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Analytical Approximations for Pricing Derivatives on Realized Volatility

Global Derivatives

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Introduction: Volatility Derivatives

How do contracts on the realized volatility look like?

- Any traded asset can be chosen as an underlying spot process
- The time period over which the volatility is calculated is specified
- The spot value of the underlying is recorded on every business day
- After the expiration, the *annualized* realized volatility of the asset is calculated as a square root of the annualized realized variance:

$$V = \sqrt{\frac{255}{N-1}\sum_{i=1}^{N}(R_i - \bar{R})^2}$$

where R_i are returns of the underlying asset

$$R_i = \log\left(\frac{S_i}{S_{i-1}}\right)$$

Introduction: Pricing Considerations

Demeterfy, Derman, Kamal & Zou (1999) have derived an exact replication for *variance* swaps using vanilla products. This result allows pricing these products in the model-independent way

Unfortunately, no such solution has been found for *volatility* swaps and swaptions, so to solve this problem we have to give up model independence

- In the limit of small time step, the realized volatility is completely determined by the volatility of the stochastic process
- Rather than model the evolution of the underlying, we choose to model the evolution of its variance
- Since variance is additive, the realized volatility in this limit is simply the square root of the average variance over time

This approach naturally brings us to stochastic volatility models

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The Heston Model

We are using the stochastic volatility framework proposed by Steven Heston (Heston 1993). It has the following form:

$$dS = \mu S dt + \sqrt{V} S dZ_S$$

$$dV = \beta(\bar{V} - V)dt + \sigma\sqrt{V} dZ_V$$

$$\langle dZ_S, dZ_V \rangle = \rho dt$$

So, in addition to the process for the underlying, there is a stochastic process for the variance that is correlated with the underlying process.

This model has certain advantages that made it very popular among practitioners:

- Mean-reverting behavior of the variance
- Small number of intuitively understood parameters
- Analytical solution for vanilla options
- Only two factor model

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Variance Distribution in Heston Model

We are interested in the behavior of stochastic variable that follows the following process:

$$dV = \beta(\bar{V} - V)dt + \sigma\sqrt{V}dZ_V$$

We do know the distribution of the variance at time T (Feller 1951):

γV_T follows the non-central chi-square distribution:

$$\chi_n^2(\gamma V_T;\lambda) = \sum_{j=0}^{\infty} e^{-\lambda}\frac{\lambda^j}{j!}\chi_{n+2j}^2(\gamma V_T)$$

where

$$\chi_n^2(x) = \frac{1}{2^{n/2}\Gamma(n/2)}x^{n/2-1}e^{-x/2}$$

$$\gamma = \frac{4\beta}{\sigma^2\left(1-e^{-\beta T}\right)}$$

$$n = \frac{4\beta}{\sigma^2}\bar{V}$$

$$\lambda = \gamma V_0 e^{-\beta T}$$

How About the Distribution of the Average Variance?

What we need to know to price derivatives on the realized volatility is the distribution of variance *averaged over time*

Given the complexity of the non-central chi-square distribution, it was not possible to derive the closed form of the average variance distribution

But we can still obtain important information about the distribution by calculating its moments

The expected average variance (or the first moment of the average variance distribution) in the Heston model is given by:

$$M1 = \bar{V} + (V_0 - \bar{V})\frac{1 - e^{-\beta T}}{\beta T}$$

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The Analytical Expression for the Second Moment of the Average Variance

We have derived an original formula for the second moment of the average variance distribution:

$$M2 = \bar{V}^2 + 2\bar{V}(V_0 - \bar{V})\left[\frac{\beta T - 1 + e^{-\beta T}}{(\beta T)^2}\right] +$$

$$\frac{\bar{V}\sigma^2}{\beta}\left[\frac{\beta T - 1 + e^{-\beta T}}{(\beta T)^2}\right] +$$

$$2(V_0 - \bar{V})\left(\bar{V} + \frac{\sigma^2}{\beta}\right)\left[\frac{1 - e^{-\beta T} - \beta T e^{-\beta T}}{(\beta T)^2}\right] +$$

$$\left[(V_0 - \bar{V})^2 + \frac{\sigma^2}{2\beta}(\bar{V} - 2V_0)\right]\left[\frac{1 - e^{-\beta T}}{\beta T}\right]^2$$

What Is Our Approach?

The result presented on the previous page gives us a way to rapidly and consistently calculate the first two moments of the average variance distribution once we specify the parameters of the Heston model.

Therefore we can try different model distributions that are fully characterized by two moments and see whether they provide a good fit for the actual distribution.

Our approach is as follows:

- Fit the Heston model to the market
- Calculate analytically the first two moments of the average variance distribution
- Fit a few different model distributions to the values of these two moments
- Run the simulations and see which model works better for a range of underlyings and strikes

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Fitting the Heston Model

We would like to apply this methodology to pricing derivatives on the six month realized volatility for three foreign exchange rates:|
JPY/EUR, JPY/USD and USD/EUR

In each of these cases, we fitted the Heston model to the following three parameters of the six month volatility smile:

1) The ATM volatility

2) The 25-delta butterfly (measure of the smile steepness)

3) The 25-delta risk reversal (measure of the smile skewness)

As you can see on the next page, the quality of fit is consistently high through the wide range of strikes.

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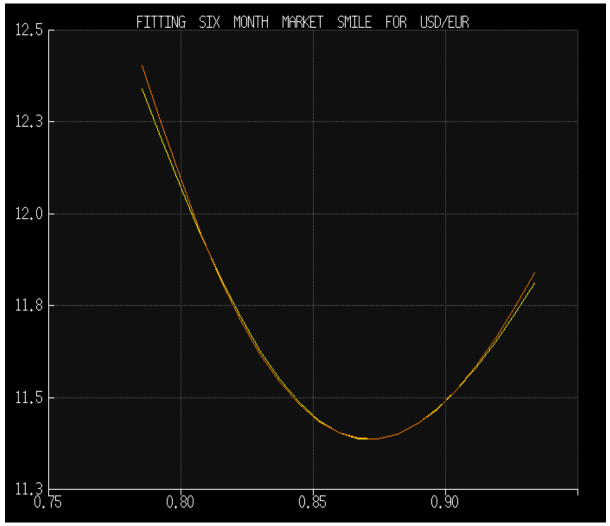
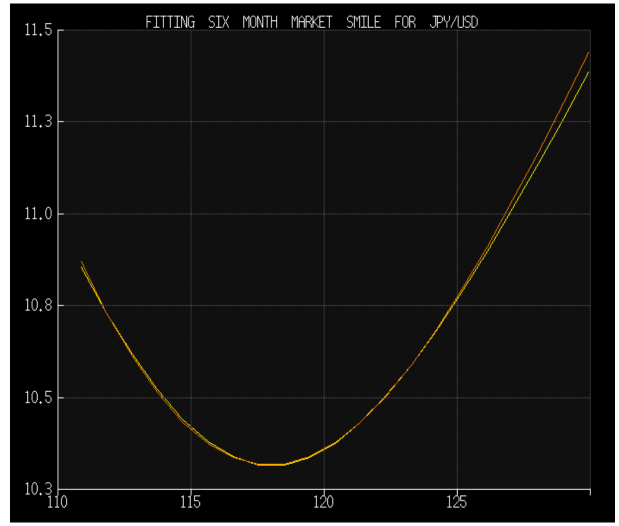
Fitting the Heston Model: Results

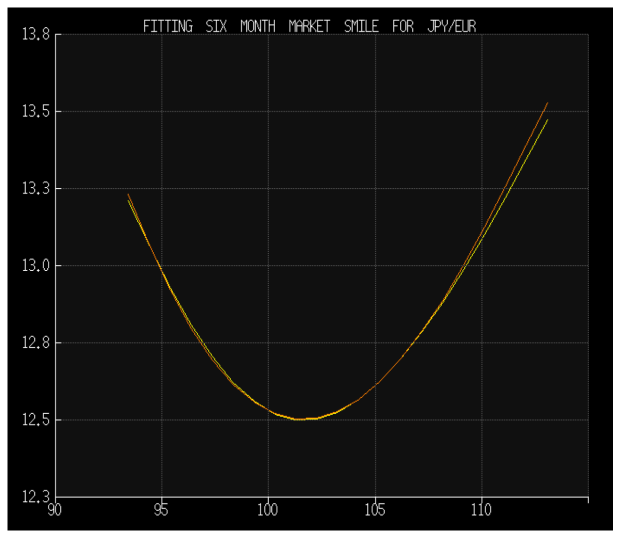
USD/EUR



JPY/USD



JPY/EUR



What Requirements Do We Impose on the Average Variance Distribution?

1) It should be defined for all non-negative values of the variance

2) It should be unimodal (have only one maximum)

3) Zero variance should be associated with zero probability

4) It should be fully specified by the values of the first two moments

There is a number of distributions that satisfy these requirements. For the purposes of this work, we have selected the following four models:

- The lognormal distribution
- The gamma distribution
- Two special distributions A & B

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How to Derive the Pricing Formulae

Once we know the functional form of the average variance distribution, deriving the pricing formulae becomes straightforward

Expected realized volatility (needed to price the volatility swaps):

$$E[\sqrt{V}] = \int_0^{\infty} f(V)\sqrt{V}dV$$

Call swaption with strike K:

$$C(K, T) = e^{-rT}\int_{K^2}^{\infty} f(V)(\sqrt{V} - K)dV$$

Put swaption with strike K:

$$P(K, T) = e^{-rT}\int_0^{K^2} f(V)(K - \sqrt{V})dV$$

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The Lognormal Model

The lognormal probability density is:

$$p^{\mu,\sigma}(V) = \frac{1}{\sqrt{2\pi}\sigma V} e^{-\frac{(\log[V]-\mu)^2}{2\sigma^2}}$$

Parameters of the distribution μ and σ can be calculated analytically from the values of the first two moments:

$$\mu = \frac{1}{2}\log\left[\frac{M1^4}{M2}\right] \qquad \sigma = \sqrt{\log\left[\frac{M2}{M1^2}\right]}$$

The expected volatility is:

$$E[\sqrt{V}] = e^{\frac{\mu}{2}+\frac{\sigma^2}{8}}$$

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The Lognormal Model

The prices of call and put swaptions are:

$$C = e^{-rT}\left[e^{\frac{\mu}{2}+\frac{\sigma^2}{8}} N\left(\frac{\mu + \sigma^2/2 - 2Log[K]}{\sigma}\right) - KN\left(\frac{\mu - 2Log[K]}{\sigma}\right)\right]$$

$$P = e^{-rT}\left[KN\left(\frac{2Log[K] - \mu}{\sigma}\right) - e^{\frac{\mu}{2}+\frac{\sigma^2}{8}} N\left(\frac{2Log[K] - \mu - \sigma^2/2}{\sigma}\right)\right]$$

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The Gamma Model

The gamma probability density is:

$$\gamma^{\alpha,\beta}(V) = \frac{1}{\beta\Gamma(\alpha)}\left(\frac{V}{\beta}\right)^{\alpha-1} e^{-\frac{V}{\beta}}$$

Parameters α and β of the distribution can be calculated analytically from the values of the first two moments:

$$\alpha = \frac{M1^2}{M2 - M1^2} \qquad \beta = \frac{M2 - M1^2}{M1}$$

Let's define the incomplete gamma function and its complement:

$$P^{\alpha,\beta}(X) = \int_0^X \gamma^{\alpha,\beta}(x)dx = \int_0^X \frac{1}{\beta\Gamma(\alpha)}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\frac{x}{\beta}} dx$$

$$Q^{\alpha,\beta}(X) = \int_X^\infty \gamma^{\alpha,\beta}(x)dx = \int_X^\infty \frac{1}{\beta\Gamma(\alpha)}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\frac{x}{\beta}} dx$$

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The Gamma Model

The prices of call and put swaptions are:

$$C = e^{-rT}\left[\frac{\sqrt{\beta}\,\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(\alpha)}Q^{\alpha+\frac{1}{2},\,\beta}(K^2) - KQ^{\alpha,\,\beta}(K^2)\right]$$

$$P = e^{-rT}\left[KP^{\alpha,\,\beta}(K^2) - \frac{\sqrt{\beta}\,\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(\alpha)}P^{\alpha+\frac{1}{2},\,\beta}(K^2)\right]$$

The expected volatility is:

$$E[\sqrt{V}] = \frac{\sqrt{\beta}\,\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(\alpha)}$$

The Power Gamma Model

Let us consider the following distribution:

$$f^{\alpha,\beta,\lambda}(V) = \frac{\lambda}{\beta\Gamma\left(\frac{\alpha}{\lambda}\right)}\left(\frac{V}{\beta}\right)^{\alpha-1} e^{-\left(\frac{V}{\beta}\right)^{\lambda}}$$

This is a distribution of the random variable such that its λ-th power follows the gamma distribution

This distribution has three parameters, and we only need two, so one parameter needs to be fixed externally

- Distribution A corresponds to λ=0.5
- Distribution B corresponds to λ=0.75

The remaining parameters of the distribution α and β are determined by numerical solution of the following system of nonlinear equations:

$$M1 = \frac{\Gamma\left(\frac{1+\alpha}{\lambda}\right)}{\beta\Gamma\left(\frac{\alpha}{\lambda}\right)} \qquad M2 = \frac{\Gamma\left(\frac{2+\alpha}{\lambda}\right)}{\beta^2\Gamma\left(\frac{\alpha}{\lambda}\right)}$$

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The Power Gamma Model

The prices of call and put swaptions are:

$$C = e^{-rT}\left[\frac{\sqrt{\beta}\,\Gamma\left(\frac{2\alpha+1}{2\lambda}\right)}{\Gamma\left(\frac{\alpha}{\lambda}\right)} Q^{\frac{2\alpha+1}{2\lambda},\,\beta}(\beta^{1-\lambda}K^{2\lambda}) - KQ^{\frac{\alpha}{\lambda},\,\beta}(\beta^{1-\lambda}K^{2\lambda})\right]$$

$$P = e^{-rT}\left[KP^{\frac{\alpha}{\lambda},\,\beta}(\beta^{1-\lambda}K^{2\lambda}) - \frac{\sqrt{\beta}\,\Gamma\left(\frac{2\alpha+1}{2\lambda}\right)}{\Gamma\left(\frac{\alpha}{\lambda}\right)} P^{\frac{2\alpha+1}{2\lambda},\,\beta}(\beta^{1-\lambda}K^{2\lambda})\right]$$

The expected volatility is:

$$E[\sqrt{V}] = \frac{\sqrt{\beta}\,\Gamma\left(\frac{2\alpha+1}{2\lambda}\right)}{\Gamma\left(\frac{\alpha}{\lambda}\right)}$$

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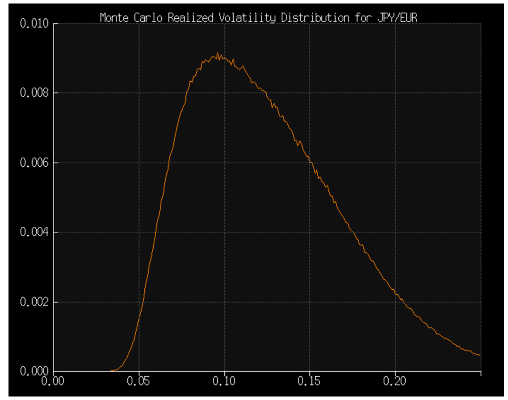
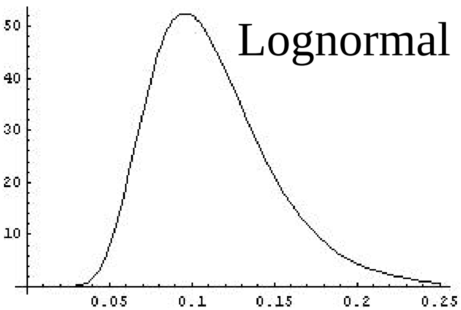
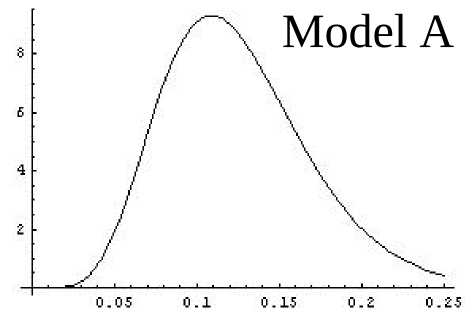
RESULTS

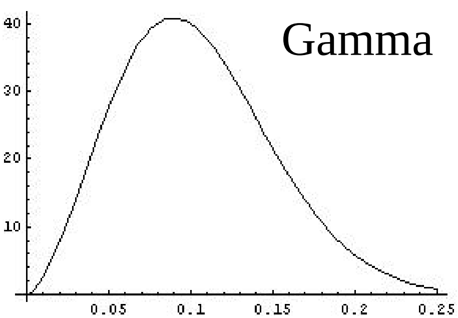
Results: JPY/EUR Volatility Distributions

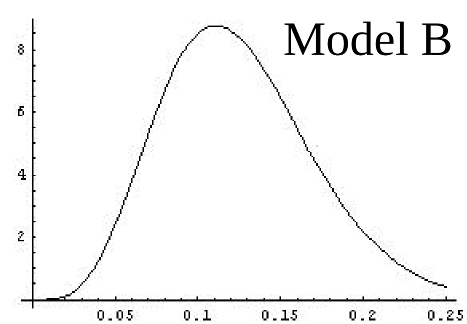
Monte-Carlo











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Results: Skewness of the Distribution

Skewness is related to the third central moment of the distribution

The following table represents the skewness of all model distributions compared to the results of the Monte Carlo simulations

Exchange Rate	Monte Carlo	Lognormal Distribution	Gamma Distribution	Distribution A	Distribution B
JPY/EUR	1.64	2.64	1.49	1.94	1.69
JPY/USD	1.78	2.92	1.60	2.11	1.83
USD/EUR	1.51	2.37	1.37	1.78	1.55

- The skewness of distribution B comes closest to the skewness of the simulated distribution: it is within 0.05 for all three crosses.
- The gamma distribution has lower skewness than Monte Carlo
- The skewness of the lognormal distribution is clearly too high.

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Results: Kurtosis of the Distribution

Kurtosis is related to the fourth central moment of the distribution

The following table represents the kurtosis of all model distributions compared to the results of the Monte Carlo simulations

Exchange Rate	Monte Carlo	Lognormal Distribution	Gamma Distribution	Distribution A	Distribution B
JPY/EUR	6.80	20.50	6.31	9.58	7.64
JPY/USD	7.45	24.25	6.86	10.79	8.42
USD/EUR	6.21	17.41	5.80	8.48	6.91

- The gamma distribution and distribution B are somewhat close to Monte Carlo, with the former having lower kurtosis than Monte Carlo and the latter having a higher one
- Distribution A has an even higher kurtosis
- The lognormal distribution has kurtosis that is on average over three times as high as the kurtosis of the Monte Carlo distribution.

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Results: Pricing Volatility Swaps

The following table represents the fair strikes of the volatility swaps (i.e. expected realized volatility):

Exchange Rate	Monte Carlo	Lognormal Distribution	Gamma Distribution	Distribution A	Distribution B
JPY/EUR	12.56	12.63	12.46	12.55	12.51
JPY/USD	10.37	10.42	10.24	10.34	10.29
USD/EUR	11.46	11.51	11.40	11.46	11.43

- Distribution A is within three basis points for all three crosses - an excellent match
- Distribution B predicts lower expected volatility than Monte Carlo
- The lognormal distribution predicts higher expected vol than Monte Carlo
- The gamma distribution has the highest deviations from Monte Carlo on this test

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Results: Pricing JPY/EUR Vol Swaptions

The following table represents the prices of JPY/EUR call swaptions for a range of strikes from 6% to 16% (covering both ITM and OTM swaptions):

Strike (in %)	Monte Carlo	Lognormal Distribution	Gamma Distribution	Distribution A	Distribution B
6.00	6.44	6.49	6.42	6.45	6.43
8.00	4.65	4.65	4.71	4.67	4.69
10.00	3.18	3.08	3.25	3.16	3.20
12.00	2.05	1.91	2.09	2.00	2.05
14.00	1.24	1.12	1.25	1.19	1.22
16.00	0.70	0.64	0.69	0.67	0.68

- Distribution B is the closest to Monte Carlo
- Distribution A is the second closest
- Lognormal distribution predicts lower prices for ITM swaptions than both Monte Carlo and the rest of the models

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Results: Pricing JPY/USD Vol Swaptions

The following table represents the prices of JPY/USD call swaptions for a range of strikes from 6% to 16%:

Strike (in %)	Monte Carlo	Lognormal Distribution	Gamma Distribution	Distribution A	Distribution B
6.00	4.39	4.40	4.41	4.40	4.40
8.00	2.86	2.78	2.92	2.85	2.89
10.00	1.75	1.62	1.78	1.70	1.75
12.00	0.99	0.88	0.99	0.94	0.97
14.00	0.52	0.46	0.51	0.49	0.50
16.00	0.25	0.24	0.23	0.24	0.24

- The results are similar to those for JPY/EUR

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Results: Pricing USD/EUR Vol Swaptions

The following table represents the prices of USD/EUR call swaptions for a range of strikes from 6% to 16%:

Strike (in %)	Monte Carlo	Lognormal Distribution	Gamma Distribution	Distribution A	Distribution B
6.00	5.37	5.41	5.38	5.38	5.38
8.00	3.64	3.61	3.69	3.64	3.66
10.00	2.27	2.16	2.31	2.24	2.27
12.00	1.29	1.19	1.30	1.25	1.28
14.00	0.67	0.61	0.66	0.64	0.65
16.00	0.31	0.30	0.30	0.30	0.30

- The results are similar to those shown previously

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Pricing Volatility Swaptions: Summary

- Distribution B has the closest match with Monte Carlo across different crosses and strikes: in 16 out of 18 cases, prices calculated using this distribution are within two basis points of Monte Carlo prices.
- This model distribution offers the most robust approximation to pricing swaptions on realized volatility
- Distribution A offers the second best match overall (within 5bp everywhere), but it systematically predicts lower prices for in-the-money swaptions
- The gamma distribution predicts higher prices for at-the-money swaptions
- The lognormal distribution provides the worst overall match, with considerably lower (up to 14bp) in-the-money prices and higher out-of-the-money prices

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CONCLUSION

Conclusions: Methodology

- We have developed a methodology for pricing derivatives on the realized volatility within the Heston model

- While we have tested this approach on the foreign exchange markets, it can be applicable to other markets where the volatility smile is realistically represented by the Heston model

- We have suggested a few approximate functional forms for the risk-neutral average variance distribution, and have derived analytical formulae for pricing volatility derivatives for all models used

- This methodology had been tested on different foreign exchange markets and have been found to have a very good overall agreement with the results of the large-scale Monte Carlo simulations

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CONCLUSION

Conclusions: Foreign Exchange Examples

- The best overall fit is provided by model distribution B
- It gives the best match for volatility swaption prices and the skewness of the distribution, and very good results for kurtosis and volatility swap prices.
- Model distribution A was the most accurate for pricing volatility swaps.
- The lognormal distribution is the least accurate of all models tested, in terms of both swaption prices and distribution statistics