An Analytical Implementation of the Hull and White Model

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Ho and Lee introduced the first no-arbitrage model of the evolution of the spot interest rate. Hull and White extended this work to include mean reversion of the spot interest rate. When writing about the implementation of their model in discrete time, Hull and White use a search process at each date and forward induction to identify the level of interest rates in a trinomial lattice.

We derive an analytical solution for the level of interest rates. We apply our analytical solution to an example created by Hull and White.

Vasicek [1977] initiated an important stream of research relating to the evolution of interest rates. Cox, Ingersoll, and Ross [1985] added an equilibrium model of the evolution of interest rates. While analytically tractable, these models do not provide valuations consistent with the absence of arbitrage.

Ho and Lee [1986] address this disadvantage with a no-arbitrage model that incorporates the market term and volatility structure of interest rates. Heath, Jarrow, and Morton [1992] extend the basic Ho-Lee model to incorporate the entire structure of forward rates; Ho and Lee assume interest rates are normally distributed. Black, Derman, and Toy [1990] and Black and Karasinski [1991] contribute no-arbitrage models that assume interest rates are lognormally distributed. This group of models relies on numerical methods for their implementation.

Hull and White (HW) [1990, 1993, 1994, and 1996] extend the Ho and Lee approach by adding Vasicek's idea of a mean–reverting interest rate. In addition to their theoretical modeling, HW [1996] propose a numerical implementation of their model requiring a search process at each date to identify the level of interest rates. This, in turn, requires implementation through forward induction.

We complement HW's work by deriving an analytical solution for the level of interest rates and illustrate the solution by applying it to an example used by HW [1996].

I. THE ANALYTICAL IMPLEMENTATION

The HW model expresses the continuous-time evolution of the instantaneous spot rate as:

$$dr(t) = \left\{ \mu(t) + \alpha \big(\gamma(t) - r(t) \big) \right\} dt + \sigma(t) dz(t) \qquad (1)$$

In Equation (1), the spot interest rate at date t is $r(t)$. The drift in the spot rate is composed of two terms, a "pure" drift term $\mu(t)$, and a mean reversion term, $\alpha[\gamma(t) - r(t)]$. The mean reversion term causes the interest rate to revert to a time-varying "normal" value, $\gamma(t)$, at the instantaneous rate α. We write the instantaneous volatility of the spot interest rate, $\sigma(t)$, in terms of a standard Wiener process for which $dz(t) \sim N(0,\ 1)\sqrt{dt}$.[1]

The discrete-time analogue of Equation (1) for the change in the spot rate over the time interval Δt, i.e., for the time period $[t, t + \Delta t]$, is:[2]

$$\Delta r(t) = \left(\mu(t) + (1-k)[\gamma(t) - r(t)]\right)\Delta t + b\sigma(t)\sqrt{\Delta t}\,\Delta z(t) \quad (2)$$

where $r(t)$ and $\sigma(t)$ are the spot rate and the volatility of the spot rate at time t for the time interval from t to $t + \Delta t$; α is a positive constant (< 1); $\Delta z(t)$ is a unit-normal random variable:

$$k = e^{-\alpha \Delta t} \quad (2\text{-A})$$

$$b = \left(\frac{1 - e^{-2\alpha \Delta t}}{2\alpha}\right)^{\frac{1}{2}} \quad (2\text{-B})$$

and the time-varying normal rate of interest at date t is:

$$\begin{aligned}
\gamma(t) &= \gamma(t-1) + \mu(t-1) \\
&= \gamma(t-2) + \mu(t-2) + \mu(t-1) \\
&= \gamma(0) + \sum_{j=0}^{t-1} \mu(j)
\end{aligned} \quad (2\text{-C})$$

Without loss of generality, we can set $t = 0$ and $\Delta t = 1$, and rewrite the equation for the evolution of the spot rate as:

$$\begin{aligned}
\Delta r(0) &\equiv r(1) - r(0) \\
&= \mu(0) + (1-k)[\gamma(0) - r(0)] + b\sigma(0)\Delta z(0)
\end{aligned} \quad (3)$$

This yields, for example, for date 1:

$$\begin{aligned}
r(1) &= r(0) + \mu(0) + (1-k)[\gamma(0) - r(0)] + b\sigma(0)\Delta z(0) \\
&= r(0) + \mu(0) + \gamma(0) - r(0) - k\gamma(0) + kr(0) + b\sigma(0)\Delta z(0) \\
&= \gamma(0) + \mu(0) + k[r(0) - \gamma(0)] + b\sigma(0)\Delta z(0)
\end{aligned} \quad (4)$$

The mean reversion effect causes the deviation of the spot interest rate from the time-varying normal value at date 0, $r(0) - \gamma(0)$, to decrease to $k[r(0) - \gamma(0)]$ at date 1.

Similarly, the spot rate at date 2 is

$$\begin{aligned}
r(2) &= r(1) + \mu(1) + (1-k)[\gamma(1) - r(1)] + b\sigma(1)\Delta z(1) \\
&= \gamma(1) + \mu(1) + k[r(1) - \gamma(1)] + b\sigma(1)\Delta z(1)
\end{aligned}$$

At date 1, the time-varying normal rate is $\gamma(1) = \gamma(0) + \mu(0)$, which gives $r(1) - \gamma(1) = k[r(0) - \gamma(0)] + b\sigma(0)\Delta z(0)$. Substituting this expression into the equation above, we obtain

$$\begin{aligned}
r(2) &= \gamma(0) + \mu(0) + \mu(1) + \\
&\quad k\{k[r(0) - \gamma(0)] + b\sigma(1)\Delta z(1)\} + \\
&\quad b\sigma(1)\Delta z(1) \\
&= \gamma(0) + \mu(0) + \mu(1) + \\
&\quad k^2[r(0) - \gamma(0)] + kb\sigma(0)\Delta z(0) + \\
&\quad b\sigma(1)\Delta z(1)
\end{aligned} \quad (5)$$

and in general:

$$\begin{aligned}
r(t) &= \gamma(0) + \sum_{j=0}^{t-1} \mu(j) + \\
&\quad k^t[r(0) - \gamma(0)] + \\
&\quad \sum_{j=0}^{t-1} k^{t-j-1} b\sigma(j)\Delta z(j)
\end{aligned} \quad (6)$$

Equation (6) indicates that the spot rate is the sum of a set of non-stochastic drift terms and a set of stochastic terms; the latter are all normally distributed. Consequently, the spot interest rates are normally distributed as follows:

$$r(1) \sim N\left(\gamma(0) + \mu(0) + k[r(0) - \gamma(0)],\ \sigma^2(r(1))\right)$$

$$\begin{aligned}
r(2) \sim N\big(&\gamma(0) + \mu(0) + \mu(1) + \\
&k^2[r(0) - \gamma(0)], \\
&\sigma^2(r(1) + r(2))\big)
\end{aligned}$$

and in general:

$$r(t) \sim N\left(\gamma(0) + \sum_{j=0}^{t-1}\mu(j-1) + k^t\left[r(0) - \gamma(0)\right], \ \sigma^2\left(\sum_{j=1}^{t}r(j)\right)\right)$$

(7)

The inputs for an HW no-arbitrage interest rate model in discrete time are 1) a set of known prices of pure discount bonds that mature at dates 1, 2, 3, …, n, {$P(1)$, $P(2)$, $P(3)$, ..., $P(n)$}, and 2) the volatility (standard deviation) of future one-period normally distributed spot interest rates, {$\sigma(0)$, $\sigma(1)$, ..., $\sigma(n-1)$}.[3]

An evolution of the spot interest rate that precludes arbitrage must satisfy the local expectations condition that all bonds, regardless of maturity, offer the same expected rate of return in a given period under the equivalent martingale probability (EMP) distribution, Q. This is equivalent to the expectation that the discounted value of each bond's terminal payment is equal to its given market (initial) value.[4]

Let the present value, at date $t = 0$, of a bond's terminal payment be given by $p(n) = \exp[-\sum_{j=0}^{n-1}r(j)]$. Therefore, the no-arbitrage conditions will be stated as

$$P(1) = e^{-f(0)} \equiv E_0^Q[p(1)] = E_0^Q\left[e^{-r(0)}\right] = e^{-r(0)}$$

$$P(2) = e^{-\{f(0)+f(1)\}} \equiv E_0^Q[p(2)] = E_0^Q\left[e^{-\{r(0)+r(1)\}}\right]$$

and, in general:

$$P(n) = \exp\left(-\sum_{j=0}^{n-1}f(j)\right) \equiv E_0^Q[p(n)]$$

$$= E_0^Q\left[\exp\left(-\sum_{j=0}^{n-1}r(j)\right)\right]$$

(8)

$E_0^Q[\cdot]$ is the expectation at date 0 under the EMP distribution Q, and $f(j)$ is the forward rate for the interval from j to $j + 1$.

From statistics, we know that if x is normally distributed, $N(\mu, \sigma^2)$, then:[5]

$$E\left[e^{-x}\right] = e^{-\mu + \frac{1}{2}\sigma^2}$$

(9)

Therefore, for date $t = 2$:

$$P(2) = E_0^Q[p(2)]$$

$$= E_0^Q\left[e^{-\{r(0)+r(1)\}}\right]$$

$$= e^{-\{r(0)+E_0^Q[r(1)]\}+\frac{1}{2}\sigma^2(r(1))}$$

Upon simplification:

$$\ln P(2) = -r(0) - E_0^Q[r(1)] + \frac{1}{2}\sigma^2(r(1))$$

or

$$E_0^Q[r(1)] = -\ln P(2) - r(0) + \frac{1}{2}\sigma^2(r(1))$$

Because $\ln P(2) = -f(0) - f(1) = -r(0) - f(1)$, upon substitution in the equation above:

$$E_0^Q[r(1)] = f(1) + \frac{1}{2}\sigma^2(r(1))$$

(10)

The expectation of the spot rate at date 1 is the forward rate plus a term determined by the variance of the spot rate, $\frac{1}{2}\sigma^2[r(1)]$.

Taking the expectation of Equation (4), we have:[6]

$$E_0^Q[r(1)] = r(0) + \mu(0)$$

(11)

From Equations (10) and (11), we derive:

$$\mu(0) = f(1) - r(0) + \frac{1}{2}\sigma^2(r(1))$$

(12)

Thus, the drift term, $\mu(0)$, is equal to the sum of two effects: 1) $f(1) - r(0)$ is the difference between the forward rate and the spot rate, i.e., the spot interest rate drifts up or down toward the forward rate, and 2) $\frac{1}{2}\sigma^2[r(1)]$ is a positive drift adjustment term (DAT) that is required to preclude arbitrage.[7]

Let $\delta(t)$ denote the DAT for date t. Then:

$$\delta(0) = \frac{1}{2}\sigma^2(r(1))$$

(13)

Now, we can work out the details for date $t = 3$.

$$P(3) = E_0^Q\left[p(3)\right]$$
$$= E_0^Q\left[e^{-\{r(0)+r(1)+r(2)\}}\right]$$
$$= e^{-E_0^Q[r(0)+r(1)+r(2)]+\frac{1}{2}\sigma^2(r(1)+r(2))}$$

Simplifying:

$$\ln P(3) = -r(0) - E_0^Q\left[r(1)\right] - E_0^Q\left[r(2)\right] +$$
$$\frac{1}{2}\sigma^2\left(r(1)+r(2)\right)$$
$$= -r(0) - f(1) - \frac{1}{2}\sigma^2\left(r(1)\right) -$$
$$E_0^Q\left[r(2)\right] + \frac{1}{2}\sigma^2\left(r(1)+r(2)\right)$$

or

$$E_0^Q\left[r(2)\right] = -\ln P(3) - r(0) - f(1) +$$
$$\frac{1}{2}\sigma^2\left(r(1)+r(2)\right) - \frac{1}{2}\sigma^2\left(r(1)\right)$$

We know that $\ln P(3) = -f(0) - f(1) - f(2) = -r(0) - f(1) - f(2)$. Upon substitution in the equation above:

$$E_0^Q\left[r(2)\right] = f(2) + \frac{1}{2}\sigma^2\left(r(1)+r(2)\right) - \frac{1}{2}\sigma^2\left(r(1)\right) \quad (14)$$

The expectation at date $t = 0$ of the spot rate at date 2 is the forward rate plus a term determined by the variance: $\frac{1}{2}\sigma^2[r(1) + r(2)] - \frac{1}{2}\sigma^2[r(1)]$.

Taking the expectation of Equation (5), we have:

$$E_0^Q\left[r(2)\right] = r(0) + \mu(0) + \mu(1) \quad (15)$$

From Equations (14) and (15) we derive:

$$\mu(1) = f(2) - r(0) - \mu(0) +$$
$$\frac{1}{2}\sigma^2\left(r(1)+r(2)\right) - \frac{1}{2}\sigma^2\left(r(1)\right)$$

Substitute Equation (12) into the above to get:

$$\mu(1) = f(2) - f(1) +$$
$$\frac{1}{2}\sigma^2\left(r(1)+r(2)\right) - \sigma^2\left(r(1)\right) \quad (16)$$

The drift term, $\mu(1)$, is equal to the sum of two effects: 1) $f(2) - f(1)$ is the difference between the forward rate at date 2 and the forward rate at date 1, and 2) $\frac{1}{2}\sigma^2[r(1) + r(2)] - \sigma^2[r(1)]$ is the positive DAT required to preclude arbitrage.

Let $\delta(1)$ denote the DAT for date 1. Then:

$$\delta(1) = \frac{1}{2}\sigma^2\left(r(1)+r(2)\right) - \sigma^2\left(r(1)\right) \quad (17)$$

If we add equations for $\delta(0)$ and $\delta(1)$ [Equations (13) and (17)], we get

$$\sum_{t=0}^{1}\delta(t) = \frac{1}{2}\sigma^2\left(r(1)\right) + \frac{1}{2}\sigma^2\left(r(1)+r(2)\right) - \sigma^2\left(r(1)\right)$$
$$= \frac{1}{2}\sigma^2\left(r(1)+r(2)\right) - \frac{1}{2}\sigma^2\left(r(1)\right)$$

If we add equations for $\mu(0)$ and $\mu(1)$ [Equations (12) and (16)], we get

$$\mu(0) + \mu(1) = f(1) - r(0) + \frac{1}{2}\sigma^2\left(r(1)\right) +$$
$$f(2) - f(1) + \frac{1}{2}\sigma^2\left(r(1)+r(2)\right) - \sigma^2\left(r(1)\right)$$
$$= f(2) - r(0) + \frac{1}{2}\sigma^2\left(r(1)+r(2)\right) -$$
$$\frac{1}{2}\sigma^2\left(r(1)\right)$$

which can be simplified to

$$\sum_{t=0}^{1}\mu(t) = f(2) - r(0) + \sum_{t=0}^{1}\delta(t) \quad (18)$$

The results of the first two dates can be generalized for the case of date t:

E X H I B I T 1
Initial Prices and Yields in the Hull and White Example

Maturity	1	2	3	4	5	6	7	8	9
Price	$0.9503	$0.8906	$0.8277	$0.7639	$0.7065	$0.6536	$0.6010	$0.5573	$0.5139
Yield	5.093%	5.795%	6.305%	6.733%	6.948%	7.087%	7.274%	7.308%	7.397%

$$E_0^Q\left[r(t)\right] = f(t) + \frac{1}{2}\sigma^2\left(\sum_{j=1}^{t} r(j)\right) -$$

$$\frac{1}{2}\sigma^2\left(\sum_{j=1}^{t-1} r(j)\right) \; \forall \; 1 < t \le T - 1 \qquad (19)$$

$$\mu(t-1) = f(t) - f(t-1) + \frac{1}{2}\sigma^2\left(\sum_{j=1}^{t} r(j)\right) -$$

$$\sigma^2\left(\sum_{j=1}^{t-1} r(j)\right) + \frac{1}{2}\sum_{n=1}^{t-2}\sigma^2\left(\sum_{j=1}^{n} r(j)\right) \; \forall \; t \ge 3 \qquad (20)$$

In addition:

$$\sum_{n=0}^{t} \delta(n) = \frac{1}{2}\sigma^2\left(\sum_{j=1}^{t+1} r(j)\right) -$$

$$\frac{1}{2}\sigma^2\left(\sum_{j=1}^{t} r(j)\right) \; \forall \; t \ge 1 \qquad (21)$$

$$\sum_{n=0}^{t} \mu(n) = f(t+1) - r(0) +$$

$$\sum_{n=1}^{t} \delta(n) \; \forall \; t \ge 1 \qquad (22)$$

Equations (19)–(22) give the necessary recursive relations to evolve the HW no-arbitrage model of spot interest rates. The inputs are the set of market prices of (pure) discount bonds, a structure of volatilities for the spot rates, and other parametric values.

Our discussion is general in the sense that it applies equally well to implementation based on interest rate trees and Monte Carlo simulation.[8]

II. IMPLEMENTATION EXAMPLE

We illustrate implementation with reference to an example developed by Hull and White [1996]. They illustrate implementation of their model by pricing a three-year put option on a zero-coupon bond that pays $100 in nine years. The exercise price is $63; the volatility, *s*, is constant at 1% per year for all dates; and the speed of reversion to the mean, α, is 0.10. Exhibit 1 displays the prices and yields of the zero-coupon bonds.

The HW implemention uses a trinomial lattice with upper and lower bounds. First, HW identify the step sizes and the probabilities necessary to achieve the desired volatility, around zero, of interest rates. Second, they find the expected value of the interest rate at each date that is consistent with the initial conditions. This requires the use of search process and forward induction. We display HW's results in Exhibit 2.

Our first task is to illustrate how to eliminate a numerical search procedure and forward induction to identify the mean value of the interest rates. Specifically, given the initial one-period interest rate, 5.0928%, how do we analytically derive the subsequent mean values of 6.5026%, 7.3393%, and 8.0538%?

Recall that:

$$E_0^Q\left[r(1)\right] = f(1) + \frac{1}{2}\sigma^2\left(r(1)\right) \qquad (23\text{-A})$$

$$E_0^Q\left[r(2)\right] = f(2) + \frac{1}{2}\left(\sigma^2\left(r(1) + r(2)\right) - \sigma^2\left(r(1)\right)\right) \qquad (23\text{-B})$$

$$E_0^Q\left[r(3)\right] = f(3) + \frac{1}{2}\left(\sigma^2\left(r(1) + r(2) + r(3)\right) - \sigma^2\left(r(1) + r(2)\right)\right) \qquad (23\text{-C})$$

E X H I B I T   2
Intermediate Calculations for Four-Epoch Trinomial Lattice—Hull-White Short Rate Example

Calibrating Variances in Lattice

Steps	Transition Probabilities P_u	P_m	Pd	Node Rates $i = 0$	$i = 1$	$i = 2$	$i = 3$
2	0.8993	0.0110	0.0897			3.2979%	3.2979%
1	0.1236	0.6576	0.2188		1.6489%	1.6490%	1.6490%
0	0.1667	0.6667	0.1667	0.0000%	0.0000%	0.0000%	0.0000%
−1	0.2188	0.6576	0.1236		−1.6490%	−1.6489%	−1.6489%
−2	0.0897	0.0110	0.8993			−3.2979%	−3.2979%

Calibrating Prices in Lattice

Steps	Transition Probabilities P_u	P_m	Pd	Node Rates $i = 0$	$i = 1$	$i = 2$	$i = 3$
2	0.8993	0.0110	0.0897			10.6372%	11.3517%
1	0.1236	0.6576	0.2188		8.1515%	8.9883%	9.7028%
0	0.1667	0.6667	0.1667	5.0928%	6.5026%	7.3393%	8.0538%
−1	0.2188	0.6576	0.1236		4.8536%	5.6904%	6.4049%
-2	0.0897	0.0110	0.8993			4.0414%	4.7559%

from Equations (11), (15), and (19). Also:

$$r(1) = r(0) + \mu(0) + bs\Delta z(0) \tag{24-A}$$

$$r(2) = r(0) + \mu(0) + \mu(1) + bs\big(k\Delta z(0) + \Delta z(1)\big) \tag{24-B}$$

$$r(3) = r(0) + \mu(0) + \mu(1) + \mu(2) + bs\big(k^2\Delta z(0) + k\Delta z(1) + \Delta z(2)\big) \tag{24-C}$$

from Equations (4), (5), and (6). The expressions for k and b are given in Equations (2-A) and (2-B).

With this information, we can calculate the variances of the sum of the spot rates:

$$\sigma^2[r(1)] = \sigma^2\big(bs[\Delta z(0)]\big)$$
$$= b^2 s^2 = 0.9520^2 (0.0001)$$
$$= 0.00009063$$

$$\sigma^2[r(1) + r(2)] = \sigma^2\big(bs\{\Delta z(0) + k\Delta z(0) + \Delta z(1)\}\big)$$
$$= b^2 s^2\{(1+k)^2 + 1\}$$
$$= (0.00009063)(4.6284) = 0.00041949$$

$$\sigma^2[r(1) + r(2) + r(3)] = \sigma^2\Big(bs\big\{[1 + k + k^2]\Delta z(0) + \Delta z(1)(1+k) + \Delta z(2)\big\}\Big)$$
$$= b^2 s^2\Big([1 + k + k^2]^2 + (1+k)^2 + 1\Big)$$

$$= (0.00009063)(12.0462) = 0.0010918$$

With this information, we can derive analytically the values that HW derive through searches:

$$E_0^Q[r(1)] = -\ln 0.8906 + \ln 0.9503 + (0.5)(0.00009063)$$
$$= 6.5026\%$$

$$E_0^Q\big[r(2)\big] = -\ln 0.8277 + \ln 0.8906 +$$
$$(0.5)(0.00041949 - 0.0009063)$$
$$= 7.3393\%$$

$$E_0^Q\big[r(3)\big] = -\ln 0.7639 + \ln 0.8277 +$$
$$(0.5)(0.0010918 - 0.00041949)$$
$$= 8.0538\%$$

These results match those produced by HW's search method, as shown in Exhibit 2.

III. CONCLUSION

HW develop an attractive no-arbitrage model of the evolution of the spot interest rate that incorporates mean reversion. Their implementation of the model requires both the use of a search method to identify the expected value of the spot interest rate at each date and forward induction. The analytical expression for the expected value of the future spot rates that we derive eliminates need for the search process.

Our implementation applies equally well to interest rate binomial trees, trinomial lattices, and Monte Carlo simulation implementation of the model, and it can be adapted to incorporate additional complexities such as time-varying volatility.

ENDNOTES

[1] Hull and White [1996, p. 26] write the first term as $\alpha(\theta(t) - r)dt$; therefore, $a = \alpha$ and $\theta(t) = \mu(t) + \alpha\gamma(t)$. We write the model as we have because the derivation of $\mu(t)$ is a central result.

[2] See Hull and White [1996, p. 29], Jamshidian [1989], and Arnold [1974] for the use and development of the components k and b used in this discrete-time expression.

[3] For simplicity, we assume that $\gamma(0) = r(0)$ for the reminder of the article.

[4] We can illustrate the equivalence with respect to the expected rate of return on the two-period bond from date zero to date 1. $E_0^Q[\cdot]$ is the expectations operator under the equivalent martingale probability distribution Q:

$$\ln\left\{\frac{E_0^Q\big[e^{-r(1)}\big]}{P(2)}\right\} = r(0) \text{ or } \frac{E_0^Q\big[e^{-r(1)}\big]}{P(2)} = e^{-r(0)} \text{ or } P(2) = E_0^Q\big[e^{-r(0)-r(1)}\big]$$

[5] See Mood, Graybill, and Boes [1974, p. 117] for a discussion of this result.

[6] Recall that for simplicity we have set $\gamma(0) = r(0)$.

[7] Boyle [1978] was the first to point out this general result.

[8] Monte Carlo implementation of the HW model may be important for the valuation of path-dependent securities. At least one derivatives firm values amortizing index swaps using Monte Carlo implementation of the HW model.

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