Construction of interest rate trinomial tree for Hull-White model

We shall give a description on how to construct an interest rate trinomial tree for Hull-White model

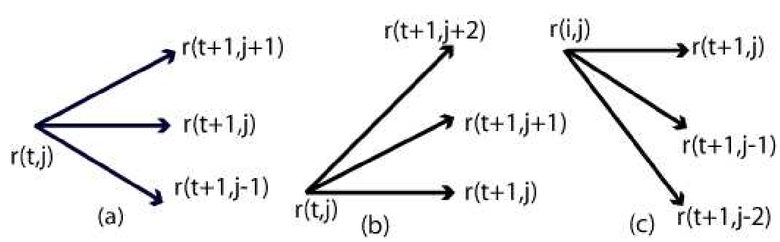
$$dr = (\theta(t) - ar)dt + \sigma dW \quad (1)$$

using Arrow-Debreu prices (see Appendix). To start, lets define some notation. For $t = 1, 2, 3, \dots,$ let

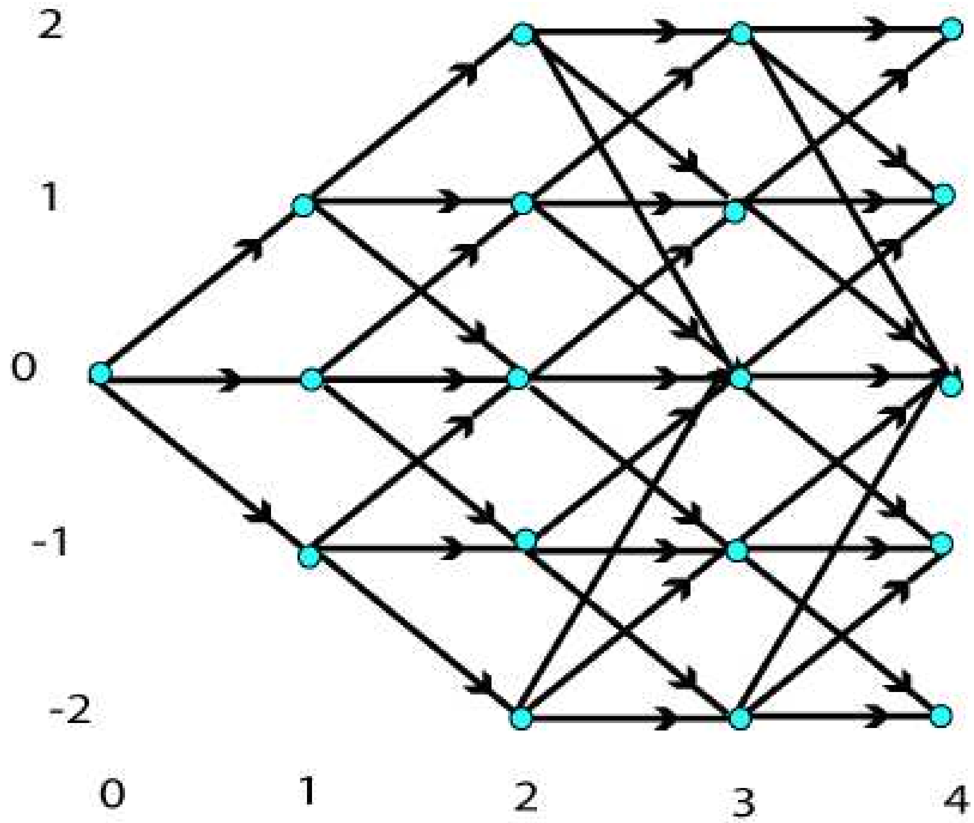
- $D(t)$ be the discount factor over time period $[0, t]$. $D(t)$ could be thought of as the value at $t = 0$ of a \$1 face value default free zero bond that matures at time t.
- $r(t)$ the interest rate over $[0, t]$. We shall use continuously compounded interest. Hence $D(t) = e^{-t \cdot r(t)}$.
- $\sigma > 0$ be the volatility, with respect to the risk neutral probability, of the interest rate at time t.
- $a > 0$.
- $D(t, j)$ be the discount factor at time t and state j, at (t, j) for short, over the time period $[t, t + 1]$.
- $r(t, j)$ be the spot interest rate at (t, j) over time period $[t, t + 1]$. Note that $D(t, j) = e^{-r(t,j)}$
- j_{max} = the smallest integer equal or greater than $0.184/a$. (The choice of j_{max} is to ensure the risk neutral probabilities at each node are positive. Please see [1, p 581] for details.)
- $\Delta R = \sigma\sqrt{3}$. This choice of ΔR is to minimise the approximation error to the continuous case. (See [1, p 581] for details.)
- $n_t = min(t, j_{max})$.

Note that $r(0, 0) = r(1)$. To construct a trinomial tree inductively, we start at $(0, 0)$. If we are at node (t, j), then

- Branch (a) is used if $-j_{max} < j < j_{max}$.
- Branch (b) is used if $j = -j_{max}$.
- Branch (c) is used if $j = j_{max}$.

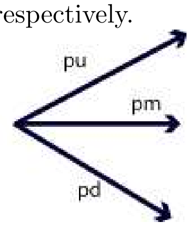


Note that the constructed trinomial tree is symmetric about $t = 0$, and n_t is the number of nodes on either side of $t = 0$ at t . If $a = 0.1$, then $j_{max} = 2$, the trinomial tree would look like the following.



When the interest rate is "sufficiently high", a downward branch is used. Likewise when interest rate is "sufficiently low". This mimics the mean reversion process suggested by (1) .

At each node (t, j), let p_u, p_m, p_d be the risk neutral probabilities of up, middle and down direction respectively.



For the rest of this article assume that

$$r(t, j) = r(t, 0) + j\Delta R \quad \text{for } -n_t \leq j \leq n_t, \tag{2}$$

It turns out that at (t, j), p_u, p_m, p_d only depends on j and are given by the following (see [1, p 582] for details).For branch (a)

$$p_u = \frac{1}{6} + \frac{j^2a^2 - ja}{2} \tag{3}$$

$$p_m = \frac{2}{3} - j^2a^2 \tag{4}$$

$$p_d = \frac{1}{6} + \frac{j^2a^2 + ja}{2} \tag{5}$$

For branch (b)

$$
\begin{aligned}
p_u &= \frac{1}{6} + \frac{j^2a^2 + ja}{2} && (6)\\
p_m &= -\frac{1}{3} - j^2a^2 - 2ja && (7)\\
p_d &= \frac{7}{6} + \frac{j^2a^2 + 3ja}{2} && (8)
\end{aligned}
$$

For branch (c)

$$
\begin{aligned}
p_u &= \frac{7}{6} + \frac{j^2a^2 - 3ja}{2} && (9)\\
p_m &= -\frac{1}{3} - j^2a^2 + 2ja && (10)\\
p_d &= \frac{1}{6} + \frac{j^2a^2 - ja}{2} && (11)
\end{aligned}
$$

(The equations for p_u, p_m, p_d come from the constrains on the mean and variance of $r(t, k+1), r(t, k), r(t, k-1)$ and $p_u + p_m + p_d = 1$. See [1, p 582] for details.)

Set $r(0, 0) = r(1) = -\ln D(1)$. We now show how to find $r(t, j)$ inductively, where $1 \leq t \leq n-1$, $-n_t \leq j \leq n_t$, which satisfies (2) and the $r(t, j)$s are chosen so that there is no arbitrage opportunity. These $r(t, j)$s are discretisation of the Hull-white model

$$dr = (\theta(t) - ar)dt + \sigma dW$$

Please consult [1, Chapter 21] and [2, Chapter 9, section 5] for more details.

At time $t = 0$, consider

- portfolio A that consists of a zero bond which matures at time $t = 2$ with a face value of \$1.
- portfolio B that consists of a derivative which pays

$$
\begin{cases}
D(1,1) & \text{at } (1,1)\\
D(1,0) & \text{at } (1,0)\\
D(1,-1) & \text{at } (1,-1)
\end{cases}
$$

The value of portfolio A at time $t = 0$ is $D(2)$. The value of portfolio B at time $t = 0$ is $Q(1,-1)D(1,-1) + Q(1,0)D(1,0) + Q(1,1)D(1,1)$, where $Q(t, j)$'s are the Arrow-Debreu prices and they are known (see Appendix). As both portfolios have the same payoff at $t = 1$, by the no arbitrage argument, their value at time $t = 0$ must be the same. Hence

$$D(2) = Q(1,-1)D(1,-1) + Q(1,0)D(1,0) + Q(1,1)D(1,1) \tag{12}$$

From (2), we can express $D(1,-1), D(1,0), D(1,1)$ in terms of $r(1,0)$. Hence (12) becomes

$$
\begin{aligned}
D(2) &= Q(1,1)e^{-r(1,0)+\Delta R} + Q(1,0)e^{-r(1,0)} + Q(1,1)e^{-r(1,0)-\Delta R} && (13)\\
r(1,0) &= \ln\left(\frac{Q(1,1)e^{\Delta R} + Q(1,0) + Q(1,1)e^{-\Delta R}}{D(2)}\right) && (14)
\end{aligned}
$$

Then $r(1,-1), r(1,1)$ follows from (2).

Now that we have worked out the spot rates at time $t = 1$, we move on to time $t = 2$. Recall $n_2 = min(2, j_{max})$, the number of nodes at time 2. At time $t = 0$, consider (new portfolios)

- portfolio A that consists of a zero bond which matures at time $t = 3$ with a face value of \$1.
- portfolio B that consists of a derivative which at time 2 has payoff $D(2, j)$ at $(2, j)$ for $-n_2 \leq j \leq n_2$.

Both portfolios A and B have the same payoff at time $t = 2$. By the no arbitrage argument they must have the same value at time $t = 0$. This gives

$$D(3) = \sum_{j=-n_2}^{n_2} Q(2, j)D(2, j) \tag{15}$$

From (2), we can express the $D(2, j)$s in terms of $r(2, 0)$. Hence (15) would become

$$D(3) = \sum_{j=-n_2}^{n_2} Q(2, j)e^{-r(2,0)-j\Delta R} \tag{16}$$

$$r(2, 0) = \ln\left(\frac{\sum_{j=-n_2}^{n_2} Q(2, j)e^{-j\Delta R}}{D(3)}\right) \tag{17}$$

Then $r(2, j)$ for $-n_2 \leq j \leq n_2$ follows from (2).

In general, suppose $t \geq 1$ and we have worked out $r(t-1, j)$ and $Q(t-1, j)$ for $-n_{t-1} \leq j \leq n_{t-1}$. (Note that $r(0, 0) = r(1)$ and $Q(0, 0) = 1$.) For $-n_t \leq j_0 \leq n_t$, we have (see (24))

$$Q(t, j_0) = \sum_{j=-n_{t-1}}^{n_{t-1}} (\text{Prob of}\,(t-1, j) \rightarrow (t, j_0))D(t-1, j)Q(t-1, j) \tag{18}$$

Note that "Prob of $(t-1, j) \rightarrow (t, j_0)$" is the risk neutral probability of going from node $(t-1, j)$ to (t, j_0). It might be zero or one of p_u, p_m, p_d defined earlier. The no arbitrage argument described above gives

$$D(t+1) = \sum_{j=-n_t}^{n_t} Q(t, j)D(t, j) \tag{19}$$

Then by rewriting the $D(t, j)$s in terms of $r(t, 0)$ using (2), (19) becomes

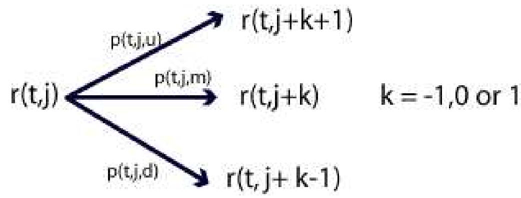
$$D(t+1) = \sum_{j=-n_t}^{n_t} Q(t, j)e^{-r(t,0)-j\Delta R} \tag{20}$$

$$r(t, 0) = \ln\left(\frac{\sum_{j=-n_t}^{n_t} Q(t, j)e^{-j\Delta R}}{D(t+1)}\right) \tag{21}$$

Once we have worked out $r(t, 0)$, the other $r(t, j)$s could be deduced from (2).

Appendix Arrow-Debreu prices

Let $r(t,j)$ be the interest rate at time t and state j, at (t,j) for short, over time period $[t,t+1]$ on a trinomial tree. At node (t,j), let $p(t-1,j,d), p(t-1,j,m), p(t-1,j,u)$ be the risk neutral probabilities of the up, middle, down direction respectively.



For $0 \le t_0$, let $Q(t_0, j_0)$ be the value of a derivative at time 0 and the payoff at $t = t_0$ is given by

$$\delta_{j_0 j} \text{ where j is the state reached at time } t_0 \tag{22}$$

(We also use $Q(t_0, j_0)$ to denote the above defined derivative.) Note that $Q(0,0)$ is 1. The $Q(t,j)$'s are known as the Arrow-Debreu prices.

Let $V(t,j)$ be the value (payoff) of an arbitrary derivative at (t,j). It can be easily verified that $V(0,0)$, the value of the derivative at time $t=0$ is given by

$$V(0,0) = \sum_s V(t,s)Q(t,s) \tag{23}$$

Let t_0, j_0 be given. The value of $Q(t_0, j_0)$ at node $(t_0 - 1, j)$ is

$$(\text{Prob of } (t_0 - 1, j) \to (t_0, j_0)) \cdot D(t_0 - 1, j)$$

where Prob of $(t_0 - 1, j) \to (t_0, j_0)$ is $p(t_0-1,j,d), p(t_0-1,j,m), p(t_0-1,j,u)$ depending on the values of j, j_0.

For $t \ge 1$, by applying (23) to the value of $Q(t,j)$ at $t-1$, we have

$$Q(t,j) = \sum_s (\text{Prob of } (t-1,s) \to (t,s)) \cdot D(t-1,s)Q(t-1,s) \tag{24}$$

Note that we define $Q(t,j) = 0$ if (t,j) is not part of the tree. From (24), we see that $Q(t,j)$ could be calculated inductively.

S H Man 8/4/03

References

[1] J Hull, *Options, Futures and Other Derivatives*, fourth edition, Prentice-Hall International

[2] L Clewlow and C Strickland *Implementing Derivatives Models*, Wiley