

Analytical Implementation of the Ho and Lee Model for the Short Interest Rate

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Abstract

Ho and Lee introduced the first no-arbitrage model of the evolution of the short interest rate. When writing about the Ho and Lee model, other authors adopted the Black, Derman and Toy approach to implementation, namely numerical solutions and forward induction. This method of implementation is relatively complex and time-consuming when applied to cases that permit the use of an interest rate lattice. Under many assumptions, the Ho and Lee model can generate an interest rate tree. For those cases, implementation via numerical methods and forward induction appear to be impractical. In this paper we have shown how to implement the model analytically. It is relatively straightforward to identify analytical expressions for all interest rates at all dates. This method of implementation applies equally well to interest rate trees and Monte Carlo simulation.

Analytical Implementation of the Ho and Lee Model for the Short Interest Rate

Ho and Lee (1986) (HL henceforth) pioneered the use of no-arbitrage computational lattices for the evolution of the short interest rate. They were following Vasicek (1977) and Cox, Ingersoll and Ross (1985) in giving new direction to the research in modelling interest rates. Black, Derman and Toy (1990) (BDT henceforth) quickly followed HL with an innovative no-arbitrage model of interest-rate evolution. Whereas HL assumed that the interest rates are normally distributed, BDT assumed that they are log-normally distributed. These computational lattices have become very popular because of (1) their built-in ability to price exactly a given vector of bond-prices and (2) their resemblance to the binomial approach of Cox, Ross and Rubinstein (1979) which made the pricing of options a much simpler process. In addition, BDT introduced an elegant numerical, albeit search, method of implementation such that both the correct expectations (of discounted value of bonds) and variances obtain simultaneously. Rebonato (1998) states that “the procedure can be shown to be equivalent to determining the change in drift required by Girsanov’s theorem if arbitrage is to be avoided. ... This equivalent measure can differ from the real world measure by a drift transformation.”

When writing about the Ho-Lee model, other researchers adopted the numerical approach in order to extend the original lattice models of HL and BDT. (See, for example, Hull and White (1996), Jarrow and Turnbull (JT) (1996), or Ritchken(1996).) Under many realistic circumstances, the numerical approach results in binomial trees rather than lattices. These trees grow exponentially and are not practical for many problems. For those problems, Monte Carlo simulation is the preferred numerical method. The current method of calibration, namely forward induction with a search at each date for the appropriate drift term, would be extremely difficult and time consuming to implement with Monte Carlo simulation and we are not aware of any

efforts to find efficient ways around the problem. The results presented in this paper apply equally well to Monte Carlo simulation thereby expanding significantly the capacity to model realistic specifications of the evolution of interest rates.

In this paper we develop an analytical solution to the implementation of the HL model of the short interest rate.¹ This solution obviates the need for setting up the evolution as a “goal seek” math-program subject to the constraints of conditional expectations and volatility. We illustrate the method via three examples: (1) Time-varying volatility i.e., the volatility is fixed for a particular maturity. This means that the volatilities for the short-rate one period hence, for the short-rate two periods hence, for the short-rate three periods hence, and so on, are different but fixed. The volatility structure allows the volatility of the short rate to vary across short rates but to be constant for each maturity as time elapses. This specification yields a short-rate tree. This is different specification than that assumed in example 3 which yields a short-rate lattice. (2) The canonical case of constant volatility, i.e., the volatility is constant for all terms-to-maturity. This specification yields a short-rate lattice. (3) Time-varying volatility which changes as time elapses. This specification of volatility is taken from Jarrow and Turnbull (1996). They assume an evolution in which the volatility of the short rate varies across short rates but is constant for each short rate as time passes. This specification yields a short-rate lattice. In terms of difficulty of implementation, the case of a constant-volatility is the easiest. The cases of examples 1 and 3, involving time-varying volatilities, are more complex.

¹ Kijima and Nagayama (1994) may be considered to have done something similar in their development of what they call the *shift function*. Their results are derived in the continuous-time setting and then implemented for the Hull and White trinomial tree. Whether their results are generalizable and applicable in a wider context is unknown. Hull and White (1990a, 1990b) explore extensions of the Vasicek model that provide an exact fit to the initial term-structure of interest rates. When the speed of mean-reversion is set to zero, the Hull-White model reduces to the HL model. Hull and White provide two implementations of their continuous-time model, viz., trinomial-tree building and explicit finite-difference method, based on a demonstration by Brennan and Schwartz (1978). In a series of subsequent articles, they expound upon the trinomial-tree method. While their model does provide elegant mathematical tractability to subsume earlier models as special cases of the Hull-White model, they do not provide an analytical model to obviate the need for a numerical trial-and-error method for computing the values on a trinomial lattice. Hull (2000) provides software-based help in building a trinomial-tree with up to ten steps.

I. Derivation of the Analytical Implementation Method

Ritchken (1996) notes that the Ho and Lee model of the evolution of the short interest rate at time t , $r(t)$, is given in continuous time as

$$dr(t) = \mathbf{m}(t)dt + \mathbf{s}(t)dz(t) \quad \text{for } t > 0, \quad (1)$$

where $\mathbf{m}(t)$ is the drift, $\mathbf{s}(t)$ is the instantaneous volatility of the short rate, and $dz(t) \sim N(0,1)$.

This expression permits both the drift and volatility to be functions of time and it produces an instantaneous short interest rate that is normally distributed.

The corresponding expression for the change in the short interest rate over the discrete time interval Δt , i.e., for the time period $[t, t + \Delta t]$, is

$$\Delta r(t) = \mathbf{m}(t)\Delta t + \mathbf{s}(t)\Delta z(t) \quad \text{for } t \geq 0 \quad (2)$$

where $r(t)$ and $\mathbf{s}(t)$ are the short rate and the volatility of the short rate at time t for the interval from t to $t + \Delta t$ and $\Delta z(t)$ is a unit normal random variable. Without loss of generality, let $\Delta t = 1$ and let $t = 0$. We can write the evolution of the short rate as

$$\Delta r(0) \equiv r(1) - r(0) = \mathbf{m}(0) + \mathbf{s}(0)\Delta z(0). \quad (3)$$

This yields, for example,

$$r(1) = r(0) + \mathbf{m}(0) + \mathbf{s}(0)\Delta z(0).$$

$$\begin{aligned} r(2) &= r(1) + \mathbf{m}(1) + \mathbf{s}(1)\Delta z(1) \\ &= r(0) + \mathbf{m}(0) + \mathbf{s}(0)\Delta z(0) + \mathbf{m}(1) + \mathbf{s}(1)\Delta z(1) \\ &= r(0) + \{\mathbf{m}(0) + \mathbf{m}(1)\} + \{\mathbf{s}(0)\Delta z(0) + \mathbf{s}(1)\Delta z(1)\}. \end{aligned}$$

$$r(3) = r(0) + \{\mathbf{m}(0) + \mathbf{m}(1) + \mathbf{m}(2)\} + \{\mathbf{s}(0)\Delta z(0) + \mathbf{s}(1)\Delta z(1) + \mathbf{s}(2)\Delta z(2)\}.$$

In general, then,

$$\begin{aligned}
 r(t) &= r(t-1) + \mathbf{m}(t-1) + \mathbf{s}(t-1)\Delta z(t-1) \\
 &= r(0) + \sum_{j=1}^{t-1} \mathbf{m}(j-1) + \sum_{j=1}^{t-1} \mathbf{s}(j-1)\Delta z(j-1).
 \end{aligned} \tag{4}$$

Expression (4) shows that the short rate is the sum of a set of non-stochastic drift terms and a set of stochastic terms; all of the latter are normally distributed. Consequently, all short interest rates are normally distributed (albeit with changing parametric values). For example,

$$r(1) \sim N\left(r(0) + \mathbf{m}(0), \mathbf{s}^2(r(1))\right).$$

$$r(2) \sim N\left(r(0) + \mathbf{m}(0) + \mathbf{m}(1), \mathbf{s}^2(r(1) + r(2))\right).$$

$$r(3) \sim N\left(r(0) + \mathbf{m}(0) + \mathbf{m}(1) + \mathbf{m}(2), \mathbf{s}^2(r(1) + r(2) + r(3))\right).$$

In general, then,

$$r(t) \sim N\left(r(0) + \sum_{j=1}^{t-1} \mathbf{m}(j-1), \mathbf{s}^2\left(\sum_{j=1}^t r(j)\right)\right). \tag{5}$$

The inputs for a Ho and Lee no-arbitrage interest rate model in discrete time are (1) a set of known (pure) discount bond prices, $\{P(1), P(2), P(3), \dots, P(n)\}$,² and (2) the volatility (standard deviation) of future one-period short rates, $\{\mathbf{s}(0), \mathbf{s}(1), \dots, \mathbf{s}(n-1)\}$.

An evolution of the short rate that precludes arbitrage must satisfy the local expectations condition that bonds of any maturity offer the same expected rate of return in a given period. This is equivalent to the expectation of the discounted value of each bond's terminal payment

² This is a short-form notation for $P(0, T)$ where the price is given at date $t = 0$ for maturity of T periods (years). T can be designated as $T_1 = 1$, $T_2 = 2$, $T_3 = 3$, etc. The long-form notation will be useful in the later sections.

being equal to its given market value.³ Let the present values, at date 0, of a bond's terminal

$$\text{payments be given by } p(n) = \exp\left(-\sum_{j=0}^{n-1} r(j)\right).$$

Therefore, the no-arbitrage conditions will be stated as

$$P(1) = e^{-f(0)} \equiv E_0^Q [p(1)] = E_0^Q [e^{-r(0)}] = e^{-r(0)}.$$

$$P(2) = e^{-\{f(0)+f(1)\}} \equiv E_0^Q [p(2)] = E_0^Q [e^{-\{r(0)+r(1)\}}].$$

$$P(3) = e^{-\{f(0)+f(1)+f(2)\}} \equiv E_0^Q [p(3)] = E_0^Q [e^{-\{r(0)+r(1)+r(2)\}}].$$

In general, then,

$$P(n) = \exp\left(-\sum_{j=0}^{n-1} f(j)\right) \equiv E_0^Q [p(n)] = E_0^Q \left[\exp\left(-\sum_{j=0}^{n-1} r(j)\right) \right], \quad (6)$$

where $E_0^Q [\cdot]$ is the expectation at date $t=0$ under the equivalent martingale probability

distribution Q and $f(j)$ is the one-period forward rate observed at date j .

From statistics we know that if $x : N(\mathbf{m}, \mathbf{s}^2)$, then⁴

$$E[e^{-x}] = e^{-\mathbf{m} + \frac{\mathbf{s}^2}{2}}. \quad (7)$$

Therefore, for date $t=2$,

$$\begin{aligned} P(2) &= E_0^Q [e^{-\{r(0)+r(1)\}}] = e^{-r(0)} E_0^Q [e^{-r(1)}] \\ &= e^{-r(0)} e^{-E_0^Q[r(1)] + \frac{1}{2}\mathbf{s}^2(r(1))}. \end{aligned}$$

Further,

³ For example, we can illustrate that the equivalence with respect to the expected rate of return on the two-period bond from time 0 (zero) to time 1:

$$\ln \left\{ \frac{E[e^{-r(1)}]}{P_2} \right\} = r(0) \quad \text{or} \quad \frac{E[e^{-r(1)}]}{P_2} = e^{r(0)} \quad \text{or} \quad P_2 = E[e^{-r(0)-r(1)}].$$

⁴ See Mood, Graybill and Boes (1974, p. 117) for a discussion of this result.

$$\ln P(2) = -r(0) - E_0^Q[r(1)] + \frac{1}{2}\mathbf{s}^2(r(1)) \quad \text{or}$$

$$E_0^Q[r(1)] = -\ln P(2) - r(0) + \frac{1}{2}\mathbf{s}^2(r(1)).$$

We know that $\ln P(2) = -f(0) - f(1) = -r(0) - f(1)$. Therefore, upon substitution,

$$E_0^Q[r(1)] = f(1) + \frac{1}{2}\mathbf{s}^2(r(1)). \quad (8)$$

Thus, the expectation at date 0 of the short rate at date 1 is the forward rate plus a term determined by the variance, $\frac{1}{2}\mathbf{s}^2(r(1))$.

Further, applying the expectations operator to expression (4), we get a second expression for the expectation of the short rate,

$$E_0^Q[r(1)] = r(0) + \mathbf{m}(0). \quad (9)$$

From expressions (8) and (9), we get

$$\mathbf{m}(0) = f(1) - r(0) + \frac{1}{2}\mathbf{s}^2(r(1)). \quad (10)$$

Expression (10) tells us that the drift term, $\mathbf{m}(0)$, is given by the combination of two effects: (1) $f(1) - r(0)$ is the difference between the forward rate and the short rate, i.e., the short rate drifts up or down towards the forward rate. (2) $\frac{1}{2}\mathbf{s}^2(r(1))$ is a positive drift adjustment term (DAT) that is required to preclude arbitrage.⁵

Let $\mathbf{d}(t)$ denote the DAT for date t . Then,

$$\mathbf{d}(0) = \frac{1}{2}\mathbf{s}^2(r(1)). \quad (11)$$

Now we can work out the details for $t = 3$.

⁵ Boyle (1978) was the first one to point out this general result.

$$\begin{aligned} P(3) &= E_0^Q \left[e^{-\{r(0)+r(1)+r(2)\}} \right] = e^{-r(0)} E_0^Q \left[e^{-\{r(1)+r(2)\}} \right] \\ &= e^{-r(0)} e^{-E_0^Q[r(1)+r(2)] + \frac{1}{2} \mathbf{s}^2(r(1)+r(2))}. \end{aligned}$$

Further,

$$\begin{aligned} \ln P(3) &= -r(0) - E_0^Q[r(1)] - E_0^Q[r(2)] + \frac{1}{2} \mathbf{s}^2(r(1)+r(2)) \\ &= -r(0) - f(1) - \frac{1}{2} \mathbf{s}^2(r(1)) - E_0^Q[r(2)] + \frac{1}{2} \mathbf{s}^2(r(1)+r(2)) \quad \text{or} \\ E_0^Q[r(2)] &= -\ln P(3) - r(0) - f(1) + \frac{1}{2} \mathbf{s}^2(r(1)+r(2)) - \frac{1}{2} \mathbf{s}^2(r(1)). \end{aligned}$$

We know that $\ln P(3) = -f(0) - f(1) - f(2) = -r(0) - f(1) - f(2)$. Therefore, upon substitution,

$$E_0^Q[r(2)] = f(2) + \frac{1}{2} \mathbf{s}^2(r(1)+r(2)) - \frac{1}{2} \mathbf{s}^2(r(1)). \quad (12)$$

Thus, the expectation at date 0 of the short rate at date 2 is the forward rate plus a term determined by the variance, $\frac{1}{2} \mathbf{s}^2(r(1)+r(2)) - \frac{1}{2} \mathbf{s}^2(r(1))$.

Further, applying the expectations operator to expression (4), we get a second expression for the expectation of the short rate,

$$E_0^Q[r(2)] = r(0) + \mathbf{m}(0) + \mathbf{m}(1). \quad (13)$$

From expressions (12) and (13), we get

$$\mathbf{m}(1) = f(2) - r(0) - \mathbf{m}(0) + \frac{1}{2} \mathbf{s}^2(r(1)+r(2)) - \frac{1}{2} \mathbf{s}^2(r(1)).$$

Substitute expression (10) above to get:

$$\mathbf{m}(1) = f(2) - f(1) + \frac{1}{2} \mathbf{s}^2(r(1)+r(2)) - \mathbf{s}^2(r(1)). \quad (14)$$

Expression (14) tells us that the drift term, $\mathbf{m}(1)$, is given by the combination of two effects: (1) $f(2) - f(1)$ is the difference between the forward rate at date 2 and the forward rate

at date 1, i.e., the nearby forward short rate drifts up or down towards the distant forward rate.

(2) $\frac{1}{2}\mathbf{s}^2(r(1)+r(2))-\mathbf{s}^2(r(1))$ is a positive drift adjustment term (DAT) that is required to preclude arbitrage.

Let $\mathbf{d}(1)$ denote the DAT for date 1. Then,

$$\mathbf{d}(1) = \frac{1}{2}\mathbf{s}^2(r(1)+r(2)) - \mathbf{s}^2(r(1)). \quad (15)$$

If we add expressions for $\mathbf{d}(0)$ and $\mathbf{d}(1)$ (expressions (11) and (15)) we get

$$\begin{aligned} \sum_{t=0}^1 \mathbf{d}(t) &= \frac{1}{2}\mathbf{s}^2(r(1)) + \frac{1}{2}\mathbf{s}^2(r(1)+r(2)) - \mathbf{s}^2(r(1)) \\ &= \frac{1}{2}\mathbf{s}^2(r(1)+r(2)) - \frac{1}{2}\mathbf{s}^2(r(1)). \end{aligned}$$

If we add expressions for $\mathbf{m}(0)$ and $\mathbf{m}(1)$ (expressions (10) and (14)) we get

$$\begin{aligned} \mathbf{m}(0) + \mathbf{m}(1) &= f(1) - r(0) + \frac{1}{2}\mathbf{s}^2(r(1)) + f(2) - f(1) + \frac{1}{2}\mathbf{s}^2(r(1)+r(2)) - \mathbf{s}^2(r(1)) \\ &= f(2) - r(0) + \frac{1}{2}\mathbf{s}^2(r(1)+r(2)) - \frac{1}{2}\mathbf{s}^2(r(1)), \end{aligned}$$

which can be simplified to

$$\sum_{t=0}^1 \mathbf{m}(t) = f(2) - r(0) + \sum_{t=0}^1 \mathbf{d}(t). \quad (16)$$

Now we can work out the details for $t = 4$.

$$\begin{aligned} P(4) &= E_0^Q \left[e^{-\{r(0)+r(1)+r(2)+r(3)\}} \right] = e^{-r(0)} E_0^Q \left[e^{-\{r(1)+r(2)+r(3)\}} \right] \\ &= e^{-r(0)} e^{-E_0^Q[r(1)+r(2)+r(3)] + \frac{1}{2}\mathbf{s}^2(r(1)+r(2)+r(3))}. \end{aligned}$$

Further,

$$\begin{aligned}\ln P(4) &= -r(0) - E_0^Q[r(1)] - E_0^Q[r(2)] - E_0^Q[r(3)] + \frac{1}{2}\mathbf{s}^2(r(1)+r(2)+r(3)) \\ &= -r(0) - f(1) - \frac{1}{2}\mathbf{s}^2(r(1)) - f(2) - \frac{1}{2}\mathbf{s}^2(r(1)+r(2)) + \frac{1}{2}\mathbf{s}^2(r(1)) - E_0^Q[r(3)] \\ &\quad + \frac{1}{2}\mathbf{s}^2(r(1)+r(2)+r(3)) \quad \text{or}\end{aligned}$$

$$E_0^Q[r(3)] = -\ln P(4) - r(0) - f(1) - f(2) + \frac{1}{2}\mathbf{s}^2(r(1)+r(2)+r(3)) - \frac{1}{2}\mathbf{s}^2(r(1)+r(2)).$$

We know that $\ln P(4) = -f(0) - f(1) - f(2) - f(3) = -r(0) - f(1) - f(2) - f(3)$.

Therefore, upon substitution,

$$E_0^Q[r(3)] = f(3) + \frac{1}{2}\mathbf{s}^2(r(1)+r(2)+r(3)) - \frac{1}{2}\mathbf{s}^2(r(1)+r(2)). \quad (17)$$

Thus, the expectation at date 0 of the short rate at date 3 is the forward rate plus a term determined by the variance, $\frac{1}{2}\mathbf{s}^2(r(1)+r(2)+r(3)) - \frac{1}{2}\mathbf{s}^2(r(1)+r(2))$.

Further, applying the expectations operator to expression (4), we get a second expression for the expectation of the short rate,

$$E_0^Q[r(3)] = r(0) + \mathbf{m}(0) + \mathbf{m}(1) + \mathbf{m}(2). \quad (18)$$

From expressions (17) and (18), we get

$$\mathbf{m}(2) = f(3) - r(0) - \mathbf{m}(0) - \mathbf{m}(1) + \frac{1}{2}\mathbf{s}^2(r(1)+r(2)+r(3)) + \frac{1}{2}\mathbf{s}^2(r(1)+r(2)) - \frac{1}{2}\mathbf{s}^2(r(1)).$$

Substitute expressions (10) and (14) above to get:

$$\mathbf{m}(2) = f(3) - f(2) + \frac{1}{2}\mathbf{s}^2(r(1)+r(2)+r(3)) - \mathbf{s}^2(r(1)+r(2)) + \frac{1}{2}\mathbf{s}^2(r(1)). \quad (19)$$

Expression (19) tells us that the drift term, $\mathbf{m}(2)$, is given by the combination of two effects: (1) $f(3) - f(2)$ is the difference between the forward rate at date 3 and the forward rate at date 2, i.e., the second nearby forward short rate drifts up or down towards the distant forward

rate. (2) $\frac{1}{2}\mathbf{s}^2(r(1)+r(2)+r(3))-\mathbf{s}^2(r(1)+r(2))+\frac{1}{2}\mathbf{s}^2(r(1))$ is a positive drift adjustment term (DAT) that is required to preclude arbitrage.

Let $\mathbf{d}(2)$ denote the DAT for date 2. Then,

$$\mathbf{d}(2) = \frac{1}{2}\mathbf{s}^2(r(1)+r(2)+r(3)) - \mathbf{s}^2(r(1)+r(2)) + \frac{1}{2}\mathbf{s}^2(r(1)). \quad (20)$$

If we add expressions for $\mathbf{d}(0)$, $\mathbf{d}(1)$ and $\mathbf{d}(2)$ (expressions (11), (15) and (20)) we get

$$\begin{aligned} \sum_{t=0}^2 \mathbf{d}(t) &= \frac{1}{2}\mathbf{s}^2(r(1)+r(2)) - \frac{1}{2}\mathbf{s}^2(r(1)) + \frac{1}{2}\mathbf{s}^2(r(1)+r(2)+r(3)) \\ &\quad - \mathbf{s}^2(r(1)+r(2)) + \frac{1}{2}\mathbf{s}^2(r(1)) \\ &= \frac{1}{2}\mathbf{s}^2(r(1)+r(2)+r(3)) - \frac{1}{2}\mathbf{s}^2(r(1)+r(2)). \end{aligned}$$

If we add expressions for $\mathbf{m}(0)$, $\mathbf{m}(1)$ and $\mathbf{m}(2)$ (expressions (10), (14) and (19)) we get

$$\begin{aligned} \mathbf{m}(0) + \mathbf{m}(1) + \mathbf{m}(2) &= f(2) - r(0) + \frac{1}{2}\mathbf{s}^2(r(1)+r(2)) - \frac{1}{2}\mathbf{s}^2(r(1)) + f(3) - f(2) \\ &\quad + \frac{1}{2}\mathbf{s}^2(r(1)+r(2)+r(3)) - \mathbf{s}^2(r(1)+r(2)) + \frac{1}{2}\mathbf{s}^2(r(1)) \\ &= f(3) - r(0) + \frac{1}{2}\mathbf{s}^2(r(1)+r(2)+r(3)) - \frac{1}{2}\mathbf{s}^2(r(1)+r(2)), \end{aligned}$$

which can be simplified to

$$\sum_{t=0}^2 \mathbf{m}(t) = f(3) - r(0) + \sum_{t=0}^2 \mathbf{d}(t). \quad (21)$$

The results of the first three dates can be generalized for the general case of date t .

$$E_0^Q[r(t)] = f(t) + \frac{1}{2}\mathbf{s}^2\left(\sum_{j=1}^t r(j)\right) - \frac{1}{2}\mathbf{s}^2\left(\sum_{j=1}^{t-1} r(j)\right) \quad \forall 1 < t \leq T-1. \quad (22)$$

$$\mathbf{m}(0) = f(1) - r(0) + \frac{1}{2}\mathbf{s}^2(r(1)), \quad (23-a)$$

$$\mathbf{m}(1) = f(2) - f(1) + \frac{1}{2}\mathbf{s}^2\left(\sum_{j=1}^2 r(j)\right) - \mathbf{s}^2(r(1)), \quad (23-b)$$

$$\mathbf{m}(t-1) = f(t) - f(t-1) + \frac{1}{2} \mathbf{s}^2 \left(\sum_{j=1}^t r(j) \right) - \mathbf{s}^2 \left(\sum_{j=1}^{t-1} r(j) \right) + \frac{1}{2} \sum_{n=1}^{t-2} \mathbf{s}^2 \left(\sum_{j=1}^n r(j) \right) \quad \forall t \geq 3. \quad (23-c)$$

In addition,

$$\sum_{n=0}^t \mathbf{d}(n) = \frac{1}{2} \mathbf{s}^2 \left(\sum_{j=1}^{t+1} r(j) \right) - \frac{1}{2} \mathbf{s}^2 \left(\sum_{j=1}^t r(j) \right) \quad \forall t \geq 1. \quad (24)$$

$$\sum_{n=0}^t \mathbf{m}(n) = f(t+1) - r(0) + \sum_{n=1}^t \mathbf{d}(n) \quad \forall t \geq 1. \quad (25)$$

Equations (22)–(25) give the necessary recursive relations to evolve the HL no-arbitrage model of short interest rate. The inputs are the set of market prices of (pure) discount bonds and a structure of volatilities for the short rates.

The above discussion is general in the sense that it applies equally well to implementation based on the binomial models and Monte Carlo simulation. If we adopt the tree approach to depict the evolution, we would write the evolutionary equation as

$$r(t) = \begin{cases} r(t-\Delta t) + \mathbf{m}(t-\Delta t)\Delta t + \mathbf{s}(t-\Delta t)\sqrt{\Delta t} & \text{with probability } \frac{1}{2} \\ r(t-\Delta t) + \mathbf{m}(t-\Delta t)\Delta t - \mathbf{s}(t-\Delta t)\sqrt{\Delta t} & \text{with probability } \frac{1}{2}, \end{cases} \quad (26)$$

or in the case of $\Delta t = 1$,

$$r(t) = \begin{cases} r(t-1) + \mathbf{m}(t-1) + \mathbf{s}(t-1) & \text{with probability } \frac{1}{2} \\ r(t-1) + \mathbf{m}(t-1) - \mathbf{s}(t-1) & \text{with probability } \frac{1}{2}. \end{cases} \quad (27)$$

Thus, for $t = 1$,

$$\begin{aligned} r_1(1) &= r(0) + \mathbf{m}(0) + \mathbf{s}(0) & \text{with probability } \frac{1}{2}, \\ r_0(1) &= r(0) + \mathbf{m}(0) - \mathbf{s}(0) & \text{with probability } \frac{1}{2}, \end{aligned} \quad (28)$$

where $r_n(t)$ denotes the n th node at date t .⁶

⁶ If the evolution can be depicted as a lattice, then the n th node means n up-moves. On the other hand, if the evolution is depicted as a tree, then the n th node is an ordinal rank, starting with $n = 0$ at the bottom of the tree and ending with $n = t^2$ at the top of the tree at date t . Depending upon the context, one must infer whether the n th node shows n up-moves or shows the ordinal rank.

And, for $t = 2$,

$$\begin{aligned}
 r_3(2) &= r_0(1) + \mathbf{m}(1) + \mathbf{s}(1) \quad \text{with probability } \frac{1}{2} \\
 &= r(0) + \mathbf{m}(0) + \mathbf{m}(1) + \mathbf{s}(0) + \mathbf{s}(1), \\
 r_2(2) &= r_0(1) + \mathbf{m}(1) - \mathbf{s}(r(1)) \quad \text{with probability } \frac{1}{2} \\
 &= r(0) + \mathbf{m}(0) + \mathbf{m}(1) + \mathbf{s}(0) - \mathbf{s}(1).
 \end{aligned} \tag{29-a}$$

and

$$\begin{aligned}
 r_1(2) &= r_1(1) + \mathbf{m}(1) + \mathbf{s}(1) \quad \text{with probability } \frac{1}{2} \\
 &= r(0) + \mathbf{m}(0) + \mathbf{m}(1) - \mathbf{s}(0) + \mathbf{s}(1), \\
 r_0(2) &= r_1(1) + \mathbf{m}(1) - \mathbf{s}(r(1)) \quad \text{with probability } \frac{1}{2} \\
 &= r(0) + \mathbf{m}(0) + \mathbf{m}(1) - \mathbf{s}(0) - \mathbf{s}(1).
 \end{aligned} \tag{29-b}$$

The progression to the next date should be clear. See Figure 1-A for an example of a tree and Figure 2-A for an example of a lattice.

Substituting for $\sum \mathbf{m}(t)$ we can develop an alternative to the above evolutionary scheme.

This alternative may be preferable. From expressions (10) and (11), for $t = 1$,

$$\begin{aligned}
 r_1(1) &= f(1) + \mathbf{d}(0) + \mathbf{s}(0) \quad \text{with probability } \frac{1}{2}, \\
 r_0(1) &= f(1) + \mathbf{d}(0) - \mathbf{s}(0) \quad \text{with probability } \frac{1}{2},
 \end{aligned} \tag{30}$$

where, as before, $r_n(t)$ denotes the n th node at date t and $f(t)$ denotes the one-period forward rate at date t .

And, for $t = 2$,

$$\begin{aligned}
 r_3(2) &= f(2) + \mathbf{d}(0) + \mathbf{d}(1) + \mathbf{s}(0) + \mathbf{s}(1) \quad \text{with probability } \frac{1}{2}, \\
 r_2(2) &= f(2) + \mathbf{d}(0) + \mathbf{d}(1) + \mathbf{s}(0) - \mathbf{s}(1) \quad \text{with probability } \frac{1}{2}.
 \end{aligned} \tag{31-a}$$

and

$$\begin{aligned}
 r_1(2) &= f(2) + \mathbf{d}(0) + \mathbf{d}(1) - \mathbf{s}(0) + \mathbf{s}(1) \quad \text{with probability } \frac{1}{2}, \\
 r_0(2) &= f(2) + \mathbf{d}(0) + \mathbf{d}(1) - \mathbf{s}(0) - \mathbf{s}(1) \quad \text{with probability } \frac{1}{2}.
 \end{aligned} \tag{31-b}$$

The progression to the next date should be clear. See Figure 1-A for an example of a tree and Figure 2-A for an example of a lattice.

Depending on the data at hand and ease of computation, one or the other approach may be preferred. This point will become clear from the illustrations given in the next section.

Under both approaches, however, we recognize from expression (24) that we need the variances of the sums of short rates, i.e., $\mathbf{s}^2(r(1) + r(2))$, $\mathbf{s}^2(r(1) + r(2) + r(3))$, and so on.

For a quick reference, expression (4) is reproduced below:

$$r(t) = r(0) + \sum_{j=1}^{t-1} \mathbf{m}(j-1) + \sum_{j=1}^{t-1} \mathbf{s}(j-1) \Delta z(j-1). \quad (32)$$

For the ease of exposition, let the (time) indexes in the parentheses be designated as a subscript.

Then,⁷

$$\mathbf{s}^2(r_1) = \mathbf{s}^2(r_0 + \mathbf{m}_0 + \mathbf{s}_0 \Delta z_0) = \mathbf{s}^2(\mathbf{s}_0 \Delta z_0) = \mathbf{s}_0^2. \quad (33-a)$$

$$\begin{aligned} \mathbf{s}^2(r_1 + r_2) &= \mathbf{s}^2(r_0 + \mathbf{m}_0 + \mathbf{s}_0 \Delta z_0 + r_0 + \mathbf{m}_0 + \mathbf{m}_1 + \mathbf{s}_0 \Delta z_0 + \mathbf{s}_1 \Delta z_1) \\ &= \mathbf{s}^2(\mathbf{s}_0 \Delta z_0 + \mathbf{s}_0 \Delta z_0 + \mathbf{s}_1 \Delta z_1) = \mathbf{s}^2(2\mathbf{s}_0 \Delta z_0 + \mathbf{s}_1 \Delta z_1) \\ &= \mathbf{s}^2(2\mathbf{s}_0 \Delta z_0) + \mathbf{s}^2(\mathbf{s}_1 \Delta z_1) + 2\text{Cov}(2\mathbf{s}_0 \Delta z_0, \mathbf{s}_1 \Delta z_1) \\ &= 4\mathbf{s}_0^2 + \mathbf{s}_1^2. \end{aligned} \quad (33-b)$$

$$\begin{aligned} \mathbf{s}^2(r_1 + r_2 + r_3) &= \mathbf{s}^2(\mathbf{s}_0 \Delta z_0 + \mathbf{s}_0 \Delta z_0 + \mathbf{s}_1 \Delta z_1 + \mathbf{s}_0 \Delta z_0 + \mathbf{s}_1 \Delta z_1 + \mathbf{s}_2 \Delta z_2) \\ &= \mathbf{s}^2(3\mathbf{s}_0 \Delta z_0 + 2\mathbf{s}_1 \Delta z_1 + \mathbf{s}_2 \Delta z_2) \\ &= \mathbf{s}^2(3\mathbf{s}_0 \Delta z_0) + \mathbf{s}^2(2\mathbf{s}_1 \Delta z_1) + \mathbf{s}^2(\mathbf{s}_2 \Delta z_2) \\ &\quad + 2\text{Cov}(3\mathbf{s}_0 \Delta z_0, 2\mathbf{s}_1 \Delta z_1) + 2\text{Cov}(3\mathbf{s}_0 \Delta z_0, \mathbf{s}_2 \Delta z_2) \\ &\quad + 2\text{Cov}(2\mathbf{s}_1 \Delta z_1, \mathbf{s}_2 \Delta z_2) \\ &= 9\mathbf{s}_0^2 + 4\mathbf{s}_1^2 + \mathbf{s}_2^2. \end{aligned} \quad (33-c)$$

Therefore, in general,

⁷ Note the distinction between $\mathbf{s}^2(r(t))$ and $\mathbf{s}(t)$. The former is the variance of the short rate, the latter is the specified volatility structure of the short rates.

$$\begin{aligned}
 \mathbf{s}^2 \left(\sum_{j=1}^t r(j) \right) &= \mathbf{s}^2 \left(\sum_{k=1}^t (t-k+1) \mathbf{s}_{k-1} \Delta z_{k-1} \right) \\
 &= \sum_{k=1}^t (t-k+1)^2 \mathbf{s}_{k-1}^2.
 \end{aligned} \tag{34}$$

For example, expression (34) will yield for $t = 4$,

$$\mathbf{s}^2 (r_1 + r_2 + r_3 + r_4) = 16\mathbf{s}_0^2 + 9\mathbf{s}_1^2 + 4\mathbf{s}_2^2 + \mathbf{s}_3^2.$$

Implementation of expression (34) can be made easier if we use matrix notation. Let \mathbf{D}_t denote a *diagonal* $t \times t$ matrix whose elements are $a_{jj} = \mathbf{s}_{j-1}^2 \quad \forall j \leq t-1$. Let \mathbf{w}_t denote a t -dimensional column vector whose elements are the integer values of the index t in reverse order. Then, expression (34) can be written as

$$\mathbf{s}^2 \left(\sum_{j=1}^t r(j) \right) = \mathbf{w}_t^T \mathbf{D}_t \mathbf{w}_t \quad \forall t, \tag{35}$$

where T denotes transposition.

For example, expression (35) will yield for $t = 4$,

$$\begin{aligned}
 \mathbf{s}^2 \left(\sum_{j=1}^4 r_j \right) &= \mathbf{w}_4^T \mathbf{D}_4 \mathbf{w}_4 = [4 \quad 3 \quad 2 \quad 1] \begin{bmatrix} \mathbf{s}_0^2 & 0 & 0 & 0 \\ 0 & \mathbf{s}_1^2 & 0 & 0 \\ 0 & 0 & \mathbf{s}_2^2 & 0 \\ 0 & 0 & 0 & \mathbf{s}_3^2 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} \\
 &= 16\mathbf{s}_0^2 + 9\mathbf{s}_1^2 + 4\mathbf{s}_2^2 + \mathbf{s}_3^2.
 \end{aligned}$$

II. Implementation Examples

In this section we provide three examples to demonstrate the implementation. These examples differ in volatility structure assumed for evolution.⁸

The first example follows the exposition closely. The volatility of the short rate is not constant, i.e., it differs as time changes. For example, the volatility of the short rate at any given time can be $\{\mathbf{s}(0), \mathbf{s}(1), \mathbf{s}(2), \dots, \mathbf{s}(T-1)\}$ where T denotes the horizon of the analysis.

When we compute the evolution of the short interest rate as a binomial model, this example produces a short-rate tree.

The second example is the canonical Ho and Lee model where the volatility of the short rate is constant at all times. When we compute the evolution of the short interest rate as a binomial model, the canonical example produces a short-rate lattice.

Note that the assumption of constant volatility (as in HL) is not necessary for producing a lattice. Jarrow and Turnbull (1996, p. 456–459) assume non-constant volatility structure, employ a search algorithm and produce a short-rate lattice. The volatility structure allows the volatility of the short rate to vary across short rates but to be constant for each short rate as time elapses. For example, the volatility of the short rate from date 3 to date 4 can differ from the volatility of the short rate from date 4 to date 5 but those two different volatilities do not change as time elapses. The effect of this non-constant volatility structure is quite different from that of the first example. The final trees evolved in the first and third examples are quite dissimilar.

The figures show the evolution of the short interest rate as well as the satisfaction of no-arbitrage conditions. These conditions are that the bond prices are recovered at date 0 and that volatility of interest rates obtains at every date. In addition, the equality of one-period rates of returns is illustrated, thereby satisfying the interpretation of no-arbitrage as equality of local expectations. In other words, at any vertex (except those on the last date), we can calculate the expectation of the rate of return on a two-year bond, as we can for any longer-term bonds. This expectation of the rate of return should equal the short rate evolved at that vertex. If this equality does not obtain, then arbitrage profits are possible.

⁸ Annotated spreadsheets for these examples are available from authors upon request.

II.A. Time-Varying Volatility Structure

As demonstrated in Section I (see expressions (30)–(32)), a decline in the short rate followed by an increase is not equal to an increase in the short rate followed by a decline. The magnitude of the interest-rate change differs in each period. Thus, recombination of branches is not possible. Therefore, the number of nodes in the tree increases exponentially, namely, at date t the tree will have 2^t nodes. Figure I-A shows the short-rate tree in an extensive form for four dates.

--- Figure I-A goes here ---

Table 1 shows the initial data, consisting of pure discount bond prices and volatility structure, used to produce the tree shown in Figure I-B. Table 1 contains the relevant calculations of $\mathbf{d}(t)$ and $\mathbf{m}(t)$ as well.

--- Figure I-B goes here ---

Consider date $t = 2$, node $n = 0$. Here $r_0(2) = 4.664900\%$. The expectation of the value of the two-year bond is

$$\begin{aligned} E_2^Q [P(2,4)] &= \frac{1}{2}(\$0.946741 + \$0.967800) e^{-0.046649} \\ &= \$0.957271 \times 0.954422 = \$0.913641. \end{aligned}$$

The expectation of the rate of return on the two-year bond is ($R(t,T)$ denotes the rate of return at date t for the total maturity of T years):

$$E_2^Q [R(2,4)] = \ln \left(\frac{\frac{1}{2}(\$0.946741 + \$0.967800)}{\$0.913641} \right) = 4.6649\%,$$

which is the same as the short rate at the node. Thus the local expectations requirement is satisfied. This requirement holds at every node for bonds of longer terms also.

II.B. Constant Volatility

The short-rate tree produced in Figure I-A and Figure I-B is unappealing for many realistic problems because it grows exponentially. In the style of HL, if we assume that volatility is constant, we will get a lattice. Figure II-A shows the short-rate tree in an extensive form for four dates.

--- Figure II-A goes here ---

Let the volatility structure be given as $\mathbf{s}(t) = \mathbf{s}_c$ for all t . Then, from expression (34) will simplify to

$$\begin{aligned} \mathbf{s}^2 \left(\sum_{j=1}^t r(j) \right) &= \mathbf{s}^2 \left(\sum_{k=1}^t (t-k+1) \mathbf{s}_c \Delta z_{k-1} \right) \\ &= \mathbf{s}_c^2 \cdot \sum_{k=1}^t (t-k+1)^2. \end{aligned} \quad (36)$$

From expression (36), the variance of the sums of the short rates can be calculated as

$$\mathbf{s}^2(r_1) = \mathbf{s}_c^2. \quad (37-a)$$

$$\mathbf{s}^2(r_1 + r_2) = 5\mathbf{s}_c^2. \quad (37-b)$$

$$\mathbf{s}^2(r_1 + r_2 + r_3) = 14\mathbf{s}_c^2. \quad (37-c)$$

$$\mathbf{s}^2(r_1 + r_2 + r_3 + r_4) = 30\mathbf{s}_c^2. \quad (37-d)$$

Or, of course, one can use expression (35) to get the same results.

Table 2 shows the initial data, consisting of pure discount bond prices and volatility structure, used to produce the tree show in Figure II-B. Table 2 contains the relevant calculations of $\mathbf{d}(t)$ and $\mathbf{m}(t)$ as well.

--- Figure II-B goes here ---

Consider date $t = 2$, node $n = 0$. Because this is a lattice, n th node corresponds to n up-moves. Here $r_0(2) = 4.468$ 100%. The expectation of the value of the two-year bond is

$$\begin{aligned} E_2^Q [P(2,4)] &= \frac{1}{2}(\$0.942792 + \$0.975398)e^{-0.044681} \\ &= \$0.959095 \times 0.956302 = \$0.917185. \end{aligned}$$

The expectation of the rate of return on the two-year bond is

$$E_2^Q [R(2,4)] = \ln \left(\frac{\frac{1}{2}(\$0.942792 + \$0.975398)}{\$0.917185} \right) = 4.4681\%,$$

which is the same as the short rate at the node. Thus the local expectations requirement is satisfied. This requirement holds at every node for bonds of longer terms also.

II.C. Time-Varying Volatility Structure

Jarrow and Turnbull (1996, p. 456) “explain how to construct a lattice of future short interest rates”. Their explanation requires a numerical solution for the short interest rate one period ahead and forward induction. We demonstrate how the method developed in this paper can be applied to the same problem. The method, however, will need to be placed in the context of their volatility structure and its underlying mode of evolution.⁹

They assume an evolution in which the volatility of the short rate varies across short rates but is constant for each short rate as time passes. This evolution can be specified as:

$$r(1) = r(0) + \mathbf{m}(0) + \mathbf{s}(0)\Delta z(0).$$

$$\begin{aligned} r(2) &= r(1) + \mathbf{m}(1) + \mathbf{s}(1)(\Delta z(0) + \Delta z(1)) - \mathbf{s}(0)\Delta z(0) \\ &= r(0) + \mathbf{m}(0) + \mathbf{m}(1) + \mathbf{s}(1)(\Delta z(0) + \Delta z(1)). \end{aligned}$$

$$\begin{aligned} r(3) &= r(2) + \mathbf{m}(2) + \mathbf{s}(2)(\Delta z(0) + \Delta z(1) + \Delta z(2)) - \mathbf{s}(1)(\Delta z(0) + \Delta z(1)) \\ &= r(0) + \mathbf{m}(0) + \mathbf{m}(1) + \mathbf{m}(2) + \mathbf{s}(2)(\Delta z(0) + \Delta z(1) + \Delta z(2)). \end{aligned}$$

⁹ Hull (2000) provides a discussion of time-varying volatility and provide an Excel-based pedagogy-oriented software handling a limited number of models and dates. He, however, does not comment on the example or implementation method provided by Jarrow and Turnbull.

$$\begin{aligned} r(4) &= r(3) + \mathbf{m}(3) + \mathbf{s}(3)(\Delta z(0) + \Delta z(1) + \Delta z(2) + \Delta z(3)) - \mathbf{s}(1)(\Delta z(0) + \Delta z(1) + \Delta z(2)) \\ &= r(0) + \mathbf{m}(0) + \mathbf{m}(1) + \mathbf{m}(2) + \mathbf{m}(3) + \mathbf{s}(3)(\Delta z(0) + \Delta z(1) + \Delta z(2) + \Delta z(3)). \end{aligned}$$

In general, then,

$$r(t) = r(0) + \sum_{j=0}^{t-1} \mathbf{m}(j) + \mathbf{s}(t-1) \sum_{j=0}^{t-1} \Delta z(j). \quad (38)$$

Now, we need the variance of the sums of short rates. For the ease of exposition, let the (time) indexes in the parentheses be designated as a subscript. Then,

$$\mathbf{s}^2(r_1) = \mathbf{s}^2(r_0 + \mathbf{m}_0 + \mathbf{s}_0 \Delta z_0) = \mathbf{s}^2(\mathbf{s}_0 \Delta z_0) = \mathbf{s}_0^2.$$

$$\begin{aligned} \mathbf{s}^2(r_1 + r_2) &= \mathbf{s}^2(\mathbf{s}_0 \Delta z_0 + \mathbf{s}_1(\Delta z_0 + \Delta z_1)) \\ &= \mathbf{s}^2(\mathbf{s}_0 \Delta z_0) + \mathbf{s}^2(\mathbf{s}_1(\Delta z_0 + \Delta z_1)) + 2\text{Cov}(\mathbf{s}_0 \Delta z_0, \mathbf{s}_1(\Delta z_0 + \Delta z_1)) \\ &= \mathbf{s}_0^2 + 2\mathbf{s}_1^2 + 2\mathbf{s}_0 \mathbf{s}_1. \end{aligned}$$

$$\begin{aligned} \mathbf{s}^2(r_1 + r_2 + r_3) &= \mathbf{s}^2(\mathbf{s}_0 \Delta z_0 + \mathbf{s}_1(\Delta z_0 + \Delta z_1) + \mathbf{s}_2(\Delta z_0 + \Delta z_1 + \Delta z_2)) \\ &= \mathbf{s}^2(\mathbf{s}_0 \Delta z_0) + \mathbf{s}^2(\mathbf{s}_1(\Delta z_0 + \Delta z_1)) + \mathbf{s}^2(\mathbf{s}_2(\Delta z_0 + \Delta z_1 + \Delta z_2)) \\ &\quad + 2\text{Cov}(\mathbf{s}_0 \Delta z_0, \mathbf{s}_1(\Delta z_0 + \Delta z_1)) + 2\text{Cov}(\mathbf{s}_0 \Delta z_0, \mathbf{s}_2(\Delta z_0 + \Delta z_1 + \Delta z_2)) \\ &\quad + 2\text{Cov}(\mathbf{s}_1(\Delta z_0 + \Delta z_1), \mathbf{s}_2(\Delta z_0 + \Delta z_1 + \Delta z_2)) \\ &= \mathbf{s}_0^2 + 2\mathbf{s}_1^2 + 3\mathbf{s}_2^2 + 2\mathbf{s}_0 \mathbf{s}_1 + 2\mathbf{s}_0 \mathbf{s}_2 + 4\mathbf{s}_1 \mathbf{s}_2. \end{aligned}$$

Therefore, in general,

$$\mathbf{s}^2\left(\sum_{j=1}^t r_j\right) = \sum_{j=0}^{t-1} (j+1) \mathbf{s}_j^2 + \sum_{j=0}^{t-2} \sum_{k=j+1}^{t-1} 2(j+1) \mathbf{s}_j \mathbf{s}_k. \quad (39)$$

For example, expression (39) will yield for $t = 4$,

$$\begin{aligned} \mathbf{s}^2\left(\sum_{j=1}^4 r_j\right) &= \mathbf{s}_0^2 + 2\mathbf{s}_1^2 + 3\mathbf{s}_2^2 + 4\mathbf{s}_3^2 + 2\mathbf{s}_0 \mathbf{s}_1 + 2\mathbf{s}_0 \mathbf{s}_2 + 2\mathbf{s}_0 \mathbf{s}_3 \\ &\quad + 4\mathbf{s}_1 \mathbf{s}_2 + 4\mathbf{s}_1 \mathbf{s}_3 + 6\mathbf{s}_2 \mathbf{s}_3. \end{aligned}$$

Implementation of expression (39) can be made easier if we use matrix notation. Let $\mathbf{V}_{n \times n}$

denote a $n \times n$ matrix of cross-product terms, i.e., a matrix whose elements are $a_{jk} = \mathbf{s}_{j-1} \mathbf{s}_{k-1}$

$\forall j, k \geq 1$. Let $\mathbf{S}_{m \times m}$ denote a $m \times m$ the principal submatrix of the matrix $\mathbf{V}_{n \times n}$ and is formed by deleting rows and columns of $\mathbf{V}_{n \times n}$ simultaneously, e.g., row 1 and column 1; deletions always start with row 1 and column 1, thereby leaving a submatrix of southeast elements. Let \mathbf{e}_n denote a n -dimensional column vector whose elements are all equal to unity. Then, expression (39) can be written as

$$\mathbf{s}^2 \left(\sum_{j=1}^t r_j \right) = \sum_{m=1}^t \mathbf{e}_m^T \mathbf{V}_{m \times m} \mathbf{e}_m. \quad (40)$$

The evolutionary equation for this case is given as

$$\begin{aligned} r_1(1) &= f(1) + \mathbf{d}(0) + \mathbf{s}(0) \quad \text{with probability } \frac{1}{2}, \\ r_0(1) &= f(1) + \mathbf{d}(0) - \mathbf{s}(0) \quad \text{with probability } \frac{1}{2}, \end{aligned} \quad (41)$$

where, as before in the context of the lattice, $r_n(t)$ denotes the n th up-move in the short rate at date t and $f(t)$ denotes the one-period forward rate at date t .

And, for $t = 2$,

$$\begin{aligned} r_2(2) &= f(2) + \mathbf{d}(0) + \mathbf{d}(1) + 2\mathbf{s}(1) \quad \text{with conditional probability } \frac{1}{4}, \\ r_1(2) &= f(2) + \mathbf{d}(0) + \mathbf{d}(1) \quad \text{with conditional probability } \frac{1}{2}, \\ r_0(2) &= f(2) + \mathbf{d}(0) + \mathbf{d}(1) - 2\mathbf{s}(1) \quad \text{with conditional probability } \frac{1}{4}, \end{aligned} \quad (42)$$

where the probabilities represent the conditional probability from date 0. The equivalent martingale probability still remains at $\frac{1}{2}$ from the relevant vertex at the previous date.

And, for $t = 3$,

$$\begin{aligned}
 r_3(3) &= f(3) + \sum_{j=0}^2 \mathbf{d}(j) + 3\mathbf{s}(2) && \text{with conditional probability } \frac{1}{8}, \\
 r_2(3) &= f(3) + \sum_{j=0}^2 \mathbf{d}(j) + \mathbf{s}(2) && \text{with conditional probability } \frac{3}{8}, \\
 r_1(3) &= f(3) + \sum_{j=0}^2 \mathbf{d}(j) - \mathbf{s}(2) && \text{with conditional probability } \frac{3}{8}, \\
 r_0(3) &= f(3) + \sum_{j=0}^2 \mathbf{d}(j) - 3\mathbf{s}(2) && \text{with conditional probability } \frac{1}{8},
 \end{aligned} \tag{43}$$

where the interpretation of the probability is as given for the preceding expression.

The progression to the next date should be clear. Figure III-A shows the short-rate tree in an extensive form for four dates.

--- Figure III-A goes here ---

Table 3 shows the initial data, consisting of pure discount bond prices and volatility structure, used to produce the tree show in Figure III-B. These data are the same as used by Jarrow and Turnbull. Table 3 contains the relevant calculations of $\mathbf{d}(t)$ and $\mathbf{m}(t)$ as well.

--- Figure III-B goes here ---

Consider date $t = 2$, node $n = 0$. Because this is a lattice, n th node corresponds to n up-moves. Here $r_0(2) = 4.858300\%$. The expectation of the value of the two-year bond is

$$\begin{aligned}
 E_2^Q [P(2,4)] &= \frac{1}{2}(\$0.937717 + \$0.958575)e^{-0.048583} \\
 &= \$0.948146 \times 0.952578 = \$0.903183.
 \end{aligned}$$

The expectation of the rate of return on the one-year bond is

$$E_2^Q [R(2,4)] = \ln \left(\frac{\frac{1}{2}(\$0.937717 + \$0.958575)}{\$0.903183} \right) = 4.8583\%,$$

which is the same as the short rate at the node. Thus the local expectations requirement is satisfied. This requirement holds at every node for bonds of longer terms.

Conclusions

Ho and Lee's interest-rate model retains the distinction of being the first no-arbitrage model that can be calibrated to market data. One of the major short-comings, however, has been the complexity of its discrete time implementation. In general, it has required numerical methods and forward induction. In this paper we have analytically demonstrated its implementation. It is relatively straightforward to identify recursive expressions for short rates at all (nodes and) dates. Armed with a set of expressions, we can map out the entire evolution. It is advisable to remember that the objective of the paper was to demonstrate the implementation of the HL model in discrete time and not necessarily discuss the evolution of interest rates under different specifications of the volatility function. Whether the evolution will result in a tree or a lattice will depend on the volatility structure assumed for short rates. This implementation has an added advantage of being scalable, such that once the longest-maturity is known we can use matrix algebra for intermediate calculations. Lastly we reemphasize that this method of implementation applies to both binomial models and Monte Carlo simulation of interest rates.

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Table 1
Data and Intermediate Calculations for the Interest Rate Tree Shown in Figure 1-B (Non-Constant Volatility)

Maturity Years	Bond Price (at date 0)	Short Rate Volatility	Bond Yield	Forward Rate	Var (Sum (Short Rate))	Sum (Delta)	DAT = Delta	Drift	Sum (Drift)	
t	$P(t)$	$s(t)$	$y(t)$	$f(t)$	$s^2 \left(\sum_{j=1}^t r(j) \right)$	$\sum_{j=0}^{t-1} d(j)$	$d(t)$	$m(t)$	$\sum_{j=0}^{t-1} m(j)$	$E_0^Q [r(t)]$
0		1.7%		6.198 2%		0.014 450%	0.014 450%	0.424 050%	0.424 050%	6.198 200%
1	\$0.939 900	1.5%	6.198 2%	6.607 8%	0.000 289	0.054 600%	0.040 150%	1.242 650%	1.666 700%	6.622 250%
2	\$0.879 801	1.1%	6.403 0%	7.810 3%	0.001 381	0.112 050%	0.057 450%	-0.291 950%	1.374 750%	7.864 900%
3	\$0.813 700		6.872 1%	7.460 9%	0.003 622	0.178 363%	0.066 312%			7.572 950%
4	\$0.755 201		7.019 3%		0.007 189					

Notes: Bond prices are given exogenously. The face value is \$1. Bond yields are calculated as $y(t) = -(\ln P(t))/t$. All rates are annual.

Table 2
Data and Intermediate Calculations for the Interest Rate Lattice Shown in Figure 2-B (Constant Volatility: Canonical HL Model)

Maturity Years	Bond Price (at date 0)	Short Rate Volatility	Bond Yield	Forward Rate	Var (Sum (Short Rate))	Sum (Delta)	DAT = Delta	Drift	Sum (Drift)	
t	$P(t)$	$s(t)$	$y(t)$	$f(t)$	$s^2 \left(\sum_{j=1}^t r(j) \right)$	$\sum_{j=0}^{t-1} d(j)$	$d(t)$	$m(t)$	$\sum_{j=0}^{t-1} m(j)$	$E_0^Q [r(t)]$
0		1.7%		6.198 2%		0.014 450%	0.014 450%	0.424 050%	0.424 050%	6.198 200%
1	\$0.939 900	1.5%	6.198 2%	6.607 8%	0.000 289	0.057 800%	0.043 350%	1.245 850%	1.669 900%	6.622 250%
2	\$0.879 801	1.1%	6.403 0%	7.810 3%	0.001 445	0.130 050%	0.072 250%	-0.277 150%	1.392 750%	7.868 100%
3	\$0.813 700		6.872 1%	7.460 9%	0.004 046	0.231 200%	0.101 150%			7.590 950%
4	\$0.755 201		7.019 3%		0.008 670					

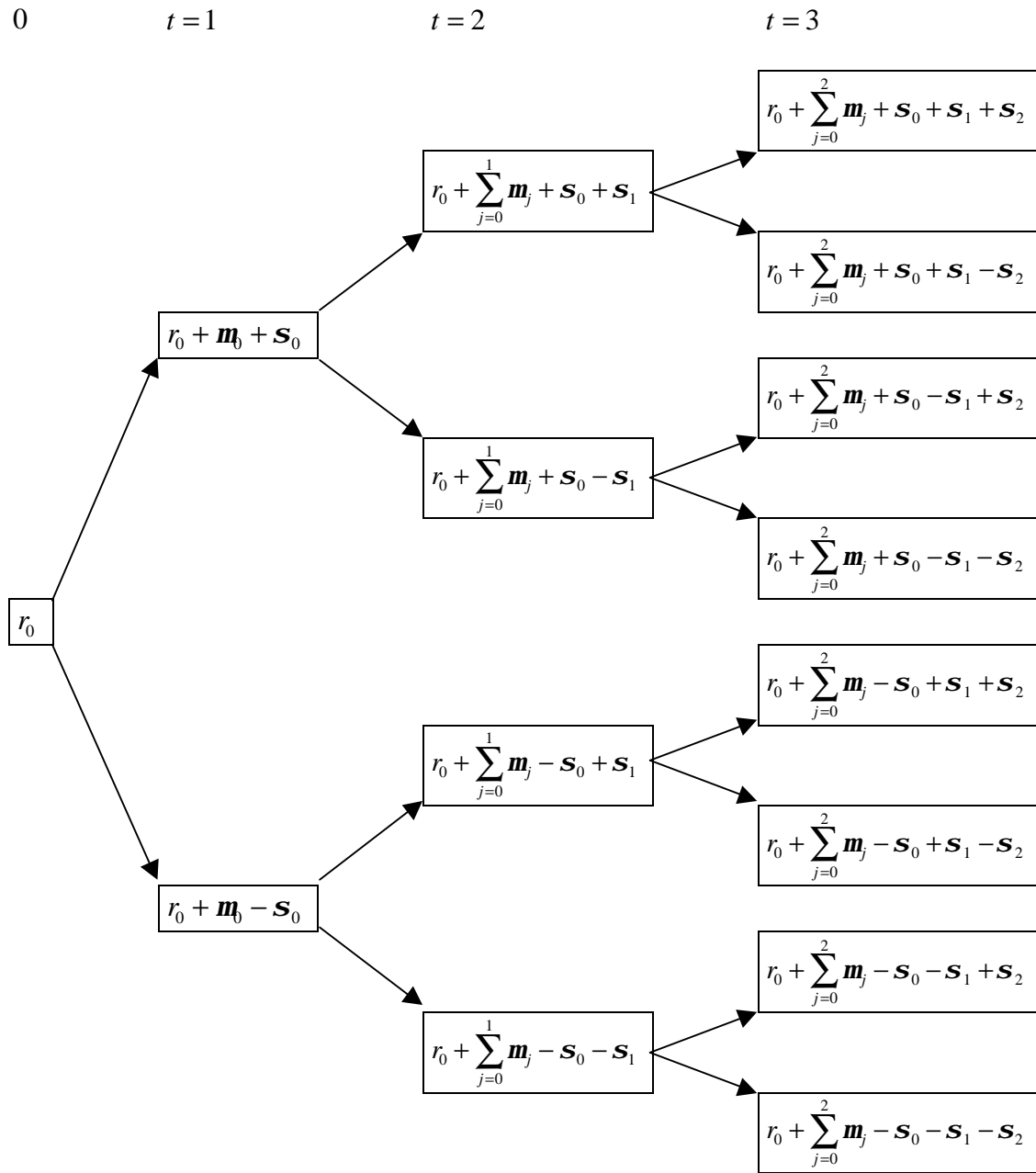
Notes: Bond prices are given exogenously. The face value is \$1. Bond yields are calculated as $y(t) = -(\ln P(t))/t$. All rates are annual.

Table 3
Data and Intermediate Calculations for the Interest Rate Lattice Shown in Figure 3-B (Jarrow-Turnbull Volatility)

Maturity Years	Bond Price (at date 0)	Short Rate Volatility	Bond Yield	Forward Rate	Var (Sum (Short Rate))	Sum (Delta)	DAT = Delta	Drift	Sum (Drift)	
t	$P(t)$	$s(t)$	$y(t)$	$f(t)$	$s^2 \left(\sum_{j=1}^t r(j) \right)$	$\sum_{j=0}^{t-1} d(j)$	$d(t)$	$m(t)$	$\sum_{j=0}^{t-1} m(j)$	$E_0^Q [r(t)]$
0		1.7%		6.198 2%		0.014 450%	0.014 450%	0.424 050%	0.424 050%	6.198 200%
1	\$0.939 900	1.5%	6.198 2%	6.607 8%	0.000 289	0.048 000%	0.033 550%	1.236 050%	1.660 100%	6.622 250%
2	\$0.879 801	1.1%	6.403 0%	7.810 3%	0.001 249	0.069 850%	0.021 850%	-0.327 550%	1.332 550%	7.858 300%
3	\$0.813 700		6.872 1%	7.460 9%	0.002 646	0.071 250%	0.001 400%			7.530 750%
4	\$0.755 201		7.019 3%		0.004 071					

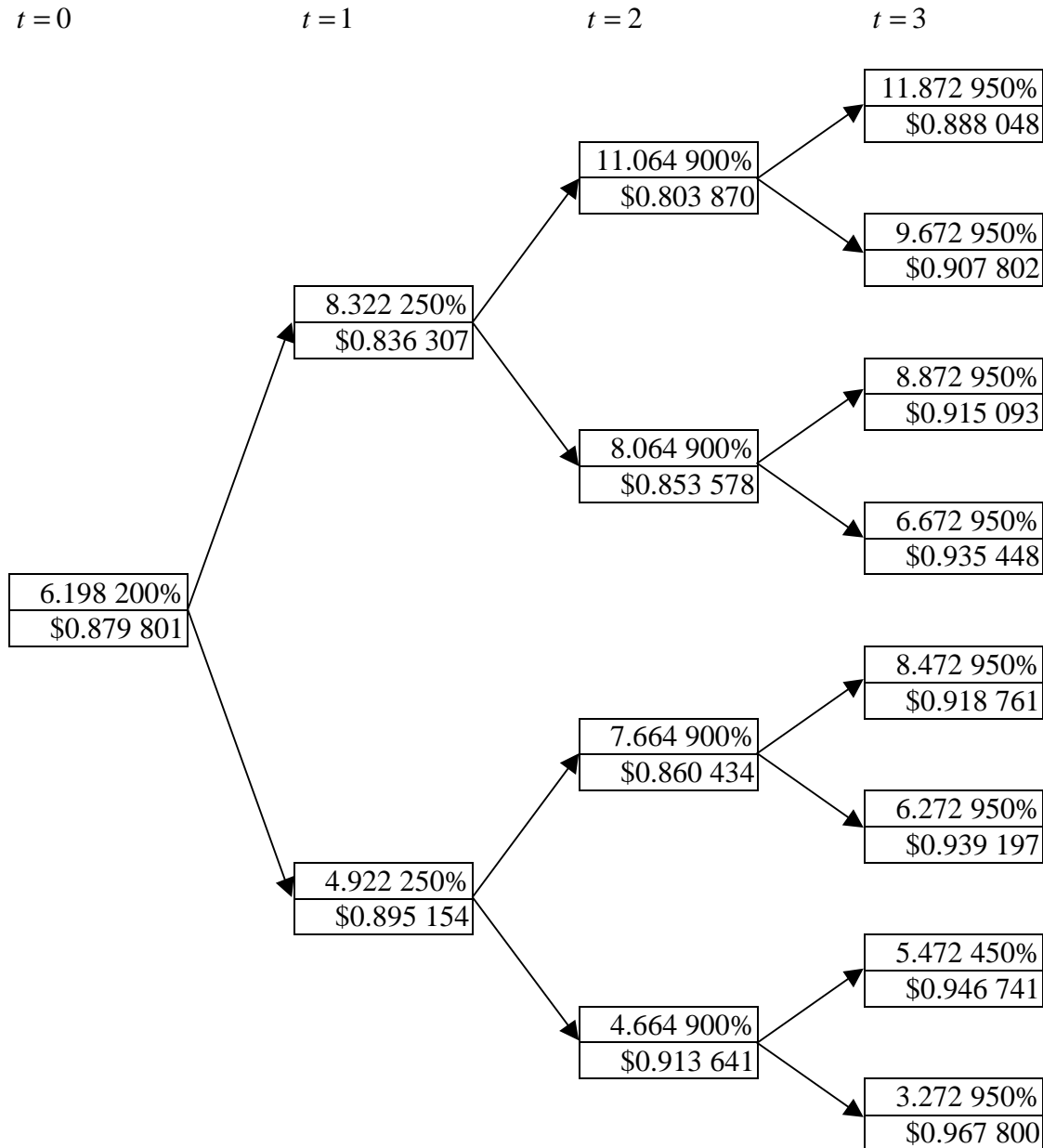
Notes: Bond prices are given exogenously. The face value is \$1. Bond yields are calculated as $y(t) = -(\ln P(t))/t$. All rates are annual.

Figure I-A
Extended-Form Short Interest Rate Tree (Non-Constant Volatility)



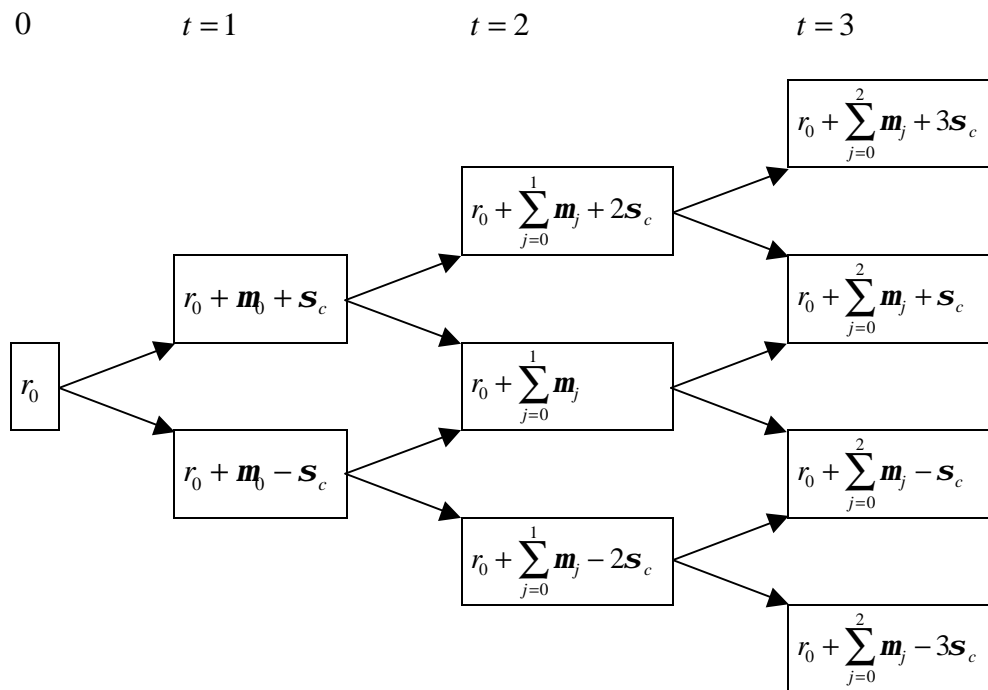
Notes: $r(t) \equiv r_t$ is the short rate at date t , $m(t) \equiv m_t$ is the drift term at date t ,
 $s(t) \equiv s_t$ is the given volatility for short-rate for date t .

Figure I-B
Numerical Short Interest Rate Tree (Non-Constant Volatility)



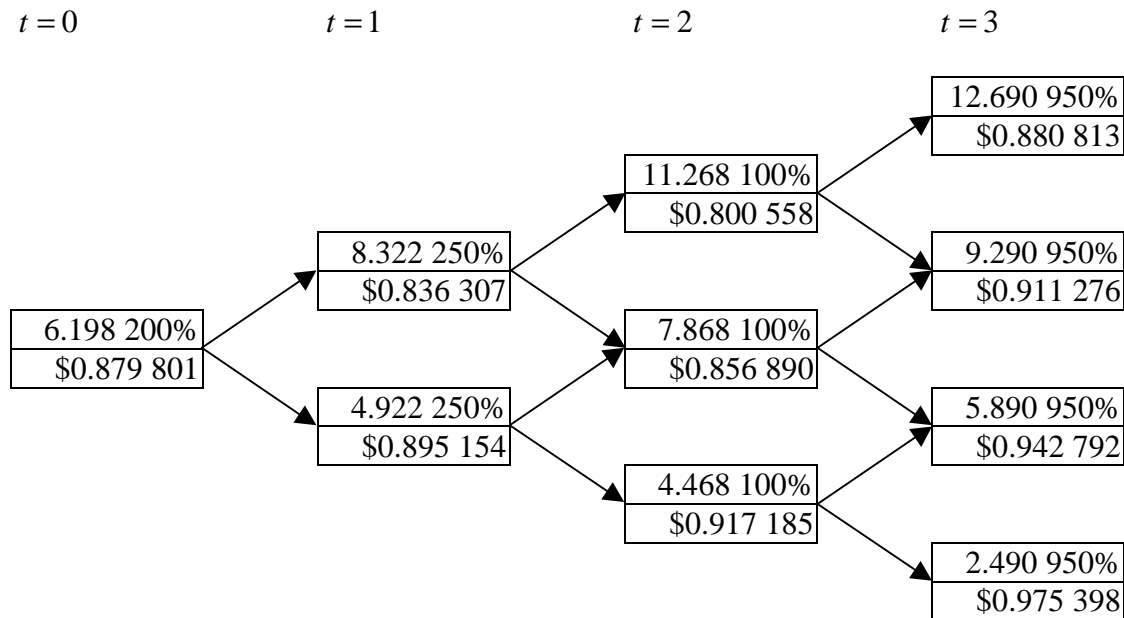
Notes: The first number is the short interest rate. The second number is the expectation of the value of a two-year bond, except at $t = 3$ where it is one-year bond.
 Data: Bond prices are given as $P(0,1) = \$0.939\ 000$, $P(0,2) = \$0.879\ 801$, $P(0,3) = \$0.813\ 700$, $P(0,4) = \$0.755\ 201$, for bonds with face value of \$1; the volatility structure is given as: $\mathbf{s}(0) \equiv \mathbf{s}_0 = 1.7\%$, $\mathbf{s}(1) \equiv \mathbf{s}_1 = 1.5\%$, $\mathbf{s}(2) \equiv \mathbf{s}_2 = 1.1\%$, $\mathbf{s}(3) \equiv \mathbf{s}_3 = 0.75\%$. Continuous compounding is used.

Figure II-A
Extended-Form Short Interest Rate Lattice (Constant Volatility)



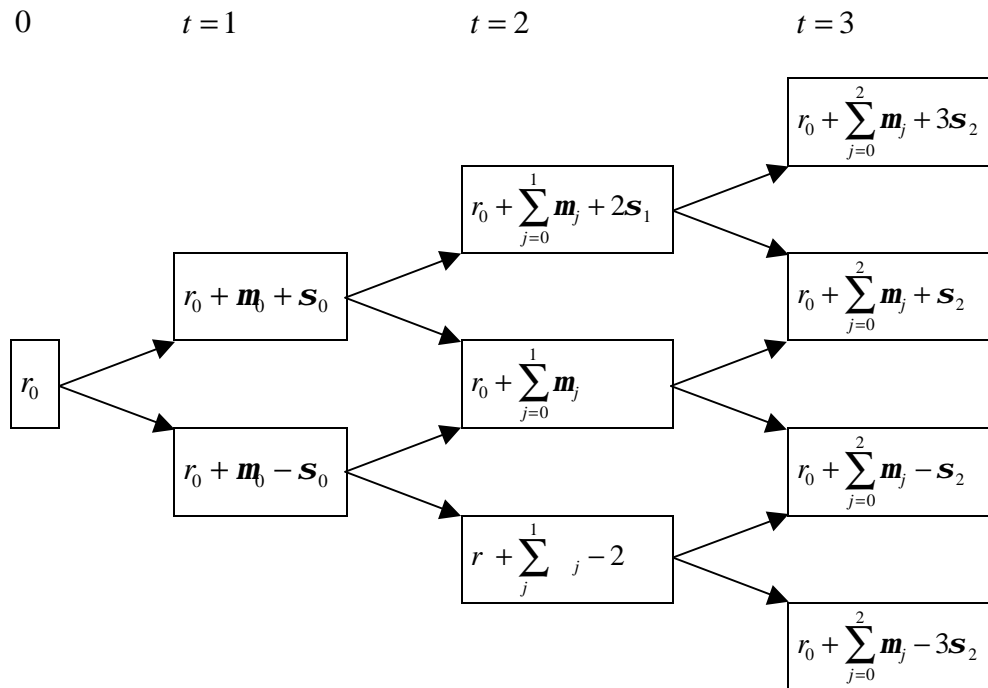
Notes: $r(t) \equiv r_t$ is the short rate at date t , $m(t) \equiv m_t$ is the drift term at date t , $s(t) \equiv s_t$ is the given volatility for short-rate for date t .

Figure II-B
Numerical Short Interest Rate Lattice (Constant Volatility)



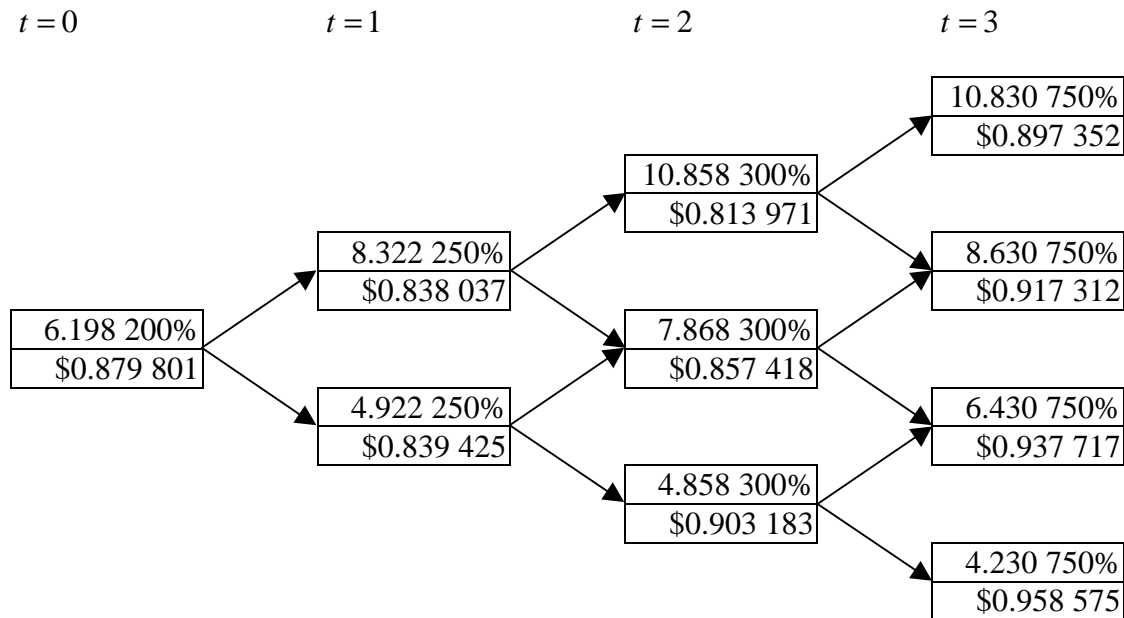
Notes: The first number is the short interest rate. The second number is the expectation of the value of a two-year bond, except at $t = 3$ where it is one-year bond.
 Data: Bond prices are given as $P(0,1) = \$0.939\ 000$, $P(0,2) = \$0.879\ 801$,
 $P(0,3) = \$0.813\ 700$, $P(0,4) = \$0.755\ 201$, for bonds with face value of \$1; the
 volatility structure is given as: $\mathbf{s}(0) \equiv \mathbf{s}_0 = 1.7\%$, $\mathbf{s}(1) \equiv \mathbf{s}_1 = 1.7\%$,
 $\mathbf{s}(2) \equiv \mathbf{s}_2 = 1.7\%$, $\mathbf{s}(3) \equiv \mathbf{s}_3 = 1.7\%$. Continuous compounding is used.

Figure III-A
Extended-Form Short Interest Rate Lattice (Jarrow-Turnbull
Volatility)



Notes: $r(t) \equiv r_t$ is the short rate at date t , $m(t) \equiv m_t$ is the drift term at date t , $s(t) \equiv s_t$ is the given volatility for short-rate for date t .

Figure III-B
Numerical Short Interest Rate Lattice (Jarrow-Turnbull Volatility)



Notes: The first number is the short interest rate. The second number is the expectation of the value of a two-year bond, except at $t = 3$ where it is one-year bond.
 Data: Bond prices are given as $P(0,1) = \$0.939\ 000$, $P(0,2) = \$0.879\ 801$,
 $P(0,3) = \$0.813\ 700$, $P(0,4) = \$0.755\ 201$, for bonds with face value of \$1; the
 volatility structure is given as: $\mathbf{s}(0) \equiv \mathbf{s}_0 = 1.7\%$, $\mathbf{s}(1) \equiv \mathbf{s}_1 = 1.5\%$,
 $\mathbf{s}(2) \equiv \mathbf{s}_2 = 1.1\%$, $\mathbf{s}(3) \equiv \mathbf{s}_3 = 0.75\%$. Continuous compounding is used.