Heath-Jarrow-Morton: a simple implementation in C

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1 Heath-Jarrow-Morton [1992]

1.1 The Model

The trajectory of the forward rate is given by the following SDE:

$$r_t(T-t) - r_s(T-s) = \int_s^t \mu(\tau,T)d\tau + \int_s^t \sigma(\tau,T)dW(\tau) \tag{1}$$

with $\sigma(\tau,T)^\top = \begin{pmatrix} \sigma_1(\tau,T) \\ \vdots \\ \sigma_n(\tau,T) \\ \vdots \\ \sigma_N(\tau,T) \end{pmatrix}$ for risk sources, $W(\tau) = \begin{pmatrix} W_1(\tau) \\ \vdots \\ W_n(\tau) \\ \vdots \\ W_N(\tau) \end{pmatrix}$ an N-dimensional $\mathbb{P} - Wiener$ process with independence property $\mathbb{E}(W(\tau)W(\tau)^\top) = I_N\tau$. Then, the dynamics of the pure discount rate $r_t(0)$ are such that:

$$\begin{cases} r_t(0) = r_s(t-s) + \int_s^t \mu(\tau,t)d\tau + \int_s^t \sigma(\tau,t)dW(\tau) \\ \mathbf{E}_{\mathbb{P}}(W(\tau)W(\tau)^\top) = I_N\tau \end{cases} \tag{2}$$

Heath, Jarrow and Morton demonstrate that under the risk-adjusted measure $\mathbb{P}'$,

$$\begin{cases} r_t(0) = r_s(t-s) + \sum_{n=1}^{N} \int_s^t \sigma_n(u,t) \left[\int_u^t \sigma_n(u,v)dv\right] du + \int_s^t \sigma(u,t)dW'(u) \\ \mathbf{E}_{\mathbb{P}'}(W'(u)W'(u)^\top) = I_N u \\ W'(.) \text{ is a } \mathbb{P}' - Wiener \text{ process} \end{cases} \tag{3}$$

Example 1 $\begin{cases} N = 1 \\ \sigma(u,t) = \sigma \end{cases}$ *from HJM [1992]. The solution is trivial:*

under $\mathbb{P}'$

$$\begin{aligned} r_t(0) &= r_s(t-s) + \int_s^t \sigma \left[\int_u^t \sigma dv\right] du + \int_s^t \sigma dW'(u) \\ &= r_s(t-s) + \tfrac{1}{2}\sigma^2(t-s)^2 + \sigma(W'(t) - W'(s)) \end{aligned} \tag{4}$$

Example 2 $\begin{cases} N = 2 \\ \sigma(u,t)^\top = \begin{pmatrix} \sigma_1 \\ \sigma_2 \exp(-\frac{c}{2}(t-u)) \end{pmatrix} \end{cases}$ *from HJM [1992]. The solution is:*

under $\mathbb{P}'$

$$\begin{aligned} \int_s^t \sigma_1(u,t) \left[\int_u^t \sigma_1(u,v)dv\right] du &= \tfrac{1}{2}\sigma_1^2(t-s)^2 \\ \int_s^t \sigma_2(u,t) \left[\int_u^t \sigma_2(u,v)dv\right] du &= \sigma_2^2 \int_s^t \exp(-\tfrac{c}{2}(t-u)) \left[\int_u^t \exp(-\tfrac{c}{2}(v-u))dv\right] du \\ &= \tfrac{4\sigma_2^2}{c^2}\left[1 - \exp(-\tfrac{c}{2}(t-s))\right] + \tfrac{2\sigma_2^2}{c^2}\left[\exp(-c(t-s)) - 1\right] \end{aligned}$$

Then,

$$\begin{aligned} r_t(0) &= r_s(t-s) + \tfrac{1}{2}\sigma_1^2(t-s)^2 + \tfrac{4\sigma_2^2}{c^2}\left[1 - \exp(-\tfrac{c}{2}(t-s))\right] + \tfrac{2\sigma_2^2}{c^2}\left[\exp(-c(t-s)) - 1\right] \\ &+ \sigma_1(W_1'(t) - W_1'(s)) + \sigma_2 \int_s^t \exp(-\tfrac{c}{2}(t-u))dW'(u) \end{aligned}$$

1.2 The Program

```
#include <randlib.h>
#include <stdio.h>
#include <math.h>
```

DIM is the dimension of the sigma vector, NSIM the number of interest rate simulations

```
#define DIM 2
#define NSIM 5
```

Here is the specification of a general form for the sigma function

```
double sigmafunc(double x, double y, double sigma, double c)
{
  return (sigma*exp(-0.5*c*(y-x)));
}
```

Integration of the functions using trapeze method

```
double trapzd(double (*func)(double,double,int,double,double),
              double a, double b, int nstep, double sigma, double c, int integralno)
{
  double bound,x,step,sum;
  int i;
```

This is an important point to define the bound

```
  if (integralno==1) bound=b;
  else bound=a;
  sum = 0.0;
  step=(b-a)/nstep;
  for (i=0,x=a;i<nstep;i++,x+=step)
    sum += step*((*func)(x,bound,nstep,sigma,c)
          -0.5*fabs((*func)(x+step,bound,nstep,sigma,c)-(*func)(x,bound,nstep,sigma,c)));
  return sum;
}
```

The inside integral whose lower bound depends on the integrated variable of the global integral

```
double integral2(double v, double u, int nstep, double sigma, double c)
{
  return (sigmafunc(u,v,sigma,c));
}
```

The term inside the global deterministic integral

```
double integral1(double u, double t,int nstep, double sigma, double c)
{
  return (sigmafunc(u,t,sigma,c)*trapzd(integral2,u,t,nstep,sigma,c,2));
}
```

The global deterministic integral

```
double Integral(double a, double b, int nstep, double sigma, double c)
{
  return trapzd(integral1,a,b,nstep,sigma,c,1);
}
```

This function returns one simulation for the interest rate

```
double HeathJarrowMorton(double s, double t, double f0, double sigma[],double c[],
                         double m)
{
  int n,i;
  double I,y,step,nstep,x,HJM;

  nstep = pow(2,m);
  step = (t-s)/nstep;
  for (n=1,y=0.0;n<=DIM;n++) {
    for (i=0,x=s;i<=nstep;i++,x+=step) {
      y += sigmafunc(x,t,sigma[n],c[n])*gennor(0,1)*sqrt(step);
    }
    I = Integral(s,t,nstep,sigma[n],c[n]);
    HJM = f0 + I + y;
  }
  return HJM;
}
```

Numerical Solution of Example 2

```
main()
{
  double f0,s,t;
  int m,j,n;
  double I[DIM+1];
  double sigma[DIM+1];
  double c[DIM+1];
  double hjm[NSIM+1];
  f0 = 0.10;
  sigma[1] = 0.02;
  sigma[2] = 0.05;
  c[1] = 0;
  c[2] = 0.125;
  s = 1;
  t = 5;
  m = 6;

  for (n=1;n<=DIM;n++) {
    I[n] = Integral(s,t,pow(2,m),sigma[n],c[n]);
    printf(''I[%d] = %1.8f\n'',n,I[n]);
  }

  setall(1234,4321);
  for (j=1;j<=NSIM;j++) {
    hjm[j] = HeathJarrowMorton(s,t,f0,sigma,c,m);
    printf(''hjm[%d] = %lf\n'',j,hjm[j]);
  }
  return 0;
}
```

The output is:

```
I[1] = 0.00320000
I[2] = 0.01565720
hjm[1] = 0.085220
hjm[2] = 0.177335
hjm[3] = 0.097375
```

```
hjm[4] = 0.224052
hjm[5] = 0.234484
```

References

[1] BLACK F. AND M. SCHOLES [1973], The pricing of options and corporate liabilities, *Journal of Political Economy*, **81**, 637-59.

[2] CHALASANI P. AND S. JHA [1997], Steven Shreve: Stochastic Calculus and Finance, *Postscript*, Department of Mathematical Sciences, Carnegie Mellon University.

[3] EL KARAOUI N. [1996], Modèles stochastiques en Finance, *Lecture Notes*, Université Paris VI.

[4] HEATH D., R. JARROW AND A. MORTON [1990], Bond pricing and the term structure of interest rates: A discrete time approximation, *Journal of Financial and Quantitative Analysis* , **25**, 419-39.

[5] HEATH D., R. JARROW AND A. MORTON [1992], Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation, *Econometrica* , **60**, 77-105.

[6] HO T.S.Y. AND S.-B. LEE [1986], Term structure movements and pricing interest rate contingent claims, *The Journal of Finance*, 41, 1011-29.

[7] MERMET B. [1996], C, Algorithmique et Programmation, *Lecture Notes*, DEA de Chimie Informatique et Theorique.

[8] RONCALLI T. [1998], La structure par terme des taux zéros: modélisation et implémentation numérique, *Thèse ès Sciences Economiques.*