THE HULL AND WHITE MODEL OF THE SHORT RATE: AN ALTERNATIVE ANALYTICAL REPRESENTATION

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Abstract

Hull and White extend Ho and Lee's no-arbitrage model of the short interest rate to include mean reversion. This addition eliminates the problem of negative interest rates and has found wide application. To implement their model, Hull and White employ a sequential search process to identify the mean interest rate in a trinomial lattice at each date. In this paper we extend Hull and White's work by developing an analytical solution for the mean interest rate at each date. This solution applies equally well to trinomial lattices, interest rate trees, and Monte Carlo simulation. We illustrate the analytical result by applying it to an example originally used by Hull and White and then for valuing an option on a bond.

JEL classification: C15, C63, G12, G13

I. Introduction

Vasicek (1977) initiated a stream of research into the evolution of interest rates that produced analytically tractable valuation expressions. The values produced by this model are not consistent with market prices. Ho and Lee (1986) address this disadvantage by introducing a no-arbitrage model of the evolution of interest rates based on market interest rates and their volatilities. Heath, Jarrow, and Morton (1992) extend the Ho-Lee model of the normally distributed spot interest rate to incorporate the entire structure of forward rates. Black, Derman, and Toy (1990) and Black and Karasinski (1991) add a second class of no-arbitrage models, based on the assumption that interest rates are log-normally distributed. Hull and White (1990, 1993, 1994, 1996) combine Ho and Lee's model with Vasicek's idea of mean-reverting interest rates to eliminate the probability of negative rates. As Hull and White describe this extended model it requires a search process to identify the level of interest rates at each date and forward induction. Nevertheless, the Hull and White approach has attracted much attention in the literature and is widely used in industry (e.g., Carverhill (1995), Rebonato (1998)).[1]

Our primary contribution is the derivation of an analytical expression for the expected value of the spot interest rate at each date for the Hull and White model. This result eliminates the need to use a search process at each date and forward induction when implementing the Hull and White model. The result also facilitates implementation of the Hull and White model through Monte Carlo simulation. This is desirable when trying to value path-dependent derivatives such as index amortizing swaps.[2] After developing the analytical solution for the Hull and White model of the short rate, we apply it to an example used by Hull and White. This example

[1] As Pelsser (1996) points out several LIBOR-market and swap-market models have been developed to provide a balance of tractability and realism; yet these models are best implemented as Monte Carlo simulation for almost any security other than simple pure discount bonds and their options. Pelsser himself extends squared Gaussian model of Constantinides (1992) and provides a richer (non-negative) interest-rate model for evolution and pricing purposes.

illustrates both the implementation of model as a trinomial lattice as well as a binomial tree methods and option valuation on a bond. These option values are benchmarked against the Monte Carlo simulation.

II. Derivation of the Analytical Implementation Method

In this section we sketch[3] the analytical results necessary to implement the Hull and White model without a search process and forward induction. To do that, we (a) illustrate the evolution of the spot interest rate and identify the distribution of future spot rates, and (b) employ the no-arbitrage condition to derive analytical expressions for the expectation of the spot rate under a pseudo-probability distribution at each date, as a function of the forward rate and the volatility of the spot short rate.

The Evolution and Distribution of the Spot Interest Rate

The following equation expresses the continuous-time evolution of the instantaneous spot rate:

$$dr(t) = \left\{ m(t) + a\left(g(t) - r(t)\right) \right\} dt + s(t)dz(t). \tag{1}$$

In equation (1) the spot interest rate at date t is $r(t)$. The drift in the spot rate is composed of two terms: a "pure" drift term, $m(t)$, and a mean-reversion term, $a\left(g(t) - r(t)\right)$. The mean-reversion term causes the interest rate to revert to a time-varying "normal" value, $g(t)$, at the instantaneous rate a, often called the speed of reversion. We write the instantaneous volatility of the spot interest rate, $s(t)$, in terms of a standard Wiener process for which $dz(t) \sim N(0,1)$. Hull and White (1996, p.26) write the first term as $a\left(q(t) - r\right)dt$; therefore, $a = a$ and

[2] It is possible to extend this result by deriving the corresponding analytical solution for an extension of the Hull and White model that incorporates the desirable property of volatility proportional to the spot rate.
[3] A more detailed derivation is available from the authors in the form of a technical appendix.

$q(t) = m(t) + a\,g(t)$. We write the model as we have because the derivation of $m(t)$ is a central result.

The discrete-time analogue of equation (1) for the change in the spot rate over the time interval Δt, that is, for the period $[t,\ t+\Delta t]$, is[4]

$$\Delta r(t) = \left(m(t) + (1-k)\left[g(t) - r(t)\right]\right)\Delta t + bs(t)\sqrt{\Delta t}\Delta z(t), \tag{2}$$

where $r(t)$ and $s(t)$ are respectively the spot rate and volatility of the spot rate at time t for the interval from t to $t+\Delta t$, a is a positive constant (< 1), $\Delta z(t)$ is a unit independently and identically distributed (i.i.d.) normal random variable,

$$k = e^{-a\,\Delta t}, \tag{2a}$$

$$b = \left(\frac{1-e^{-2a\,\Delta t}}{2a}\right)^{\frac{1}{2}}, \tag{2b}$$

and the time-varying normal rate of interest at date t is

$$g(t) = g(t-1) + m(t-1) = g(t-2) + m(t-2) + m(t-1) = g(0) + \sum_{j=0}^{t-1} m(j). \tag{2c}$$

Without loss of generality, we can set $t = 0$ and $\Delta t = 1$ and rewrite the equation of the evolution of the spot rate as

$$\Delta r(0) \equiv r(1) - r(0) = m(0) + (1-k)\left[g(0) - r(0)\right] + bs(0)\Delta z(0). \tag{3}$$

This yields the spot rate for date 1

$$\begin{aligned} r(1) &= r(0) + m(0) + (1-k)\left[g(0) - r(0)\right] + bs(0)\Delta z(0) \\ &= g(0) + m(0) + k\left[r(0) - g(0)\right] + bs(0)\Delta z(0), \end{aligned} \tag{4}$$

and the spot rate for date 2

[4] See Hull and White (1996, p. 29), Jamshidian (1989), and Arnold (1974) for the use and development of the components *k* and *b* used in this discrete time expression.

$$
\begin{aligned}
r(2) &= r(1)+m(1)+(1-k)\left[g(1)-r(1)\right]+bs(1)\Delta z(1) \\
&= g(0)+m(0)+m(1)+k^2\left[r(0)-g(0)\right]+k\,bs(0)\Delta z(0)+bs(1)\Delta z(1).
\end{aligned} \tag{5}
$$

Note that the magnitude of the deviation of the spot interest rate from the time-varying normal value at date 0, $r(0)-g(0)$, decreases to $k\left[r(0)-g(0)\right]$ at date 1 and to $r(1)-g(1)=k\left[r(0)-g(0)\right]+bs(0)\Delta z(0)$. This is the "mean-reversion" effect.

In general, then

$$
r(t)=g(0)+\sum_{j=0}^{t-1} m(j)+k^t\left[r(0)-g(0)\right]+\sum_{j=0}^{t-1} k^{t-j-1} bs(j)\Delta z(j). \tag{6}
$$

Equation (6) shows that the spot rate is the sum of a set of non-stochastic drift terms and a set of normally distributed stochastic terms. Consequently, the spot interest rates are normally distributed. For example, $r(1) \sim N\left(g(0)+m(0)+k\left[r(0)-g(0)\right],\ s^2\left(r(1)\right)\right)$ and $r(2) \sim N\left(g(0)+m(0)+m(1)+k^2\left[r(0)-g(0)\right],\ s^2\left(r(1)+r(2)\right)\right)$.

In general, then

$$
r(t) \sim N\left(g(0)+\sum_{j=0}^{t-1} m(j-1)+k^t\left[r(0)-g(0)\right],\ s^2\left(\sum_{j=1}^{t} r(j)\right)\right). \tag{7}
$$

Equation (7) shows that the short rates are normally distributed with changing parameters where the change is a deterministic function of "historical" changes in the mean and the variance. The change in the mean fulfills an additional important role, namely, making the price-generating process such that arbitrage becomes impossible.

No-Arbitrage and the Expectation of the Spot Interest Rate

The inputs for a Hull and White no-arbitrage interest rate model in discrete time are (1) a set of known prices of pure discount bonds that mature at dates 1, 2, 3, …, *n*, $\left\{P(1), P(2), P(3), \ldots, P(n)\right\}$, and (2) the volatility (standard deviation) of future one-period normally distributed spot interest rates, $\{s\ (0),\ s\ (1),\ ...,\ s\ (n-1)\}$. For simplicity, we assume that $g\ (0) = r\ (0)$ for the reminder of this paper.

An evolution of the spot interest that precludes arbitrage must satisfy the local expectations condition that all bonds, regardless of maturity, offer the same expected rate of return in a given period under the equivalent martingale probability (EMP) distribution, Q. This is equivalent to the expectation of the discounted value of each bond's terminal payment being equal to its given market (initial) value. For example, we can illustrate the equivalence with respect to the expected rate of return on the two-period bond from date zero to date 1. $E_0^Q[\ \cdot\ ]$ is the expectations operator under the equivalent martingale probability distribution Q:

$$\ln\left\{\frac{E_0^Q\left[e^{-r(1)}\right]}{P(2)}\right\} = r(0) \quad \text{or} \quad \frac{E_0^Q\left[e^{-r(1)}\right]}{P(2)} = e^{r(0)} \quad \text{or} \quad P(2) = E_0^Q\left[e^{-r(0)-r(1)}\right].$$

Let the present value, at date $t = 0$, of a bond's terminal payment be given by $p(n) = \exp\left(-\sum_{j=0}^{n-1} r\ (j)\right)$. Therefore, the no-arbitrage conditions will be stated as

$$P(n) = \exp\left(-\sum_{j=0}^{n-1} f\ (j)\right) \equiv E_0^Q\left[p(n)\right] = E_0^Q\left[\exp\left(-\sum_{j=0}^{n-1} r\ (j)\right)\right], \tag{8}$$

where $E_0^Q[\ \cdot\ ]$ is the expectation at date 0 under the EMP distribution Q and $f\ (j)$ is the forward rate for the interval from j to $j + 1$.

We know that if x is normally distributed, $N\left(m,\ s^{\,2}\right)$, then (Mood, Graybill, and Boes (1974, p. 117))

$$E\left[e^{-x}\right] = e^{-m+\frac{1}{2}s^{\,2}}. \tag{9}$$

Therefore, for date $t = 2$,

$$P(2)=E_0^Q\left[p(2)\right]=E_0^Q\left[e^{-\{r(0)+r(1)\}}\right]=e^{-\left\{r(0)+E_0^Q[r(1)]\right\}+\frac{1}{2}s^2(r(1))}.$$

Upon simplification,

$$E_0^Q\left[r(1)\right]=-\ln P(2)-r(0)+\frac{1}{2}s^2\left(r(1)\right)=f(1)+\frac{1}{2}s^2\left(r(1)\right), \tag{10}$$

because $\ln P(2)=-f(0)-f(1)=-r(0)-f(1)$ Thus, expectation of the spot rate at date 1 is the forward rate plus a term determined by the variance of the spot rate, $\frac{1}{2}s^2\left(r(1)\right)$.

Taking the expectation of equation (4) and using equation (10), we derive the drift term and the drift-adjustment term (DAT). The drift term, $m(0)=f(1)-r(0)+\frac{1}{2}s^2\left(r(1)\right)$ $=f(1)-r(0)+d(0)$, is equal to the sum of two effects: (1) $f(1)-r(0)$ is the difference between the forward rate and the spot rate, that is, the spot interest rate drifts up or down toward the forward rate, (2) $d(0)=\frac{1}{2}s^2\left(r(1)\right)$ is a positive DAT that is required to preclude arbitrage.[5]

A similar argument for date $t = 3$ produces

$$\begin{aligned}E_0^Q\left[r(2)\right]&=-\ln P(3)-r(0)-f(1)+\frac{1}{2}s^2\left(r(1)+r(2)\right)-\frac{1}{2}s^2\left(r(1)\right)\\&=f(2)+\frac{1}{2}s^2\left(r(1)+r(2)\right)-\frac{1}{2}s^2\left(r(1)\right),\end{aligned} \tag{11}$$

because $\ln P(3)=-f(0)-f(1)-f(2)=-r(0)-f(1)-f(2)$.

The expectation at date $t = 0$ of the spot rate at date 2 is the forward rate plus a term determined by the variance, $\frac{1}{2}s^2\left(r(1)+r(2)\right)-\frac{1}{2}s^2\left(r(1)\right)$. Taking the expectation of equation (5) and using equation (11), we derive the drift term and the DAT. The drift term, $m(1)=f(2)-f(1)+\frac{1}{2}s^2\left(r(1)+r(2)\right)-s^2\left(r(1)\right)$, is equal to the sum of two effects: (1)

[5] Boyle (1978) was the first to point out this general result.

$f(2) - f(1)$ is the difference between the forward rate at date 2 and the forward rate at date 1, and (2) $d(1) = \frac{1}{2}s^2\big(r(1)+r(2)\big) - s^2\big(r(1)\big)$ is the positive DAT required to preclude arbitrage.

If we add $d(0)$ and $d(1)$, we get

$$\sum_{t=0}^{1} d(t) = \frac{1}{2}s^2\big(r(1)\big) + \frac{1}{2}s^2\big(r(1)+r(2)\big) - s^2\big(r(1)\big) = \frac{1}{2}s^2\big(r(1)+r(2)\big) - \frac{1}{2}s^2\big(r(1)\big).$$

If we add $m(0)$ and $m(1)$, we get

$$\begin{aligned} m(0)+m(1) &= f(1) - r(0) + \frac{1}{2}s^2\big(r(1)\big) + f(2) - f(1) + \frac{1}{2}s^2\big(r(1)+r(2)\big) - s^2\big(r(1)\big) \\ &= f(2) - r(0) + \frac{1}{2}s^2\big(r(1)+r(2)\big) - \frac{1}{2}s^2\big(r(1)\big), \end{aligned}$$

which can be simplified to $\sum_{t=0}^{1} m(t) = f(2) - r(0) + \sum_{t=0}^{1} d(t)$.

The results of these dates can be generalized for date *t*.

$$E_0^Q\big[r(t)\big] = f(t) + \frac{1}{2}s^2\left(\sum_{j=1}^{t} r(j)\right) - \frac{1}{2}s^2\left(\sum_{j=1}^{t-1} r(j)\right) \;\forall\; 1 < t \le T-1. \tag{12}$$

$$m(0) = f(1) - r(0) + \frac{1}{2}s^2\big(r(1)\big), \tag{13a}$$

$$m(1) = f(2) - f(1) + \frac{1}{2}s^2\left(\sum_{j=1}^{2} r(j)\right) - s^2\big(r(1)\big), \tag{13b}$$

$$m(t-1) = f(t) - f(t-1) + \frac{1}{2}s^2\left(\sum_{j=1}^{t} r(j)\right) - s^2\left(\sum_{j=1}^{t-1} r(j)\right) + \frac{1}{2}\sum_{n=1}^{t-2} s^2\left(\sum_{j=1}^{n} r(j)\right) \;\forall\; t \ge 3. \tag{13c}$$

In addition,

$$\sum_{n=0}^{t} d(n) = \frac{1}{2}s^2\left(\sum_{j=1}^{t+1} r(j)\right) - \frac{1}{2}s^2\left(\sum_{j=1}^{t} r(j)\right) \;\forall\; t \ge 1. \tag{14}$$

$$\sum_{n=0}^{t} m(n) = f(t+1) - r(0) + \sum_{n=1}^{t} d(n) \;\forall\; t \ge 1. \tag{15}$$

Equations (12)–(15) give the necessary recursive relations to evolve the Hull and White no-arbitrage model of spot interest rate. The inputs are the set of market prices of (pure) discount bonds, a structure of volatilities for the spot rates, and other parametric values.

The above discussion is general in the sense that it applies equally well to implementation based on interest rate trees and Monte Carlo simulation. We now apply these results to an example introduced by Hull and White (1996).

III. Hull and White's Example

An analytical implementation of the trinomial lattice

Hull and White illustrate the implementation of their model with the example of pricing a three-year put option on a zero-coupon bond that pays \$100 in nine years. The exercise price is \$63, the instantaneous volatility, *s*, is constant at one percent per annum for all dates, and the speed of reversion to the mean α, is 0.10. The first line of Table 1 shows the prices of the zero-coupon bonds. The valuation of an option on an interest-bearing instrument necessitates the evolution of interest rates.

Hull and White implement their model of the evolution of the short rate through a trinomial lattice[6] with upper and lower bounds. The construction of this lattice proceeds in two parts. First, Hull and White identify the step sizes and the probabilities necessary to achieve the desired volatility, around zero, of the interest rates. Second, Hull and White find the expected value of the interest rate at each date that is consistent with the initial bond prices. This requires the use of search process and forward induction. Line 2 in Table 1 shows the resulting zero rates from the trinomial lattice. Details of these calculations are given in Hull and White (1996, pp. 29 30).

[6] It is also possible to implement the Hull and White model as a binomial tree, as is done subsequently. Further details are available from the authors.

Our first task is to illustrate how to eliminate a numerical search procedure and forward induction to identify the mean value of the interest rates. Specifically, given the initial one-period interest rate of 5.0928 percent, we need to derive analytically the subsequent mean values of 6.5026 percent , 7.3393 percent, and 8.0538 percent.

Recall from the previous discussions that

$$E_0^Q\left[r(1)\right]= f(1)+\frac{1}{2}s^2\left(r(1)\right), \tag{16a}$$

$$E_0^Q\left[r(2)\right]= f(2)+\frac{1}{2}\left(s^2\left(r(1)+r(2)\right)-s^2\left(r(1)\right)\right), \tag{16b}$$

$$E_0^Q\left[r(3)\right]= f(3)+\frac{1}{2}\left(s^2\left(r(1)+r(2)+r(3)\right)-s^2\left(r(1)+r(2)\right)\right). \tag{16c}$$

Also,

$$r(1)=r(0)+m(0)+bs\Delta z(0), \tag{17a}$$

$$r(2)=r(0)+m(0)+m(1)+bs\left(k\Delta z(0)+\Delta z(1)\right), \tag{17b}$$

$$r(3)=r(0)+m(0)+m(1)+m(2)+bs\left(k^2\Delta z(0)+k\Delta z(1)+\Delta z(2)\right). \tag{17c}$$

The expressions for k and b are given in equations (2a) and (2b), respectively.

With this information, we calculate the variances of the sum of the spot rates.

$$s^2\left(r(1)\right)=s^2\left(bs\times\left(\Delta z(0)\right)\right)=b^2s^2=0.9520^2(0.0001)=0.00009063.$$

$$\begin{aligned} s^2\left(r(1)+r(2)\right)&=s^2\left(bs\times\left\{\Delta z(0)+k\Delta z(0)+\Delta z(1)\right\}\right)=b^2s^2\left\{(1+k)^2+1\right\} \\ &=(0.00009063)(4.6284)=0.00041949. \end{aligned}$$

$$\begin{aligned} s^2\left(r(1)+r(2)+r(3)\right)&=s^2\left(bs\times\left\{\left[1+k+k^2\right]\Delta z(0)+\Delta z(1)(1+k)+\Delta z(2)\right\}\right) \\ &=b^2s^2\left(\left[1+k+k^2\right]^2+(1+k)^2+1\right) \\ &=(0.00009063)(12.0462)=0.0010918. \end{aligned}$$

We can now derive analytically the values derived by Hull and White in their paper through numerical searches.

$$E_0^Q\left[r(1)\right] = -\ln 0.8906 + \ln 0.9503 + (0.5)(0.00009063) = 6.5026\%.$$

$$E_0^Q\left[r(2)\right] = -\ln 0.8277 + \ln 0.8906 + (0.5)(0.00041949 - 0.0009063) = 7.3393\%.$$

$$E_0^Q\left[r(3)\right] = -\ln 0.7639 + \ln 0.8277 + (0.5)(0.0010918 - 0.00041949) = 8.0538\%.$$

These results shown in bold in Panel B of Table 2 match the values calculated by Hull and White.

Hull and White's example on option valuation

Hull and White use their trinomial lattice to calculate a value for a three-year put, with an exercise price of $63, on a zero-coupon bond with a face value of $100 and a maturity of nine years. We do likewise for a three-year put, with an exercise price of $0.63, on a pure discount bond paying one dollar at maturity. The given initial prices of bonds are now modified accordingly. All other parametric values remain the same as before. Hull and White use the closed-form formula for the price of the bond at year three for each node on the lattice. For comparison, we calculate the value of the option in four ways, namely, Monte Carlo simulation, the Hull and White formula, a full nine-date trinomial lattice, and a full nine-date binomial tree. The value from the Monte Carlo simulation is used as a benchmark. Note that our values are based on a face value of $1 and therefore must be scaled up by a factor of 100 for comparison with the values produced by Hull and White.

Monte Carlo Simulation. The Monte Carlo simulation is straightforward to implement. It requires calculations of the expected value of the interest rate at each of the eight future dates. Each iteration requires eight normal random variates with which to calculate the stochastic interest rates at a future date. We implement the model for 5,000 iterations and extend the effective sample size to 10,000 by using the antithetic method of variance reduction. (Note, by the way, that none of the interest rates in the simulation is negative.) The estimate of the value of the put is $2.3278 and the 95 percent confidence interval is $2.2128 to $2.4427.

For a simple problem of this type, the execution speed is very fast. We anticipate that the solution will be more accurate than the lattice or tree alternatives with just a few dates because the trees, recombining or not, provide a coarser approximation of the distribution. Of course, one can always increase the number of intervals (epochs) at the cost of increasing complexity, but simulation remains unbeatable so far as the range of values sampled is concerned. Because the Monte Carlo simulation samples the entire range of interest rates at each date, this valuation method presents none of the problems associated with increasing the number of epochs in the lattice or tree.

We use this simulated value as a benchmark to compare the values obtained through other methods.

Hull-White's Formula. In their paper, Hull and White use a formula to calculate the value of the bond at date 3, the expiration date of the put, for each node on the tree. When discounted back to date zero, the value of the put is $1.8734. This valuation yields an estimate that is 19.5 percent below the simulation estimate.

Hull-White's Full Trinomial Lattice. In Figure I we show the first four dates of a complete Hull and White trinomial lattice pricing the nine-period bond and the put written on it. The estimated value of the put is $1.8799. This valuation is 19.2 percent below the simulation estimate.

Full Binomial Tree. We create a complete binomial tree for the entire nine-year period, that is, we create a nine-date tree with 256 nodes at the ninth date. Note incidentally that the lowest value the interest rate reaches in the tree is 2.753 percent, so negative interest rates are not an issue for this tree. In Figure II we show the first four dates of this tree pricing the nine-period bond and the put written on it. The estimated value of the put is \$1.9487. Thus, the binomial tree produces a value that is 16.3 percent below the simulated value. The tree does not impose upper and lower bounds on the interest rate and, in general, represents the distribution of interest rates by many more points than is possible in the trinomial lattice. This may account for its somewhat smaller underpricing when compared with the simulation estimate.

IV. Conclusions

Hull and White develop a no-arbitrage model of the evolution of the spot interest rate that incorporates mean reversion. For the simple case of constant volatility, Hull and White show how to implement their model using a trinomial lattice with upper and lower bounds on the spot interest rate. This approach has several shortcomings. It requires the use of a numerical method to identify the expectation of the value of the spot interest rate at each date. The implementation of the method also requires forward induction. Furthermore, complex versions of the model may require implementation through Monte Carlo simulation. Consequently, a numerical identification of the expectation of future spot rates is impractical. Last, it is not clear what effects the bounds on interest rate has on the estimates of the values of derivatives.

Our analytical implementation of the Hull and White model allows identification of the expectations of the future spot rates based on the initial conditions. This method of implementation applies equally well to interest rate trees and Monte Carlo simulation, a characteristic that may be especially useful for more complex and realistic versions of the model. It can accommodate different specifications of volatility structures such as those of time-varying nature. For example, the use of a specification where the volatility is proportional to the spot rate can mitigate substantially the concerns about negative interest rates.

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TABLE 1. Initial Prices and Zero-Coupon Yields in the Hull and White Example of the Evolution of the Short Rate.

Maturity	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Price	\$0.9503	\$0.8906	\$0.8277	\$0.7639	\$0.7065	\$0.6536	\$0.6010	\$0.5573	\$0.5139
Zero Rate	5.093%	5.795%	6.305%	6.733%	6.948%	7.087%	7.274%	7.308%	7.397%

Note: The data are reproduced from Hull and White (1996, p. 29, Exhibit 5) to show their use of the trinomial lattice. They use a horizon of nine years. Bond prices are in dollars and yields in percents. Line 1 gives the initial prices of zero-coupon bonds. The parameters a and s are chosen to be 0.10 and 0.01, respectively. Using a search process and forward induction to identify the "drift-adjustment term", a, to fit the bond prices at each date, Hull and White derive the zero-coupon spot rates for the various nodes in the trinomial lattice. These rates are given in Line 2.

TABLE 2. Intermediate Calculations for the Four-Epoch Trinomial Lattice in the Hull and White Example of the Short Rate.

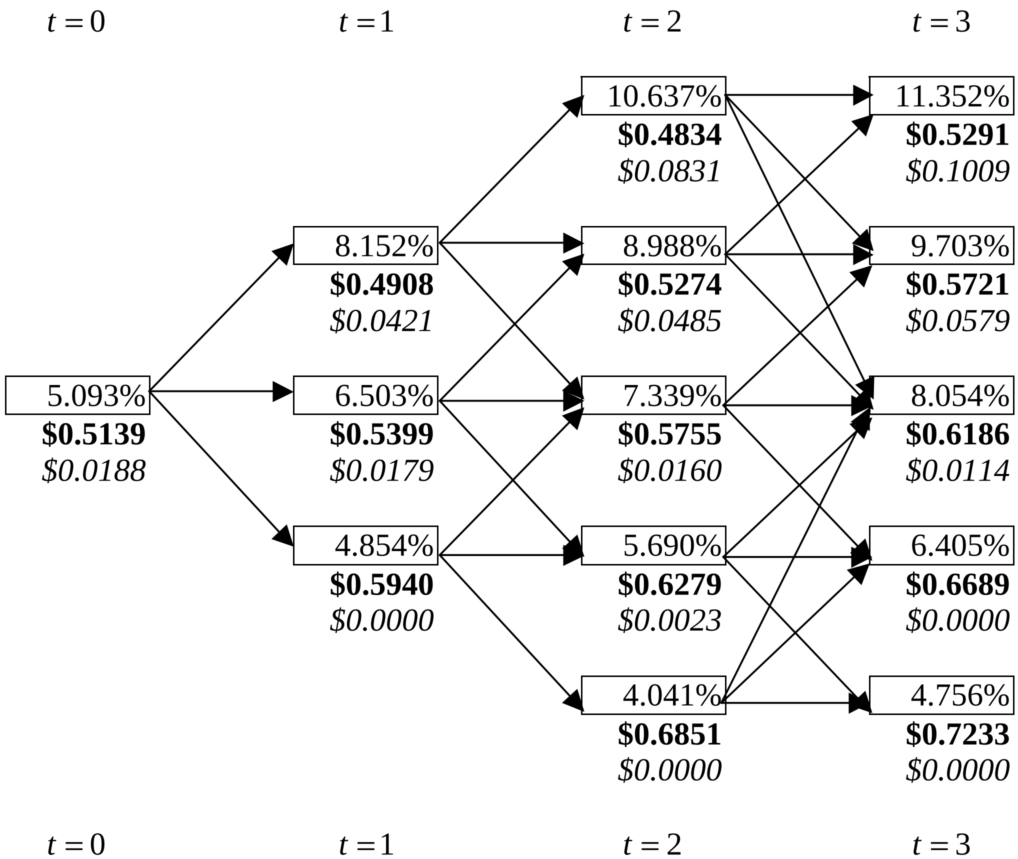
Panel A. Calibrating the Variances in the Lattice

Step	Transition Probabilities			Node Rates		
j	p_u	p_m	p_d	$i=0$	$i=1$	$i=2$
2	0.8993	0.0110	0.0897			3.2979%
1	0.1236	0.6576	0.2188		1.6489%	1.6490%
0	0.1667	0.6667	0.1667	0.0000%	0.0000%	0.0000%
–1	0.2188	0.6576	0.1236		–1.6490%	–1.6489%
–2	0.0897	0.0110	0.8993			–3.2979%

Panel B. Calibrating the Prices in the Lattice

Step	Transition Probabilities			Node Rates			
j	p_u	p_m	p_d	$i=0$	$i=1$	$i=2$	i = 3
2	0.8993	0.0110	0.0897			10.6372%	11.3517%
1	0.1236	0.6576	0.2188		8.1515%	8.9883%	9.7028%
0	0.1667	0.6667	0.1667	5.0928%	**6.5026%**	**7.3393%**	**8.0538%**
–1	0.2188	0.6576	0.1236		4.8536%	5.6904%	6.4049%
–2	0.0897	0.0110	0.8993			4.0414%	4.7559%

Note: The data are reproduced from Hull and White (1996, p. 30) Exhibits 4 and 6. At time-step i, at node j, the up-, middle-, and down-branching probabilities are given by p_u, p_m, and p_d, respectively. The probabilities of transiting from node (i, j) to nodes $(i+1, j+1)$, $(i+1, j)$, and $(i+1, j-1)$ are normally $p_u(j)$, $p_m(j)$, and $p_d(j)$, respectively. When $j=\pm 2$, Hull and White use a different branching scheme. In Panel A, the node rates are calculated to account for the volatility assumed. In Panel B, the node rates are calibrated to both the volatility and the initial bond prices. The final node rates are the result of the search method employed by Hull and White. The bolded numbers are the mean values which we derived analytically, that is without resorting to a search method, in the text.

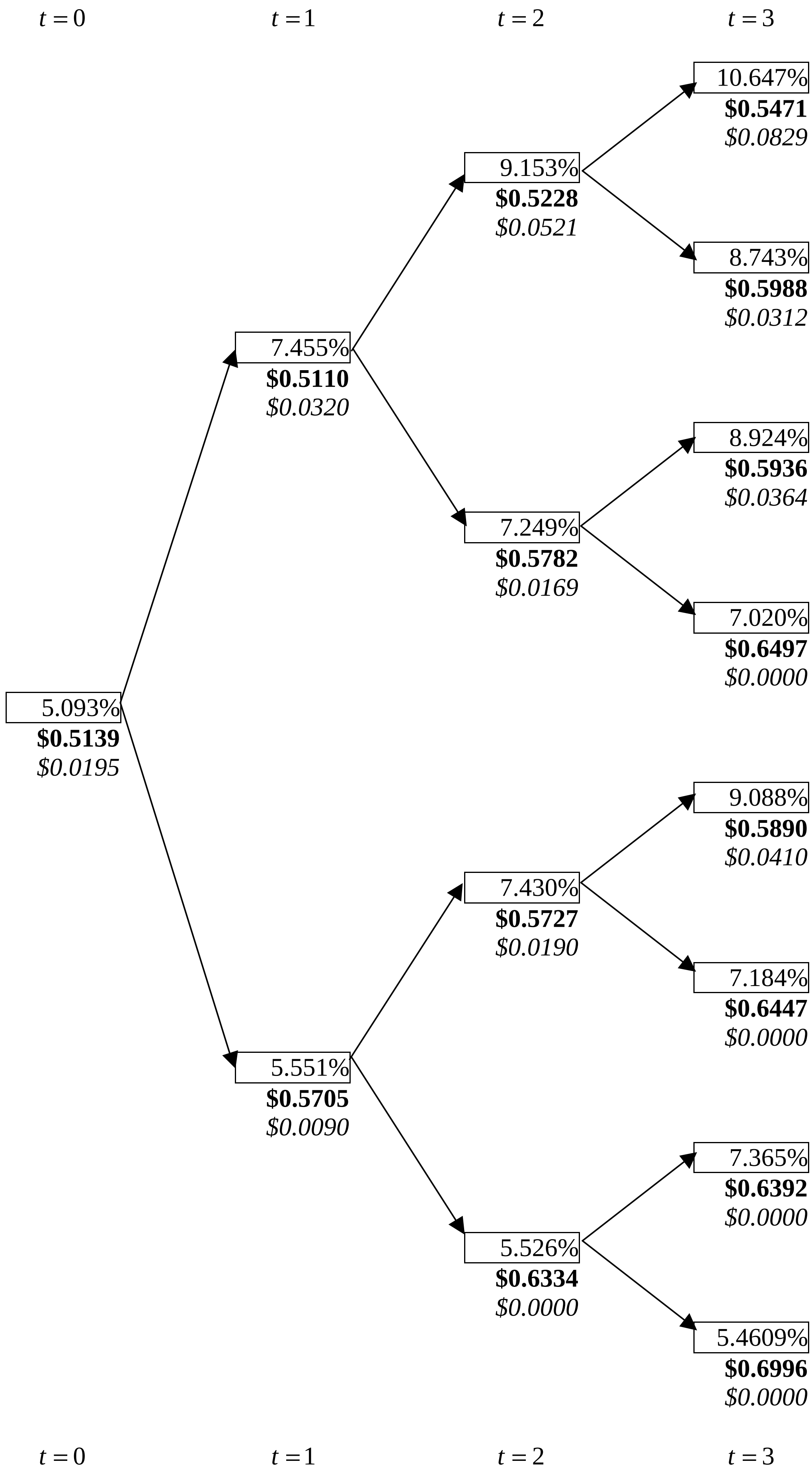


Note: Data used are the bond prices, $P(1) =$ \$0.9503, $P(2) =$ \$0.8906, …, $P(9) =$ \$0.5139; mean-reversion parameter, a = 0.10; and constant volatility, s = 1%. At each node j (at any date t), the set of three numbers denotes the node spot rate in percent, the bond price in dollars (in bold), and the put option value in dollars (in italics). The full trinomial lattice is used to value a put option with expiration of three years and an exercise price of \$0.63 on a nine-year pure-discount bond paying \$1 at maturity.

Refer to the node at date 1 with the lowest interest rate 4.854%. At this node, the discounted expectation at date 1 of \$1 at time 9 is \$0.5940. This is equal to the probability-weighted sum of the present values of the possible values of this bond at time 2: [0.2188 (\$0.5755) + 0.6576 (\$0.6279) + 0.1236 (\$0.6851)] × $e^{-0.048536}$. If we rewrite this equation, we can illustrate that the local expectations requirement is satisfied: The expected rate of return on a nine-year security at date 1 is identical to the rate of return on a 1-year security at date 1:

$$\ln\left(\frac{0.2188(\$0.5755)+0.6576(\$0.6279)+0.1236(\$0.6851)}{\$0.5940}\right)=0.048536\,.$$

Figure I. A Partial Display of the Full Trinomial Lattice Implementation to Value a Put on a Bond Using the Constant-Volatility Hull and White Model of the Short Rate



t = 0
t = 1
t = 2
t = 3
10.647%
$0.5471
$0.0829
9.153%
$0.5228
$0.0521
8.743%
$0.5988
$0.0312
7.455%
$0.5110
$0.0320
8.924%
$0.5936
$0.0364
7.249%
$0.5782
$0.0169
7.020%
$0.6497
$0.0000
5.093%
$0.5139
$0.0195
9.088%
$0.5890
$0.0410
7.430%
$0.5727
$0.0190
7.184%
$0.6447
$0.0000
5.551%
$0.5705
$0.0090
7.365%
$0.6392
$0.0000
5.526%
$0.6334
$0.0000
5.4609%
$0.6996
$0.0000
t = 0
t = 1
t = 2
t = 3

Note: Data used are the bond prices, $P(1) =$ \$0.9503, $P(2) =$ \$0.8906, …, $P(9) =$ \$0.5139; mean-reversion parameter, $a = 0.10$; and constant volatility, $s = 1\%$. At each node j (at any date t), the set of three numbers denotes the node spot rate in percent, the bond price in dollars (in bold), and the put option value in dollars (in italics). The full binomial tree is used to value a put option with expiration of three years and an exercise price of \$0.63 on a nine-year pure-discount bond paying \$1 at maturity.

Figure II. A Partial Display of the Full Binomial Tree Implementation to Value a Put on a Bond Using the Constant-Volatility Hull and White Model of the Short Rate