An Analytical Implementation of the Hull and White Model

Dwight Grant[*] and **Gautam Vora**[**]

Revised: February 8, 2001 & November 1, 2001
Do not quote.
Comments welcome.

[*] Douglas M. Brown Professor of Finance, Anderson School of Management, University of New Mexico, Albuquerque, NM 87131-1221. Tel: 505-277-5995, fax: 505-277-7108, e-mail: dgrant@swcp.com.
[**] Associate Professor of Finance, Anderson School of Management, University of New Mexico, Albuquerque, NM 87131-1221. Tel: 505-277-0669, fax: 505-277-7108, e-mail: vora@unm.edu.

An Analytical Implementation of the Hull and White Model

Abstract

Ho and Lee introduced the first no-arbitrage model of the evolution of the spot interest rate. Hull and White extended this work to include mean reversion of the spot interest rate. When writing about the implementation of their model in discrete time, they have employed a search process at each date and forward induction to identify the level of interest rates in a trinomial lattice. We derive an analytical solution for the level of interest rates. We apply analytical solution to an example created by Hull and White.

An Analytical Implementation of the Hull and White Model

Vasicek [1977] initiated an important stream of research relating to the evolution of interest rates. Cox, Ingersoll, and Ross (1985) added an equilibrium model of the evolution of interest rates. While analytically tractable these models do not provide valuations consistent with the absence of arbitrage.

Ho and Lee [1986] address this disadvantage with a no-arbitrage model that incorporates the market term and volatility structure of interest rates. Heath, Jarrow and Morton [1992] extend the basic Ho-Lee model to incorporate the entire structure of forward rates; Ho and Lee assume interest rates are normally distributed. Black, Derman, and Toy [1990] and Black and Karasinski [1991] contribute no-arbitrage models that assume interest rates are log-normally distributed. This group of models relies on numerical methods for their implementation.

Hull and White (HW) [1990, 1993, 1994, and 1996] extend the Ho and Lee approach by adding Vasicek's idea of a mean-reverting interest rate. In addition to their theoretical modeling, HW propose (1996) a numerical implementation of their model requiring a search process at each date to identify the level of interest rates. This, in turn, requires implementation through forward induction. We complement HW's work by deriving an analytical solution for the level of interest rates and illustrate the solution by applying it to an example used by HW (1996).

I. The Analytical Implementation

The HW model expresses the continuous time evolution of the instantaneous spot rate as:

$$dr(t) = \left\{ m(t) + a \left(g(t) - r(t) \right) \right\} dt + s(t) dz(t). \tag{1}$$

In equation (1) the spot interest rate at date t is $r(t)$. The drift in the spot rate is composed of two terms, a "pure" drift term $m(t)$, plus a mean reversion term, $a\left(g(t) - r(t)\right)$. The mean reversion term causes the interest rate to revert to a time-varying "normal" value, $g(t)$, at the instantaneous rate a. We write the instantaneous volatility of the spot interest rate, $s(t)$, in terms of a standard Wiener process for which $dz(t) \sim N(0,1)$.[1]

The discrete-time analogue of equation (1) for the change in the spot rate over the time-interval Δt, i.e., for the time-period $[t,\ t+\Delta t]$, is[2]

$$\Delta r(t) = \left(m(t) + (1-k)\left[g(t) - r(t)\right]\right)\Delta t + bs(t)\sqrt{\Delta t}\,\Delta z(t), \tag{2}$$

where $r(t)$ and $s(t)$ are respectively the spot rate and volatility of the spot rate at time *t* for the time-interval from *t* to $t+\Delta t$; a is a positive constant (< 1); $\Delta z(t)$ is a unit normal random variable:

$$k = e^{-a\,\Delta t}, \tag{2-A}$$

$$b = \left(\frac{1-e^{-2a\,\Delta t}}{2a}\right)^{\frac{1}{2}}, \tag{2-B}$$

and the time-varying normal rate of interest at date *t* is

$$g(t) = g(t-1) + m(t-1) = g(t-2) + m(t-2) + m(t-1) = g(0) + \sum_{j=0}^{t-1} m(j). \tag{2-C}$$

Without loss of generality, we can set $t = 0$ and $\Delta t = 1$ and rewrite equation of the evolution of the spot rate as

$$\Delta r(0) \equiv r(1) - r(0) = m(0) + (1-k)\left[g(0) - r(0)\right] + bs(0)\Delta z(0). \tag{3}$$

This yields, for example, for date 1

---

[1] Hull and White (1996, p.26) write the first term as $a(q(t) - r)dt$; therefore, $a = a$ and $q(t) = m(t) + a\,g(t)$. We write the model as we have because the derivation of $m(t)$ is a central result.

[2] See Hull and White (1996, p. 29), Jamshidian (1989) and Arnold (1974) for the use and development of the components *k* and *b* used in this discrete time expression.

$$\begin{aligned} r(1) &= r(0) + m(0) + (1-k)[g(0) - r(0)] + bs(0)\Delta z(0) \\ &= r(0) + m(0) + g(0) - r(0) - kg(0) + kr(0) + bs(0)\Delta z(0) \\ &= g(0) + m(0) + k[r(0) - g(0)] + bs(0)\Delta z(0). \end{aligned} \tag{4}$$

The mean reversion effect causes the deviation of the spot interest rate from the time-varying normal value at date 0, $r(0)-g(0)$, to decrease to $k\left[r(0)-g(0)\right]$ at date 1.

Similarly, the spot rate at date 2 is

$$\begin{aligned} r(2) &= r(1) + m(1) + (1-k)\left[g(1) - r(1)\right] + bs(1)\Delta z(1) \\ &= g(1) + m(1) + k\left[r(1) - g(1)\right] + bs(1)\Delta z(1). \end{aligned}$$

At date 1, the time-varying normal rate is $g(1) = g(0) + m(0)$ which gives $r(1) - g(1) =$ $k\left[r(0)-g(0)\right] + bs(0)\Delta z(0)$. Substituting this expression into the above equation, we obtain

$$\begin{aligned} r(2) &= g(0) + m(0) + m(1) + k\left\{k\left[r(0) - g(0)\right] + bs(1)\Delta z(1)\right\} + bs(1)\Delta z(1) \\ &= g(0) + m(0) + m(1) + k^2\left[r(0) - g(0)\right] + k\,bs(0)\Delta z(0) + bs(1)\Delta z(1), \end{aligned} \tag{5}$$

and in general,

$$r(t) = g(0) + \sum_{j=0}^{t-1} m(j) + k^t\left[r(0) - g(0)\right] + \sum_{j=0}^{t-1} k^{t-j-1} bs(j)\Delta z(j). \tag{6}$$

Equation (6) indicates that the spot rate is the sum of a set of non-stochastic drift terms and a set of stochastic terms; the latter are all normally distributed. Consequently, the spot interest rates are normally distributed as follows:

$$r(1) \sim N\left(g(0) + m(0) + k\left[r(0) - g(0)\right],\ s^2\left(r(1)\right)\right).$$

$$r(2) \sim N\left(g(0) + m(0) + m(1) + k^2\left[r(0) - g(0)\right],\ s^2\left(r(1) + r(2)\right)\right).$$

and in general,

$$r(t) \sim N\left(g(0) + \sum_{j=0}^{t-1} m(j-1) + k^t\left[r(0) - g(0)\right],\ s^2\left(\sum_{j=1}^{t} r(j)\right)\right). \tag{7}$$

The inputs for a HW no-arbitrage interest rate model in discrete time are 1) a set of known prices of pure discount bonds that mature at dates 1, 2, 3, …, n, $\{P(1), P(2), P(3), ..., P(n)\}$, and 2) the volatility (standard deviation) of future one-period normally distributed spot interest rates, $\{s\,(0), s\,(1), ..., s\,(n-1)\}$.[3]

An evolution of the spot interest that precludes arbitrage must satisfy the local expectations condition that all bonds, regardless of maturity, offer the same expected rate of return in a given period under the equivalent martingale probability (EMP) distribution, Q. This is equivalent to the expectation of the discounted value of each bond's terminal payment being equal to its given market (initial) value.[4]

Let the present value, at date t = 0, of a bond's terminal payment be given by $p(n) = \exp\left(-\sum_{j=0}^{n-1} r(j)\right)$. Therefore, the no-arbitrage conditions will be stated as

$$P(1) = e^{-f(0)} \equiv E_0^Q\left[p(1)\right] = E_0^Q\left[e^{-r(0)}\right] = e^{-r(0)},$$

$$P(2) = e^{-\{f(0)+f(1)\}} \equiv E_0^Q\left[p(2)\right] = E_0^Q\left[e^{-\{r(0)+r(1)\}}\right],$$

and, in general,

$$P(n) = \exp\left(-\sum_{j=0}^{n-1} f(j)\right) \equiv E_0^Q\left[p(n)\right] = E_0^Q\left[\exp\left(-\sum_{j=0}^{n-1} r(j)\right)\right]. \tag{8}$$

$E_0^Q[\,\cdot\,]$ is the expectation at date 0 under the EMP distribution Q and $f(j)$ is the forward rate for the interval from j to j + 1.

[3] For simplicity, we assume that $g\,(0) = r\,(0)$ for the reminder of this paper.

From statistics we know that if x is normally distributed, $N\left(m,\ s^{2}\right)$, then:[5]

$$E\left[e^{-x}\right]=e^{-m+\frac{1}{2}s^{2}}. \tag{9}$$

Therefore, for date t = 2:

$$P\left(2\right)=E_0^Q\left[p\left(2\right)\right]=E_0^Q\left[e^{-\left\{r\left(0\right)+r\left(1\right)\right\}}\right]=e^{-\left\{r\left(0\right)+E_0^Q\left[r\left(1\right)\right]\right\}+\frac{1}{2}s^{2}\left(r\left(1\right)\right)}.$$

Upon simplification:

$$\ln P\left(2\right)=-r\left(0\right)-E_0^Q\left[r\left(1\right)\right]+\frac{1}{2}s^{2}\left(r\left(1\right)\right),\ \text{ or}$$
$$E_0^Q\left[r\left(1\right)\right]=-\ln P\left(2\right)-r\left(0\right)+\frac{1}{2}s^{2}\left(r\left(1\right)\right).$$

Because $\ln P\left(2\right)=-f\left(0\right)-f\left(1\right)=-r\left(0\right)-f\left(1\right)$, upon substitution in the above equation,

$$E_0^Q\left[r\left(1\right)\right]=f\left(1\right)+\frac{1}{2}s^{2}\left(r\left(1\right)\right). \tag{10}$$

The expectation of the spot rate at date 1 is the forward rate plus a term determined by the variance of the spot rate, $\frac{1}{2}s^{2}\left(r\left(1\right)\right)$.

Taking the expectation of equation (4), we have[6]

$$E_0^Q\left[r\left(1\right)\right]=r\left(0\right)+m\left(0\right). \tag{11}$$

From equations (10) and (11), we derive:

$$m\left(0\right)=f\left(1\right)-r\left(0\right)+\frac{1}{2}s^{2}\left(r\left(1\right)\right). \tag{12}$$

---

[4] For example, we can illustrate the equivalence with respect to the expected rate of return on the two-period bond from date zero to date 1. $E_0^Q\left[\ \cdot\ \right]$ is the expectations operator under the equivalent martingale probability distribution Q: $\ln\left\{\frac{E_0^Q\left[e^{-r(1)}\right]}{P\left(2\right)}\right\}=r\left(0\right)$ or $\frac{E_0^Q\left[e^{-r(1)}\right]}{P\left(2\right)}=e^{r(0)}$ or $P\left(2\right)=E_0^Q\left[e^{-r(0)-r(1)}\right]$.

[5] See Mood, Graybill and Boes (1974, p. 117) for a discussion of this result.

[6] Recall that for simplicity we have set $g\left(0\right)=r\left(0\right)$.

Thus, the drift term, $m(0)$, is equal to the sum of two effects: 1) $f(1)-r(0)$ is the difference between the forward rate and the spot rate, i.e., the spot interest rate drifts up or down toward the forward rate, 2) $\frac{1}{2}s^2(r(1))$ is a positive drift adjustment term (DAT) that is required to preclude arbitrage.[7]

Let $d(t)$ denote the DAT for date *t*. Then:

$$d(0)=\frac{1}{2}s^2(r(1)). \tag{13}$$

Now, we can work out the details for date *t* = 3.

$$P(3)=E_0^Q\left[p(3)\right]=E_0^Q\left[e^{-\{r(0)+r(1)+r(2)\}}\right]=e^{-E_0^Q\left[r(0)+r(1)+r(2)\right]+\frac{1}{2}s^2(r(1)+r(2))}.$$

Simplifying:

$$\begin{aligned}\ln P(3)&=-r(0)-E_0^Q\left[r(1)\right]-E_0^Q\left[r(2)\right]+\frac{1}{2}s^2(r(1)+r(2))\\&=-r(0)-f(1)-\frac{1}{2}s^2(r(1))-E_0^Q\left[r(2)\right]+\frac{1}{2}s^2(r(1)+r(2)) \quad \text{or}\end{aligned}$$

$$E_0^Q\left[r(2)\right]=-\ln P(3)-r(0)-f(1)+\frac{1}{2}s^2(r(1)+r(2))-\frac{1}{2}s^2(r(1)).$$

We know that $\ln P(3)=-f(0)-f(1)-f(2)=-r(0)-f(1)-f(2)$. Upon substitution in the equation above:

$$E_0^Q\left[r(2)\right]=f(2)+\frac{1}{2}s^2(r(1)+r(2))-\frac{1}{2}s^2(r(1)). \tag{14}$$

The expectation at date *t* = 0 of the spot rate at date 2 is the forward rate plus a term determined by the variance, $\frac{1}{2}s^2(r(1)+r(2))-\frac{1}{2}s^2(r(1))$.

Taking the expectation of equation (5), we have:

$$E_0^Q\left[r(2)\right]=r(0)+m(0)+m(1). \tag{15}$$

[7] Boyle was the first to point out this general result.

From equations (14) and (15) we derive:

$$m(1) = f(2) - r(0) - m(0) + \frac{1}{2}s^2\left(r(1)+r(2)\right) - \frac{1}{2}s^2\left(r(1)\right).$$

Substitute equation (12) into the above to get:

$$m(1) = f(2) - f(1) + \frac{1}{2}s^2\left(r(1)+r(2)\right) - s^2\left(r(1)\right). \tag{16}$$

The drift term, $m(1)$, is equal to the sum of two effects: 1) $f(2) - f(1)$ is the difference between the forward rate at date 2 and the forward rate at date 1, and 2) $\frac{1}{2}s^2\left(r(1)+r(2)\right) - s^2\left(r(1)\right)$ is the positive DAT required to preclude arbitrage.

Let $d(1)$ denote the DAT for date 1. Then:

$$d(1) = \frac{1}{2}s^2\left(r(1)+r(2)\right) - s^2\left(r(1)\right). \tag{17}$$

If we add equations for $d(0)$ and $d(1)$ (equations (13) and (17)) we get

$$\begin{aligned}\sum_{t=0}^{1} d(t) &= \frac{1}{2}s^2\left(r(1)\right) + \frac{1}{2}s^2\left(r(1)+r(2)\right) - s^2\left(r(1)\right) \\ &= \frac{1}{2}s^2\left(r(1)+r(2)\right) - \frac{1}{2}s^2\left(r(1)\right).\end{aligned}$$

If we add equations for $m(0)$ and $m(1)$ (equations (12) and (16)) we get

$$\begin{aligned}m(0)+m(1) &= f(1) - r(0) + \frac{1}{2}s^2\left(r(1)\right) + f(2) - f(1) + \frac{1}{2}s^2\left(r(1)+r(2)\right) - s^2\left(r(1)\right) \\ &= f(2) - r(0) + \frac{1}{2}s^2\left(r(1)+r(2)\right) - \frac{1}{2}s^2\left(r(1)\right),\end{aligned}$$

which can be simplified to

$$\sum_{t=0}^{1} m(t) = f(2) - r(0) + \sum_{t=0}^{1} d(t). \tag{18}$$

The results of the first two dates can be generalized for the case of date *t*.

$$E_0^Q\left[r(t)\right] = f(t) + \frac{1}{2}s^2\left(\sum_{j=1}^{t} r(j)\right) - \frac{1}{2}s^2\left(\sum_{j=1}^{t-1} r(j)\right) \ \forall\ 1 < t \leq T-1. \tag{19}$$

$$m(t-1) = f(t) - f(t-1) + \frac{1}{2}s^2\left(\sum_{j=1}^{t} r(j)\right) - s^2\left(\sum_{j=1}^{t-1} r(j)\right) + \frac{1}{2}\sum_{n=1}^{t-2} s^2\left(\sum_{j=1}^{n} r(j)\right) \ \forall\ t \geq 3. \tag{20}$$

In addition:

$$\sum_{n=0}^{t} d(n) = \frac{1}{2}s^2\left(\sum_{j=1}^{t+1} r(j)\right) - \frac{1}{2}s^2\left(\sum_{j=1}^{t} r(j)\right) \ \forall\ t \geq 1. \tag{21}$$

$$\sum_{n=0}^{t} m(n) = f(t+1) - r(0) + \sum_{n=1}^{t} d(n) \ \forall\ t \geq 1. \tag{22}$$

Equations (19)–(22) give the necessary recursive relations to evolve the HW no-arbitrage model of spot interest rate. The inputs are the set of market prices of (pure) discount bonds, a structure of volatilities for the spot rates, and other parametric values.

The above discussion is general in the sense that it applies equally well to implementation based on interest-rate trees and Monte Carlo simulation.[8]

II. Implementation Example

We illustrate the implementation with reference to an example developed by HW (1996). They illustrate implementation of their model with the example of pricing a three-year put option on a zero-coupon bond that pays \$100 in 9 years. The exercise price is \$63, the volatility, *s*, is constant at 1% per annum for all dates, and the speed of reversion to the mean, α, is 0.10. Exhibit 1 displays the prices and yields of the zero-coupon bonds.

The HW implementation uses a trinomial lattice with upper and lower bounds. First, HW identify the step sizes and the probabilities necessary to achieve the desired volatility, around zero, of the interest rates. Second, they find the expected value of the interest rate at each date

[8] Monte Carlo implementation of the HW model may be important for the valuation of path-dependent securities. For example, at least one derivatives firm values amortizing index swaps using Monte Carlo implementation of the HW model.

that is consistent with the initial conditions. This requires the use of search process and forward induction. We display HW's results in Exhibit 2.

Our first task is to illustrate how to eliminate a numerical search procedure and forward induction to identify the mean value of the interest rates. Specifically, given the initial 1-period interest rate 5.0928%, how do we derive analytically the subsequent mean values of 6.5026%, 7.3393%, and 8.0538%?

Recall that:

$$E_0^Q\left[r(1)\right] = f(1) + \frac{1}{2}s^2\left(r(1)\right), \tag{23-A}$$

$$E_0^Q\left[r(2)\right] = f(2) + \frac{1}{2}\left(s^2\left(r(1)+r(2)\right) - s^2\left(r(1)\right)\right), \tag{23-B}$$

$$E_0^Q\left[r(3)\right] = f(3) + \frac{1}{2}\left(s^2\left(r(1)+r(2)+r(3)\right) - s^2\left(r(1)+r(2)\right)\right), \tag{23-C}$$

from equations (11), (15) and (19). Also:

$$r(1) = r(0) + m(0) + bs\Delta z(0), \tag{24-A}$$

$$r(2) = r(0) + m(0) + m(1) + bs\left(k\Delta z(0) + \Delta z(1)\right), \tag{24-B}$$

$$r(3) = r(0) + m(0) + m(1) + m(2) + bs\left(k^2\Delta z(0) + k\Delta z(1) + \Delta z(2)\right), \tag{24-C}$$

from equations (4), (5) and (6), respectively. The expressions for *k* and *b* are given in equations (2-A) and (2-B), respectively.

With this information, we can calculate the variances of the sum of the spot rates.

$$s^2\left(r(1)\right) = s^2\left(bs\times\left(\Delta z(0)\right)\right) = b^2s^2 = 0.9520^2(0.0001) = 0.00009063.$$

$$s^2\left(r(1)+r(2)\right) = s^2\left(bs\times\left\{\Delta z(0) + k\Delta z(0) + \Delta z(1)\right\}\right) = b^2s^2\left\{(1+k)^2+1\right\}$$
$$= (0.00009063)(4.6284) = 0.00041949.$$

$$s^2\left(r(1)+r(2)+r(3)\right)=s^2\left(bs\times\left\{\left[1+k+k^2\right]\Delta z(0)+\Delta z(1)(1+k)+\Delta z(2)\right\}\right)$$
$$=b^2s^2\left(\left[1+k+k^2\right]^2+(1+k)^2+1\right)$$
$$= (0.00009063)(12.0462) = 0.0010918.$$

With this information, we can derive analytically the values that HW derived through searches.

$$E_0^Q\left[r(1)\right]=-\ln 0.8906+\ln 0.9503+(0.5)(0.00009063) = 6.5026\%.$$

$$E_0^Q\left[r(2)\right]=-\ln 0.8277+\ln 0.8906+(0.5)(0.00041949-0.0009063) = 7.3393\%.$$

$$E_0^Q\left[r(3)\right]=-\ln 0.7639+\ln 0.8277+(0.5)(0.0010918-0.00041949) = 8.5038\%.$$

These results match those produced by HW’s search method, as shown in Exhibit 2.

III. Conclusion

HW develop an attractive no-arbitrage model of the evolution of the spot interest rate that incorporates mean reversion. Their implementation of the model requires both the use of a search method to identify the expected value of the spot interest rate at each date and forward induction. The analytical expression for the expected value of the future spot rates derived in this paper eliminates need for the search process.

Our implementation described here applies equally well to interest-rate binomial trees, trinomial lattices and Monte Carlo simulation implementation of the model and can be adapted to incorporate additional complexities such as time-varying volatility.

References

Arnold, L. *Stochastic Differential Equations*. John Wiley and Sons, Inc. New York, N. Y. 1974.

Black, F., E. Derman and W. Toy. "A One-Factor Model of Interest Rates and Its Application to Treasury Bond Options." *Financial Analysts Journal*, 46 (1990) (January/February), pp. 33–39.

Boyle, P. P. "Immunization Under Stochastic Models of the Interest Rate Structure." *Journal of the Institute of Actuaries*, 105 (1978) pp. 177–87.

Cox, J. C. and S. A. Ross. "The Valuation of Options for Alternative Stochastic Processes." *Journal of Financial Economics*, 3 (1976), pp. 145–166.

Cox, J. C., J. E. Ingersoll and S.A. Ross. "A Theory of the Term Structure of Interest Rates." *Econometrica*, 53 (1985), pp. 385–407.

Harrison, J. M. and D. M. Kreps. "Martingales and Arbitrage in Multiperiod Security Markets." *Journal of Economic Theory*, 20 (1978), pp. 381–408.

Heath, D., R. Jarrow and A. Morton. "Bond Pricing and the Term Structure of Interest Rates: A Discrete Time Approximation." *Journal of Financial and Quantitative Analysis*, 25 (1990), pp. 419–40.

Heath, D., R. Jarrow and A. Morton. "Contingent Claim Valuation with a Random Evolution of Interest Rates." *Review of Futures Markets*, (1991) pp. 54–76.

Heath, D., R. Jarrow and A. Morton. "Bond Pricing and the Term Structure of the Interest Rates: A New Methodology," *Econometrica*, 60 (1992), pp. 77–105.

Ho, T. S. Y. and S.-B. Lee. "Term Structure Movements and Pricing Interest Rate Contingent Claims." *Journal of Finance*, 41 (1986), pp. 1011–1029.

Hull, J. and A. White. "Pricing Interest Rate Derivative Securities." *Review of Financial Studies*, 3 (1990), pp. 573–92.

Hull, J. and A. White. "One-Factor Interest Rate Models and the Valuation of Interest Rate Derivative Securities." *Journal of Financial and Quantitative Analysis*, (1995), pp. 89–.

Hull, J. and A. White. "Numerical Procedures for Implementing Term Structure Models I: Single-Factor Models." *Journal of Derivatives*, 2 (Fall, 1994), pp. 7–16.

Hull, J. and A. White. "Numerical Procedures for Implementing Term Structure Models II: Two-Factor Models." *Journal of Derivatives*, 2 (Winter, 1994), pp. 37–48.

Hull, J. and A. White. "Using Hull-White Interest Rate Trees." *Journal of Derivatives*, 3 (Spring, 1996), pp. 26–36.

Hull, J. *Options, Futures, and Other Derivatives* (Third Edition). Prentice-Hall, Englewood Cliffs, NJ, 1997.

Jamshidian, F. "An Exact Bond Option Formula." *Journal of Finance*, 44 (1989), pp. 205–209.

Mood, A. M., F. A. Graybill and D. C. Boes. *Introduction to the Theory of Statistics* (Third Edition). McGraw-Hill, New York, NY, 1974.

Vasicek, O. A. "An Equilibrium Characterization of the Term Structure." *Journal of Financial Economics*, 5 (1977), pp. 177–88.

Exhibit 1 **Initial Prices and Yields in the Hull and White Example**									
Maturity	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Price	$0.9503	$0.8906	$0.8277	$0.7639	$0.7065	$0.6536	$0.6010	$0.5573	$0.5139
Yield	5.093%	5.795%	6.305%	6.733%	6.948%	7.087%	7.274%	7.308%	7.397%

Exhibit 2
Intermediate Calculations for the Four-Epoch Trinomial Lattice for the Example of the Hull and White Short Rate

Calibrating the Variances in the Lattice

Steps	Transition Probabilities			Node Rates			
J	p_u	p_m	p_d	$i = 0$	$i = 1$	$i = 2$	$i = 3$
2	0.8993	0.0110	0.0897			3.2979%	3.2979%
1	0.1236	0.6576	0.2188		1.6489%	1.6490%	1.6490%
0	0.1667	0.6667	0.1667	0.0000%	0.0000%	0.0000%	0.0000%
–1	0.2188	0.6576	0.1236		–1.6490%	–1.6489%	–1.6489%
–2	0.0897	0.0110	0.8993			–3.2979%	–3.2979%

Calibrating the Prices in the Lattice

Steps	Transition Probabilities			Node Rates			
J	p_u	p_m	p_d	$i = 0$	$i = 1$	$i = 2$	i = 3
2	0.8993	0.0110	0.0897			10.6372%	11.3517%
1	0.1236	0.6576	0.2188		8.1515%	8.9883%	9.7028%
0	0.1667	0.6667	0.1667	5.0928%	6.5026%	7.3393%	8.0538%
–1	0.2188	0.6576	0.1236		4.8536%	5.6904%	6.4049%
–2	0.0897	0.0110	0.8993			4.0414%	4.7559%