

## STABILITY ANALYSIS OF AN ODD–EVEN-LINE HOPSCOTCH METHOD FOR THREE-DIMENSIONAL ADVECTION–DIFFUSION PROBLEMS\*

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**Abstract.** A linear stability analysis is given for an odd–even-line hopscotch (OELH) method, which has been developed for integrating three-space dimensional, shallow water transport problems. Sufficient and necessary conditions are derived for strict von Neumann stability for the case of the general, constant coefficient, linear advection–diffusion model problem. The analysis is based on an equivalence with an associated scheme which is composed of the leapfrog, the Du Fort–Frankel, and the Crank–Nicolson schemes. The results appear to be rather intricate. For example, the resulting expressions for critical stepsizes reveal that the presence of horizontal diffusion generally leads to a smaller value, in spite of the fact that we have unconditional stability for pure diffusion problems. It is pointed out that this is due to the Du Fort–Frankel deficiency. On the other hand, it is also shown, by a numerical experiment, that in practice it is sufficient to obey the weaker Courant–Friedrichs–Lewy (CFL) condition associated with the case of pure horizontal advection, unless a huge number of integration steps are to be taken.

**Key words.** time-dependent advection–diffusion, numerical methods, odd–even-line hopscotch method, stability

**AMS subject classifications.** 65M06, 65M12

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**1. Introduction.** In [10] and [11] an odd–even-line hopscotch (OELH) method is developed and implemented for the efficient numerical solution of three-space dimensional advection–diffusion problems modeling the transport of pollutants and suspended material in shallow water. A special feature of this OELH method is that it is explicit for the horizontal transport and implicit for the vertical transport. The implicitness in the vertical direction is necessary to avoid a too stringent stability restriction on the time step. This implicitness gives rise to the solution of a large set of tridiagonal systems, one for every gridpoint in the horizontal plane. The solution of this large set of tridiagonal systems can be vectorized and parallelized over the horizontal grid, which results in very good performance [11]. In comparison with other techniques discussed in [10, 11], the method has been shown superior.

In neither of the aforementioned two papers is a comprehensive stability analysis given. The purpose of the present paper is to fill this gap. For the general, constant coefficient, linear advection–diffusion model problem we will derive sufficient and necessary conditions for von Neumann stability in the strict sense. Strict means that the stability property we investigate requires the absolute value of amplification factors to be less than or equal to one. The stability analysis is based on an equivalence with an associated scheme which is composed of the leapfrog, the Du Fort–Frankel, and the Crank–Nicolson schemes. The actual Fourier analysis is carried out for this associated scheme and appears to be rather intricate. For example, the resulting expressions for critical stepsizes reveal that the presence of horizontal diffusion generally leads

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to a smaller value, in spite of the fact that we have unconditional stability for pure diffusion problems.

**2. The OELH method formulated for the model problem.** We consider the 3D, constant coefficient, scalar advection–diffusion model problem

$$(2.1) \quad u_t + q_1u_x + q_2u_y + q_3u_z = \epsilon_1u_{xx} + \epsilon_2u_{yy} + \epsilon_3u_{zz}.$$

Let

$$(2.2) \quad \frac{d}{dt}U_{ijk} = L_hU_{ijk}$$

be the semidiscrete approximation resulting from the use of second-order central differences at the uniformly spaced gridpoints  $(x_i, y_j, z_k) = (ih_1, jh_2, kh_3)$ . The basic formula [1, 2, 3, 4] defining the OELH method studied in [10, 11] then reads

$$(2.3) \quad U_{\underline{i}}^{n+1} = U_{\underline{i}}^n + \tau\theta_{\underline{i}}^n L_h U_{\underline{i}}^n + \tau\theta_{\underline{i}}^{n+1} L_h U_{\underline{i}}^{n+1},$$

where  $\underline{i} = (i, j, k)$ ,  $\tau = t_{n+1} - t_n$ , and the hopscotch parameter  $\theta_{\underline{i}}^n$  is defined by

$$(2.4) \quad \theta_{\underline{i}}^n = \begin{cases} 1 & \text{for odd values of } n + i + j, \\ 0 & \text{for even values of } n + i + j. \end{cases}$$

Notice that the subscript  $k$  is not involved in this definition, i.e., all gridpoints on a vertical gridline have the same  $\theta$  value. If we consider only the odd points in the space–time grid, which means  $\theta_{\underline{i}}^n = 1$  and  $\theta_{\underline{i}}^{n+1} = 0$  in (2.3), then the forward Euler rule results,

$$(2.5) \quad U_{\underline{i}}^{n+1} = U_{\underline{i}}^n + \tau L_h U_{\underline{i}}^n,$$

and at the even points, for the same  $n$ , we have the backward Euler rule

$$(2.6) \quad U_{\underline{i}}^{n+1} = U_{\underline{i}}^n + \tau L_h U_{\underline{i}}^{n+1}.$$

Consequently, by first applying the explicit forward Euler method at all odd points, and subsequently the implicit backward Euler method at all even points, we have carried out one step with (2.3). The merit of the method lies in the fact that the implicit step is only implicit for the vertical direction. This follows from the three-point coupling in the horizontal directions and from the definition of the  $\theta_{\underline{i}}^n$ . If we remove the third dimension, then we recover the odd–even-hopscotch (OEH) scheme which is scalarly implicit. Note that the OEH scheme for the 3D problem results if we replace  $(n + i + j)$  in  $\theta_{\underline{i}}^n$  by  $(n + i + j + k)$ . The stability of the OEH scheme applied to (2.1) has been studied in [12].

The von Neumann stability approach cannot be carried out for (2.3) as it stands. Following [3, 12], we therefore derive an equivalent formula which does admit Fourier analysis. First introduce, for  $m = 1, 2, 3$ , the advection parameter  $c_m$  and the diffusion parameter  $\sigma_m$ ,

$$(2.7) \quad c_m = \frac{\tau q_m}{h_m}, \quad \sigma_m = \frac{\tau \epsilon_m}{h_m^2},$$

and the difference operators  $H_m$  and  $\delta_m^2$ ,

$$(2.8) \quad H_1 U_{\underline{i}} = U_{i+1jk} - U_{i-1jk}, \text{ etc.},$$

$$(2.9) \quad \delta_1^2 U_{\underline{i}} = U_{i+1jk} - 2U_{ijk} + U_{i-1jk}, \text{ etc.}$$

We then may express  $\tau L_h U_{\underline{i}}$  as

$$(2.10) \quad \tau L_h U_{\underline{i}} = \sum_{m=1}^3 \left( -\frac{1}{2} c_m H_m + \sigma_m \delta_m^2 \right) U_{\underline{i}}.$$

Next introduce, in addition to (2.3), the OELH formula for the next time step

$$(2.11) \quad U_{\underline{i}}^{n+2} = U_{\underline{i}}^{n+1} + \tau \theta_{\underline{i}}^{n+1} L_h U_{\underline{i}}^{n+1} + \tau \theta_{\underline{i}}^{n+2} L_h U_{\underline{i}}^{n+2}.$$

Using (2.3), (2.4), and (2.11), for the odd points we then can write, considering time levels  $n$  and  $n+2$ ,

$$(2.12) \quad U_{\underline{i}}^{n+2} = U_{\underline{i}}^n + \tau L_h \left( U_{\underline{i}}^n + U_{\underline{i}}^{n+2} \right).$$

Likewise, for the even points we find

$$(2.13) \quad U_{\underline{i}}^{n+2} = 2U_{\underline{i}}^{n+1} - U_{\underline{i}}^n.$$

Next we elaborate the odd-point formula (2.12). Using (2.13) to eliminate variables at even points, an elementary calculation with (2.10) shows that (2.12) can be written as

$$(2.14) \quad \begin{aligned} (1 + \sigma) U_{\underline{i}}^{n+2} &= (1 - \sigma) U_{\underline{i}}^n + (4\sigma_1 \mu_1 + 4\sigma_2 \mu_2) U_{\underline{i}}^{n+1} \\ &- (c_1 H_1 + c_2 H_2) U_{\underline{i}}^{n+1} + \left( -\frac{1}{2} c_3 H_3 + \sigma_3 \delta_3^2 \right) \left( U_{\underline{i}}^n + U_{\underline{i}}^{n+2} \right), \end{aligned}$$

where  $\mu_m$  is the averaging operator

$$(2.15) \quad \mu_1 U_{\underline{i}} = \frac{1}{2} (U_{i+1jk} + U_{i-1jk}), \text{ etc.},$$

and

$$(2.16) \quad \sigma = 2(\sigma_1 + \sigma_2).$$

It is important to note that in (2.14) only variables at odd-numbered points appear. This means that the solution defined by (2.3) can first be computed by means of (2.14) at the complete set of odd points, and thereafter at the complete set of even points by means of (cf. (2.13))

$$(2.17) \quad U_{\underline{i}}^{n+1} = \frac{1}{2} \left( U_{\underline{i}}^n + U_{\underline{i}}^{n+2} \right).$$

Hence for the stability analysis we may proceed with the odd-point scheme (2.14) because the sets of even and odd points are decoupled.

We see that this odd-point scheme is composed of the leapfrog scheme for the horizontal advection part,

$$(2.18) \quad U_{\underline{i}}^{n+2} = U_{\underline{i}}^n - (c_1 H_1 + c_2 H_2) U_{\underline{i}}^{n+1},$$

of the Du Fort–Frankel scheme for the horizontal diffusion part,

$$(2.19) \quad (1 + \sigma) U_{\underline{i}}^{n+2} = (1 - \sigma) U_{\underline{i}}^n + (4\sigma_1 \mu_1 + 4\sigma_2 \mu_2) U_{\underline{i}}^{n+1},$$

and of the Crank–Nicolson scheme, with stepsize  $2\tau$ , for the vertical advection and diffusion part,

$$(2.20) \quad U_{\underline{i}}^{n+2} = U_{\underline{i}}^n + \left(-\frac{1}{2}c_3 H_3 + \sigma_3 \delta_3^2\right) \left(U_{\underline{i}}^n + U_{\underline{i}}^{n+2}\right).$$

Consequently, in view of the unconditional stability of the Crank–Nicolson and Du Fort–Frankel scheme, at first sight one might expect that the critical stepsize for stability equals that of the leapfrog scheme (2.18). In the next section we will prove that this is indeed true if there is no horizontal diffusion. However, if horizontal diffusion terms are present, then the situation turns out to be more complicated. We will show that in this case the critical stepsize is generally smaller.

**3. Strict von Neumann stability.** Substitution of the Fourier mode

$$(3.1) \quad U_{\underline{i}}^n = \xi^n e^{I(\omega_1 x_i + \omega_2 y_j + \omega_3 z_k)}, \quad I^2 = -1,$$

into scheme (2.14) leads to the characteristic equation  $f(\xi) = 0$ , with  $f$  given by

$$(3.2) \quad f(\xi) = a_0 + a_1 \xi + a_2 \xi^2$$

and

$$(3.3) \quad \begin{aligned} a_0 &= -1 + \sigma - 2\sigma_3 (\cos \theta_3 - 1) + I c_3 \sin \theta_3, \\ a_1 &= \sum_{m=1}^2 (-4\sigma_m \cos \theta_m + 2I c_m \sin \theta_m), \\ a_2 &= 1 + \sigma - 2\sigma_3 (\cos \theta_3 - 1) + I c_3 \sin \theta_3, \end{aligned}$$

where  $\theta_m = \omega_m h_m$  denotes the phase angle. The specific stability property we will investigate is von Neumann stability in the strict sense.

DEFINITION 1. *Method (2.14) is called von Neumann stable if the zeros  $\xi_1, \xi_2$  of the characteristic polynomial (3.2) satisfy*

$$(3.4) \quad |\xi_1|, |\xi_2| \leq 1 \text{ for all } |\theta_m| \leq \pi, \quad m = 1, 2, 3.$$

Hence strict means that the stability property we investigate requires the absolute value of amplification factors to be less than or equal to one. In the literature, this is also called “practical” or “modified” von Neumann stability [8, 7, 5]. Note that the original von Neumann condition is weaker as it requires  $|\xi| \leq 1 + O(\tau)$  [8]. As is well known, for advection–diffusion problems this weaker condition can lead to unacceptably large errors [7]. Strict stability is also more natural here, since Fourier modes of the true solution cannot grow in time either.

For the von Neumann analysis we will use results from [6]. We therefore introduce the polynomial

$$(3.5) \quad f^*(\xi) = \bar{a}_2 + \bar{a}_1 \xi + \bar{a}_0 \xi^2,$$

and the so-called first reduced polynomial

$$(3.6) \quad f_1(\xi) = \bar{a}_2 a_1 - \bar{a}_1 a_0 + (\bar{a}_2 a_2 - \bar{a}_0 a_0) \xi,$$

where

$$(3.7) \quad \begin{aligned} \bar{a}_2 a_1 - \bar{a}_1 a_0 &= -8 \sum_{m=1}^2 \sigma_m \cos \theta_m + I \left( 8c_3 \sin \theta_3 \sum_{m=1}^2 \sigma_m \cos \theta_m \right) \\ &+ I \left( 4(\sigma + 2\sigma_3 - 2\sigma_3 \cos \theta_3) \sum_{m=1}^2 c_m \sin \theta_m \right) \end{aligned}$$

and

$$(3.8) \quad \bar{a}_2 a_2 - \bar{a}_0 a_0 = 4(\sigma + 2\sigma_3 - 2\sigma_3 \cos \theta_3).$$

Note that in the pure advection case the first reduced polynomial vanishes because then  $\sigma_m = 0$  for  $m = 1, 2, 3$ .

In the remainder of this section we will prove and discuss two stability theorems. Theorem 1 deals with the case where horizontal diffusion is absent ( $\epsilon_1 = 0, \epsilon_2 = 0$ , and  $\epsilon_3 \geq 0$ ). In Theorem 2 we consider the remaining cases where diffusion exists in at least one of the two horizontal directions ( $\epsilon_1 \geq 0, \epsilon_2 \geq 0, \epsilon_3 \geq 0$ , and  $\epsilon_1 + \epsilon_2 > 0$ ). In both theorems all velocities  $c_m$  may take on arbitrary values, including zero.

**THEOREM 1.** *Suppose  $\epsilon_1 = 0, \epsilon_2 = 0$ , and  $\epsilon_3 \geq 0$ . Then we have von Neumann stability if and only if*

$$(3.9) \quad |c_1| + |c_2| \leq 1.$$

*Proof.* We distinguish the two cases  $\epsilon_3 = 0$  and  $\epsilon_3 > 0$ . First suppose  $\epsilon_3 = 0$ . Then the first reduced polynomial  $f_1 \equiv 0$ , so that according to case (ii) of Theorem 6.1 from [6], there holds  $|\xi_1|, |\xi_2| \leq 1$  if and only if the root  $\xi_0$  of the derivative polynomial  $f'$  satisfies  $|\xi_0| \leq 1$ . Since  $\xi_0 = -a_1/2a_2$  we find

$$(3.10) \quad |\xi_0|^2 = \frac{\left( \sum_{m=1}^2 c_m \sin \theta_m \right)^2}{1 + c_3^2 \sin^2 \theta_3},$$

which immediately proves the theorem for the case  $\epsilon_3 = 0$ . Next suppose  $\epsilon_3 > 0$ . Two subcases then must be distinguished, viz., phase angle  $\theta_3 = 0$  and  $\theta_3 \neq 0$ . If  $\theta_3 = 0$ , then again  $f_1 \equiv 0$  and the proof goes the same as above. If  $\theta_3 \neq 0$ , then  $f_1$  does not vanish so that now case (i) of Theorem 6.1 from [6] applies. That is,  $|\xi_1|, |\xi_2| \leq 1$  if and only if

- (a)  $|f^*(0)| > |f(0)|$  and
- (b) the root  $\xi_0$  of  $f_1$  satisfies  $|\xi_0| \leq 1$ .

Condition (a) means  $|\bar{a}_2| > |a_0|$  or, according to (3.8),

$$(3.11) \quad |a_2|^2 - |a_0|^2 = \bar{a}_2 a_2 - \bar{a}_0 a_0 = 4(\sigma + 2\sigma_3 - 2\sigma_3 \cos \theta_3) > 0.$$

We immediately conclude that condition (a) holds unconditionally because the diffusion parameter  $\sigma_3$  is positive and  $\sigma = 0$ . Generally, condition (b) holds if and only if

$$(3.12) \quad \left| -2 \sum_{m=1}^2 \sigma_m \cos \theta_m + I \left( 2c_3 \sin \theta_3 \sum_{m=1}^2 \sigma_m \cos \theta_m \right) + I \left( (\sigma + 2\sigma_3 - 2\sigma_3 \cos \theta_3) \sum_{m=1}^2 c_m \sin \theta_m \right) \right| \leq \sigma + 2\sigma_3 - 2\sigma_3 \cos \theta_3.$$

Because  $\sigma_1 = \sigma_2 = 0$  and  $\sigma_3 > 0$ , this inequality simply means that

$$\left| \sum_{m=1}^2 c_m \sin \theta_m \right| \leq 1,$$

which immediately proves the theorem also for the case  $\epsilon_3 > 0$ .  $\square$

In the situation of Theorem 1 the Du Fort–Frankel scheme is absent in (2.14), so that only the leapfrog scheme and the Crank–Nicolson scheme as combined in (2.14) play a role. Theorem 1 nicely shows this. We see that the critical stepsize for von Neumann stability is determined by the familiar CFL condition of the leapfrog scheme (2.18),

$$(3.13) \quad \tau \left( \frac{|q_1|}{h_1} + \frac{|q_2|}{h_2} \right) \leq 1.$$

This is an optimal result in the sense that the vertical velocity  $q_3$  and the vertical mesh width  $h_3$  are absent in the stability condition, which is due to the unconditional stability of the Crank–Nicolson scheme. It is especially important that  $h_3$  should be absent, since in shallow water transport problems  $h_3$  is significantly smaller than  $h_1$  and  $h_2$ . This, in fact, was the motivation for developing the OELH method [10, 11]. Also note that in the case of pure advection ( $\epsilon_m = 0, m = 1, 2, 3$ ) the characteristic polynomial  $f$  is conservative ( $|\xi_1| = |\xi_2| = 1$ ) as long as (3.13) holds (Theorem 6.4 in [6]). If we impose strict inequality, then  $f$  is simple conservative (conservative and  $\xi_1 \neq \xi_2$ ; see [6, Cor. 6.5]). This means that in the case of pure advection the OELH scheme does not damp Fourier modes, which is a natural property because the true Fourier modes are not damped either. If  $\epsilon_3 > 0$ , then one of the amplification factors must lie in the open unit disk as long as (3.13) holds, since  $f_1$  does not vanish. If we impose strict inequality in (3.13), then both factors lie in the open unit disk, which means damping of Fourier modes similarly as for the true solution.

Before we present Theorem 2, we first give a result due to [5] and repeat its proof here for reasons of self-containedness.

LEMMA 1. *Consider the finite, real-valued series*

$$S = 1 - \sum_{m=1}^M \alpha_m \theta_m^2 + \left( \sum_{m=1}^M c_m \theta_m \right)^2.$$

*Suppose  $\alpha_m \geq 0$  for all  $m = 1, \dots, M$ . Then we have  $S \leq 1$  for all  $\theta_m$  if and only if*

$$\sum_{m=1}^M \frac{c_m^2}{\alpha_m} \leq 1.$$

*Proof.* Denote  $\alpha = \text{diag}(\alpha_1, \dots, \alpha_M)$ ,  $\vec{c} = (c_1, \dots, c_M)^T$ ,  $\vec{\theta} = (\theta_1, \dots, \theta_M)^T$ . Then  $S$  can be expressed as

$$S = 1 - \vec{\theta}^T (\alpha - \vec{c} \vec{c}^T) \vec{\theta}.$$

Thus, we have  $S \leq 1$  for all  $\vec{\theta}$  if and only if the matrix  $\beta = \alpha - \vec{c} \vec{c}^T$  is nonnegative definite. In particular, its diagonal elements  $\alpha_m - c_m^2$  must be nonnegative, so that  $\alpha_m = 0$  implies  $c_m = 0$  and the  $m$ th dimension can be dropped. Hence in the remainder of the proof we may assume all  $\alpha_m > 0$ . If we then define

$$\gamma = \alpha^{-1/2} = \text{diag}(\alpha_1^{-1/2}, \dots, \alpha_M^{-1/2}),$$

we have  $\beta = \alpha^{1/2}(I_M - \gamma\bar{c}\bar{c}^T\gamma)\alpha^{1/2}$  and the matrix

$$\beta' = I_M - \gamma\bar{c}\bar{c}^T\gamma = I_M - (\gamma\bar{c})(\gamma\bar{c})^T = I_M - \vec{d}\vec{d}^T,$$

where  $\vec{d} = \gamma\bar{c}$ , must also be nonnegative. This, in turn, means nonnegativity of

$$\vec{z}^T\beta'\vec{z} = \vec{z}^T\vec{z} - (\vec{d}^T\vec{z})^2$$

for all  $\vec{z}$ . We can deduce that this is true if and only if

$$\vec{d}^T\vec{d} \leq 1.$$

Sufficiency follows immediately from the Cauchy–Schwarz inequality  $(\vec{d}^T\vec{z})^2 \leq (\vec{d}^T\vec{d})(\vec{z}^T\vec{z})$  and necessity by selecting  $z_m = cd_m$  for  $m = 1, \dots, M$ , where  $c$  is an arbitrary constant. Since  $\vec{d}^T\vec{d} = \sum c_m^2/\alpha_m$ , the proof is complete.  $\square$

This lemma is used to prove necessity of inequality (3.14) in Theorem 2. Note that in certain cases the sum in (3.14) is infinite (division by  $\sigma_m = 0$ ), implying that the interval for von Neumann stability is empty. This situation is discussed in more detail later on. We wish to emphasize that the proof of this theorem is inspired by the proof of the stability theorem in [5], which also uses the result of Lemma 1.

**THEOREM 2.** *Suppose  $\epsilon_1, \epsilon_2, \epsilon_3 \geq 0$  and  $\epsilon_1 + \epsilon_2 > 0$ . Then we have von Neumann stability if and only if*

$$(3.14) \quad \sum_{m=1}^3 \frac{c_m^2}{2\sigma_m/\sigma} \leq 1.$$

*Proof.* Because  $\sigma > 0$ , the first reduced polynomial  $f_1$  does not vanish, so that case (i) of Theorem 6.1 from [6] applies, similarly as in the second part of the proof of Theorem 1 above. Hence  $|\xi_1|, |\xi_2| \leq 1$  if and only if inequalities (3.11) and (3.12) hold. We immediately conclude that inequality (3.11) holds unconditionally because  $\sigma > 0$  and  $\sigma_3 \geq 0$ . So our task is to check inequality (3.12). Denote

$$\begin{aligned} \sigma^* &= \sigma + 2\sigma_3 - 2\sigma_3 \cos \theta_3, \\ \sigma_m^* &= 2\sigma_m/\sigma^*, \quad m = 1, 2, \\ c_1^* &= c_1, \quad c_2^* = c_2, \quad c_3^* = c_3 \sum_{m=1,2} \sigma_m^* \cos \theta_m. \end{aligned}$$

Inequality (3.12) is equivalent to  $|\mu| \leq 1$ , where

$$(3.15) \quad \mu = \frac{\sigma}{\sigma^*} - \sum_{m=1}^2 \sigma_m^*(1 - \cos \theta_m) - \sum_{m=1}^3 Ic_m^* \sin \theta_m.$$

Introduce the new diffusion parameter  $\sigma_3^*$  by writing

$$(3.16) \quad \frac{\sigma}{\sigma^*} = 1 - \sigma_3^*(1 - \cos \theta_3),$$

which implies the same expression as for  $\sigma_1^*$  and  $\sigma_2^*$ ,

$$(3.17) \quad \sigma_3^* = \frac{2\sigma_3}{\sigma^*}.$$

Note that for zero phase angle  $\theta_3$  the definition of  $\sigma_3^*$  through (3.16) is meaningless. However, from the limiting case

$$\sigma^* = \sigma + \sigma_3\theta_3^2 + O(\theta_3^4), \quad \theta_3 \rightarrow 0$$

it follows, by substitution of (3.17) into (3.16), that expression (3.17) is also valid for  $\theta_3 = 0$ . Hence for all phase angles we can write

$$(3.18) \quad \mu = 1 - \sum_{m=1}^3 \sigma_m^* (1 - \cos \theta_m) - \sum_{m=1}^3 I c_m^* \sin \theta_m,$$

so that inequality (3.12) holds if and only if

$$(3.19) \quad |\mu|^2 = \left(1 - \sum_{m=1}^3 \sigma_m^* (1 - \cos \theta_m)\right)^2 + \left(\sum_{m=1}^3 c_m^* \sin \theta_m\right)^2 \leq 1.$$

Our task is now to prove that (3.14) is necessary and sufficient for (3.19). We will first establish necessity. Consider the limiting case  $\theta_m \rightarrow 0$  with  $|\theta_m| \leq \theta$  for  $m = 1, 2, 3$ . For  $\theta_3 \rightarrow 0$  we have

$$\sigma_m^* = \frac{2\sigma_m}{\sigma} + O(\theta_3^2) \text{ for } m = 1, 2, 3 \text{ and } c_3^* = c_3 + O(\theta^2),$$

so that in the limiting case  $|\mu|^2$  satisfies

$$(3.20) \quad |\mu|^2 = 1 - \sum_{m=1}^3 \frac{2\sigma_m}{\sigma} \theta_m^2 + \left(\sum_{m=1}^3 c_m \theta_m\right)^2 + O(\theta^4).$$

Set  $\alpha_m = 2\sigma_m/\sigma$ . Because  $\sigma > 0$ , we have  $\alpha_m \geq 0$  for  $m = 1, 2, 3$  and application of Lemma 1 immediately reveals the necessity of (3.14). In particular, if  $\alpha_m = 0$ , then the corresponding  $c_m$  must be zero too, which means that the dimension is dropped. Hence in the sufficiency part of the proof we will assume that all  $\alpha_m$  are positive and observe that for a lower dimension the proof of sufficiency goes entirely similar.

To prove sufficiency of (3.14) we proceed as follows. Write

$$(3.21) \quad \sum_{m=1}^3 c_m^* \sin \theta_m = \sum_{m=1}^2 \frac{c_m}{\sqrt{\alpha_m}} \sqrt{\alpha_m} \sin \theta_m + \frac{c_3}{\sqrt{\alpha_3}} \sqrt{\alpha_3} \left(\sum_{m=1}^2 \sigma_m^* \cos \theta_m\right) \sin \theta_3.$$

The Cauchy-Schwarz inequality then yields

$$(3.22) \quad \left(\sum_{m=1}^3 c_m^* \sin \theta_m\right)^2 \leq \left(\sum_{m=1}^3 \frac{c_m^2}{\alpha_m}\right) \cdot \left(\sum_{m=1}^2 \alpha_m \sin^2 \theta_m + \alpha_3 \left(\sum_{m=1}^2 \sigma_m^* \cos \theta_m\right)^2 \sin^2 \theta_3\right).$$

Set  $y_m = \cos \theta_m$  and invoke (3.14). Using  $\alpha_1 + \alpha_2 = 1$ , we then can write

$$(3.23) \quad \left(\sum_{m=1}^3 c_m^* \sin \theta_m\right)^2 \leq 1 - \alpha_1 y_1^2 - \alpha_2 y_2^2 + \alpha_3 (\sigma_1^* y_1 + \sigma_2^* y_2)^2 (1 - y_3^2).$$



Further, using  $\sigma^* = \sigma + 2\sigma_3(1 - y_3)$ , we have

$$(3.24) \quad \left( 1 - \sum_{m=1}^3 \sigma_m^* (1 - \cos \theta_m) \right)^2 = \frac{1}{\sigma^{*2}} (2\sigma_1 y_1 + 2\sigma_2 y_2)^2,$$

so that there remains to prove

$$(3.25) \quad |\mu|^2 \leq 1 + \frac{1}{\sigma^{*2}} (2\sigma_1 y_1 + 2\sigma_2 y_2)^2 + \alpha_3 (\sigma_1^* y_1 + \sigma_2^* y_2)^2 (1 - y_3)^2 - \alpha_1 y_1^2 - \alpha_2 y_2^2 \leq 1$$

for all  $y_m \in [-1, 1]$ ,  $m = 1, 2, 3$ . Define  $\vec{y} = (y_1, y_2)^T$  and  $Y = \alpha_3(1 - y_3^2)$ . Then the second inequality can be rewritten as

$$(3.26) \quad \vec{y}^T A \vec{y} \leq 0,$$

where  $A$  is a symmetric two-by-two matrix with the entries

$$(3.27) \quad A_{11} = \frac{4(Y+1)}{\sigma^{*2}} \sigma_1^2 - \frac{2\sigma_1}{\sigma}, \quad A_{12} = \frac{4(Y+1)}{\sigma^{*2}} \sigma_1 \sigma_2, \quad A_{22} = \frac{4(Y+1)}{\sigma^{*2}} \sigma_2^2 - \frac{2\sigma_2}{\sigma}.$$

Note that the entries do depend on  $y_3$  but not on  $\vec{y}$ . Hence it is sufficient that  $A$  is nonpositive definite for all  $y_3 \in [-1, 1]$ . Because  $A_{12} > 0$ ,  $A$  is nonpositive definite if

$$A_{11} + A_{12} \leq 0 \quad \text{and} \quad A_{22} + A_{12} \leq 0.$$

A trivial calculation shows that this is indeed the case for all  $y_3 \in [-1, 1]$ , which completes the proof of the theorem.  $\square$

Any case covered by Theorem 2 involves the Du Fort–Frankel scheme in (2.14) since  $\sigma > 0$ . We emphasize that this gives rise to curious and unexpected stability results. Substitution of  $\sigma_m, c_m$  in (3.14) shows that the critical stepsize for von Neumann stability in all cases covered by Theorem 2 is determined by

$$(3.28) \quad \tau^2 \left( \sum_{m=1}^3 \frac{q_m^2}{\epsilon_m} \sum_{l=1}^2 \frac{\epsilon_l}{h_l^2} \right) \leq 1.$$

First, we see that the vertical meshwidth  $h_3$  is absent, which is advantageous as we explained in the discussion of Theorem 1. Second, for zero velocities (the pure diffusion case) we have unconditional stability, which is in complete agreement with the unconditional stability of the Du Fort–Frankel scheme (2.19) and the Crank–Nicolson scheme (2.20). However, if a velocity is not zero, then the corresponding diffusion parameter plays a role. Surprisingly, the critical stepsize determined by (3.28) is generally smaller than the one determined by the CFL condition (3.13) and in fact can be zero.

To see this, let us first suppose that  $\epsilon_1, \epsilon_2, \epsilon_3$  are positive. Application of the Cauchy–Schwarz inequality to the CFL condition (3.13) then leads to (3.28) as follows:

$$(3.29) \quad \left( \sum_{l=1}^2 \frac{\tau |q_l|}{h_l} \right)^2 = \left( \sum_{l=1}^2 \frac{\tau |q_l| \sqrt{\epsilon_l}}{h_l \sqrt{\epsilon_l}} \right)^2 \leq \sum_{l=1}^2 \frac{\tau^2 q_l^2}{\epsilon_l} \sum_{l=1}^2 \frac{\epsilon_l}{h_l^2} \leq \sum_{m=1}^3 \frac{\tau^2 q_m^2}{\epsilon_m} \sum_{l=1}^2 \frac{\epsilon_l}{h_l^2} \leq 1.$$

Generally (3.28) appears to be more restrictive, implying a smaller critical stepsize. We consider this curious because it means, for example, that adding artificial diffusion to the advection problem can have a destabilizing effect for the time integration rather than working out stabilizing. A similar curious situation has been observed earlier in [9, 12]. Also note that if the three diffusion parameters are equal, then they cancel out in (3.28) so that the critical stepsize then even is independent of the diffusion but yet smaller than in the case of the CFL condition. Of course, the difference between the two conditions is minor if

$$(3.30) \quad \frac{|q_1|h_1}{\epsilon_1} \approx \frac{|q_2|h_2}{\epsilon_2} \quad \text{and} \quad \frac{q_3^2}{\epsilon_3} \ll \min \left( \frac{q_1^2}{\epsilon_1}, \frac{q_2^2}{\epsilon_2} \right).$$

The observation that for cases covered by Theorem 2 the critical stepsize can even be zero follows directly from inspection of (3.28). For example, if we take  $q_1, q_2, q_3 \neq 0$ ,  $\epsilon_1, \epsilon_2$  fixed, and  $\epsilon_3 \rightarrow 0$ , then  $\tau \rightarrow 0$  when satisfying the stability inequality. By also taking into account Theorem 1, we thus can formulate the following.

**THEOREM 3.** *For von Neumann stability it is necessary that either  $\epsilon_1$  and  $\epsilon_2$  are zero or positive and if they are both positive, then it is required to have  $\epsilon_3 > 0$  too.*

**4. The Du Fort–Frankel deficiency.** We will further explain this curious stability result by relating it to the well-known Du Fort–Frankel deficiency, which describes the situation that for parabolic problems this method is only conditionally convergent, in spite of its unconditional stability (see [8, Sect. 7.5]).

The necessity of (3.14) or (3.28) has been established from the asymptotic relation (3.20) where all three phase angles  $\theta_m \rightarrow 0$ . This suggests to compute for this limiting case the maximum of the absolute value of the two amplification factors  $\xi_1, \xi_2$  directly from the polynomial (3.2). Denote  $\xi_{max} = \max(|\xi_1|, |\xi_2|)$ . An elementary calculation then yields

$$(4.1) \quad \xi_{max} = 1 - \sum_{m=1}^3 \sigma_m \theta_m^2 + \frac{1}{2} \sigma \left( \sum_{m=1}^3 c_m \theta_m \right)^2 + O(\theta^3).$$

Indeed, use of Lemma 1 shows again the necessity of (3.14). However, expression (4.1) also reveals a link with the aforementioned convergence deficiency. To see this, consider the modified equation for scheme (2.14) (cf. [8, Sect. 7.5]),

$$(4.2) \quad u_t + q_1 u_x + q_2 u_y + q_3 u_z = \epsilon_1 u_{xx} + \epsilon_2 u_{yy} + \epsilon_3 u_{zz} - \frac{1}{2} \sigma \tau u_{tt}.$$

This modified equation shows the convergence deficiency through the additional term  $-\frac{1}{2} \sigma \tau u_{tt}$ . To establish the link between our stability deficiency and the convergence deficiency, it now suffices to substitute a Fourier mode into (4.2) and to compute the associated continuous amplification factor for vanishing phase angles, similar to what we did in the derivation of (4.1). We then find that the continuous amplification factor just equals (4.1), up to  $O(\theta^3)$ . Further, it then follows that the term which causes the instability, that is,

$$(4.3) \quad \frac{1}{2} \sigma \left( \sum_{m=1}^3 c_m \theta_m \right)^2,$$

originates from the deficiency term  $-\frac{1}{2} \sigma \tau u_{tt}$ , although this term itself is independent of the velocities  $c_m$ . This means that, also, the modified equation is unstable if (3.14)

is violated in the sense that it admits growing Fourier modes in the low frequency range. This obviously implies that this then also must happen for scheme (2.14) when subjected to the von Neumann stability test.

It is noteworthy that if we bound the phase angles from below, say  $\theta_m \geq \theta_0 > 0$ , then an interval  $0 < \tau \leq \tau_0$  exists for which the amplification factors  $\xi_1, \xi_2$  are strictly less than one. This follows from expression (3.18), since its real part is independent of  $\tau$  and can be made less than one by taking  $\theta_0$  sufficiently small, while the imaginary part can be made sufficiently small by taking  $\tau_0$  small enough. Hence if we consider a fixed grid, then we can always achieve stability, but of course  $\tau_0$  becomes smaller if the grid is refined.

**5. Practical considerations.** Strict von Neumann stability is known to have great practical relevance. There is no doubt that the von Neumann method is the best single technique (cf. [5]) for finding necessary conditions for stability if we are in a nonmodel situation, which in practice, of course, always happens. In this connection a natural question is, “How bad is the stability deficiency for the OELH scheme?” In other words, should we in practice consider the CFL condition (3.13) as a “practical restriction,” or should we take the more stringent condition (3.28) seriously?

Let  $\tau_{cfl}$  and  $\tau_{(3.28)}$  denote the critical stepsizes. Because the necessity of condition (3.14) shows up in the limiting case  $\theta_m \rightarrow 0$ , the maximum  $\xi_{max}$  as derived in (4.1) will be only marginally larger than one if  $\tau_{(3.28)} < \tau \leq \tau_{cfl}$ . However, there is a possibility that other critical combinations of phase angles exist, away from zero, which also lead to (3.14). Therefore, we have computed approximate values of  $\xi_{max}$  (the maximum taken over all discrete  $\theta$  values) as a function of  $\tau$  for several choices of  $\epsilon_m, q_m, h_m$ . We indeed observed other critical  $\theta$  combinations away from zero. Yet, in all tests  $\xi_{max}$  appeared to become only marginally larger than one in the stepsize range  $\tau_{(3.28)} < \tau \leq \tau_{cfl}$ , similar to the limiting case which led to (3.14).

Figure 1 shows a plot of  $\xi_{max}(\tau)$  which is typical for the tests considered. We see that the overshoot due to violating (3.28) is practically insignificant. In the interval  $\tau_{(3.28)} < \tau \leq \tau_{cfl}$  the overshoot of  $\xi_{max}(\tau)$  is  $\leq 0.001$ . However, as expected, we also see that  $\tau > \tau_{cfl}$  will quickly result in severe instability. The fact that the CFL condition should be satisfied in general, thus, also in all cases covered by Theorem 2, can be understood by computing (3.18) for special choices of the  $\theta_m$ . For example, for  $\theta_m = \frac{\pi}{2}, m = 1, 2, 3$ , we get

$$(5.1) \quad \mu = 1 - \sum_{m=1}^3 \sigma_m^* - \sum_{m=1}^3 I c_m^* = -I(c_1 + c_2),$$

which trivially yields the CFL condition (3.13) for positive  $c_1, c_2$  (cf. (3.19)).

We conclude that the more stringent condition (3.28) is only a theoretical curiosity. In actual practice it will be of little importance since the instability that will occur by violation is so small that it will not be observed in actual computation, of course, as long as the CFL condition (3.13) is satisfied. This condition is highly relevant in practice and should always be obeyed. On the other hand, violation of (3.28) will only be noticeable after an unrealistically large number of time steps. To illustrate this in actual integration, we applied the OELH integrator to the model equation (2.1), discretized on a uniform  $40 \times 40 \times 10$  grid, using periodic boundary conditions. The parameters and the grid sizes in this experiment were set to the same values as in Figure 1. Obviously,  $u \equiv 1$  is an exact solution for the test model. To study the long-term stability behavior of the OELH method, we slightly perturbed the initial

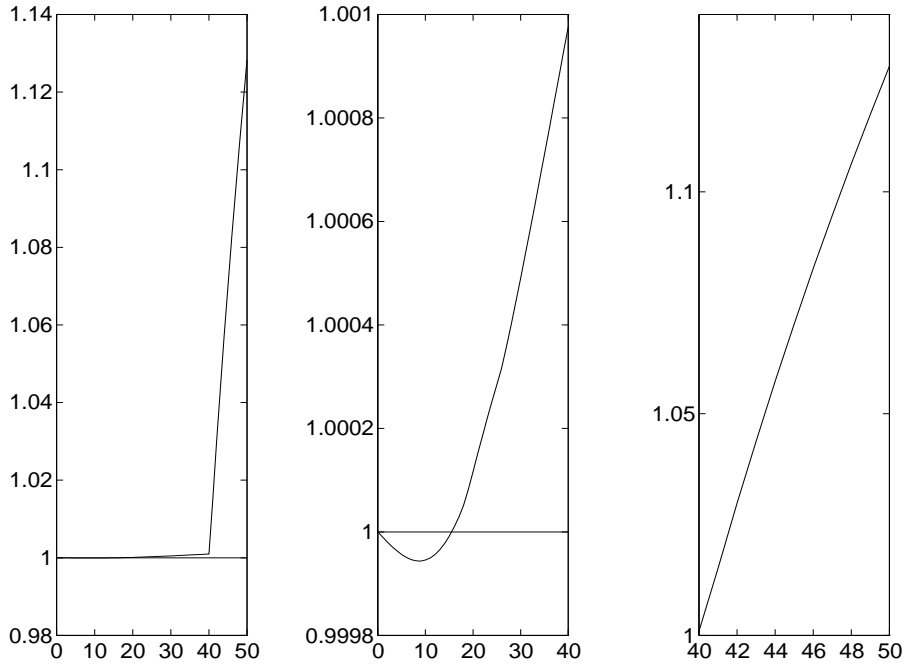


FIG. 1. Plots of  $\xi_{max}(\tau)$  for the parameters  $(\epsilon_1, \epsilon_2, \epsilon_3) = (1.0, 0.5, 0.01)$ ,  $(q_1, q_2, q_3) = (3, 2, 1)$ . The grid sizes are  $(h_1, h_2, h_3) = (200, 200, 1)$ . This yields  $\tau_{(3.28)} \approx 15.1$  and  $\tau_{cfl} = 40.0$ . The left plot covers the  $\tau$ -interval  $0 \leq \tau \leq 50$ , the middle plot  $0 \leq \tau \leq \tau_{cfl}$ , and the right plot  $\tau_{cfl} \leq \tau \leq 50$ . The middle and the right plots show a finer scale in the vertical.

TABLE 1  
Experimental amplification factors (5.2).

	$\tau = 15$	$\tau = 40$	$\tau = 40.04$
$N = 10^4$	0.659	1.52	$10^{177}$
$N = 10^5$	0.433	0.40	
$N = 10^6$	0.258	$.15 \cdot 10^9$	

condition to  $u(x, y, z) = 1.0 + \delta g(x, y, z)$ , with  $\delta = 10^{-5}$  and  $g$  a smooth function with maximum modulus equal to 1.0,

$$g(x, y, z) = \sin\left(\frac{\pi x}{40h_1}\right) \sin\left(\frac{\pi y}{40h_2}\right) \sin\left(\frac{\pi z}{10h_3}\right).$$

Table 1 contains the values of the experimental amplification factors

$$(5.2) \quad \delta^{-1} \max_{\underline{i}} |U_{\underline{i}}^N - 1|$$

for various values of  $\tau$  and  $N$ . Here  $U_{\underline{i}}^N$  denotes the numerical solution at gridpoint  $\underline{i}$  after  $N$  steps of length  $\tau$ . The results are self-evident. Violation of the CFL condition with only one promille is disastrous, whereas violation of (3.28) leads to error growth but only destroys the solution after an unrealistically large number of time steps.

Finally, it is also of interest to recall the convergence deficiency, from which the OELH scheme also suffers. Presumably, this convergence deficiency is also of little relevance for the shallow water transport application. In this application the regular

temporal and spatial truncation errors are expected to be larger than the error induced by the parasitic, nonphysical term  $\frac{1}{2}\sigma\tau u_{tt}$ . For example, in the experiments reported in [10, 11] this error plays no role. Experiments where this error is shown, though, can be found in [12].

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