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RESEARCH ARTICLE

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Consistency conditions for affine term structure models

II. Option pricing under diffusions with embedded jumps

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Abstract Sufficient conditions for the application of the Feynman-Kac formula for option pricing for wide classes of affine term structure models in the jump-diffusion case are derived by generalizing earlier results for bond pricing in the pure-diffusion case

Keywords Affine term structure models · Feynman-Kac formula · Processes with jumps

JEL Classification Numbers D81 · C61

1 Introduction

Consider a financial market under several sources of uncertainty represented by a multi-variate Markov process X. The price of an interest rate derivative of the European type, maturing at date T, with the terminal pay-off $g(X(T))$, can be expressed as

$$f(X(t), t) = E_t \left[\exp\left(-\int_t^T r(X(s))\mathrm{d}s \right) g(X(T)) \right], \tag{1}$$

where the expectation is taken under the risk-neutral measure chosen by the market. In pure-diffusion affine term structure models (ATSM), X is modelled as the solution to the stochastic differential equation

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$$dX_j(t) = b_j(X(t), t)dt + \sum_{k=1}^{n} \beta_{jk}\sqrt{S_j(X(t), t)}dW_j, \tag{2}$$

$j = 1, \ldots, n$, where $\beta_{jk} \in \mathbf{R}$, b_j and S_j are affine functions of $X(t)$, and dW is the increment of the standard n-dimensional Brownian motion. The instantaneous interest rate r is modelled as an affine function of the state variable:

$$r(X(s)) = \langle d, X(s)\rangle + d_0, \tag{3}$$

where $d \in \mathbf{R}^n$ is a constant vector and $d_0 \in \mathbf{R}$ is a scalar; $\langle\cdot, \cdot\rangle$ denotes the standard inner product in $\mathbf{R}^n$. The Feynman-Kac formula and the Fourier transform can be used to reduce the calculation of $f(X(t), t)$ to the solution of a parabolic equation, and then to the Cauchy problem for a system of ODE (Riccati equations), with the initial data depending on a parameter. The reduction to the Riccati equations in the case $g = 1$ (bond pricing) was suggested in Cox et al. (1981); the idea to use the Fourier transform to price bond and currency options is due to Heston (1993). Heston's approach was generalized by Duffie and Kan (1996), who coined the term *Affine Term Structure Models*. For the classification of ATSM under diffusion processes, see Dai and Singleton (2000), and for the extension of ATSM to some jump-diffusion processes and extensive bibliography on different families of ATSM for both pure jump and jump-diffusion cases, see Björk et al. (1997), Duffie et al. (2000), and Chacko and Das (2002). Notice that the presence of jumps imposes additional restrictions on the parameters of the model. For instance, in the one-dimensional case, one must ensure that jumps cannot move $X(t)$ in the region where the volatility coefficient becomes negative. Thus, either the volatility is independent of the state variable, or an appropriate restriction on the direction of jumps must be imposed. For very general classes of affine Markov models with jumps, under conditions which ensure the non-negativity of r, see Duffie et al. (2002).

The first step of the solution of an ATSM, namely, the reduction to the backward parabolic problem

$$(\partial_t + L - r)f(x, t) = 0, \quad t < T, \tag{4}$$

$$f(x, T) = g(x), \tag{5}$$

where L is the infinitesimal generator of X, cannot be easily deduced from the general Feyman-Kac theorem although in the majority of papers on ATSM, the applicability of this theorem is taken for granted. The standard and relatively easy justification can be made only when the short rate remains bounded from below. For the standard pure diffusion ATSM, this means that all the factors must be of the CIR-type (i.e. each of them must live on $\mathbf{R}_+$, as in Cox–Ingersoll–Ross (Cox et al. 1985) one-factor model). In the classification of Dai and Singleton (2000), these are $A_n(n)$-models. The justification is also easy (albeit different) at the other extreme of the family of $A_m(n)$-models, namely, for $A_0(n)$-models, the simplest example being the one-factor model of Vasicek (1977) (for details, see e.g. Levendorskiĭ 2004). For the intermediate families $A_m(n)$, $1 \leq m < n$, sufficient conditions for the applicability of the Feynman-Kac formula, and hence, the justification of the standard solution of ATSM-models, were derived in Levendorskiĭ (2004) for bounded pay-offs, and in the pure- diffusion case only. Thus, the bond pricing in

ATSM can be justified by referring to the results in Levendorskiĭ (2004), but not the pricing of options on yields and bonds. Indeed, in an ATSM, the price of the zero-coupon bond is of the form

$$P(x, \tau) = \exp\left[\langle B(\tau), x\rangle + C(\tau)\right], \tag{6}$$

where $\tau > 0$ is the time to maturity. Hence, the pay-off of the European call option written on the zero-coupon bond which matures at time $U > T$ equals

$$g_{\text{call}}(X(T)) = \left(\exp\left[\langle B(U-T), X(T)\rangle + C(U-T)\right] - K\right)_+, \tag{7}$$

where $a_+ = \max\{a, 0\}$ and K is the strike price. We see that the RHS, as a function of the factors, grows exponentially in a certain direction, which depends on $U-T$; in the case of a call option on a basket of bonds, the pay-off will grow exponentially in several directions. Similarly, the pay-off of the call option written on a yield

$$y(U-T) = -(U-T)^{-1}\left[\langle B(U-T), X(T)\rangle + C(U-T)\right] \tag{8}$$

exhibits a linear growth.

In the present paper, we derive sufficient conditions for the applicability of the Feynman-Kac theorem for pay-offs which grow in some directions but not faster than an exponential function, both in the pure-and jump- diffusion cases. Since for applications, an addition of the jump component with an integrable density of jumps provides quite a satisfactory extension of diffusion models, we restrict ourselves to this case (the author is grateful to Mikhail Chernov for a discussion about this issue). The results and proofs hold if the jump component is of finite variation. We also show that under the same conditions, the formal solution obtained by the reduction to the Riccati equations is the price of the contingent claim. For simplicity, we impose conditions on the rate of the growth of the pay-off, which are satisfied for bonds, options on yields and bonds, and baskets of yields and bonds, and forward contracts on yields and bonds; the generalization for more general pay-off functions, with different rates of growth, is straightforward.

Even in the case of bond pricing, we extend the results of Levendorskiĭ (2004), where the conditions imposed guarantee that the bond price does not increase as the CIR-factors increase. The conditions in the present paper allow for the bond price to be non-monotone with respect to time and any of the state variables. (The author is indebted to Mikhail Chernov and Yacine Aït-Sahalia for the indication that in many empirical studies, bond prices are not monotone.)

To clarify the main ideas, we consider the family $A_1(2)$ first (subsection 2.1), then families $A_1(n)$ and $A_2(3)$ (subsections 2.2 and 2.3, respectively; other families $A_m(n)$ can be studied in the same manner), and finally, in section 3, we introduce jumps into $A_1(n)$, $n \geq 2$, and $A_2(3)$ models. Similarly, jumps can be introduced into other $A_m(n)$, $2 \leq m < n$, models.

2 Pure diffusion case

2.1 Family $A_1(2)$

The state space is $\mathbf{R}_+ \times \mathbf{R}$, the short rate r is given by equation (3) with $d_2 > 0$, and the infinitesimal generator of the process is of the form

$$L = (\theta_1 - \kappa_{11}x_1)\partial_1 + (\theta_2 - \kappa_{21}x_1 - \kappa_{22}x_2)\partial_2 + \frac{1}{2}x_1\partial_1^2 + \frac{\alpha + \beta x_1}{2}\partial_2^2, \quad (9)$$

where $\kappa_{11}, \kappa_{22}, \theta_1, \alpha, \beta$ are positive. In Levendorskiĭ (2004), we also assumed that $d_1 \geq 0$ but this condition was useful for the proof of the monotonicity of the bond price, and not used in the proof of the Feynman-Kac formula. Set $\gamma = \kappa_{22}^{-1}d_2$.

Theorem 2.1 *Assume that the following conditions hold:*

(i) $\kappa_{11}, \kappa_{22}, \theta_1, \alpha, \beta$ *and* d_2 *are positive;*
(ii) for $y = 0, -\gamma$,

$$d_1 + \frac{\kappa_{11}^2}{2} + \kappa_{21}y - \frac{\beta}{2}y^2 > 0; \quad (10)$$

(iii) the pay-off g *is continuous, and it satisfies the bound*

$$|g(x)| \leq C \exp[\mu_1 x_1 + (\mu_2^- x_2)_+ + (\mu_2^+ x_2)_+], \quad (11)$$

where $0 \leq \mu_1 < \kappa_{11}$, $\mu_2^- < -\gamma < 0 < \mu_2^+$ *and*

$$d_1 + \kappa_{11}\mu_1 - \frac{1}{2}\mu_1^2 + \kappa_{21}\mu_2^\pm - \frac{\beta}{2}(\mu_2^\pm)^2 > 0. \quad (12)$$

Then the expressions (1), (2) and (3) are the unique solution to the problems (4) and (5) in the class of continuous functions which admit the bound

$$|f(x, \tau)| \leq C_\epsilon \exp[\mu_1 x_1 + (\mu_2^- x_2)_+ + (\mu_2^+ x_2)_+ + \epsilon|x|], \quad (13)$$

for any $\epsilon > 0$*; the constant* C_ϵ *depends on* ϵ *but not on* $x \in \mathbf{R}_+ \times \mathbf{R}$.

Proof First, note that the set of $(\mu_1, \mu_2^\pm)$ that satisfy the conditions of the theorem is non-empty due to equation (10). Second, this theorem was proved in Levendorskiĭ (2004) but formulated in a simpler (albeit weaker) form (Theorem 3.3 in op.cit.). Subsequently, we explain necessary modifications. The main trick of the proof in Levendorskiĭ (2004) is the conjugation of the operator in equation (4) with an appropriate exponential function, and the representation of the operator

$$A_\nu := e^{-\langle \nu, x\rangle}(L - \langle d, x\rangle - d_0)e^{\langle \nu, x\rangle}$$

in the form

$$A_\nu = L^\nu - \langle d^\nu, x\rangle - d_0^\nu, \quad (14)$$

where L^ν is the infinitesimal generator of another process, X^ν, without killing and birth, and properties similar to the ones of X. The crucial requirements are $\kappa^\nu \geq 0$, $\theta_1^\nu > 0$,

$$\langle d^\nu, x\rangle \geq c|x|^\delta \tag{15}$$

and

$$|g(x)|\mathrm{e}^{-\langle \nu,x\rangle} \leq C\mathrm{e}^{-\rho|x|}, \tag{16}$$

where $c, \delta, C, \rho > 0$ are independent of $x \in \mathbf{R}_+ \times \mathbf{R}$. The exponential weight $e^{\langle \nu,x\rangle}$ determines the class of functions in which the problems (4) and (5) have a unique solution given by the expressions (1), (2) and (3).

We have $\theta_1^\nu = \theta_1 > 0$ and $\kappa_{11}^\nu = \kappa_{11} - \nu_1 \geq 0$ iff $\nu_1 \leq \kappa_{11}$, but certainly, it is impossible to obtain equations (15) and (16) on $\mathbf{R}_+ \times \mathbf{R}$ by using a constant vector $\nu = (\nu_1, \nu_2)$. In Levendorskiǐ (2004), different ν_2 (call them $\nu_2^\pm$) are used on the quadrants $x_1 > 0, x_2 > 0$ and $x_1 > 0, x_2 < 0$ (outside a strip adjacent to the half-axis $x_2 = 0$). The two linear functions $\langle \nu^\pm, x\rangle = \nu_1 x_1 + \nu_2^\pm x_2$ are smoothly matched by an appropriate construction on a strip adjacent to the half-axis $x_2 = 0$.

To satisfy the bound (16), it suffices to take $\nu_1 \in (\mu_1, \kappa_{11})$, $\nu_2^- < \mu_2^-$, $\nu_2^+ > \mu_2^+$, but $\nu_2^\pm$ cannot deviate too far from $\mu_2^\pm$ without violating equation (15). Direct calculations show that

$$d_1^\nu = d_1 + \kappa_{11}\nu_1 - \frac{1}{2}\nu_1^2 + \kappa_{21}\nu_2 - \frac{\beta}{2}\nu_2^2,$$
$$d_2^\nu = d_2 + \kappa_{22}\nu_2.$$

The exact value of d_0^μ is not needed, and so we do not show it here. We must have $d_1^{\nu^\pm} > 0$ and $\pm d_2^{\nu^\pm} > 0$. Since $d_2 > 0$ and $\kappa_{22} > 0$, we have $d_2 + \kappa_{22}\nu_2^+ > 0$ for any $\nu_2^+ > 0$ and $d_2 + \kappa_{22}\nu_2^- < 0$ for any $\nu_2^- < -\gamma$. If equation (12) holds, we can choose $\nu_1 \in (\mu_1, \kappa_{11})$, $\nu_2^- < \mu_2^-$ and $\nu_2^+ > \mu_2^+$ so that $d_1^{\nu^\pm} > 0$, and we can choose them arbitrary close to μ_1 and $\mu_2^\pm$, respectively. This finishes the proof of the theorem. □

In a moment, we show that under an additional condition on the parameters of the process and d_1, d_2, and with an appropriate choice of μ_1 and $\mu_2^\pm$, the pay-offs of the bond, and options on yields and bonds, and baskets of yields and bonds satisfy the conditions of Theorem 2.1. Therefore, the Feynman-Kac theorem is applicable, and $f(g; x, \tau)$ given by equations (1)–(3) solves the problems (4)–(5). However, for applications, the Feynman-Kac theorem is the means but not the end. We need to know that the formal solution obtained after the reduction to the Riccati equation, call it $f_0(g; x, \tau)$, coincides with $f(g; x, \tau)$.

Theorem 2.2 *Let $\kappa_{11}, \kappa_{22}, \theta_1, \alpha, \beta$ and d_2 be positive, and let equation (10) hold. Let g be the pay-off function of one of the following contingent claims:*

(i) the bond;
(ii) an European call or put option on a yield and bond, or basket of yields and/or bonds;
(iii) a forward contract on a yield or bond.

Then the formal solution $f_0(g; x, \tau)$ is the price $f(g; x, \tau)$ of the corresponding contingent claim.

Proof Denote by $Q(y)$ in LHS of equation (10). Since Q is concave, equation (10) holds for all $y \in (-\gamma-\epsilon, \epsilon)$ provided $\epsilon > 0$ is sufficiently small. Since the pay-off of a bond and an option on a yield grows slower than any exponential function, it satisfies (11) with any $\mu_1 \in [0, \kappa_{11})$, $\mu_2^- < 0$ and $\mu_2^+ > 0$. If μ_1 is sufficiently close to κ_{11}, $\mu_2^- = -\gamma - \epsilon/2$ and $\mu_2^+ = \epsilon/2$, where $\epsilon > 0$ is sufficiently small, then equation (12) follows from equation (10). Hence, equation (10) is a sufficient condition for the applicability of the Feynman-Kac theorem to bonds and options on yields.

The argument for the case of options on bonds is similar. Suppose that equation (10) holds the pay-off of the bond satisfies equation (11) with any $\mu_1 < 0$, $\mu_2^- \le -\gamma$ and $\mu_2^+ > 0$ (see the calculations subsequently), and we can choose them so that equation (12) holds. Applying Theorem 2.1, we conclude that the bond price satisfies equation (13), and therefore, the pay-off of an option on the bond (or on a basket of bonds) satisfies equation (13) as well. Hence, it satisfies equation (11) with a bit larger μ_1 and μ_2^+ and smaller μ_2^-, which satisfy all the sufficient conditions for the applicability of the Feynman-Kac theorem.

It remains to show that $f_0(g; x, \tau)$, which is a continuous solution to the problems (4) and (5), satisfies the bound equation (13) with $\mu_1 and \mu_2^\pm$ that satisfy the conditions of Theorem 2.1. First consider the bond pricing problem. By substituting the formal solution (6) into (4) and (5), we obtain the system of Riccati equations on $(0, T)$:

$$B_1' = -\kappa_{11}B_1 + \frac{1}{2}B_1^2 - \kappa_{21}B_2 + \frac{\beta}{2}B_2^2 - d_1, \tag{17}$$

$$B_2' = -\kappa_{22}B_2 - d_2, \tag{18}$$

$$C' = -d_0 + \theta_1 B_1 + \frac{\alpha}{2}B_2^2, \tag{19}$$

subject to boundary conditions $B_1(0) = B_2(0) = C(0) = 0$. We solve equation (18), subject to $B_2(0) = 0$:

$$B_2(\tau) = -\gamma\left(1 - \mathrm{e}^{-\kappa_{22}\tau}\right), \tag{20}$$

where $\gamma := \kappa_{22}^{-1}d_2 > 0$. Set $Q_1(\mu) = (\mu - \kappa_{11})^2/2$, and rewrite equation (17) as

$$B_1' = Q_1(B_1(\tau)) - Q(B_2(\tau)). \tag{21}$$

Due to equations (10) and (20), $Q(B_2(\tau))$ is positive on $[0, +\infty)$. Hence, the solution of ODE (21) that starts at 0 cannot reach κ_{11} because the derivative of B_1 becomes negative when $B_1(\tau)$ arrives in a sufficiently small left neighborhood of κ_{11}. Thus, there exists $\mu_1 \in [0, \kappa_{11})$ such that $B_1(\tau) \le \mu_1$ for all τ. It follows that $P(x, \tau)$ satisfies the bound (13) with this μ_1, $\mu_2^- = -\gamma$ and $\mu_2^+ = 0$. Hence, $P(x, \tau) = f(1; x, \tau)$.

For the proof of the equality $f_0(g; x, \tau) = f(g; x, \tau)$ for (call and put) options on yields and bonds, and baskets of yields and bonds, it is important that, for a fixed $\tau_0 > 0$, the bond price $P(x, \tau_0)$ satisfies equation (11) with $\mu_2^- > -\gamma$: on the strength of equation (20), any $\mu_2^- \in (-\gamma, -\left[1 - e^{-\kappa_{22}\tau_0}\right]\gamma]$ will do. We conclude that the pay-off of an European call or put option on a yield or bond satisfies an estimate of the form

$$|g(x)| < C\exp[\mu_1^g x_1 + \mu_2^g x_2 - \rho|x|], \tag{22}$$

where $\rho > 0$, $\mu_1^g \in [0, \kappa_{11})$, $\mu_2^g \in (-\gamma, \epsilon)$ and $\epsilon > 0$ is sufficiently small so that equation (10) holds with $y = \epsilon$. In addition, there exists M such that

$$g(x) = 0, \quad \text{where} \quad \mu_1^g x_1 + \mu_2^g x_2 < -M. \tag{23}$$

Set $g_\mu(x) = \exp[-\mu_1^g x_1 - \mu_2^g x_2]g(x)$. On the strength of equations (22) and (23), the Fourier transform of g_μ, denoted $\hat{g}_\mu(\xi)$:

$$\hat{g}_\mu(\xi) = \int\int_{\mathbf{R}^2} \mathrm{e}^{-\mathrm{i}\langle x,\xi\rangle} g_\mu(x)\mathrm{d}x,$$

is well defined on $\mathbf{R}^2$; equivalently, $\hat{g}(\xi)$ is well defined on the plane $\mathrm{Im}\,\xi_1 = -\mu_1^g$, $\mathrm{Im}\,\xi_2 = -\mu_2^g$ in $\mathbf{C}^2$. Using the definition of the call and put options and the fact that the bond price is infinitely smooth, with all the derivatives admitting the bound of the same form as the one for the price, we obtain that $\hat{g}$ is integrable over this plane. Applying the inverse Fourier transform, we obtain a representation for the price of the option:

$$g(x) = (2\pi)^{-1} \int\limits_{\mathrm{Im}\,\xi_1=-\mu_1^g} \int\limits_{\mathrm{Im}\,\xi_2=-\mu_2^g} \mathrm{e}^{\mathrm{i}\langle x,\xi\rangle} \hat{g}(\xi)\mathrm{d}\xi. \tag{24}$$

Let $(B(\xi;\tau), C(\xi;\tau))$ be the solution to equations (17), (18) and (19), subject to

$$B_1(0) = \mathrm{i}\xi_1; \quad B_2(0) = \mathrm{i}\xi_2; \quad C(0) = 0. \tag{25}$$

Then the option price is

$$f_0(g; x, \tau) = (2\pi)^{-1} \int\limits_{\mathrm{Im}\,\xi_1=-\mu_1^g} \int\limits_{\mathrm{Im}\,\xi_2=-\mu_2^g} e^{\langle x, B(\xi;\tau)\rangle + C(\xi;\tau)} \hat{g}(\xi)\mathrm{d}\xi, \tag{26}$$

and therefore, to prove that $f_0(g; x, \tau)$ satisfies the bound equation (13) with $\mu_1 \in [0, \kappa_{11})$, $\mu_2^- < 0 < \mu_2^+$ such that equation (12) holds, it suffices to show that for any τ,

$$\mathrm{Re}\,B_1(\xi;\tau) < \kappa_{11}, \quad -\gamma < \mathrm{Re}\,B_2(\xi;\tau) < \epsilon. \tag{27}$$

The first inequality is immediate from the explicit formula

$$B_2(\xi;\tau) = \gamma(\mathrm{e}^{-\kappa_{22}\tau} - 1) + \mathrm{i}\,\xi_2 \mathrm{e}^{-\kappa_{22}\tau}. \tag{28}$$

To derive the second one, we write down the equation for $\mathrm{Re}\,B_1(\xi;\tau)$, the real part of $B_1(\xi;\tau)$ (the derivative is taken w.r.t. τ):

$$\begin{aligned}\mathrm{Re}\,B_1' = &-\kappa_{11}\mathrm{Re}\,B_1 + ((\mathrm{Re}\,B_1)^2 - (\mathrm{Im}\,B_1)^2)/2 \\ &-\kappa_{21}\mathrm{Re}\,B_2 + \beta((\mathrm{Re}\,B_2)^2 - (\mathrm{Im}\,B_2)^2)/2 - d_1\end{aligned}$$

or

$$\operatorname{Re} B_1' = Q_1(\operatorname{Re} B_1) - Q(\operatorname{Re} B_2) - (\operatorname{Im} B_1)^2/2 - \beta(\operatorname{Im} B_2)^2/2.$$

On the strength of equations (10) and (28), the RHS becomes negative as $B_1(\tau)$ approaches κ_{11}. Hence, if $\operatorname{Re} B_1(0) < \kappa_{11}$, then $\operatorname{Re} B_1(\tau) < \kappa_{11}$ for all $\tau > 0$.

The proof of Theorem 2.2 for European, put and call options on a yield or bond is complete. In the case of an option on a basket of bonds or yields, it may be necessary to represent the pay-off in the form of sufficiently smooth pay-offs each of which enjoys properties used aforementioned – with different μ_2s! – and use the result aforementioned for each term.

The pay-off of the forward contract on the bond is an exponential function of the form $g_\mu(x) = C\exp[\langle\mu, x\rangle]$, where $\mu_1 < \kappa_{11}$ and $\mu_1 \in (-\gamma, 0)$, therefore the argument used previously applies. Finally, the pay-off of the forward contract on a yield can be represented as the limit of the form

$$\lim_{\epsilon\to 0} C(e^{\epsilon\langle\mu,x\rangle} - 1)/\epsilon,$$

where $\mu_1 < \kappa_{11}$ and $\mu_1 \in (-\gamma, 0)$, hence the statement of the theorem for forward contract on yields can be obtained as the limit of the results for forward contracts on the bond. □

Remark 2.1 It is natural to ask how close is the sufficient condition equation (10) to necessary ones. For $y = -\gamma$, equation (10) is almost necessary. Indeed, if

$$d_1 + \frac{\kappa_{11}^2}{2} - \kappa_{21}\gamma - \frac{\beta}{2}\gamma^2 < 0, \tag{29}$$

then in a neighborhood of $+\infty$, $Q(B_2(\tau))$ is positive, hence, starting from a certain $\tau = \tau_0$, the derivative $B_1'(\tau)$ is positive and separated from zero. It means that, eventually, $B_1(\tau) > \kappa_{11}$, and from equation (21), we conclude that the solution to the Riccati equations reaches $+\infty$ in finite time.

The remaining condition $d_1 + \kappa_{11}^2/2 > 0$ does not seem to be too restrictive (and of course, is satisfied if $d_1 \geq 0$ as it was assumed in Levendorskiĭ 2004) but it is, probably, unnecessary. The conjecture, which we were unable to prove, is

BK The standard reduction to Riccati equations is justified provided $B_1(\tau)$is bounded by κ_{11} from above.

The next argument explains how to construct a series of examples when one of the conditions of Theorem 2.2 is not satisfied but according to conjecture BK, the standard scheme should work. If $d_1 + \kappa_{11}^2/2$ is negative but its absolute value is sufficiently small, then under condition equation (10) with $y = -\gamma$, $Q(B_2(\tau))$ becomes positive before $B_1(\tau)$ reaches κ_{11} from below, and remains positive ever since. The argument in the proof of Theorem 2.2 shows that $B_1(\tau)$ will never reach κ_{11}.

Finally, we also conjecture that the condition on B_1 in conjecture BK is necessary or close to a necessary one.

2.2 Family $A_1(n)$

The state space is $\mathbf{R}_+ \times \mathbf{R}^{n-1}$, the r is given by equation (3) with $d_j > 0$, $j = 2, \dots, n$, and the infinitesimal generator of the process is of the form

$$L = \langle \theta - \kappa x, \partial \rangle + \frac{1}{2} x_1 \partial_1^2 + \frac{1}{2} \sum_{j,k=2}^{n} (\alpha_{jk} + x_1 \beta_{jk}) \partial_j \partial_k, \tag{30}$$

where $\theta_1, \kappa_{jj}, j = 1, \dots, n$, are positive, $\kappa_{jl} \le 0, 1 \le l < j, \kappa_{jk} = 0, j < k$, and $\alpha = [\alpha_{jk}]_{j,k=2}^n$, $\beta = [\beta_{jk}]_{j,k=2}^n$ are positive definite matrices (these restrictions can be relaxed). By using the family of transformations described in Dai and Singleton (2000), it is possible to reduce any $A_1(n)$-model to a model, where the only non-zero entries of the matrix κ are κ_{j1} and κ_{jj}, $j = 1, 2, \dots, n$, and the next two theorems about the justification of the use of the Feynman-Kac theorem and the formal solution will be formulated under this assumption. Set $\gamma_j = \kappa_{jj}^{-1} d_j$, $\kappa^{21} = (\kappa_{21}, \dots, \kappa_{n1})$, and introduce a quadratic polynomial

$$Q^0(y) = d_1 + \langle y, \kappa^{21} \rangle - \frac{1}{2} \langle \beta y, y \rangle$$

in $n - 1$ variables. The next theorem and its proof are straightforward generalizations of the ones in the two-factor case aforementioned (and generalizations of Theorem 3.7 and its proof in Levendorskiǐ 2004).

Theorem 2.3 *Suppose that the following conditions are satisfied:*

(i) the conditions on the parameters of the model formulated previously hold;

(ii)
$$\frac{\kappa_{11}^2}{2} + Q^0(y) > 0, \quad \forall\, y \in \{-\gamma_2, 0\} \times \cdots \times \{-\gamma_n, 0\}; \tag{31}$$

(iii) the pay-off g is continuous, and it satisfies the bound

$$|g(x)| \le C \exp\left[\mu_1 x_1 + \sum_{j=2}^{n} ((\mu_j^- x_j)_+ + (\mu_j^+ x_j)_+) \right], \tag{32}$$

for some $0 \le \mu_1 < \kappa_{11}$ *and* $\mu_j^- < -\gamma_j < 0 < \mu_j^+$, $j = 2, 3, \dots, n$, *such that*

$$\kappa_{11}\mu_1 - \frac{1}{2}\mu_1^2 + Q^0(y) > 0, \quad \forall\, y \in \{\mu_2^-, \mu_2^+\} \times \cdots \times \{\mu_n^-, \mu_n^+\}. \tag{33}$$

Then the expressions (1), (2) and (3) are the unique solution to the problems (4) and (5) in the class of continuous functions, which admit the bound

$$|f(x, \tau)| \le C_\epsilon \exp\left[\mu_1 x_1 + \sum_{j=2}^{n} ((\mu_j^- x_j)_+ + (\mu_j^+ x_j)_+) + \epsilon |x| \right], \tag{34}$$

for any $\epsilon > 0$, *where* C_ϵ *depends on* ϵ *but not on* x.

Theorem 2.4 *Let the conditions on the parameters of the model in Theorem 2.3 hold, and let g be the pay-off function of one of the interest rate derivatives considered in Theorem 2.2.*

Then the formal solution $f_0(g; x, \tau)$ is the price $f(g; x, \tau)$ of the corresponding contingent claim.

Proof Since the pay-off of an option on a yield grows slower than any exponential function, it satisfies equation (32) with any positive μ_1, μ_j^+, $j = 2, \dots, n$, and negative μ_j^-, $j = 2, \dots, n$; if $\mu_1 = \kappa_{11} - \epsilon/2$, $\mu_j^- = -\gamma_j - \epsilon/2$, and $\mu_j^+ = \epsilon/2$, $j = 2, \dots, n$, where $\epsilon > 0$ is sufficiently small, then equation (33) follows from equation (31). Hence, equation (33) is a sufficient condition for the applicability of the Feynman-Kac theorem to pricing of bonds and European call and put options on yields. The reader recognizes the same argument as in the beginning of the proof of Theorem 2.2, and the remaining part of the proof is also a straightforward modification of the proof of Theorem 2.2. □

Remark 2.2 The condition equation (31) with $y = (-\gamma_2, \dots, -\gamma_n)$ is almost necessary: if the inequality is of the opposite sign, then $B_1(\tau)$ reaches $+\infty$ in finite time. For the other y, the condition (31) is not too restrictive but it is unnecessary. The more involved techniques (similar to the one used in Levendorskiǐ 2004) but without the direct use of the representation theorem for the analytic semigroups and the localization in the x-space – the localization in the (x, t)-space must be used instead) allow one to replace the condition (31) with a weaker version

$$\frac{\kappa_{11}^2}{2} + Q^0(B_2(\tau), \dots, B_n(\tau)) > 0, \quad \forall\, \tau \in [0, +\infty]. \tag{35}$$

Condition (35) is quite similar to condition (10), and may be unnecessary, as the latter. We make the same conjecture BK as in the case $n = 2$.

2.3 Family $A_2(3)$

By using the family of transformations described in Dai and Singleton (2000), it is possible to reduce any three-factor ATSM with two factors of the CIR-type to a model of the form

$$\mathrm{d}X(t) = (\theta - \kappa X(t))\mathrm{d}t + \sqrt{[S_{jj}(t)]}\mathrm{d}B(t), \tag{36}$$

where $\theta \in \mathbf{R}^3$ is the vector with components $\theta_3 = 0$,

$$\theta_1 > 0, \quad \theta_2 > 0, \tag{37}$$

and the entries of matrix $\kappa = [\kappa_{jl}]$ satisfy

$$\kappa_{11}, \kappa_{22}, \kappa_{33} > 0, \tag{38}$$

$$\kappa_{21}, \kappa_{12} \le 0, \tag{39}$$

$$\kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21} > 0, \tag{40}$$

$$\kappa_{13} = \kappa_{23} = 0. \tag{41}$$

Further,

$$S_{jj}(t) = \beta_{jj} X_j(t), \quad j = 1, 2, \tag{42}$$

$$S_{33}(t) = \alpha_3 + \beta_{31} X_1(t) + \beta_{32} X_2(t), \tag{43}$$

where

$$\alpha_3, \beta_{11}, \beta_{22} > 0, \quad \beta_{31}, \beta_{32} \geq 0, \tag{44}$$

and finally, the short rate process is given by equation (3) with $d_3 > 0$. Notice that under conditions, (37), (38), (39), (40) and (41), any trajectory of the process X, which starts in the region $x_1 \geq 0,\ x_2 \geq 0$, remains in this region, a.s. Notice that the infinitesimal generator of the process is

$$L = (\theta - \kappa x)^T \partial_x + \frac{1}{2} \sum_{j=1,2} \beta_{jj} x_j \partial_j^2 + \frac{1}{2}(\alpha_3 + \beta_{31} x_1 + \beta_{32} x_2) \partial_3^2, \tag{45}$$

and set $\gamma = d_3/\kappa_{33}$, $l_j = \kappa_{jj}/\beta_{jj}$. The next theorem generalizes Theorem 3.11 in Levendorskiǐ (2004); the proof is a modification of the proof of Theorem 3.11 along the lines indicated in subsection 2.1.

Theorem 2.5 *Assume that the following conditions hold:*

(i) parameters of the model satisfy the conditions stated previously;
(ii) for $y = 0, -\gamma$,

$$d_1 + \frac{\kappa_{11}^2}{2\beta_{11}} + \frac{\kappa_{21}\kappa_{22}}{\beta_{22}} + \kappa_{31} y - \frac{\beta_{31}}{2} y^2 > 0, \tag{46}$$

$$d_2 + \frac{\kappa_{22}^2}{2\beta_{22}} + \frac{\kappa_{12}\kappa_{11}}{\beta_{11}} + \kappa_{32} y - \frac{\beta_{32}}{2} y^2 > 0; \tag{47}$$

(iii) the pay-off g is continuous, and it satisfies the bound

$$|g(x)| \leq C \exp\left[\mu_1 x_1 + \mu_2 x_2 + (\mu_3^- x_3)_+ + (\mu_3^+ x_3)_+\right], \tag{48}$$

where $0 \leq \mu_j < l_j$ *and* $\mu_3^- < -\gamma < 0 < \mu_3^+$ *are such that*

$$d_1 + \kappa_{11}\mu_1 - \frac{\beta_{11}\mu_1^2}{2} + \kappa_{21}\mu_2 + \kappa_{31} y - \frac{\beta_{31}}{2} y^2 > 0, \tag{49}$$

$$d_2 + \kappa_{22}\mu_2 - \frac{\beta_{22}\mu_2^2}{2} + \kappa_{12}\mu_1 + \kappa_{32} y - \frac{\beta_{32}}{2} y^2 > 0, \tag{50}$$

for $y = \mu_3^{\pm}$.

Then the expressions (1), (2) and (3) are the unique solution to the problems (4) and (5) in the class of continuous functions which admit the bound

$$|f(x, \tau)| \leq C_\epsilon \exp\left[\mu_1 x_1 + \mu_2 x_2 + (\mu_3^- x_3)_+ + (\mu_3^+ x_3)_+ + \epsilon |x|\right], \tag{51}$$

for any $\epsilon > 0$, *where* C_ϵ *depends on* ϵ *but not on* x.

Theorem 2.6 *Let the conditions on the parameters of the process in Theorem 2.5 hold, and let g be the pay-off function of one of the interest rate derivatives in Theorem 2.2.*

Then the formal solution $f_0(g; x, \tau)$ is the price $f(g; x, \tau)$ of the corresponding contingent claim.

Proof The proof is essentially the same as for family $A_1(n)$, the new details being caused by a different structure of the Riccati equations, which now are

$$B_1' = -\kappa_{11}B_1 - \kappa_{21}B_2 + \frac{\beta_{11}}{2}B_1^2 - d_1 - \kappa_{31}B_3 + \frac{\beta_{31}}{2}B_3^2, \tag{52}$$

$$B_2' = -\kappa_{12}B_1 - \kappa_{22}B_2 + \frac{\beta_{22}}{2}B_2^2 - d_2 - \kappa_{32}B_3 + \frac{\beta_{32}}{2}B_3^2, \tag{53}$$

$$B_3' = -\kappa_{33}B_1 - d_3, \tag{54}$$

$$C' = -d_0 + \theta_1 B_1 + \theta_2 B_2 + \frac{\alpha_3}{2}B_3^2. \tag{55}$$

We need to ensure that the trajectory of the subsystems (52), (53) and (54) that start at (μ_1, μ_2, μ_3), where $\mu_j \in [0, l_j)$, $j = 1, 2$, and $\mu_3 \in (-\gamma, \epsilon)$ (where $\epsilon > 0$ is sufficiently small), remains in the region $B_1(\tau) < l_1, B_2(\tau) < l_2, B_3(\tau) \in (-\gamma, \epsilon)$; the modification for the case of complex initial data is straightforward. We can easily find B_3:

$$B_3(\tau) = -\gamma(1 - \mathrm{e}^{-\kappa_{33}\tau}) + \mu_3\mathrm{e}^{-\kappa_{33}\tau}, \tag{56}$$

therefore $B_3(\tau)$ remains in the interval $(-\gamma, \epsilon)$. To show that $B_1(\tau) < l_1$, $B_2(\tau) < l_2$ for all $\tau > 0$, we denote the LHS of equations (46) and (47) by $Q_{13}(y)$ and $Q_{23}(y)$, and rewrite equations (52) and (53) as

$$B_1' = \kappa_{21}[l_2 - B_2] + \frac{\beta_{11}}{2}[l_1 - B_1]^2 - Q_{13}(B_3), \tag{57}$$

$$B_2' = \kappa_{12}[l_1 - B_1] + \frac{\beta_{22}}{2}[l_2 - B_2]^2 - Q_{23}(B_3). \tag{58}$$

Due to equations (46), (47) and (56) $Q_{3k}(B_3(\tau)), k = 1, 2$, are negative for all $\tau > 0$, and the first terms on the RHS of equations (57) and (58) are non-positive in the region $B_1 < l_1$, $B_2 < l_2$, since $\kappa_{21} \le 0$ and $\kappa_{12} \le 0$. Therefore, as long as B_2 remains below l_2, B_1 cannot reach l_1 from below: the RHS of equation (57) will become negative. Hence, B_1 must remain below l_1. Similarly, as long as B_1 is below l_1, B_2 cannot cross the level l_2. Finally, B_k cannot reach l_k, $k = 1, 2$, simultaneously because then the RHS in the both equations (57) and (58) will become negative. □

Remark 2.3 As in the case of $A_1(n)$ families, the conditions aforementioned are unnecessary. We conjecture that a sufficient condition, which is close to a necessary one, is: $(B_1(\tau), B_2(\tau))$ never leaves $(-\infty, l_1) \times (-\infty, l_2)$.

3 Jump-diffusion case

3.1 Family $A_1(2)$ with a jump component

Now we assume that, in addition to the diffusion component of the process, with the infinitesimal generator given by equation (9), there is a compound Poisson jump component, so that the infinitesimal generator of the process X can be represented as a sum

$$L = L_{\mathrm{D}} + L^0 + x_1 L^1, \tag{59}$$

where L^k acts as follows:

$$L^k u(x) = \int\limits_{(\mathbf{R}_+ \times \mathbf{R})\setminus\{0\}} (u(x+y) - u(x)) F^k(\mathrm{d}y).$$

Notice that we do not allow for jumps outside the state space $\mathbf{R}_+ \times \mathbf{R}$ of the model. In an ATSM model, we must allow for functions to grow exponentially in some directions. The infinitesimal generator of a jump component can act in a space of exponentially growing functions only if the Lévy density decays exponentially. Hence, we assume that there are open sets $U^k \in \mathbf{R}^2$, $k = 0, 1$, containing 0, such that

$$\Psi^k(\mu) := \int\limits_{(\mathbf{R}_+ \times \mathbf{R})\setminus\{0\}} (\mathrm{e}^{\langle \mu, y \rangle} - 1) F^k(\mathrm{d}y) < \infty, \quad \forall\, \mu \in U^k. \tag{60}$$

We also assume that the densities are integrable: $F^k \in L_1, k = 0, 1$, and the density $F^1(dy)$ is concentrated on the axis $y_1 = 0$:

$$F^1(\mathrm{d}y) = \delta \otimes F^1(\mathrm{d}y_2),$$

so that U^1 is a strip of the form $U^1 = \{\mu_2 \mid \lambda^- < \mu_2 < \lambda^+\}$. To understand how large U^1 and U^2 must be in order that an ATSM model with jumps had the simplest basic applications, consider the formal solution to the bond pricing problem. [In Chacko and Das (2002), it is assumed that $\Psi^k(\mu)$ is defined for all μ, which excludes exponentially decaying Lévy densities; Duffie et al. (2002) conduct thorough analysis for general Markov processes but under conditions which ensure the positivity of the short rate] Let τ be the time to expiry. One looks for the bond price in the form (6). In the case of $A_1(2)$-model with jumps, $B(\tau)$ and $C(\tau)$ satisfy the generalized system of the Riccati equations

$$B_1' = -\kappa_{11} B_1 + \frac{1}{2} B_1^2 - \kappa_{21} B_2 + \frac{\beta}{2} B_2^2 - d_1 + \Psi^1(B_2), \tag{61}$$

$$B_2' = -\kappa_{22} B_2 - d_2, \tag{62}$$

$$C' = -d_0 + \theta_1 B_1 + \frac{\alpha}{2} B_2^2 + \Psi^0(B), \tag{63}$$

subject to boundary conditions $B_1(0) = B_2(0) = C(0) = 0$. We solve equation (62) subject to $B_2(0) = 0$ – see equation (20)– and we observe that if we want to be able to price the bond far from maturity *by using the standard reduction to the Riccati equations*, we need to impose the conditions $\lambda^- < -\gamma$, and $\{0\} \times (0, -\gamma) \in U^0$. Certainly, these conditions are by no means sufficient. A general necessary and sufficient condition is: the trajectory of the dynamic systems (61) and (62), that start at (0, 0), remains in $U^0 \cap U^1$. The following lemma provides relatively simple sufficient conditions for $(B_1(\tau), B_2(\tau))$ to remain in $U^0 \cap U^1$ for all $\tau > 0$.

Lemma 3.1 *Let there exist* $\epsilon > 0$ *such that*

$$[0, \kappa_{11}) \times [-\gamma, \epsilon] \subset U^1 \times U^0, \tag{64}$$

and let, for $y = 0, -\gamma$,

$$d_1 + \frac{\kappa_{11}^2}{2} + \kappa_{21} y - \frac{\beta}{2} y^2 - \Psi^1(y) \geq 0. \tag{65}$$

Then the trajectory of the systems (61) and (62), that start at $(\mathrm{i}\xi_1, \mathrm{i}\xi_2)$ *with* $-\mathrm{Im}\,\xi_1 = \mu_1 < \kappa_{11}$, $-\mathrm{Im}\,\xi_2 = \mu_2 \in (-\gamma, \epsilon)$ *satisfies* $\mathrm{Re}\, B_1(\tau) < \kappa_{11}$, $\mathrm{Re}\, B_2(\tau) \in (-\gamma, \epsilon)$ *for all* $\tau > 0$.

Proof The proof is the same as in the no-jump case. The part of the function $V(y) = \beta y^2/2$ is played now by $V(y) = \beta y^2/2 + \Psi^1(y)$, and the properties of V, namely, $-V$ is concave, and $\mathrm{Re}\, V(y) \leq V(\mathrm{Re}\, y)$, which formed the basis of the proof of Theorem 2.2, hold in the presence of jumps as well. □

Theorem 3.2 *Let X be the process with the infinitesimal generator (59), and let the short rate process be defined by equation (3). Let g satisfy the bound (11) with* $\mu_1 \in [0, \kappa_{11})$ *and* $\mu_2^- < -\gamma < 0 < \mu_2^+$ *such that*

$$(0, \mu_2^\pm), (\mu_1, \mu_2^\pm) \in U^0 \cap U^1 \tag{66}$$

and

$$d_1 + \kappa_{11}\mu_1 - \frac{1}{2}\mu_1^2 + \kappa_{21}\mu_2^\pm - \frac{\beta}{2}(\mu_2^\pm)^2 - \Psi^1(\mu_2^\pm) > 0. \tag{67}$$

Then the expression (1) is the unique solution to the problems (4) and (5) in the class of continuous functions which admit the bound (13).

Proof The main trick of the proof is as in the proof of Theorem 2.1. However, now we may use only $\nu \in U^0 \cap U^1$. Thus, the first condition on $\nu^\pm$ is

$$\nu^\pm \in U^0 \cap U^1. \tag{68}$$

Direct calculations give

$$\begin{aligned} A_\nu = {} & (\theta_1 - \kappa_{11})(\partial_1 + \nu_1) + (\theta_2 - \kappa_{21}x_1 - \kappa_{22}x_2)(\partial_2 + \nu_2) \\ & + \frac{1}{2}x_1(\partial_1 + \nu_1)^2 + \frac{\alpha + \beta x_1}{2}(\partial_2 + \nu_2)^2 \\ & + L_\nu^0 + x_1 L_\nu^1 + \Psi^0(\nu) + x_1\Psi^1(\nu) - \langle d, x\rangle - d_0, \end{aligned}$$

where L^k_ν is the infinitesimal generator of the compound Poisson process with the Lévy density $\mathrm{e}^{\langle \nu, y\rangle} F^k(\mathrm{d}y)$, $k = 0, 1$. Multiplying out and rearranging, we obtain equation (14) with

$$\begin{aligned} L^\nu = &(\theta_1 - (\kappa_{11} - \nu_1)x_1)\partial_1 + \frac{1}{2}\partial_1^2 + L^0_\nu + x_1 L^1_\nu \\ &+[\theta_2 + \alpha\nu - (\kappa_{21} - \beta\nu_2)]\partial_2 + \frac{\alpha + \beta x_1}{2}\partial_2^2 \end{aligned}$$

and

$$\begin{aligned} d^\nu_1 &= d_1 + \kappa_{11}\nu_1 - \frac{1}{2}\nu_1^2 + \kappa_{21}\nu_2 - \frac{\beta}{2}\nu_2^2 - \Psi^1(\nu), \\ d^\nu_2 &= d_2 + \kappa_{22}\nu_2. \end{aligned}$$

The exact value of d^ν_0 is not needed, and so we do not show it here.

We need $\kappa^\nu_{11} := \kappa_{11} - \nu_1$ and d^ν_1 to be non-negative and positive, respectively, for both $\nu = \nu^\pm$ and $d^{\nu^\pm}_2$ to be positive (respectively, negative) for the sign "+" (respectively, "−"), therefore we impose the following conditions on ν_1 and $\nu_2^\pm$:

$$\mu_1 < \nu_1 < \kappa_{11}; \tag{69}$$

$$\nu_2^- < \mu_2^-, \quad \nu_2^+ > \mu_2^+; \tag{70}$$

$$d_1 + \kappa_{11}\nu_1 - \frac{1}{2}\nu_1^2 + \kappa_{21}\nu_2^\pm - \frac{\beta}{2}(\nu_2^\pm)^2 - \Psi^1(\nu_2^\pm) > 0. \tag{71}$$

Under conditions of the theorem, ν_1 and $\nu_2^\pm$ with these properties exist.

In the most technical part of the proof of Theorem 2.1, which is not reproduced here because it is the same as in Levendorskiĭ (2004), the representation theorem for analytic semigroup (see Yosida 1977) and the standard construction of the resolvent from the theory of boundary problems for degenerate elliptic operators (Levendorskiĭ 1993; Levendorskiĭ and Paneyakh 1990) are used. An addition of $L^0 + x_1 L^1$ to the infinitesimal generator does not spoil the proof: since the density F^0 is integrable and it exponentially decays at infinity, we have that L^0 is a bounded operator in L_2 with an appropriate exponential weight. Therefore L^0 is a weak perturbation, which is essentially irrelevant for the construction of the resolvent. (The same argument applies if the jump component is of finite variation: the infinitesimal generator is a pseudo-differential operator of order less than 1 then, and the perturbation is weak as well.) The term $x_1 L^1$ requires no changes in the construction because we assumed that L^1 acts only w.r.t. x_2, and the construction of the resolvent uses the Fourier transform w.r.t. x_2; after the Fourier transform is made, there is no essential difference between the term $x_1(\mathrm{i}\kappa_{21}\xi_2 + \beta\xi_2^2/2)$ in the pure-diffusion case, and the term $x_1(\mathrm{i}\kappa_{21}\xi_2 + \beta\xi_2^2/2 - \Psi^1(\mathrm{i}\xi_2))$ in the case of diffusions with embedded jumps. □

Corollary 3.3 *Assume that*

(i) X is the process with the infinitesimal generator (59);
(ii) the short rate process is defined by equation (3);

(iii) the parameters of the model satisfy conditions equation (64) and (65);
(iv) g is the pay-off function of one of the interest rate derivatives in Theorem 2.2.

Then the formal solution $f_0(g; x, \tau)$ is the price $f(g; x, \tau)$ of the corresponding contingent claim.

Proof Under conditions (64) and (65), we can choose μ_1 and $\mu_2^{\pm}$ so that the conditions of Theorem 3.2 are satisfied; hence, $f(g; x, \tau)$ is the unique solution to the problems (4) and (5), that satisfies the bound (13), and it remains to show that the formal solution $f_0(g; x, \tau)$ satisfies the bound (13) provided μ_1 and $\mu_2^{\pm}$ are chosen appropately. This is deduced from Lemma 3.1 as in the proof of Theorem 2.2. □

3.2 Family $A_1(n)$ with a jump component

We consider a process with the infinitesimal generator (59), where L_{D} is given by (30), and L^k is the generator of the compound Poisson process with the integrable Lévy density $F^k(\mathrm{d}y)$, $k = 0, 1$. The density F^1 is supported on the hyperplane $x_1 = 0$, and both densities decay exponentially at infinity, so that there exists open sets U^k, containing the origin, such that $\Psi^k(\mu) < +\infty$ for $\mu \in U^k$, $k = 0, 1$. As in subsection 2.2, we assume that the only non-zero entries of the matrix κ are κ_{j1} and κ_{jj}, $j = 1, \ldots, n$, and set $\gamma_j = \kappa_{jj}^{-1} d_j$, $\kappa^{21} = (\kappa_{21}, \ldots, \kappa_{n1})$. Instead of a quadratic polynomial in subsection 2.2, we need to introduce the real-valued function

$$Q^0(y) = d_1 + \langle y, \kappa^{21} \rangle - \frac{1}{2}\langle \beta y, y \rangle - \Psi^1(y)$$

with the domain $\mathbf{R}^{n-1}$. The next two theorems and their proofs are straightforward generalizations of the ones in the two-factor case aforementioned.

Theorem 3.4 *Assume that*

(i) X is the process with the infinitesimal generator (59);
(ii) the short rate process is defined by equation (3);
(iii) the parameters of the model satisfy conditions

$$[0, \kappa_{11}) \times [-\gamma_2, 0] \times [-\gamma_n, 0] \subset U^1 \times U^0; \tag{72}$$

(iv) for all $y \in \{-\gamma_2, 0\} \times \cdots \times \{-\gamma_n, 0\}$,

$$\frac{\kappa_{11}^2}{2} + Q^0(y) > 0; \tag{73}$$

(v) the pay-off g is continuous, and it satisfies equation (32) for some $0 \le \mu_1 < \kappa_{11}$ and $\mu_j^- < -\gamma_j < 0 < \mu_j^+$, $j = 2, 3, \ldots, n$, such that for all $y \in \{\mu_2^-, \mu_2^+\} \times \cdots \times \{\mu_n^-, \mu_n^+\}$,

$$(0, y), (\mu_1, y) \in U^0 \cap U^1 \tag{74}$$

and

$$\kappa_{11}\mu_1 - \frac{1}{2}\mu_1^2 + Q^0(y) > 0. \tag{75}$$

Then the expression (1) defines the unique solution to the problems (4) and (5) in the class of continuous functions which admit the bound (34).

Theorem 3.5 *Assume that*

(i) the conditions on the parameters of the model in Theorem 3.4 hold;
(ii) g is the pay-off function of one of the interest rate derivatives in Theorem 2.2.

Then the formal solution $f_0(g; x, \tau)$ is the price $f(g; x, \tau)$ of the corresponding contingent claim.

3.3 Family $A_2(3)$ with a jump component

Now we assume that, in addition to the diffusion component of the process, with the infinitesimal generator given by equation (45), there is a compound Poisson jump component, so that the infinitesimal generator of the process X can be represented as a sum

$$L = L_D + L^0 + x_1 L^1 + x_2 L^2, \tag{76}$$

We also assume that the densities F^k are integrable: $F^k \in L_1$, $k = 0, 1, 2$, and they exponentially decay at infinity. Finally, the densities $F^1(\mathrm{d}y)$ and $F^2(\mathrm{d}y)$ are concentrated on the axis $x_1 = 0$.

Theorem 3.6 *Assume that*

(i) X is the process with the infinitesimal generator equation (45);
(ii) the short rate process is given by equation (3) with $d_3 > 0$;
(iii) the parameters of the diffusion component satisfy the conditions (38), (39), (40), (41), (42), (43) and (44);
(iv)

$$[0, l_1) \times [0, l_2) \times [-\gamma, \epsilon] \subset U^0 \cap U^1 \cap U^2, \tag{77}$$

for some $\epsilon > 0$;
(v) for $y = \epsilon, -\gamma$,

$$d_1 + \frac{\kappa_{11}^2}{2\beta_{11}} + \frac{\kappa_{21}\kappa_{22}}{\beta_{22}} + \kappa_{31} y - \frac{\beta_{31}}{2} y^2 - \Psi^1(y) > 0, \tag{78}$$

$$d_2 + \frac{\kappa_{22}^2}{2\beta_{22}} + \frac{\kappa_{12}\kappa_{11}}{\beta_{11}} + \kappa_{32} y - \frac{\beta_{32}}{2} y^2 - \Psi^2(y) > 0; \tag{79}$$

(vi) the pay-off g is continuous, and it satisfies the bound (48), where $0 \le \mu_j < l_j$ and $\mu_3^- < -\gamma < 0 < \mu_3^+$ are such that

$$[0, \mu_1] \times [0, \mu_2] \times [\mu_3^-, \mu_3^+] \subset U^0 \cap U^1 \cap U^2, \tag{80}$$

and for $y = \mu_3^\pm$,

$$d_1 + \kappa_{11}\mu_1 - \frac{\beta_{11}\mu_1^2}{2} + \kappa_{21}\mu_2 + \kappa_{31} y - \frac{\beta_{31}}{2} y^2 - \Psi^1(y) > 0, \tag{81}$$

$$d_2 + \kappa_{22}\mu_2 - \frac{\beta_{22}\mu_2^2}{2} + \kappa_{12}\mu_1 + \kappa_{32} y - \frac{\beta_{32}}{2} y^2 - \Psi^2(y) > 0. \tag{82}$$

Then the expression (1) defines the unique solution to the problems (4) and (5) in the class of continuous functions which admit the bound (51) for any $\epsilon > 0$, where C_ϵ depends on ϵ but not on x.

Theorem 3.7 *Let the parameters of the process satisfy the conditions of Theorem 3.6, and let g be the pay-off of one of the interest rate derivatives in Theorem 2.2.*

Then the formal solution $f_0(g; x, \tau)$ is the price $f(g; x, \tau)$ of the corresponding derivative.

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