Interest Rate Modelling

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Contents

References: Bjork (1998); Dewynne, Howison & Wilmott (1995); Baxter & Rennie (1997), Musiela & Rutkowski (1997).

Definitions

A **zero-coupon bond** (ZCB) with maturity date T is a contract that pays nothing until T and then guarantees its holder a payoff of \$1. The price of a ZCB, i.e. time-t value of \$1 at T, is written $P(t,T)$. The **yield** $Y(t,T)$ on a ZCB $P(t,T)$ is the effective compound interest paid on the bond until maturity:

$$P(t,T) = \exp\{-Y(t,T)(T-t)\}$$

The **instantaneous forward rate** $f(t,T)$ is the time-t cost of instantaneous borrowing at a future time T, i.e. the cost at time t of borrowing over the infinitesimal time interval $[T, T+\Delta T]$.

$$P(t,T) = \exp\left\{-\int_t^T f(t,s)\,ds\right\};\Rightarrow f(t,T) = -\frac{\partial}{\partial T}\log P(t,T)$$

The **spot rate** or **short rate** $r(t)$ at time t is the cost of instantaneous borrowing at time t, i.e. $r(t) = f(t,t)$. The rational expectations hypothesis states that the present value of an asset is equal to its risk-adjusted discounted expected future value, where expectations are formed based on all information till time t. This means that

$$P(t,T) = \tilde{\mathbb{E}}_t\left[\exp\left\{-\int_t^T r(s)\,ds\right\}\right]$$

Note: There is a 1-1 correspondence between bond prices, yields and forward rates. Note also that these are parametrised by 2 time-parameters: calendar time t that changes from day to day, and maturity time T which is fixed. It is empirically observed that $P(t,T)$ (and hence each of the others) is a smooth function of T but a stochastic (jagged) function of t. This corresponds to the fact that a dollar in, say, 15 months, is worth about the same today as a dollar in 14 or 16 months (smooth in T), but the price of a dollar in 15 months can change unpredictably from day to day (stochastic in t).

The **term structure of interest rates** is the name given to the fact that the interest on a bond depends on its maturity. The **yield curve** is a graphical representation of the term structure of interest rates, i.e. a plot of $Y(t,T)$ as a function of T (for fixed t). Typically, long-maturity bonds have higher yields than short-maturity bonds, corresponding to an upward-sloping yield curve. However, real yield curves can be of any shape and in practice downward-sloping ("inverted"), concave, convex and wavy yield curves have all been observed. Given the 1-1 correspondence between bond prices, yields and forward rates, the term structure of interest rates at a given date can be described using any of the following:

$$\{Y(t,T) : 0 \le T \le \tau\};\{P(t,T) : 0 \le T \le \tau\};\{f(t,T) : 0 \le T \le \tau\}$$

The spot rate $r(t)$ corresponds to the initial point of the yield curve.

*The goal of interest rate modelling is to model the term structure of interest rates***.** There are 3 main classes of interest rate models: Spot rate models specify the process driving the short end of the yield curve, and extrapolate the evolution of the entire yield curve from it through certain no-arbitrage assumptions and restrictions. At the other end of the scale, the Heath-Jarrow-Morton class of interest rate models directly models the evolution of the entire yield curve as a multidimensional stochastic process. The third type of interest rate models focus on interest rates that are directly observable in the market, such as the 6-month LIBOR, and so are called market models.

Spot Rate Models

Modelling the term structure of interest rates is equivalent to modelling an arbitrage-free family of bond prices $\{P(t,T)\}_{T\geq t}$. The idea of short rate models is to treat a bond as an interest rate derivative $P(t,T;r(t))$, specify the process driving $r(t)$, derive the dynamics of P from this, and then calibrate the parameters of the $r(t)$-process to match an observed term structure of bond prices. The following sections formalise these ideas.

Equivalent Martingale Measure (Reference Only)

Risk and Probability: Consider a traded asset whose price is V. Future values of V cannot be predicted with certainty: the best we can do is assign probabilities to various possible future values of V. The consensus set of probabilities in a market is called the market measure or objective probability measure $\mathbb{P}$. The expected rate of growth of the asset under $\mathbb{P}$ is, say, μ and reflects the market's perception of the riskiness of the asset. If r is the risk-free rate, then typically $\mu > r$ for a risky asset. Altering the probability measure $\mathbb{P}$ amounts to re-evaluating μ, the expected rate of growth of the asset.

Equivalent Martingale Measure: No-arbitrage $\Rightarrow$ there exists a probability measure $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ (i.e. $\tilde{\mathbb{P}}(A) = 0 \Leftrightarrow \mathbb{P}(A) = 0$) under which the expected rate of growth of V is r, the riskless rate. The result goes further to state that if the number of risky assets in the market is equal to the number of sources of risk, then the measure $\tilde{\mathbb{P}}$ is unique (and all assets grow at the rate r under this measure). This measure $\tilde{\mathbb{P}}$ is often called the risk-adjusted measure. It is also called the equivalent martingale measure (EMM) for the following reason: if B is the compounded value of money in a bank account, then under $\tilde{\mathbb{P}}$, the expected rate of growth of S/B is zero, i.e. S/B is a constant-expectation process under $\tilde{\mathbb{P}}$, or a $\tilde{\mathbb{P}}-$ martingale.

Result: In the context of interest rate models, no-arbitrage $\Rightarrow$ there exists a measure $\tilde{\mathbb{P}}$ equivalent to the objective measure $\mathbb{P}$ under which the expected rate of growth of a ZCB is r, i.e. $\tilde{\mathbb{E}}_t\left[dP\right] = rPdt$.

Spot Rate Models: Ideas

Short rate models specify the process driving the riskless instantaneous continuously compounded rate. Under the objective probability measure $\mathbb{P}$,

$$dr = \alpha(t,r)dt + \sigma(t,r)dW$$

The interest rate market consists of a riskless money market account $B(t) = \exp\left\{\int_0^t r(s)\,ds\right\}$ whose dynamics are governed by r, and a continuum of T-maturity bonds $P(t,T;r)$, viewed as derivatives of the short rate r. Consequently the market contains an infinite number of risky assets and only the dynamics of the risk free asset (the money market account) are exogeneously specified.

The fact that the number of risky assets exceeds the number of driving random processes has the following important consequence: *The price of a particular bond is **not** completely determined by the specification of $r(t)$ and the no-arbitrage requirement.* The reason for this is that arbitrage pricing is always a case of pricing a derivative in terms of some traded underlying assets. Interest rates are not traded assets!

For this reason, prices of bonds with different maturities will need to satisfy certain consistency conditions in order to avoid arbitrage. If the price of one particular bond is used as a benchmark, the no-arbitrage prices of all bonds can be expressed in terms of the price of the benchmark bond and the $r(t)-$process.

The Bond Price Equation

Assume the dynamics of the spot rate process are given in the objective measure $\mathbb{P}$ as

$$dr = \alpha(t,r)dt + \sigma(t,r)dW$$

and the T-maturity bond price is a smooth function of 3 real variables: $P(t,T,r)$ with $P(T,T,r) = 1$. By Ito's lemma,

$$\begin{aligned}\frac{dP}{P} &= \alpha_P dt + \sigma_P dW\\ \sigma_P &= \frac{\sigma P_r}{P};\alpha_P = \frac{P_t + \alpha P_r + \frac{1}{2}\sigma^2 P_{rr}}{P}\end{aligned}$$

Consider 2 bonds of different maturities: $P^T = P(t,T,r)$ and $P^S = P(t,S,r)$. Construct a portfolio consisting of x of the first bond and y of the second. Then the value of this portfolio is $V = xP^T + yP^S$ and

$$\frac{dV}{V} = \frac{xP^T}{V}\frac{dP^T}{P^T} + \frac{yP^S}{V}\frac{dP^S}{P^S} = u_T\frac{dP^T}{P^T} + u_S\frac{dP^S}{P^S} = (u_T\alpha_T + u_S\alpha_S)\,dt + (u_T\sigma_T + u_S\sigma_S)\,dW$$

Choose the coefficients to knock out the dW term:

$$\left.\begin{array}{r} u_T\sigma_T + u_S\sigma_S = 0\\ u_T + u_S = 1\end{array}\right\} \Rightarrow u_T = -\frac{\sigma_S}{\sigma_T - \sigma_S};u_S = \frac{\sigma_T}{\sigma_T - \sigma_S}$$

With this choice,

$$\frac{dV}{V} = (u_T\alpha_T + u_S\alpha_S)\,dt = \left(\frac{\alpha_S\sigma_T - \alpha_T\sigma_S}{\sigma_T - \sigma_S}\right)dt$$

By no-arbitrage (or uniqueness of the risk-free rate r), it follows that

$$\frac{\alpha_S\sigma_T - \alpha_T\sigma_S}{\sigma_T - \sigma_S} = r :\Rightarrow \frac{\alpha_T - r}{\sigma_T} = \frac{\alpha_S - r}{\sigma_S} = \lambda(t,r)$$

Thus in an arbitrage-free market, bonds of all maturities have the same market price of risk, which does not depend on maturity. The subscripts S and T can therefore be dropped and the equation becomes

$$\frac{\alpha_P - r}{\sigma_P} = \lambda\left(t, r\right)$$

and substituting for α_P and σ_P gives the bond pricing equation:

$$\begin{aligned} P_t + \frac{1}{2}\sigma^2 P_{rr} + \left(\alpha - \lambda\sigma\right) P_r - rP = 0 \\ P\left(T, T, r\right) = 1 \end{aligned}$$

The bond price equation is a standard PDE but the problem is that $\lambda\left(t, r\right)$ is not determined within the model. To specify a short rate model in the objective measure $\mathbb{P}$, it is therefore necessary to explicitly specify the functional form of the market price of risk. For this reason, all the established short rate models directly specify $\alpha - \lambda\sigma$, the risk-adjusted drift. Under this specification, a calculation similar to the one above shows that $\lambda_1\left(t, r\right) = 0$, i.e. the bond grows at the risk-free rate r. In other words, the spot rate model is specified in the risk-adjusted measure!

Some of the well-known spot rate models are listed below. All specify the spot rate process under the martingale measure $\tilde{\mathbb{P}}$.

Vasicek:	$dr = \left(a - br\right) dt + \sigma d\tilde{W}$
Ho-Lee:	$dr = \theta\left(t\right) dt + \sigma d\tilde{W}$
Cox-Ingersoll-Ross:	$dr = \theta\left(t\right) r dt + \sigma\left(t\right) r d\tilde{W}$
Black-Derman-Toy:	$dr = a\left(b - r\right) dt + \sigma\sqrt{r} d\tilde{W}$

Inversion of the Yield Curve

As mentioned in the previous section, spot rate models are all specified in the martingale measure $\tilde{\mathbb{P}}$. However, the parameters of these models cannot be estimated from, say, time-series observations, since these will be made under the objective measure $\mathbb{P}$. Information about the risk-adjusted drift is inferred by looking at the observed term structure of interest rates and matching parameters of the spot rate process to these observations. The motivation for this is the assumption that the market prices bonds correctly, and so the matching process will endow the spot rate model with the correct parameters.

The process of calibrating a spot rate model to match an observed term structure is called **inversion of the yield curve**. Inversion of the yield curve is particularly simple if we assume that yields are affine in the short rate, i.e.$P(t, T) = \exp\{A(t, T) - rB(t, T)\}$. Models that make this assumption are called **affine term structure models** and many established models fall under this category.

Affine Term Structure Models

A spot rate model under the risk-adjusted measure $\tilde{\mathbb{P}}$

$$dr = \mu(t, r)dt + \sigma(t, r)d\tilde{W} \tag{1}$$

corresponds to the bond price PDE

$$P_t + \frac{1}{2}\sigma^2 P_{rr} + \mu P_r - rP = 0 \qquad (2)$$
$$P(T, T, r) = 1$$

This section derives the term structure model consistent with [1] and the assumption of affine yields, i.e.

$$P(t,T) = \exp\{A(t,T) - rB(t,T)\} \qquad (3)$$

Differentiating [3] and substituting it into [2] gives

$$A_t - rB_t + \frac{1}{2}\sigma^2 B^2 - \mu B - r = 0 \qquad (4)$$
$$A(T, T) = 0 = B(T, T)$$

There is a non-trivial question about exactly which μ and σ are compatible with such a model. It can be proved that for [4] to hold for all choices of A and B, the coefficients μ and σ^2 must be affine functions of r, i.e.

$$\mu(t, r) = u(t) + rv(t)$$
$$\sigma(t, r)^2 = g(t) + rh(t) \qquad (5)$$

Substituting [5] into [4] and collecting powers of r gives the following pair of ODE's:

$$B_t = \frac{1}{2}hB^2 - vB - 1; B(T, T) = 0$$
$$A_t = uB - \frac{1}{2}gB^2; A(T, T) = 0 \qquad (6)$$

These equations can be solved to get A and B and therefore the entire term structure corresponding to the spot rate model [1]. Thus we have the following result:

If the coefficients μ and σ^2 are affine, the spot rate model [1] admits an affine term structure of the form [3], where the coefficients of the term structure satisfy the pair of ODE's [6].

Calibration of Affine Term Structure Models

Formally, the process of calibration of a given term structure model involves solving the system of equations

$$P(t, T; \zeta) = P^*(t, T)$$

where ζ is a parameter vector, P is predicted term structure of bond prices (from the model) and P^* is the observed term structure of bond prices. This is an infinite-dimensional system of equations (one for each T) and so spot rate models with finite parameter vectors like the Vasicek model can never be perfectly calibrated. Some models (e.g. Ho-Lee, generalised Vasicek, etc.) have time-dependent and therefore infinite dimensional parameters, so they can be made to fit an observed term structure.

The Ho-Lee Model

The spot rate process under $\tilde{\mathbb{P}}$ is given by

$$dr = dr = \theta(t)\, dt + \sigma d\tilde{W}$$

The corresponding affine term structure equations are

$$\begin{aligned} B_t &= -1; B(T,T) = 0 \\ A_t &= \theta(t)\, B - \frac{1}{2}\sigma^2 B^2; A(T,T) = 0 \end{aligned}$$

These can be solved to give

$$\begin{aligned} B(t,T) &= T - t \\ A(t,T) &= \frac{1}{6}\sigma^2 (T-t)^3 - \int_t^T \theta(s)(T-s)\, ds \end{aligned}$$

The Ho-Lee model has an infinite-dimensional parameter $\theta(t)$ and so it can be calibrated exactly to an observed term structure. Let the observed term structure be $\{P^*(0,T)\}_{T\geq 0}$. Then we must have

$$\begin{aligned} \log P^*(0,T) &= A(0,T) - rB(0,T) \\ &= \frac{1}{6}\sigma^2 T^3 - \int_0^T \theta(s)(T-s)\, ds - rT \end{aligned}$$

¿From the definition of forward rates (see first section),

$$f^*(0,T) = -\frac{\partial}{\partial T}\log P^*(0,T) = -\frac{1}{2}\sigma^2 T^2 + \int_0^T \theta(s)\, ds + r$$

$$\begin{aligned} &\Rightarrow \frac{\partial}{\partial T} f^*(0,T) = -\sigma^2 T - \theta(T) \\ &\Rightarrow \theta(t) = \frac{\partial}{\partial T} f^*(0,t) + \sigma^2 t \end{aligned}$$

Thus the Ho-Lee model can be exactly fitted to any given initial term structure. Bond prices under Ho-Lee are calculated as follows:

$$\begin{aligned} P(t,T) &= \exp\{A(t,T) - rB(t,T)\} \\ &= \exp\left\{\frac{1}{6}\sigma^2 (T-t)^3 - \int_t^T \left(\frac{\partial}{\partial T} f^*(0,s) + \sigma^2 s\right)(T-s)\, ds - r(T-t)\right\} \\ &= \exp\left\{\frac{1}{6}\sigma^2 (T-t)^3 - \int_t^T \sigma^2 s (T-s)\, ds - \int_t^T \left(\frac{\partial}{\partial T} f^*(0,s)\right)(T-s)\, ds - r(T-t)\right\} \\ &= \exp\left\{-\frac{1}{2}\sigma^2 t (T-t)^2 + [(s-T) f^*(0,s)]_t^T - \left(\int_0^T f^*(0,s)\, ds - \int_0^t f^*(0,s)\, ds\right) - r(T-t)\right\} \\ &= \frac{P^*(0,T)}{P^*(0,t)} \exp\left\{(f^*(0,t) - r)(T-t) - -\frac{1}{2}\sigma^2 t (T-t)^2\right\} \end{aligned}$$

Comments on Spot Rate Models

Spot rate models lead to 1-dimensional PDE's that can be solved in real-time. Theoretical properties of interest rates, such as mean-reversion (Vasicek) and positivity (CIR) can also be incorporated into the process driving the spot rate. Multifactor models (e.g. Duffie & Kan) allow the spot rate to be driven by a multidimensional Brownian motion, corresponding to several underlying macroeconomic factors (though the factors in the Duffie-Kan paper are yields of specified maturities). However, spot rate modelling has some theoretical as well as practical drawbacks.

Spot rate models derive the evolution of the entire yield curve in terms of a single 1-dimensional state (or explanatory) variable, the short rate $r(t)$. Since the yield curve exists in an infinite-dimensional space (at least in principle), it seems reasonable that spot rate modelling will work (i.e. provide a model for arbitrage-free bond prices over time) only in very special cases. The assumptions about affine term structures, a unique market price of risk for bonds of all maturities, and invertibility of the yield curve are all consequences of this fact. This also means that yields of all maturities are perfectly correlated.

On the practical side, it is not always possible to invert the observed yield curve, and even when it is, spot rate models have to be recalibrated frequently since the modelling assumptions are likely to be valid only over short time-horizons. Additionally, it has been observed (e.g. Ait-Sahalia et al.) that none of the established spot rate models provides a good fit to empirical data.

These drawbacks have motivated models where the state variable is the entire yield curve, whose evolution is driven by a multidimensional Brownian motion. Such models can be used to price products (such as caps) that derive value from imperfect correlation between different parts of the yield curve. They also, by construction, can be made to fit an arbitrary initial term structure of interest rates. Offsetting these theoretical advantages are some very real computational problems, such as exactly how to disentangle and estimate the underlying sources of randomness. This class of models, proposed by Heath, Jarrow and Morton in 1992, is studied in the following sections.

Heath-Jarrow-Morton (HJM) Term Structure Models

The HJM approach is to specify the evolution of the entire forward rate curve as a multidimensional diffusion process under the objective measure $\mathbb{P}$:

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T,\omega)\,ds + \sum_{j=1}^{n} \int \sigma_j(s,T,\omega)\,dW_j$$

or in differential form:

$$df(t,T) = \alpha(t,T,\omega)\,dt + \sum_{j=1}^{n} \sigma_j(t,T,\omega)\,dW_j$$

with $f(0,T)$ specified. This means that it is possible to exactly match any given initial term structure. More importantly, it turns out that under the risk-adjusted measure $\tilde{\mathbb{P}}$, no-arbitrage implies that the drift coefficient must be related to the diffusion coefficients by the following relation:

$$\alpha(t,T,\omega) = \sum_{j=1}^{n} \left\{ \sigma_j(t,T,\omega) \left(\int_t^T \sigma(t,s,\omega)\, ds \right) \right\}$$

Since $r(t) = f(t,t)$, every HJM model corresponds to some spot rate model. The drift restriction above means that the HJM approach leads to arbitrage-free spot rate models that are independent of market price of risk under the risk-adjusted measure. However, the drift restriction also shows that we need information about the future term structure of volatilities $\{\sigma(t,s,\omega) : t \leq s \leq T\}_{t\geq 0}$ which in general is not known at time t.

The Brownian motions W_j are assumed to be independent, i.e. $\mathbb{E}[dW_i dW_j] = \delta_{ij} dt$. The state variable ω can be any quantity that is $\mathcal{F}_t-$ adapted, i.e. can depend on information generated until time t but not afterwards. The explicit dependence of forward rate models on ω means that they can be made path-dependent: ω could be the entire history $\{f(s,T) : 0 \leq s \leq t\}$ or just the current forward rate $f(t,T)$.

The following sections describe the HJM framework in greater detail.

Mathematical Preliminaries: Martingale Pricing

A **probability space** $(\Omega, \mathcal{F}, \mathbb{P})$ consists of a state space Ω, a family $\mathcal{F}$ of (measurable subsets of Ω called) events, and a probability measure $\mathbb{P}$ that maps $\mathcal{F}$ into $[0,1]$. A **filtration** is an increasing family of event spaces: $s \leq t \Rightarrow \mathcal{F}_s \subseteq \mathcal{F}_t$. Intuitively, the filtration $\mathcal{F}_t$ represents all information upto time t. A **stochastic process** is a collection of random variables $\{X(t,\omega) : t \geq 0, \omega \in \Omega\}$. Fixed ω corresponds to a realisation of the process. A process X_t is said to be $\mathcal{F}_t-$ **adapted**, written $X_t \in \mathcal{F}_t$, if (informally) the value of X_t can be determined completely by information unto time t.

A $\mathbb{P}$-**Brownian motion** is a stochastic process $\{W_t : t \geq 0\}$ with the following properties:

- W_t is continuous and $W_0 = 0$;
- $W_t \sim N(0,t)$, i.e. normally distributed with zero mean under $\mathbb{P}$: $\mathbb{E}[W_t] = 0$, $\mathbb{E}[W_s W_t] = \min\{s,t\}$; and
- For $t > s$, $W_t - W_s \sim N(0, t-s)$ and is independent of $\mathcal{F}_s$, the history of the process till time s.

Thus Brownian motion has stationary independent increments.

A process X_t is called a $\mathbb{P}$-**martingale** if

- $\mathbb{E}[|X_t|] < \infty$; and

- $\mathbb{E}_t[X_{t+h}] = X_t$ for any $h \geq 0$, where $\mathbb{E}_t[.]$ denotes expectation under $\mathbb{P}$, conditional on $\mathcal{F}_t$.

The following results on martingales are particularly useful in the context of financial modelling:

1. **Martingale Representation Theorem:** If W_t is a $\mathbb{P}$-Brownian motion and $\sigma(t,\omega) \in \mathcal{F}_t$ then the process
$$X_t = X_0 + \int_0^t \sigma(s,\omega)\, dW_s$$
is a $\mathbb{P}$-martingale and conversely, any $\mathbb{P}$-martingale can be written as an Ito integral of an adapted process with respect to a $\mathbb{P}$-Brownian motion.

2. **Exponential Martingales:** If $dX_t = \sigma_t X_t dW_t$ and $\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^t \sigma_s^2 ds\right\}\right] < \infty$, then X_t is an exponential martingale.

Two probability measures $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are said to be **equivalent** if $\mathbb{P}(A) = 0 \Leftrightarrow \tilde{\mathbb{P}}(A) = 0$. $\tilde{\mathbb{P}}$ is said to be **absolutely continuous** with respect to $\mathbb{P}$ if there exists a non-negative random variable R such that $\tilde{\mathbb{P}}(A) = \int_{\omega \in A} R(\omega)\, d\mathbb{P}(\omega)$. The random variable R is called the density of $\tilde{\mathbb{P}}$ relative to $\mathbb{P}$, or the **Radon-Nikodym derivative** $d\tilde{\mathbb{P}}/d\mathbb{P}$. In particular the following useful property holds:
$$\tilde{\mathbb{E}}[X] = \mathbb{E}\left[\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} X\right]$$
where $\mathbb{E}$ is expectation under $\mathbb{P}$ and $\tilde{\mathbb{E}}$ is under $\tilde{\mathbb{P}}$. The measures $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are equivalent if each is absolutely continuous relative to the other.

The main result in **martingale pricing** was developed by Harrison and Pliska in 1981. Given a risky asset whose price is $X(t)$, the relative price process is defined as $Z(t) = B(t)^{-1} X(t)$ where $B(t)$ is the value of a money market account continuously compounded at the riskless rate r, i.e. $dB = rBdt$. The martingale pricing result says that a necessary and sufficient condition for the absence of arbitrage is the existence of a unique probability measure $\tilde{\mathbb{P}}$ equivalent to the market measure $\tilde{\mathbb{P}}$ under which the relative price process is a martingale. This is a very important result, so we highlight it as a slogan:

$$\boxed{\text{No-Arbitrage} \Leftrightarrow \exists \text{ Unique EMM for Relative Prices}}$$

Under the martingale measure, all assets grow at the riskless rate.

The **application to bond pricing** is as follows: Under the martingale measure, $Z(t,T) = B(t)^{-1} P(t,T)$ is a martingale. This means that

$$\begin{aligned} &\tilde{\mathbb{E}}_t Z(T,T) = Z(t,T)\,; \quad \text{i.e. } \tilde{\mathbb{E}}_t B(T)^{-1} = B(t)^{-1} P(t,T) \\ \text{i.e. } &P(t,T) = \tilde{\mathbb{E}}_t\left[e^{-\int_t^t r(s)ds}\right] \end{aligned} \tag{7}$$

The main technical trick of bond pricing is therefore to find a unique equivalent martingale measure. Equation (7) is also called the rational expectations hypothesis.

Under the market measure, suppose the bond price process is

$$dP/P = \mu dt + \sigma dW$$

The relative price process is then given by

$$dZ/Z = (\mu - r)\, dt + \sigma dW = \sigma \left\{ dW + \frac{\mu - r}{\sigma} dt \right\}$$

For dZ/Z to be a martingale, we would like $dW + \frac{\mu - r}{\sigma} dt$ to be equal to $d\tilde{W}$ under some probability measure. The conditions necessary for the existence of such an equivalent measure are contained in the **Cameron-Martin-Girsanov Theorem** which states the following**:** If W_t is a $\mathbb{P}$-Brownian motion and $\theta_t \in \mathcal{F}_t$ with $\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^t \theta_s^2 ds\right\}\right] < \infty$, then there exists a probability measure $\tilde{\mathbb{P}}$ such that

1. $\mathbb{P}$ is equivalent to $\tilde{\mathbb{P}}$;
2. $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp\left\{-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right\}$; and
3. $\tilde{W}_t = W_t + \int_0^t \theta_s ds$ is a $\tilde{\mathbb{P}}-$Brownian motion.

The converse is also true.

We now have a toolbox for HJM. The idea is to specify the evolution of forward rates, deduce the bond price process from it, and then search for a unique equivalent martingale measure.

1-Factor HJM

The general HJM framework involves a multidimensional Brownian motion. However, the special 1-dimensional case encapsulates most of the concepts of the framework. Since the transition from one to many dimensions is (in this case) fairly straightforward, we will first focus on at the 1-d case and then study the general framework.

The Forward Rate Process

The interest rate market is assumed to consist of a continuous-trading economy and a continuum of default-free zero-coupon bonds $\{P(t,T) : t \geq 0, t \leq T \leq \tau\}$. Uncertainty in the economy is characterised by the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and evolving information is represented by the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by a single $\mathbb{P}$-Brownian motion $\{W_t\}_{t\geq 0}$. Given an initial (observed) term structure of interest rates represented by the forward rate curve $\{f(0,T)\}_{T\geq 0}$, HJM models $\{f(t,T) : 0 \leq t \leq T\}$ as follows:

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T,\omega)\, ds + \int_0^t \sigma(s,T,\omega)\, dW_s \tag{8}$$

or in differential form

$$df(t,T) = \alpha(t,T,\omega)\,dt + \sigma(t,T,\omega)\,dW_t \tag{9}$$

where the coefficients α and σ are $\mathcal{F}_t-$ adapted. In this 1-factor model, yields of all maturities are perfectly correlated. This is not the case with the multifactor HJM framework discussed later. From [8], the spot rate process is given under the objective probability measure by

$$r(t) = f(0,t) + \int_0^t \alpha(s,t,\omega)\,ds + \int_0^t \sigma(s,t,\omega)\,dW_s \tag{10}$$

Bond Price SDE and Equivalent Martingale Measure

Recall that $P(t,T) = \exp\left\{-\int_t^T f(t,s)\,ds\right\}$. Using Ito's lemma and a stochastic analogue of Fubini's theorem, it can be shown that the forward rate process [8] leads to the following bond price dynamics under the objective measure $\mathbb{P}$:

$$\frac{dP(t,T)}{P(t,T)} = a(t,T)\,dW_t + \{r(t) + b(t,T)\}\,dt$$

where

$$\begin{aligned} a(t,T) &= -\int_t^T \sigma(t,s,\omega)\,ds \\ b(t,T) &= -\int_t^T \alpha(t,s,\omega)\,ds + \frac{1}{2}a(t,T)^2 \end{aligned}$$

Now consider the relative bond price process $Z(t,T) = B(t)^{-1} P(t,T)$. $Z(t,T)$ expresses the price of a ZCB in terms of a numeraire $B(t)$, i.e. it measures the performance of the risky asset relative to a risklessly growing asset. Intuitively, since no new sources of risk are introduced into the model (all noise is generated by a single Brownian motion), we expect the average risk-adjusted relative performance of a risky asset with respect to a bank account to be constant over time. In other words, we would like the relative price process to be a martingale under the risk-adjusted measure. From a modelling perspective, defining a relative price process means that the short rate does not have to be modelled explicitly.

A straightforward application of Ito's lemma shows that the SDE for the relative price process is

$$\frac{dZ(t,T)}{Z(t,T)} = a(t,T)\,dW_t + b(t,T)\,dt$$

Now we seek a martingale measure. For a given maturity date T, assume there exists a function $\lambda_T(t) \in \mathcal{F}_t$ such that

$$b(t,T) - a(t,T)\,\lambda_T(t) = 0$$

Then, as long as $a(t,T) \neq 0$,

$$\frac{dZ(t,T)}{Z(t,T)} = a(t,T)\{dW_t + \lambda_T(t)\,dt\}$$

By the Cameron-Martin-Girsanov theorem, there exists a probability measure $\tilde{\mathbb{P}}$ equivalent to the objective measure $\mathbb{P}$ under which $\tilde{W}_t = W_t + \int_0^t \lambda_T(s)\,ds$ is a $\tilde{\mathbb{P}}$-Brownian motion. Under this new measure, the SDE for Z is

$$dZ(t,T) = a(t,T)\,Z(t,T)\,d\tilde{W}_t$$

From the characterisation of exponential martingales, it follows that Z is a $\tilde{\mathbb{P}}$-martingale.

Before moving on to pricing bonds using HJM, the following sections discuss two important points.

1. **The No-Arbitrage Drift Restriction:** If the martingale measure $\tilde{\mathbb{P}}$ is unique, it must follow that λ_T is independent of maturity date T (this is similar to the situation in spot rate models), i.e.

$$\lambda_T(t) = \frac{b(t,T)}{a(t,T)} = \frac{b(t,T')}{a(t,T')} = \lambda_{T'}(t) = \lambda(t)$$

Thus $b(t,T) - a(t,T)\lambda(t) = 0$ for all T. Substituting the definitions of a and b gives

$$-\int_t^T \alpha(t,s,\omega)\,ds + \frac{1}{2}a(t,T)^2 + \left(\int_t^T \sigma(t,s,\omega)\,ds\right)\lambda(t) = 0$$

Partial differentiation with respect to T gives

$$\alpha(t,T,\omega) = \sigma(t,T,\omega)\left\{\lambda(t) + \int_t^T \sigma(t,s,\omega)\,ds\right\} \tag{11}$$

This is the restriction on the drift condition that is necessary for the absence of arbitrage. If [11] holds, the martingale measure $\tilde{\mathbb{P}}$ is unique and this condition is necessary and sufficient for the existence of an arbitrage-free complete market.

2. **Market Price of Risk:** λ_T can be interpreted as the market price of risk in the following sense: If $\mu(t,T) = r(t) + b(t,T)$, then the bond price equation in the market measure can be written as

$$\frac{dP}{P} = a\,dW + \mu\,dt$$

In this context, $\frac{\mu-r}{a}$ represents the bond's excess return over the risk-free rate, per unit volatility. This quantity is called the market price of risk. Market volatility a is an index of the riskiness of the market, so that the price of risk is proportional to $1/a$. Since $\mu - r$ is the bond's excess return, it quantifies the amount of risk inherent in the bond and hence $\frac{\mu-r}{a}$ is called the market price of risk. It is reasonable to expect that the market price of risk for a $T-$maturity bond depends on T, but in the 1-factor case, absence of arbitrage (i.e. the existence of an EMM) force bonds of all maturities to have the same market price of risk.

Bond Pricing

In the risk-adjusted measure, the market price of risk is zero. Using the no-arbitrage drift restriction (11), the spot rate process corresponding to a 1-factor HJM model is given in this measure by

$$r(t) = f(0,t) + \int_0^t \sigma(s,t,\omega)\left(\int_s^t \sigma(s,u,\omega)\,du\right)ds + \int_0^t \sigma(s,t,\omega)\,d\tilde{W}_s$$

and the corresponding bond prices are given by (7). HJM therefore has 2 crucial advantages over spot rate models:

1. **Model Specification:** Under HJM, the volatility specification determines the spot rate model uniquely. The drift is derived from the volatility via a no-arbitrage restriction. This is an advantage over spot rate models which require both the drift and the volatility terms to be explicitly specified, and therefore also involve implicit assumptions about the market price of risk.

2. **Calibration:** HJM spot rate models can be calibrated to fit any given initial term structure of interest rates by construction. This means that inversion of the yield curve is no longer necessary and thus the HJM framework can handle a much wider class of interest rate models than affine terms structures.

HJM and Spot Rate Models

Since spot rate models specify the evolution of one point on the yield curve and HJM specifies the evolution of the entire yield curve, all spot rate models are in fact special cases of appropriately chosen HJM models.

All the spot rate models discussed in previous sections can be written in the following form:

$$dr = \rho(t,r)\,d\tilde{W} + \nu(t,r)\,dt$$

Bond prices are given by (7). Thus

$$\int_t^T f(t,s)\,ds = -\log \tilde{\mathbb{E}}_t\left[\exp\left\{-\int_t^T r(s)\,ds\right\}\right] = g(r,t,T)$$

Given a spot rate model, therefore, the implied forward rate is

$$f_{\text{implied}}(t,T) = \frac{\partial}{\partial T} g(r,t,T)$$

Using Ito's lemma,

$$df_{\text{implied}}(t,T) = \rho\frac{\partial^2 g}{\partial r\partial T}d\tilde{W} + \left\{\nu\frac{\partial^2 g}{\partial r\partial T} + \frac{\partial^2 g}{\partial t\partial T} + \frac{1}{2}\rho^2\frac{\partial^3 g}{\partial r^2\partial T}\right\}dt$$

Comparing this with the HJM forward rate process under $\tilde{\mathbb{P}}$ and matching volatilities gives

$$\sigma(t,T,\omega) = \rho \frac{\partial^2 g}{\partial r \partial T}$$
$$\int_t^T \sigma(t,s,\omega)\, ds = \rho \frac{\partial g}{\partial r}$$

and so we can find an HJM model corresponding to a given spot rate model.

In particular, consider the Vasicek model:

$$dr = \sigma d\tilde{W} + (\theta - \alpha r)\, dt$$

This SDE is linear, so we can solve it like an ODE. Rearranging terms and multiplying by a factor of $e^{\alpha t}$ gives

$$d\left(e^{\alpha t} r\right) = \sigma e^{\alpha t} d\tilde{W} + \theta e^{\alpha t} dt$$

Integrating from t_0 to t and using $r(t) = r_0$ gives

$$r(t) = \frac{\theta}{\alpha} + e^{-\alpha(t-t_0)}\left(r_0 - \frac{\theta}{\alpha}\right) + \sigma e^{-\alpha t} \int_{t_0}^t e^{\alpha s} d\tilde{W}_s$$

Writing $J = \int_{t_0}^t e^{\alpha s} d\tilde{W}_s$, we have $\tilde{\mathbb{E}}[J] = 0$ and $\tilde{\mathbb{E}}[J^2] = \int_{t_0}^t e^{2\alpha s} ds = \frac{1}{2\alpha}\left(e^{2\alpha t} - e^{2\alpha t_0}\right)$ (these are standard properties of Ito integrals) and so the integral above is just a time-change of a standard Brownian motion. Thus

$$r(t) = \frac{\theta}{\alpha} + e^{-\alpha(t-t_0)}\left(r_0 - \frac{\theta}{\alpha}\right) + \sigma \tilde{W}\left(\frac{1 - e^{-2\alpha(t-t_0)}}{2\alpha}\right)$$

To find g, we need to find $\int_t^T r(s)\, ds$.

$$\int_t^T r(s)\, ds = \frac{\theta}{\alpha}(T-t) - \frac{1}{\alpha}\left(e^{-\alpha(T-t)} - 1\right)\left(r - \frac{\theta}{\alpha}\right) + \sigma \int_t^T \tilde{W}\left(\frac{1 - e^{-2\alpha(s-t)}}{2\alpha}\right) ds$$

Substituting into the definition of g, we have

$$g(r,t,T) = \frac{\theta}{\alpha}(T-t) - \frac{1}{\alpha}\left(e^{-\alpha(T-t)} - 1\right)\left(r - \frac{\theta}{\alpha}\right) - \log \tilde{\mathbb{E}}_t\left[e^{\sigma \int_t^T \tilde{W}\left(\frac{1-e^{-2\alpha(s-t)}}{2\alpha}\right) ds}\right]$$

The first and third terms are independent of r, and so

$$\frac{\partial g}{\partial r} = -\frac{1}{\alpha}\left(e^{-\alpha(T-t)} - 1\right); \frac{\partial^2 g}{\partial r^2} = e^{-\alpha(T-t)}$$

The HJM parameters corresponding to the Vasicek model are therefore

$$\sigma_{\text{HJM}}(t,T) = \sigma e^{-\alpha(T-t)}; \alpha_{\text{HJM}}(t,T) = \frac{\sigma^2}{\alpha}\left(e^{-\alpha(T-t)} - 1\right) e^{-\alpha(T-t)}$$

Path-Dependent Spot Rates

The spot rate processes studied so far assume that the coefficients of dr depend on t and r but not on the past history of r. Empirical evidence suggests that AR(1) models, of the form $r_t = \theta r_{t-1} + \varepsilon_t$, suggesting at least some path-dependence.

Under HJM, the spot rate process is (suppressing dependence on ω)

$$r(t) = f(0,t) + \int_0^t \alpha(s,t)\,ds + \int_0^t \sigma(s,t)\,dW_s$$

Using a stochastic analogue of Fubini's theorem, we can write

$$\begin{aligned}\int_0^t \sigma(s,t)\,dW_s &= \int_0^t \sigma(s,s)\,ds + \int_0^t \{\sigma(s,t) - \sigma(s,s)\}\,dW_s \\ &= \int_0^t \sigma(s,s)\,ds + \int_0^t \left\{\int_s^t \sigma_T(s,u)\,du\right\} dW_s = \int_0^t \sigma(s,s)\,ds + \int_0^t \left\{\int_0^u \sigma_T(s,u)\,dW_s\right\} du\end{aligned}$$

Similarly,

$$\int_0^t \alpha(s,t)\,ds = \int_0^t \alpha(s,s)\,ds + \int_0^t \left\{\int_0^u \alpha_T(s,u)\,ds\right\} du$$

$$f(0,t) = r_0 + \int_0^t f_T(0,u)\,du$$

Thus the spot rate process can be written in the form

$$\begin{aligned}dr &= \zeta(t)\,dt + \sigma(t,t)\,dW_t \\ r(0) &= r_0 \\ \zeta(t) &= f_T(0,t) + \alpha(t,t) + \int_0^t \alpha_T(s,t)\,ds + \int_0^t \sigma_T(s,t)\,dW_s\end{aligned}$$

The last term depends on the entire history of the Brownian motion driving r. This makes the spot rate process highly path-dependent.

Markovian HJM Models

Since the general HJM formulation leads to path-dependent models and these are harder to study analytically, this section asks under what conditions HJM gives Markovian spot rate models.

Result: Spot rates under HJM are Markovian if and only if $\sigma(t,T) = p(t)\,q(T)$.
Proof: Consider a finite time-horizon economy with trading dates in $[0,\tau]$. Writing the spot rate process under HJM

$$r(t) = f(0,t) + \int_0^t \alpha(s,t)\,ds + \int_0^t \sigma(s,t)\,d\tilde{W}_s$$

it follows that spot rates are Markovian if and only if $D_t = \int_0^t \sigma(s,t)\, d\tilde{W}_s$ is Markovian, i.e. $\tilde{\mathbb{E}}_{\mathcal{F}(D_t)}[h(D_T)] = \tilde{\mathbb{E}}_{D_t}[h(D_T)]$ for any bounded measurable function h. Note that

$$\begin{aligned} D_T &= \int_0^T \sigma(s,T)\, d\tilde{W}_s = D_t + \int_0^T \sigma(s,T)\, d\tilde{W}_s - \int_0^t \sigma(s,t)\, d\tilde{W}_s \\ &= D_t + \int_t^T \sigma(s,T)\, d\tilde{W}_s + \left\{ \int_0^t \sigma(s,T)\, d\tilde{W}_s - \int_0^t \sigma(s,t)\, d\tilde{W}_s \right\} \end{aligned}$$

Observe that D_t has the Markov property if and only if the term in brackets is zero, i.e. $J(t,T) = \int_0^t \sigma(s,T)\, d\tilde{W}_s$ is completely determined by D_t. Treating them as random variables, this means they are perfectly correlated and initialised to zero. From the definition of perfect correlation, this means that $\tilde{\mathbb{E}}[J(t,T)\, D_t]^2 = \tilde{\mathbb{E}}\left[J(t,T)^2\right]\tilde{\mathbb{E}}[D_t^2]$. Since these are Ito integrals, this amounts to

$$\left(\int_0^t \sigma(s,t)\,\sigma(s,T)\, ds\right)^2 = \left(\int_0^t |\sigma(s,t)|^2\, ds\right)\left(\int_0^t |\sigma(s,T)|^2\, ds\right)$$

Written in terms of the standard inner product on $L^2(0,t)$, this just says that

$$\langle \sigma(s,t), \sigma(s,T)\rangle = ||\sigma(s,t)||_2\, ||\sigma(s,T)||_2$$

This means that for any $t \in [0,T]$ and $T \in [0,\tau]$, the function $\sigma(\cdot,t)$ lies in the linear span of the function $\sigma(\cdot,T)$. Thus $\sigma(t,T) = q(T)\,\sigma(t,\tau)\ \forall t \in [0,T]$. Choosing $p(t) = \sigma(t,\tau)$ completes the proof.

The General HJM Framework

While 1-factor HJM is considerably more general than spot rate modelling, it still implies that yield curve movements are perfectly correlated across maturities. Additionally, the no-arbitrage restriction between drift and volatility essentially comes from the same restriction in spot rate modelling, i.e. that bonds of all maturities have the same market price of risk. An obvious generalisation of the 1-factor version is to allow forward rates to be driven by several independent Brownian motions. This is the crux of the general HJM framework.

The interest rate market is assumed to consist of a continuous-trading economy with trading dates $t \in [0,\tau]$. Default-free bonds of all maturities $\{P(t,T) : t \in [0,T], T \in [0,\tau]\}$ trade in this market and uncertainty is characterised by the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathcal{F}$ is the information generated by n independent Brownian motions $\mathbf{W}(t) = \{(W_1(t), ..., W_n(t)) : 0 \le t \le \tau\}$. Information generated till time t is denoted by $\mathcal{F}_t$.

Given an initial forward rate curve $\{f(0,T) : T \in [0,\tau]\}$, HJM specifies the evolution of each forward rate under the objective measure $\mathbb{P}$ as

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T,\omega)\, ds + \sum_{i=1}^n \int_0^t \sigma_i(s,T,\omega)\, dW_i(s)$$

It can be shown as before (modulo certain technical conditions) that bond price dynamics under the objective or market measure $\mathbb{P}$ are given by

$$\frac{dP(t,T)}{P(t,T)} = \sum_{i=1}^{n} a_i(t,T)\, dW_i(t) + \{r(t) + b(t,T)\}\, dt$$

where

$$\begin{aligned} a_i(t,T) &= -\int_t^T \sigma_i(t,s,\omega)\, ds \\ b(t,T) &= -\int_t^T \alpha(t,s,\omega)\, ds + \frac{1}{2}\sum_{i=1}^{n} a_i(t,T)^2 \end{aligned}$$

The SDE for the relative bond price process $Z(t,T) = B(t)^{-1} P(t,T)$ is therefore

$$\frac{dZ(t,T)}{Z(t,T)} = \sum_{i=1}^{n} a_i(t,T)\, dW_i(t) + b(t,T)\, dt$$

Now we seek a martingale measure. As before, uniqueness of the equivalent martingale measure (which is equivalent to the absence of arbitrage) leads to a no-arbitrage relationship between the drift and volatility terms based on a unique market price of risk. The analysis is more general than but almost exactly similar to the 1-factor version.

Choose n bonds of arbitrary maturities $T_1, ..., T_{n.}$ We seek a unique measure $\tilde{\mathbb{P}}$ that is equivalent to the market measure $\mathbb{P}$ under which

$$\frac{dZ(t,T_j)}{Z(t,T_j)} = \sum_{i=1}^{n} a_i(t,T_j)\, dW_i(t) + b(t,T_j)\, dt$$

is a martingale for $i, j = 1, ..., n$, i.e. has zero drift (see characterisation of exponential martingales). Writing

$$\begin{aligned} \frac{dZ(t,T)}{Z(t,T)} &= \sum_{i=1}^{n} \left[a_i(t,T_j) \left\{ dW_i(t) + \frac{b(t,T_j)}{\sum_{i=1}^{n} a_i(t,T_j)} dt \right\} \right] \\ &\overset{\text{want}}{=} \sum_{i=1}^{n} \left[a_i(t,T_j) \{ dW_i(t) + \lambda_i(t; T_1, ..., T_n)\, dt \} \right] \end{aligned}$$

it can be seen that the existence of such a measure is equivalent to the existence of a solution $\lambda_i(\cdot; T_1, ..., T_n)$ to the system of simultaneous equations

$$\begin{bmatrix} b(t,T_1) \\ \vdots \\ b(t,T_n) \end{bmatrix} - \begin{bmatrix} a_1(t,T_1) & \cdots & a_n(t,T_1) \\ \vdots & & \vdots \\ a_1(t,T_n) & \cdots & a_n(t,T_n) \end{bmatrix} \begin{bmatrix} \lambda_1(t; T_1, ..., T_n) \\ \vdots \\ \lambda_n(t; T_1, ..., T_n) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

A solution exists as long as the matrix $A(t) = [a_i(t,T_j)]_{i,j}$ is non-singular. Nonsingularity implies that the Girsanov theorem can be used and the equivalent martingale measure is well-defined. As before, $\lambda_i(\cdot; T_1, ..., T_n)$ can be interpreted as the market price of risk associated with the $i-$th Brownian motion. To ensure uniqueness of the martingale measure, the market price of risk must be independent of the choice of maturity, i.e. for each volatility factor, there must exist a market price of risk $\lambda_i(t)$ which is independent of the vector of bonds chosen to define it. In other words, we must have $b(t,T) = \sum_{i=1}^{n} a_i(t,T)\lambda_i(t)$. Substituting the definitions of a and b into this equation gives

$$\alpha(t,T,\omega) = \sum_{i=1}^{n} \sigma_i(t,T,\omega) \left\{ \lambda_i(t) + \int_t^T \sigma_i(t,s,\omega)\, ds \right\} \tag{12}$$

Equation (12) is called the standard finance condition and ensures the absence of arbitrage.

To recap, the general HJM approach is to take as given an initial forward rate curve and specify its evolution as a stochastic process driven by several independent sources of noise. The corresponding family of predicted bond prices is arbitrage-free if and only if there exists an equivalent martingale measure. The existence of this measure corresponds to two facts:

1. Market price of risk is a vector independent of maturity. Each entry of the vector represents the market price of risk corresponding to one Brownian motion. Existence of this vector ensures the existence of a martingale measure. Independence of maturity makes the martingale measure unique.

2. The is a no-arbitrage drift restriction (12) is just a restatement of the condition that ensures uniqueness of the equivalent martingale measure.

Under the risk-adjusted measure $\tilde{\mathbb{P}}$, the spot rate process is given by

$$r(t) = f(0,t) + \sum_{i=1}^{n} \int_0^t \sigma_i(s,t,\omega) \left(\int_s^t \sigma_i(s,u,\omega)\, du \right) ds + \sum_{i=1}^{n} \int_0^t \sigma_i(s,t,\omega)\, d\tilde{W}_i(s)$$

As in the 1-factor case, the spot rate process is completely driven by the volatility specification. The market prices of risk are not present in this equation. Rational expectations can now be used to price bonds. Multifactor versions of all results about path-dependence and the conditions under which spot rates are state-dependent hold in the general HJM setting.

Comments on HJM

1. HJM is a framework, not a model. Every volatility specification corresponds to a specific model under the HJM framework.

2. Volatility drives everything in an HJM world. Given a term structure of volatilities, the corresponding arbitrage-free drift is uniquely determined via the standard finance condition (12), which is just a restatement of the condition necessary and sufficient for the existence of an equivalent martingale measure.

3. Under HJM, the risk-adjusted spot rate process can be specified uniquely without any assumptions about the market price of risk. Intuitively, this is because HJM uses the entire yield curve as the state variable, thereby utilising more market information. The market price of risk drops out of the spot rate expression because HJM incorporates an arbitrage-free initial term structure as the initial data.

4. Another consequence of modelling the evolution of the entire yield curve is that HJM models can be chosen to fit any initial observed term structure of interest rates by construction. This means HJM is a much wider class of models than, say, affine term structures.

5. In the general HJM framework, the yield curve evolution is driven by several sources of noise. This means that HJM can allow for imperfect correlation between movements of different parts of the yield curve, and can thus be used to price instruments (such as interest rate caps) which derive value from this imperfect correlation.

6. Spot rates derived from the HJM framework are strongly path-dependent, so that PDE formulation is possible only in very special cases (e.g. Markovian spot rate processes). This means that implementing HJM can be computationally demanding.

7. Estimating HJM models is a real problem, because it is typically impossible to perfectly disentangle n independent Brownian motions and then estimate the term structure of volatility for each of them. 3-factor models are usually the most complex ones used in practice.

8. Extensions to HJM include models that treat the yield curve as a random string and model its evolution as a Brownian sheet. This corresponds, roughly, to having infinitely many stochastic factors driving the yield curve. Another class of models (within the HJM framework) are the so-called market models that specify the evolution of swap rates that can be directly observed in the market.