X_1^2(0) + X_2^2(0) + \ldots + X_d^2(0) \geq 0,$$

and let $X_j$ be the solution to the stochastic differential equation

$$dX_j(t) = -\frac{1}{2} \beta X_j(t) \, dt + \frac{1}{2} \sigma \, dW_j(t).$$

$X_j$ is called the Orstein-Uhlenbeck process. It always has a drift toward the origin. The solution to this stochastic differential equation is

$$X_j(t) = e^{-\frac{1}{2} \beta t} \left[ X_j(0) + \frac{1}{2} \sigma \int_0^t e^{\frac{1}{2} \beta u} \, dW_j(u) \right].$$

This solution is a Gaussian process with mean function

$$m_j(t) = e^{-\frac{1}{2} \beta t} X_j(0)$$

and covariance function

$$\rho(s, t) = \frac{1}{4} \sigma^2 e^{-\frac{1}{2} \beta (s+t)} \int_0^{s+t} e^{\frac{1}{2} \beta u} \, du.$$ 

Define

$$r(t) \triangleq X_1^2(t) + X_2^2(t) + \ldots + X_d^2(t).$$

If $d = 1$, we have $r(t) = X_1^2(t)$ and for each $t$, $\mathbb{P}\{r(t) > 0\} = 1$, but (see Fig. 31.1)

$$\mathbb{P}\{\text{There are infinitely many values of } t > 0 \text{ for which } r(t) = 0\} = 1$$

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If $d \geq 2$, (see Fig. 31.1)

$$IP\{\text{There is at least one value of } t > 0 \text{ for which } r(t) = 0\} = 0.$$ 

Let $f(x_1, x_2, \ldots, x_d) = x_1^2 + x_2^2 + \cdots + x_d^2$. Then

$$f_{x_i} = 2x_i, \quad f_{x_ix_j} = \begin{cases} 2 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Itô's formula implies

$$dr(t) = \sum_{i=1}^{d} f_{x_i} \, dX_i + \frac{1}{2} \sum_{i=1}^{d} f_{x_ix_i} \, dX_i \, dX_i$$

$$= \sum_{i=1}^{d} 2X_i \left( -\frac{1}{2} \beta X_i \, dt + \frac{1}{2} \sigma \, dW_i(t) \right) + \sum_{i=1}^{d} \frac{1}{4} \sigma^2 \, dW_i \, dW_i$$

$$= -\beta r(t) \, dt + \sigma \sum_{i=1}^{d} X_i \, dW_i + \frac{d\sigma^2}{4} \, dt$$

$$= \left( \frac{d\sigma^2}{4} - \beta r(t) \right) \, dt + \sigma \sqrt{r(t)} \sum_{i=1}^{d} \frac{X_i(t)}{\sqrt{r(t)}} \, dW_i(t).$$

Define

$$W(t) = \sum_{i=1}^{d} \int_{0}^{t} \frac{X_i(u)}{\sqrt{r(u)}} \, dW_i(u).$$
Then $W$ is a martingale,

$$dW = \sum_{i=1}^{d} \frac{X_i}{\sqrt{r}} dW_i,$$

$$dW \cdot dW = \sum_{i=1}^{d} \frac{X_i^2}{r} dt = dt,$$

so $W$ is a Brownian motion. We have

$$dr(t) = \left( \frac{d\sigma^2}{4} - \beta r(t) \right) dt + \sigma \sqrt{r(t)} dW(t).$$

The Cox-Ingersoll-Ross (CIR) process is given by

$$dr(t) = (\alpha - \beta r(t)) \, dt + \sigma \sqrt{r(t)} \, dW(t),$$

We define

$$d = \frac{4\alpha}{\sigma^2} > 0.$$

If $d$ happens to be an integer, then we have the representation

$$r(t) = \sum_{i=1}^{d} X_i^2(t),$$

but we do not require $d$ to be an integer. If $d < 2$ (i.e., $\alpha < \frac{1}{2} \sigma^2$), then

$$\mathbb{P}\{ \text{There are infinitely many values of } t > 0 \text{ for which } r(t) = 0 \} = 1.$$

This is not a good parameter choice.

If $d \geq 2$ (i.e., $\alpha \geq \frac{1}{2} \sigma^2$), then

$$\mathbb{P}\{ \text{There is at least one value of } t > 0 \text{ for which } r(t) = 0 \} = 0.$$

With the CIR process, one can derive formulas under the assumption that $d = \frac{4\alpha}{\sigma^2}$ is a positive integer, and they are still correct even when $d$ is not an integer.

For example, here is the distribution of $r(t)$ for fixed $t > 0$. Let $r(0) \geq 0$ be given. Take

$$X_1(0) = 0, X_2(0) = 0, \ldots, X_{d-1}(0) = 0, X_d(0) = \sqrt{r(0)}.$$

For $i = 1, 2, \ldots, d-1$, $X_i(t)$ is normal with mean zero and variance

$$\rho(t, t) = \frac{\sigma^2}{4\beta} (1 - e^{-\beta t}).$$
$X_d(t)$ is normal with mean

$$m_d(t) = e^{-\frac{1}{2}\beta t} \sqrt{r(0)}$$

and variance $\rho(t, t)$. Then

$$r(t) = \rho(t, t) \sum_{i=1}^{d-1} \left( \frac{X_i(t)}{\sqrt{\rho(t, t)}} \right)^2 + \frac{X_d^2(t)}{\rho(t, t)}$$

Chi-square with $d - 1 = \frac{2\alpha - \sigma^2}{\sigma^2}$ degrees of freedom

Thus $r(t)$ has a non-central chi-square distribution.

**31.1 Equilibrium distribution of $r(t)$**

As $t \to \infty$, $m_d(t) \to 0$. We have

$$r(t) = \rho(t, t) \sum_{i=1}^{d-1} \left( \frac{X_i(t)}{\sqrt{\rho(t, t)}} \right)^2.$$

As $t \to \infty$, we have $\rho(t, t) = \frac{\sigma^2}{\sigma^2}$, and so the limiting distribution of $r(t)$ is $\frac{\sigma^2}{4\beta}$ times a chi-square with $d = \frac{4\alpha}{\sigma^2}$ degrees of freedom. The chi-square density with $\frac{4\alpha}{\sigma^2}$ degrees of freedom is

$$f(y) = \frac{1}{\Gamma\left(\frac{2\alpha}{\sigma^2}\right)} y^{\frac{2\alpha - \sigma^2}{\sigma^2}} e^{-y/2}.$$

We make the change of variable $r = \frac{\sigma^2}{4\beta} y$. The limiting density for $r(t)$ is

$$p(r) = \frac{4\beta}{\sigma^2} \frac{1}{\sigma^2/\sigma^2} \Gamma\left(\frac{2\alpha}{\sigma^2}\right) \left(\frac{r}{\sigma^2}\right)^{\frac{2\alpha - \sigma^2}{\sigma^2}} e^{-2\beta r}$$

$$= \left(\frac{2\beta}{\sigma^2}\right)^{\frac{2\alpha}{\sigma^2}} \frac{1}{\Gamma\left(\frac{2\alpha}{\sigma^2}\right)} \left(\frac{r}{\sigma^2}\right)^{\frac{2\alpha - \sigma^2}{\sigma^2}} e^{-2\beta r}.$$

We computed the mean and variance of $r(t)$ in Section 15.7.

**31.2 Kolmogorov forward equation**

Consider a Markov process governed by the stochastic differential equation

$$dX(t) = b(X(t)) \, dt + \sigma(X(t)) \, dW(t).$$
Because we are going to apply the following analysis to the case \( X(t) = r(t) \), we assume that \( X(t) \geq 0 \) for all \( t \).

We start at \( X(0) = x \geq 0 \) at time 0. Then \( X(t) \) is random with density \( p(0, t, x, y) \) (in the \( y \) variable). Since 0 and \( x \) will not change during the following, we omit them and write \( p(t, y) \) rather than \( p(0, t, x, y) \). We have

\[
\mathbb{E} h(X(t)) = \int_0^\infty h(y)p(t, y) \, dy
\]

for any function \( h \).

The Kolmogorov forward equation (KFE) is a partial differential equation in the “forward” variables \( t \) and \( y \). We derive it below.

Let \( h(y) \) be a smooth function of \( y \geq 0 \) which vanishes near \( y = 0 \) and for all large values of \( y \) (see Fig. 31.2). Itô’s formula implies

\[
dh(X(t)) = \left[ h'(X(t))b(X(t)) + \frac{1}{2}h''(X(t))\sigma^2(X(t)) \right] \, dt + h'(X(t))\sigma(X(t)) \, dW(t),
\]

so

\[
h(X(t)) = h(X(0)) + \int_0^t \left[ h'(X(s))b(X(s)) + \frac{1}{2}h''(X(s))\sigma^2(X(s)) \right] \, ds +
\int_0^t h'(X(s))\sigma(X(s)) \, dW(s),
\]

\[
\mathbb{E} h(X(t)) = h(X(0)) + \mathbb{E} \int_0^t \left[ h'(X(s))b(X(s)) \, dt + \frac{1}{2}h''(X(s))\sigma^2(X(s)) \right] \, ds,
\]

Figure 31.2: The function \( h(y) \)
or equivalently,

\[
\int_{0}^{\infty} h(y) p(t, y) \, dy = h(X(0)) + \int_{0}^{t} \int_{0}^{\infty} h'(y)b(y)p(s, y) \, dy \, ds + \frac{1}{2} \int_{0}^{t} \int_{0}^{\infty} h''(y)\sigma^2(y) p(s, y) \, dy \, ds.
\]

Differentiate with respect to \( t \) to get

\[
\int_{0}^{\infty} h(y)p_t(t, y) \, dy = \int_{0}^{\infty} h'(y)b(y)p(t, y) \, dy + \frac{1}{2} \int_{0}^{\infty} h''(y)\sigma^2(y) p(t, y) \, dy.
\]

Integration by parts yields

\[
\int_{0}^{\infty} h'(y)b(y)p(t, y) \, dy = h(y)b(y)p(t, y) \bigg|_{y=0}^{y=\infty} - \int_{0}^{\infty} h(y) \frac{\partial}{\partial y} (b(y)p(t, y)) \, dy,
\]

\[
= -h(y) \frac{\partial}{\partial y} \left( \sigma^2(y)p(t, y) \right) \bigg|_{y=0}^{y=\infty} + \int_{0}^{\infty} h(y) \frac{\partial^2}{\partial y^2} \left( \sigma^2(y)p(t, y) \right) \, dy.
\]

Therefore,

\[
\int_{0}^{\infty} h(y)p_t(t, y) \, dy = -\int_{0}^{\infty} h(y) \frac{\partial}{\partial y} (b(y)p(t, y)) \, dy + \frac{1}{2} \int_{0}^{\infty} h(y) \frac{\partial^2}{\partial y^2} \left( \sigma^2(y)p(t, y) \right) \, dy,
\]

or equivalently,

\[
\int_{0}^{\infty} h(y) \left[ p_t(t, y) + \frac{\partial}{\partial y} (b(y)p(t, y)) - \frac{1}{2} \frac{\partial^2}{\partial y^2} \left( \sigma^2(y)p(t, y) \right) \right] \, dy = 0.
\]

This last equation holds for every function \( h \) of the form in Figure 31.2. It implies that

\[
p_t(t, y) + \frac{\partial}{\partial y} ((b(y)p(t, y)) - \frac{1}{2} \frac{\partial^2}{\partial y^2} \left( \sigma^2(y)p(t, y) \right) = 0. \tag{KFE}
\]

If there were a place where (KFE) did not hold, then we could take \( h(y) > 0 \) at that and nearby points, but take \( h \) to be zero elsewhere, and we would obtain

\[
\int_{0}^{\infty} h \left[ p_t + \frac{\partial}{\partial y} (bp) - \frac{1}{2} \frac{\partial^2}{\partial y^2} (\sigma^2p) \right] \, dy \neq 0.
\]
If the process \( X(t) \) has an equilibrium density, it will be

\[
p(y) = \lim_{t \to \infty} p(t, y).
\]

In order for this limit to exist, we must have

\[
0 = \lim_{t \to \infty} p_t(t, y).
\]

Letting \( t \to \infty \) in (KFE), we obtain the equilibrium Kolmogorov forward equation

\[
\frac{\partial}{\partial y} (b(y)p(y)) - \frac{1}{2} \frac{\partial^2}{\partial y^2} \left( \sigma^2(y)p(y) \right) = 0.
\]

When an equilibrium density exists, it is the unique solution to this equation satisfying

\[
p(y) \geq 0 \quad \forall y \geq 0,
\]

\[
\int_0^\infty p(y) \, dy = 1.
\]

### 31.3 Cox-Ingersoll-Ross equilibrium density

We computed this to be

\[
p(r) = C r^{\frac{2\sigma^2 - \sigma^2}{\sigma^2}} e^{-\frac{2\beta}{\sigma^2} r},
\]

where

\[
C' = \left( \frac{2\beta}{\sigma^2} \right)^{\frac{2\sigma^2}{\sigma^2}} \frac{1}{\Gamma \left( \frac{2\sigma^2}{\sigma^2} \right)}.
\]

We compute

\[
p'(r) = \frac{2\alpha - \sigma^2}{\sigma^2} p(r) - \frac{2\beta}{\sigma^2} r p(r)
\]

\[
= \frac{2}{\sigma^2 r} \left( \alpha - \frac{1}{2} \sigma^2 - \beta r \right) p(r),
\]

\[
p''(r) = -\frac{2}{\sigma^2 r^2} \left( \alpha - \frac{1}{2} \sigma^2 - \beta r \right) p(r) + \frac{2}{\sigma^2 r} \left( -\beta \right) p(r) + \frac{2}{\sigma^2 r} \left( \alpha - \frac{1}{2} \sigma^2 - \beta r \right) p'(r)
\]

\[
= \frac{2}{\sigma^2 r} \left( -\frac{1}{r} \left( \alpha - \frac{1}{2} \sigma^2 - \beta r \right) - \beta + \frac{2}{\sigma^2 r} \left( \alpha - \frac{1}{2} \sigma^2 - \beta r \right)^2 \right) p(r)
\]

We want to verify the equilibrium Kolmogorov forward equation for the CIR process:

\[
\frac{\partial}{\partial r} (\left( \alpha - \beta r \right) p(r)) - \frac{1}{2} \frac{\partial^2}{\partial r^2} (\sigma^2 r p(r)) = 0.
\]  
(EKFE)
Now
\[
\frac{\partial}{\partial r} ((\alpha - \beta r)p(r)) = -\beta p(r) + (\alpha - \beta r)p'(r),
\]
\[
\frac{\partial^2}{\partial r^2} (\sigma^2 r p(r)) = \frac{\partial}{\partial r} (\sigma^2 p(r) + \sigma^2 r p'(r))
= 2\sigma^2 p'(r) + \sigma^2 r p''(r).
\]

The LHS of (EKFE) becomes
\[
-\beta p(r) + (\alpha - \beta r)p'(r) - \sigma^2 p'(r) - \frac{1}{2}\sigma^2 r p''(r)
= p(r) \left[ -\beta + (\alpha - \beta r - \sigma^2) \frac{2}{\sigma^2 r} (\alpha - \frac{1}{2}\sigma^2 - \beta r)
+ \frac{1}{r} (\alpha - \frac{1}{2}\sigma^2 - \beta r) + \beta - \frac{2}{\sigma^2 r} (\alpha - \frac{1}{2}\sigma^2 - \beta r)^2 \right]
= p(r) \left[ (\alpha - \frac{1}{2}\sigma^2 - \beta r) \frac{2}{\sigma^2 r} (\alpha - \frac{1}{2}\sigma^2 - \beta r)
- \frac{1}{2}\sigma^2 \frac{2}{\sigma^2 r} (\alpha - \frac{1}{2}\sigma^2 - \beta r)
+ \frac{1}{r} (\alpha - \frac{1}{2}\sigma^2 - \beta r) - \frac{2}{\sigma^2 r} (\alpha - \frac{1}{2}\sigma^2 - \beta r)^2 \right]
= 0,
\]
as expected.

### 31.4 Bond prices in the CIR model

The interest rate process \( r(t) \) is given by
\[
dr(t) = (\alpha - \beta r(t)) \, dt + \sigma \sqrt{r(t)} \, dW(t),
\]
where \( r(0) \) is given. The bond price process is
\[
B(t, T) = \mathbb{E} \left[ \exp \left\{ -\int_t^T r(u) \, du \right\} \mid \mathcal{F}(t) \right],
\]
Because
\[
\exp \left\{ -\int_0^t r(u) \, du \right\} B(t, T) = \mathbb{E} \left[ \exp \left\{ -\int_0^T r(u) \, du \right\} \mid \mathcal{F}(t) \right],
\]
the tower property implies that this is a martingale. The Markov property implies that \( B(t, T) \) is random only through a dependence on \( r(t) \). Thus, there is a function \( B(r, t, T) \) of the three dummy variables \( r, t, T \) such that the process \( B(t, T) \) is the function \( B(r, t, T) \) evaluated at \( r(t), t, T \), i.e.,
\[
B(t, T) = B(r(t), t, T).
\]
Because \( \exp \left\{ - \int_0^t r(u) \, du \right\} B(r(t), t, T) \) is a martingale, its differential has no \( dt \) term. We compute

\[
d \left( \exp \left\{ - \int_0^t r(u) \, du \right\} B(r(t), t, T) \right) = \exp \left\{ - \int_0^t r(u) \, du \right\} \left[ -r(t)B(r(t), t, T) \, dt + B_r(r(t), t, T) \, dr(t) + \frac{1}{2} B_{rr}(r(t), t, T) \, dr(t) \, dr(t) + B_t(r(t), t, T) \, dt \right].
\]

The expression in \([\ldots]\) equals

\[
= -rB \, dt + B_r(\alpha - \beta r) \, dt + B_r \sigma \sqrt{r} \, dW + \frac{1}{2} B_{rr} \sigma^2 r \, dt + B_t \, dt.
\]

Setting the \( dt \) term to zero, we obtain the partial differential equation

\[
- rB(r, t, T) + B_t(r, t, T) + (\alpha - \beta r)B_r(r, t, T) + \frac{1}{2} \sigma^2 r B_{rr}(r, t, T) = 0,
\]

\[
0 \leq t < T, \quad r \geq 0. \tag{4.1}
\]

The terminal condition is

\[
B(r, T, T) = 1, \quad r \geq 0.
\]

Surprisingly, this equation has a closed form solution. Using the Hull & White model as a guide, we look for a solution of the form

\[
B(r, t, T) = e^{-rC(t, T) - A(t, T)},
\]

where \( C(T, T) = 0 \), \( A(T, T) = 0 \). Then we have

\[
B_t = (-rC_t - A_t)B, \\
B_r = -CB, \quad B_{rr} = C^2 B,
\]

and the partial differential equation becomes

\[
0 = -rB + (-rC_t - A_t)B - (\alpha + \beta r)CB + \frac{1}{2} \sigma^2 r C^2 B \\
= rB(-1 - C_t + \beta C + \frac{1}{2} \sigma^2 C^2) - B(A_t + \alpha C).
\]

We first solve the ordinary differential equation

\[
-1 - C_t(t, T) + \beta C(t, T) + \frac{1}{2} \sigma^2 C^2(t, T) = 0; \quad C(T, T) = 0,
\]

and then set

\[
A(t, T) = \alpha \int_T^t C(u, T) \, du,
\]
so $A(T, T) = 0$ and

$$A(t, T) = -\alpha C(t, T).$$

It is tedious but straightforward to check that the solutions are given by

$$C(t, T) = \frac{\sinh(\gamma(T-t))}{\gamma \cosh(\gamma(T-t)) + \frac{1}{2} \beta \sinh(\gamma(T-t))},$$

$$A(t, T) = -\frac{2\alpha}{\sigma^2} \log \left[ \frac{\gamma e^{\frac{1}{2} \beta(T-t)}}{\gamma \cosh(\gamma(T-t)) + \frac{1}{2} \beta \sinh(\gamma(T-t))} \right],$$

where

$$\gamma = \frac{1}{2} \sqrt{\beta^2 + 2\sigma^2}, \quad \sinh u = \frac{e^u - e^{-u}}{2}, \quad \cosh u = \frac{e^u + e^{-u}}{2}.$$

Thus in the CIR model, we have

$$IE \left[ \exp \left\{ - \int_t^T r(u) \, du \right\} \bigg| F(t) \right] = B(r(t), t, T),$$

where

$$B(r, t, T) = \exp \left\{ -rC(t, T) - A(t, T) \right\}, \quad 0 \leq t < T, \quad r \geq 0,$$

and $C(t, T)$ and $A(t, T)$ are given by the formulas above. Because the coefficients in

$$dr(t) = (\alpha - \beta r(t)) \, dt + \sigma \sqrt{r(t)} \, dW(t)$$

do not depend on $t$, the function $B(r, t, T)$ depends on $t$ and $T$ only through their difference $\tau = T - t$. Similarly, $C(t, T)$ and $A(t, T)$ are functions of $\tau = T - t$. We write $B(r, \tau)$ instead of $B(r, t, T)$, and we have

$$B(r, \tau) = \exp \left\{ -rC(\tau) - A(\tau) \right\}, \quad \tau \geq 0, \quad r \geq 0,$$

where

$$C(\tau) = \frac{\sinh(\gamma \tau)}{\gamma \cosh(\gamma \tau) + \frac{1}{2} \beta \sinh(\gamma \tau)},$$

$$A(\tau) = -\frac{2\alpha}{\sigma^2} \log \left[ \frac{\gamma e^{\frac{1}{2} \beta \tau}}{\gamma \cosh(\gamma \tau) + \frac{1}{2} \beta \sinh(\gamma \tau)} \right],$$

$$\gamma = \frac{1}{2} \sqrt{\beta^2 + 2\sigma^2}.$$

We have

$$B(r(0), T) = IE \exp \left\{ - \int_0^T r(u) \, du \right\}. $$

Now $r(u) > 0$ for each $u$, almost surely, so $B(r(0), T)$ is strictly decreasing in $T$. Moreover,

$$B(r(0), 0) = 1,$$
\[
\lim_{T \to \infty} B(r(0), T) = \mathbb{E} \exp \left\{ - \int_0^\infty r(u) \, du \right\} = 0.
\]

But also,
\[
B(r(0), T) = \exp \left\{ - r(0) C(T) - A(T) \right\},
\]
so
\[
r(0) C(0) + A(0) = 0,
\]
and
\[
\lim_{T \to \infty} [r(0) C(T) + A(T)] = \infty,
\]
is strictly increasing in \( T \).

### 31.5 Option on a bond

The value at time \( t \) of an option on a bond in the CIR model is
\[
v(t, r(t)) = \mathbb{E} \left[ \exp \left\{ - \int_t^{T_1} r(u) \, du \right\} \left( B(T_1, T_2) - K \right)^+ \right| \mathcal{F}(t),
\]
where \( T_1 \) is the expiration time of the option, \( T_2 \) is the maturity time of the bond, and \( 0 \leq t \leq T_1 \leq T_2 \). As usual, \( \exp \left\{ - \int_0^t r(u) \, du \right\} v(t, r(t)) \) is a martingale, and this leads to the partial differential equation
\[
- rv + v_t + (\alpha - \beta r) v_r + \frac{1}{2} \sigma^2 r v_{rr} = 0, \quad 0 \leq t < T_1, \quad r \geq 0.
\]
(Where \( v = v(t, r) \).) The terminal condition is
\[
v(T_1, r) = (B(r, T_1, T_2) - K)^+, \quad r \geq 0.
\]

Other European derivative securities on the bond are priced using the same partial differential equation with the terminal condition appropriate for the particular security.

### 31.6 Deterministic time change of CIR model

**Process time scale:** In this time scale, the interest rate \( r(t) \) is given by the constant coefficient CIR equation
\[
dr(t) = (\alpha - \beta r(t)) \, dt + \sigma \sqrt{r(t)} \, dW(t).
\]

**Real time scale:** In this time scale, the interest rate \( \hat{r}(\hat{t}) \) is given by a time-dependent CIR equation
\[
d\hat{r}(\hat{t}) = (\hat{\alpha} (\hat{t}) - \hat{\beta} (\hat{t}) \hat{r}(\hat{t})) \, d\hat{t} + \hat{\sigma} (\hat{t}) \sqrt{\hat{r}(\hat{t})} \, d\hat{W}(\hat{t}).
\]
There is a strictly increasing time change function $t = \varphi(\hat{t})$ which relates the two time scales (See Fig. 31.3).

Let $\hat{B}(\hat{r}, \hat{t}, \hat{T})$ denote the price at real time $\hat{t}$ of a bond with maturity $\hat{T}$ when the interest rate at time $\hat{t}$ is $\hat{r}$. We want to set things up so

$$\hat{B}(\hat{r}, \hat{t}, \hat{T}) = B(r, t, T) = e^{-rC(t, T) - A(t, T)},$$

where $t = \varphi(\hat{t})$, $T = \varphi(\hat{T})$, and $C(t, T)$ and $A(t, T)$ are as defined previously. We need to determine the relationship between $\hat{r}$ and $r$. We have

$$B(r(0), 0, T) = E \exp \left\{ - \int_0^T r(t) \, dt \right\},$$

$$B(\hat{r}(0), 0, \hat{T}) = \mathbb{E} \exp \left\{ - \int_0^{\hat{T}} \hat{r}(\hat{t}) \, d\hat{t} \right\}.$$

With $T = \varphi(\hat{T})$, make the change of variable $t = \varphi(\hat{t})$, $dt = \varphi'(\hat{t}) \, d\hat{t}$ in the first integral to get

$$B(r(0), 0, T) = \mathbb{E} \exp \left\{ - \int_0^T r(\varphi(\hat{t})) \varphi'(\hat{t}) \, d\hat{t} \right\},$$

and this will be $B(\hat{r}(0), 0, \hat{T})$ if we set

$$\hat{r}(\hat{t}) = r(\varphi(\hat{t})) \varphi'(\hat{t}).$$
31.7 Calibration

\[ \hat{B}(\hat{r}(\hat{t}), \hat{t}, \hat{T}) = B \left( \frac{\hat{r}(\hat{t})}{\varphi'(\hat{t})}, \varphi(\hat{t}), \varphi(\hat{T}) \right) \]

\[ = \exp \left\{ -\hat{r}(\hat{t}) \frac{C(\varphi(\hat{t}), \varphi(\hat{T}))}{\varphi'(\hat{t})} - A(\varphi(\hat{t}), \varphi(\hat{T})) \right\} \]

\[ = \exp \left\{ -\hat{r}(\hat{t}) \hat{C}(\hat{t}, \hat{T}) - \hat{A}(\hat{t}, \hat{T}) \right\}, \]

where

\[ \hat{C}(\hat{t}, \hat{T}) = \frac{C(\varphi(\hat{t}), \varphi(\hat{T}))}{\varphi'(\hat{t})} \]

\[ \hat{A}(\hat{t}, \hat{T}) = A(\varphi(\hat{t}), \varphi(\hat{T})) \]

do not depend on \( \hat{t} \) and \( \hat{T} \) only through \( \hat{T} - \hat{t} \), since, in the real time scale, the model coefficients are time dependent.

Suppose we know \( \hat{r}(0) \) and \( \hat{B}(\hat{r}(0), 0, \hat{T}) \) for all \( \hat{T} \in [0, \hat{T}^*] \). We calibrate by writing the equation

\[ \hat{B}(\hat{r}(0), 0, \hat{T}) = \exp \left\{ -\hat{r}(0) \hat{C}(0, \hat{T}) - \hat{A}(0, \hat{T}) \right\}, \]

or equivalently,

\[ -\log \hat{B}(\hat{r}(0), 0, \hat{T}) = \frac{\hat{r}(0)}{\varphi'(0)} C(\varphi(0), \varphi(\hat{T})) + A(\varphi(0), \varphi(\hat{T})). \]

Take \( \alpha, \beta \) and \( \sigma \) so the equilibrium distribution of \( r(t) \) seems reasonable. These values determine the functions \( C, A \). Take \( \varphi'(0) = 1 \) (we justify this in the next section). For each \( \hat{T} \), solve the equation for \( \varphi(\hat{T}) \):

\[ -\log \hat{B}(\hat{r}(0), 0, \hat{T}) = \hat{r}(0) C(0, \varphi(\hat{T})) + A(0, \varphi(\hat{T})). \quad (*) \]

The right-hand side of this equation is increasing in the \( \varphi(\hat{T}) \) variable, starting at 0 at time 0 and having limit \( \infty \) at \( \infty \), i.e.,

\[ \hat{r}(0) C(0, 0) + A(0, 0) = 0, \]

\[ \lim_{T \to \infty} [\hat{r}(0) C(0, T) + A(0, T)] = \infty. \]

Since \( 0 \leq -\log \hat{B}(\hat{r}(0), 0, \hat{T}) < \infty \), (*) has a unique solution for each \( \hat{T} \). For \( \hat{T} = 0 \), this solution is \( \varphi(0) = 0 \). If \( \hat{T}_1 < \hat{T}_2 \), then

\[ -\log \hat{B}(r(0), 0, \hat{T}_1) < -\log \hat{B}(r(0), 0, \hat{T}_2), \]

so \( \varphi(\hat{T}_1) < \varphi(\hat{T}_2) \). Thus \( \varphi \) is a strictly increasing time-change-function with the right properties.
31.8 Tracking down \( \varphi'(0) \) in the time change of the CIR model

Result for general term structure models:

\[
- \frac{\partial}{\partial T} \log B(0, T) \bigg|_{T=0} = r(0).
\]

Justification:

\[
B(0, T) = \mathbb{E} \exp \left\{ - \int_0^T r(u) \, du \right\}
\]

\[
- \log B(0, T) = - \log \mathbb{E} \exp \left\{ - \int_0^T r(u) \, du \right\}
\]

\[
- \frac{\partial}{\partial T} \log B(0, T) = \frac{\mathbb{E} \left[ r(T) e^{- \int_0^T r(u) \, du} \right]}{\mathbb{E} e^{- \int_0^T r(u) \, du}}
\]

\[
- \frac{\partial}{\partial T} \log B(0, T) \bigg|_{T=0} = r(0).
\]

In the real time scale associated with the calibration of CIR by time change, we write the bond price as

\[
\hat{B}(\hat{r}(0), 0, \hat{T}),
\]

thereby indicating explicitly the initial interest rate. The above says that

\[
- \frac{\partial}{\partial T} \log \hat{B}(\hat{r}(0), 0, \hat{T}) \bigg|_{T=0} = \hat{r}(0).
\]

The calibration of CIR by time change requires that we find a strictly increasing function \( \varphi \) with \( \varphi(0) = 0 \) such that

\[
- \log \hat{B}(\hat{r}(0), 0, \hat{T}) = \frac{1}{\varphi'(0)} \hat{r}(0) C'(\varphi(\hat{T})) + A(\varphi(\hat{T})), \quad \hat{T} \geq 0,
\]

where \( \hat{B}(\hat{r}(0), 0, \hat{T}) \), determined by market data, is strictly increasing in \( \hat{T} \), starts at 1 when \( \hat{T} = 0 \), and goes to zero as \( \hat{T} \rightarrow \infty \). Therefore, \( - \log \hat{B}(\hat{r}(0), 0, \hat{T}) \) is as shown in Fig. 31.4.

Consider the function

\[
\hat{r}(0) C(T) + A(T),
\]

Here \( C(T) \) and \( A(T) \) are given by

\[
C(T) = \frac{\sinh(\gamma T)}{\gamma \cosh(\gamma T) + \frac{1}{2} \beta \sinh(\gamma T)},
\]

\[
A(T) = - \frac{2\alpha}{\sigma^2} \log \left[ \frac{\gamma e^{\frac{1}{2}\gamma T}}{\gamma \cosh(\gamma T) + \frac{1}{2} \beta \sinh(\gamma T)} \right],
\]

\[
\gamma = \frac{1}{2} \sqrt{\beta^2 + 2\sigma^2}.
\]
Figure 31.4: Bond price in CIR model

The function $\hat{r}(0)C(T) + A(T)$ is zero at $T = 0$, is strictly increasing in $T$, and goes to $\infty$ as $T \to \infty$. This is because the interest rate is positive in the CIR model (see last paragraph of Section 31.4).

To solve (cal), let us first consider the related equation

$$- \log \hat{B}(\hat{r}(0), 0, \hat{T}) = \hat{r}(0)C(\varphi(\hat{T})) + A(\varphi(\hat{T})).$$

(cal')

Fix $\hat{T}$ and define $\varphi(\hat{T})$ to be the unique $T$ for which (see Fig. 31.5)

$$- \log \hat{B}(\hat{r}(0), 0, \hat{T}) = \hat{r}(0)C(T) + A(T)$$

If $\hat{T} = 0$, then $\varphi(\hat{T}) = 0$. If $\hat{T}_1 < \hat{T}_2$, then $\varphi(\hat{T}_1) < \varphi(\hat{T}_2)$. As $\hat{T} \to \infty$, $\varphi(\hat{T}) \to \infty$. We have thus defined a time-change function $\varphi$ which has all the right properties, except it satisfies (cal') rather than (cal).
We conclude by showing that $\varphi'(0) = 1$ so $\varphi$ also satisfies (cal). From (cal') we compute

$$\hat{r}(0) = -\frac{\partial}{\partial T} \log \tilde{B}(\hat{r}(0), 0, \tilde{T}) \bigg|_{\tilde{T}=0}$$

$$= \hat{r}(0) C'(\varphi(0)) \varphi'(0) + A'(\varphi(0)) \varphi'(0)$$

$$= \hat{r}(0) C'(0) \varphi'(0) + A'(0) \varphi'(0).$$

We show in a moment that $C'(0) = 1$, $A'(0) = 0$, so we have

$$\hat{r}(0) = \hat{r}(0) \varphi'(0).$$

Note that $\hat{r}(0)$ is the initial interest rate, observed in the market, and is strictly positive. Dividing by $\hat{r}(0)$, we obtain

$$\varphi'(0) = 1.$$

Computation of $C'(0)$:

$$C'(\tau) = \frac{1}{\left( \gamma \cosh(\gamma \tau) + \frac{1}{2} \beta \sinh(\gamma \tau) \right)^2} \left[ \gamma \cosh(\gamma \tau) \left( \gamma \cosh(\gamma \tau) + \frac{1}{2} \beta \sinh(\gamma \tau) \right) \right.$$

$$- \sinh(\gamma \tau) \left( \gamma^2 \sinh(\gamma \tau) + \frac{1}{2} \beta \gamma \cosh(\gamma \tau) \right) \left. \right]$$

$$C'(0) = \frac{1}{\gamma^2} \left[ \gamma (\gamma + 0) - 0 (0 + \frac{1}{2} \beta \gamma) \right] = 1.$$

Computation of $A'(0)$:

$$A'(\tau) = -\frac{2 \alpha}{\sigma^2} \left[ \gamma \cosh(\gamma \tau) + \frac{1}{2} \beta \sinh(\gamma \tau) \right]$$

$$\times \frac{1}{\left( \gamma \cosh(\gamma \tau) + \frac{1}{2} \beta \sinh(\gamma \tau) \right)^2} \left[ \frac{\beta \gamma}{2} e^{\beta \tau/2} \left( \gamma \cosh(\gamma \tau) + \frac{1}{2} \beta \sinh(\gamma \tau) \right) \right.$$

$$- \gamma e^{\beta \tau/2} \left( \gamma^2 \sinh(\gamma \tau) + \frac{1}{2} \beta \gamma \cosh(\gamma \tau) \right) \left. \right],$$

$$A'(0) = -\frac{2 \alpha}{\sigma^2} \left[ \frac{\gamma + 0}{\gamma} \right] \frac{1}{(\gamma + 0)^2} \left[ \frac{\beta \gamma}{2} (\gamma + 0) - \gamma (0 + \frac{1}{2} \beta \gamma) \right]$$

$$= -\frac{2 \alpha}{\sigma^2} \frac{1}{\gamma^2} \left[ \frac{\beta \gamma^2}{2} - \frac{1}{2} \beta \gamma^2 \right]$$

$$= 0.$$