Construction of interest rate trinomial tree for Hull-White model

We shall give a description on how to construct an interest rate trinomial tree for Hull-White model

\[ dr = (\theta(t) - ar)dt + \sigma dW \]  

(1)

using Arrow-Debreu prices (see Appendix). To start, let’s define some notation. For \( t = 1, 2, 3, \ldots \), let

- \( D(t) \) be the discount factor over time period \([0, t]\). \( D(t) \) could be thought of as the value at \( t = 0 \) of a $1 face value default free zero bond that matures at time \( t \).
- \( r(t) \) the interest rate over \([0, t]\). We shall use continuously compounded interest. Hence \( D(t) = e^{-tr(t)} \).
- \( \sigma > 0 \) be the volatility, with respect to the risk neutral probability, of the interest rate at time \( t \).
- \( a > 0 \).
- \( D(t, j) \) be the discount factor at time \( t \) and state \( j \), at \((t, j)\) for short, over the time period \([t, t+1]\).
- \( r(t, j) \) be the spot interest rate at \((t, j)\) over time period \([t, t+1]\). Note that \( D(t, j) = e^{-r(t,j)} \).
- \( j_{\text{max}} \) = the smallest integer equal or greater than \( 0.184/a \). (The choice of \( j_{\text{max}} \) is to ensure the risk neutral probabilities at each node are positive. Please see [1, p 581] for details.)
- \( \Delta R = \sigma \sqrt{3} \). This choice of \( \Delta R \) is to minimise the approximation error to the continuous case. (See [1, p 581] for details.)
- \( n_t = \min(t, j_{\text{max}}) \).

Note that \( r(0, 0) = r(1) \). To construct a trinomial tree inductively, we start at \((0, 0)\). If we are at node \((t, j)\), then

- Branch (a) is used if \( -j_{\text{max}} < j < j_{\text{max}} \).
- Branch (b) is used if \( j = -j_{\text{max}} \).
- Branch (c) is used if \( j = j_{\text{max}} \).

Note that the constructed trinomial tree is symmetric about \( t = 0 \), and \( n_t \) is the number of nodes on either side of \( t = 0 \) at \( t \). If \( a = 0.1 \), then \( j_{\text{max}} = 2 \), the trinomial tree would look like the following.
When the interest rate is "sufficiently high", a downward branch is used. Likewise when interest rate is "sufficiently low". This mimics the mean reversion process suggested by (1).

At each node \((t,j)\), let \(p_u, p_m, p_d\) be the risk neutral probabilities of up, middle and down direction respectively.

\[
\begin{align*}
  p_u &= \frac{1}{6} + \frac{j^2a^2 - ja}{2} \\
  p_m &= \frac{2}{3} - \frac{j^2a^2}{2} \\
  p_d &= \frac{1}{6} + \frac{j^2a^2 + ja}{2}
\end{align*}
\]

For the rest of this article assume that

\[
r(t,j) = r(t,0) + j\Delta R \text{ for } -n_t \leq j \leq n_t, \tag{2}
\]

It turns out that at \((t,j)\), \(p_u, p_m, p_d\) only depends on \(j\) and are given by the following (see [1, p 582] for details). For branch \((a)\)
For branch (b)

\[ p_u = \frac{1}{6} + \frac{j^2a^2 + ja}{2} \]  
(6)

\[ p_m = -\frac{1}{3} - \frac{j^2a^2 - 2ja}{2} \]  
(7)

\[ p_d = \frac{7}{6} + \frac{j^2a^2 + 3ja}{2} \]  
(8)

For branch (c)

\[ p_u = \frac{7}{6} + \frac{j^2a^2 - 3ja}{2} \]  
(9)

\[ p_m = -\frac{1}{3} - \frac{j^2a^2 + 2ja}{2} \]  
(10)

\[ p_d = \frac{1}{6} + \frac{j^2a^2 - ja}{2} \]  
(11)

(The equations for \( p_u, p_m, p_d \) come from the constrains on the mean and variance of \( r(t, k + 1), r(t, k), r(t, k - 1) \) and \( p_u + p_m + p_d = 1 \). See [1, p 582] for details.)

Set \( r(0, 0) = r(1) = -\ln D(1) \). We now show how to find \( r(t, j) \) inductively, where \( 1 \leq t \leq n - 1, -n \leq j \leq n \), which satisfies (2) and the \( r(t, j) \)'s are chosen so that there is no arbitrage opportunity. These \( r(t, j) \)'s are discretisation of the Hull-white model

\[ dr = (\theta(t) - ar)dt + \sigma dW \]

Please consult [1, Chapter 21] and [2, Chapter 9, section 5] for more details.

At time \( t = 0 \), consider

- portfolio A that consists of a zero bond which matures at time \( t = 2 \) with a face value of \$1.
- portfolio B that consists of a derivative which pays

\[ \begin{cases} 
D(1, 1) & \text{at } (1, 1) \\
D(1, 0) & \text{at } (1, 0) \\
D(1, -1) & \text{at } (1, -1)
\end{cases} \]

The value of portfolio A at time \( t = 0 \) is \( D(2) \). The value of portfolio B at time \( t = 0 \) is \( Q(1, -1)D(1, -1) + Q(1, 0)D(1, 0) + Q(1, 1)D(1, 1) \), where \( Q(t, j) \)'s are the Arrow-Debreu prices and they are known (see Appendix). As both portfolios have the same payoff at \( t = 1 \), by the no arbitrage argument, their value at time \( t = 0 \) must be the same. Hence

\[ D(2) = Q(1, -1)D(1, -1) + Q(1, 0)D(1, 0) + Q(1, 1)D(1, 1) \]  
(12)

From (2), we can express \( D(1, -1), D(1, 0), D(1, 1) \) in terms of \( r(1, 0) \). Hence (12) becomes

\[ D(2) = Q(1, 1)e^{-r(1, 0)+\Delta R} + Q(1, 0)e^{-r(1, 0)} + Q(1, 1)e^{-r(1, 0)-\Delta R} \]  
(13)

\[ r(1, 0) = \ln \left( \frac{Q(1, 1)e^{\Delta R} + Q(1, 0) + Q(1, 1)e^{-\Delta R}}{D(2)} \right) \]  
(14)

Then \( r(1, -1), r(1, 1) \) follows from (2).
Now that we have worked out the spot rates at time $t = 1$, we move on to time $t = 2$. Recall $n_2 = \min(2, j_{\text{max}})$, the number of nodes at time 2. At time $t = 0$, consider (new portfolios)

- portfolio A that consists of a zero bond which matures at time $t = 3$ with a face value of $1$.
- portfolio B that consists of a derivative which at time 2 has payoff $D(2, j)$ at $(2, j)$ for $-n_2 \leq j \leq n_2$.

Both portfolios A and B have the same payoff at time $t = 2$. By the no arbitrage argument they must have the same value at time $t = 0$. This gives

$$D(3) = \sum_{j=-n_2}^{n_2} Q(2, j)D(2, j)$$

(15)

From (2), we can express the $D(2, j)$s in terms of $r(2, 0)$. Hence (15) would become

$$D(3) = \sum_{j=-n_2}^{n_2} Q(2, j)e^{-r(2, 0)j\Delta R}$$

(16)

$$r(2, 0) = \ln \left( \frac{\sum_{j=-n_2}^{n_2} Q(2, j)e^{-j\Delta R}}{D(3)} \right)$$

(17)

Then $r(2, j)$ for $-n_2 \leq j \leq n_2$ follows from (2).

In general, suppose $t \geq 1$ and we have worked out $r(t-1, j)$ and $Q(t-1, j)$ for $-n_{t-1} \leq j \leq n_{t-1}$. (Note that $r(0, 0) = r(1)$ and $Q(0, 0) = 1$.) For $-n_t \leq j_0 \leq n_t$, we have (see (24))

$$Q(t, j_0) = \sum_{j=-n_{t-1}}^{n_{t-1}} \text{(Prob of } (t-1, j) \rightarrow (t, j_0))D(t-1, j)Q(t-1, j)$$

(18)

Note that "Prob of $(t-1, j) \rightarrow (t, j_0)$" is the risk neutral probability of going from node $(t-1, j)$ to $(t, j_0)$. It might be zero or one of $p_u, p_m, p_d$ defined earlier. The no arbitrage argument described above gives

$$D(t+1) = \sum_{j=-n_t}^{n_t} Q(t, j)D(t, j)$$

(19)

Then by rewriting the $D(t, j)$s in terms of $r(t, 0)$ using (2), (19) becomes

$$D(t+1) = \sum_{j=-n_t}^{n_t} Q(t, j)e^{-r(t, 0)j\Delta R}$$

(20)

$$r(t, 0) = \ln \left( \frac{\sum_{j=-n_t}^{n_t} Q(t, j)e^{-j\Delta R}}{D(t+1)} \right)$$

(21)

Once we have worked out $r(t, 0)$, the other $r(t, j)$s could be deduced from (2).
Appendix Arrow-Debreu prices

Let \( r(t;j) \) be the interest rate at time \( t \) and state \( j \), at \( (t,j) \) for short, over time period \([t, t + 1]\) on a trinomial tree. At node \( (t,j) \), let \( p(t - 1, j, d), p(t - 1, j, m), p(t - 1, j, u) \) be the risk neutral probabilities of the up, middle, down direction respectively.

For \( 0 \leq t_0 \), let \( Q(t_0, j_0) \) be the value of a derivative at time 0 and the payoff at \( t = t_0 \) is given by

\[
Q(t_0, j_0) = \delta_{j_0, j} \text{ where } j \text{ is the state reached at time } t_0 \tag{22}
\]

(We also use \( Q(t_0, j_0) \) to denote the above defined derivative.) Note that \( Q(0,0) \) is 1. The \( Q(t,j) \)'s are known as the Arrow-Debreu prices.

Let \( V(t,j) \) be the value (payoff) of an arbitrary derivative at \( (t,j) \). It can be easily verified that \( V(0,0) \), the value of the derivative at time \( t = 0 \) is given by

\[
V(0,0) = \sum_s V(t,s)Q(t,s) \tag{23}
\]

Let \( t_0, j_0 \) be given. The value of \( Q(t_0,j_0) \) at node \( (t_0 - 1, j) \) is

\[
Q(t_0, j_0) = \sum_s (\text{Prob of } (t_0 - 1, s) \rightarrow (t, s)) \cdot D(t_0 - 1, s)Q(t - 1, s) \tag{24}
\]

Note that we define \( Q(t,j) = 0 \) if \( (t,j) \) is not part of the tree. From (24), we see that \( Q(t,j) \) could be calculated inductively.

References


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