CONTINUOUS TIME LIMIT OF THE BINOMIAL MODEL

PENDO KIVYIRO
Supervisor: Dr.Diane Wilcox

June 3, 2005

Contents

List of Figures

Acknowledgements

I would like to thank, God purposes all I do in my life. My greatest appreciation to my supervisor Dr. Diane Wilcox, for her assistance, suggestions and useful comments throughout the course of this essay. I would like to express my sincere gratitude to AIMS staff most especially Prof. Neil Turok the founder of AIMS, Prof. Fritz Hahne the director of the institute, Mirjam the administrative officer, Igsaan the facility manager, and all tutors Dr. Mike Pickles, Kate, Lisa and Carl for their assistance. It was such a great time to be with such wonderful tutors who are eager to assist at all time. I would like to thank Dr. Mike Pickles and Prof. Kotze for reading through my essay and making corrections, suggestions and comments, may our almighty God bless you all. I am indebted to my husband who has been very special to me by encouraging me on a daily basis and his approval towards AIMS I say a big thank you. I would also like to thank my fellow class mates most especially Naval for his assistance. I appreciate all my fellow students for making AIMS such a wonderful place to live and study.

Abstract

The limit of the Cox-Ross-Rubinstein formula as the length of the time step goes to zero is the Black-Scholes formula. The proof requires a suitable version of the central limit theorem. The convergence of binomial models involves a wider range of related problems.

We show the convergence of option price in the Binomial model to the price given by Black-Scholes-formula. We show also the convergence of the stock price in the binomial model to the geometric Wiener process.

We show some numerical implementations in analysing the convergence results of the binomial option pricing to the Black-Scholes formula. Particularly we discuss lattice methods developed by CRR(1979), Jarrow and Rudd(1983) and Tian(1993).

In order to undertake the task, we review the concepts of binomial distribution, Gaussian distribution , and Poisson distribution; Wiener process, Poisson process; Binomial model, option pricing in binomial model. Cox-Ross-Rubinstein formula; Models of convergence of sequences of random variables, relationship.

Chapter 1

INTRODUCTION

1.1 BACKGROUND

Financial market instruments can be divided into two distinct species. We have underlying assets such as stocks and bonds. And we have their derivatives such as options, forwards and futures and swaps. These are sometimes called contingent claims. There are various models which has been developed in pricing these underlying assets and their corresponding derivatives securities.

The work of Merton (1969, 1973)[12] and Black and Scholes(1973) innovated the use of diffusion process in continuous time methods in asset pricing problems. Diffusion process have become the workhorse model of the underlying asset price in option pricing models, such as Cox, Ingersoll, Ross(1985) and Morton(1992).

Early in this theoretical development, Sharpe(1978) developed a binomial model approach. This has some advantages of making the idea behind the diffusion process more widely accessible and making the implementation simple.

The idea of looking at a binomial model as a discrete time approximation to continuous-time diffusion were initially justified by Brennan and Schwartz (1978) and Cox, Ross and Rubinstein (1979). These authors (1979) showed that the binomial model of the stock price converges to a continuous process as the time interval goes to zero.

A more general characteristic of binomial asset pricing models is that they consist of two state conditions, either the price of the stocks tends to move upward by the factor μ, or move downward by the factor d. Also the one-step ahead price is always risky. At any time $t-1$ we are not sure of the value of the asset at time t. This is because the prices of the stock are random variables. Since the stock prices are considered as random variables we should have to study some probability and expectations. One way of coping with randomness in this essay will be to build on probability foundations to find the strongest possible links between derivatives security and their random underlying stocks.

1.2 Mathematical Tools

Definition 1.1 : σ-field

A *σ-field* $\mathcal{F}$ is a family of subsets of Ω such that:

- It contains the empty set; $\emptyset \in \mathcal{F}$.
- For all $\mathcal{A} \in \mathcal{F}$ then $\mathcal{A}^c \in \mathcal{F}$.
- $\mathcal{F}$ is closed under the operation of countable union; If $\mathcal{A}_1, \mathcal{A}_2, \ldots, \in \mathcal{F}$ then $\bigcup_{n>1} \mathcal{A}_n \in \mathcal{F}$

Definition 1.2 : Measurable space

A *measurable space* is a set Ω together with a collection $\mathcal{F}$ of subset Ω which is a sigma field.

Definition 1.3 : Probability Measure

Let Ω be a set and $\mathcal{F}$, a $\sigma - field$ of subsets of Ω. A *probability measure* is a function $P : \mathcal{F} \rightarrow$[0,1] such that

- $P(\emptyset)$=0;
- $P(\Omega)$ =1;
- If $A_1, A_2, \ldots, \in \mathcal{F}$ and $A_i \bigcap A_j = \emptyset$ if $i \neq j$, then $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

Where by $\mathbb{P}(A)$ = 'The probability that A occurs' and $A \in \mathcal{F}$ is called an event. The triple $(\Omega, \mathcal{F}, P)$ is called a *probability space*

Definition 1.4 : Measurable function

Let $(X, \mathcal{B}(X))$ and $(Y, \mathcal{B}(Y))$ be two measurable spaces. Then $f : X \rightarrow Y$ is measurable function if

$$f^{-1}(\mathcal{B}(Y)) \subseteq \mathcal{B}(X)$$

where $f^{-1}(\mathcal{B}(Y)) = \{f^{-1}(A)|A \in \mathcal{B}(Y)$.

Definition 1.5 : Random Variable

A *random variable* X is a measurable function from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that $X : \Omega \rightarrow \mathbb{R}$ is $\mathcal{F}$-measurable. It can be discrete and therefore

$$\{\omega \in \Omega : X(\omega) = x\} \in \mathcal{F} \quad \text{or continuous} \quad \{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F} \tag{1.1}$$

The latter defines the *probability distribution* of X denoted by $F(x) = P[X \leq x], \omega \in \Omega$ and is called a *scenario of randomness* and $X(\omega)$ represents an outcome of the random variable if the scenario happens.

Definition 1.6 : Conditional Expectation

Let X be a random variable of $(\Omega, \mathcal{F}, \mathsf{P})$ and let $\mathcal{H} \in \mathcal{F}$ be a σ- algebra. Then the *conditional expectation* of X given $\mathcal{H}$ is a random variable denoted by $\mathbb{E}[X|\mathcal{H}]$ and is characterised by the following properties

- $\mathbb{E}[X|\mathcal{H}] = X \quad$ if X is a $\mathcal{H}$-measured random variable.
- $\mathbb{E}[XY|\mathcal{H}] = X\mathbb{E}[Y|\mathcal{H}] \quad$ if X is $\mathcal{H}$-measurable.
- $\mathbb{E}[\mathbb{E}[X|\mathcal{H}]] = \mathbb{E}[X]$
- $\mathbb{E}[X|\mathcal{H}] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] \quad$ where $\mathcal{H} \subseteq \mathcal{G}$

Definition 1.7 : Characteristic Function

The *characteristic function* of a random variable X is a function $\varphi_X : \mathbb{R} \to \mathbb{C}$ defined by

$$\varphi_X(t) = \mathbb{E}[e^{itX}] = \int_{\mathbb{R}} e^{itx} dF_X(x),$$

where F_X is the distribution function of a random variable X.

Properties

- $\varphi_X(0) = 0.$
- If X and Y are independent random variables then $\varphi_{X+Y} = \varphi_X \varphi_Y$.
- For $a, b \in \mathbb{R}, \varphi_{aX+b}(t) = e^{itb}\varphi_X(at)$
- If $\mathbb{E}[X]^n < \infty$, then φ_X has continuous n^{th} derivative

$$\frac{d^k \varphi_X(t)}{dt^k} = \varphi_X^k(t) = \int_{\mathbb{R}} (ix)^k e^{itx} dF_X(x) \tag{1.2}$$

  and particularly,

$$\varphi_X^{(k)}(0) = i^k \mathbb{E}[X^k]. \tag{1.3}$$

1.2.1 Gaussian Distribution

This is the distribution with mean μ and standard deviation σ^2. It is sometimes called the *normal distribution.* The general formula for the probability density function of the normal distribution is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2}$$

where μ is the mean of the distribution and σ is the standard deviation. The graph of normal distribution is symmetric about the mean. The case where $\mu = 0$ and $\sigma = 1$ is called the *standard normal distribution* and the probability density function for the standard normal is

$$f(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

Approximately two thirds of the area under the curve lies within one standard deviation about the mean. This distribution occurs in many ways, for example, it can be obtained as a continuous time limit of the binomial distribution as $n \to \infty$. The former concept to this effect is presented by the central limit theorem that: *The sum of a large number of independent, identically distributed random variables is approximately normally distributed* (see section 1.3 below)

1.2.2 Log-normal Distribution

A random variable X has a log-normal distribution if its natural logarithm, $Y = log(X)$ has a normal distribution The probability density function of *log-normal* distribution with the mean μ and variance σ^2 is as given below

$$f(x) = \frac{\exp(-\frac{1}{2}(\frac{\ln(X)-\mu}{\sigma})^2)}{\sigma\sqrt{2\pi}}$$

Definition 1.8 : Filtration

The filtration of information flow on a time interval $[0,T]$ denoted $\mathcal{F}_{t\in[0,T]}$ on a probability space $(\Omega, \mathcal{F}, P)$ is an increasing function sequence of a σ-field containing information on the evolution of the price process up to time T such that for all $0 \leq s \leq t$ then $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ A probability space $(\Omega, \mathcal{F}, P)$ equipped with filtration is called a *filtered probability space* and is denoted as $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\in[0,T]}, P)$

Definition 1.9 : Stochastic Process

A *stochastic process* is a collection $\{X\}_{t\in[0,T]}$ of random variables $X_t : \Omega \to R^n$ where $t \in \tau$(a given parameter set). Usually $\tau = [0, \infty)$. Then X_t can be regarded as the state of some system at time t. We may write $X_t(\omega) = X(t, \omega) : \tau \times \Omega \to R^n$ and regard the process as a function of two variables.

Definition 1.10 : Martingale

Let $Y(t)$ be a stochastic process and let $\mathcal{H}_t$ be a filtration (an increasing family of σ-algebras). Then $Y(t)$ is a *martingale* with respect to $\mathcal{H}_t$ if

1. $Y(t)$ is $\mathcal{H}_t$ -adapted (i.e for all t the random variable $\omega \to Y_t(\omega)$ is $\mathcal{H}_t$- measurable)

2. $\mathbb{E}[|Y_t|] < \infty$

3. If $s > t$ then $\mathbb{E}[Y_s|Y_t] = Y_t$

Definition 1.11 : Markov property

This is a particular type of stochastic process where only the present value is relevant for predicting the future. The past history reflects the present price. Stock prices are usually assumed to follow a Markov process.

Definition 1.12 : Wiener process

This is the basic building block for modelling in continuous time and is also called Brownian motion. Models of stock price behaviour are usually expressed in terms of Wiener processes, which consist of properties such that W_t is a random variable, drawn from a normal distribution with mean zero and variance t.

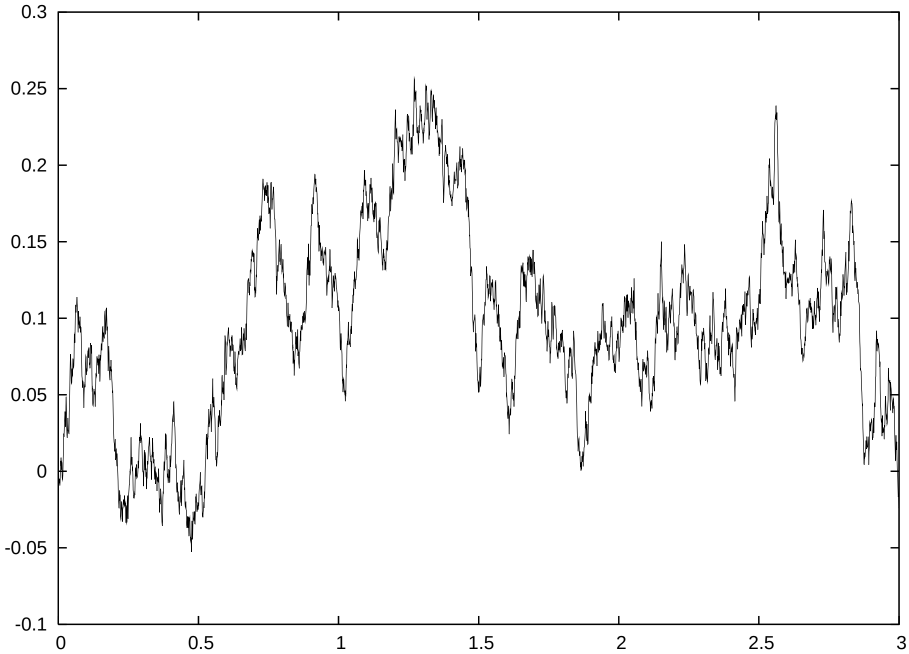

Figure 1.1: A generalised Wiener process

1.3 The Central Limit Theorem

Theorem 1.1 : *Let $X_1, \ldots, X_n$ be independent, identically distributed random variables with finite mean μ and variance σ^2. Then for n large, the distribution sample total $T = X_1 + X_2 + \ldots + X_n$ is approximately normal with mean $n\mu$ and standard deviation $\sigma(T) = \sigma\sqrt{n}$.*

Firstly we show that the mean and the variance are finite.
Proof

$$\begin{aligned}\mathbb{E}[T] &= \mathbb{E}[X_1 + X_2 + \ldots + X_n] \\ &= \mathbb{E}[X_1] + \mathbb{E}[X_2] + \ldots + \mathbb{E}[X_n] \\ &= \sum_{i=1}^{n} \mathbb{E}[X_i] = n\mu \end{aligned} \tag{1.4}$$

$$\begin{aligned}\mathrm{Var}[T] &= \mathrm{Var}[X_1 + X_2 + \ldots + X_n] \\ &= \sum_{i=1}^{n} \mathrm{Var}\,[X_i] = n\sigma^2 \end{aligned} \tag{1.5}$$

$$\text{Standard deviation } \; [T] = \sqrt{\mathrm{Var}[T]} = \sigma\sqrt{n} \tag{1.6}$$

Proof of the Theorem
The proof of the theorem is based on the concept of the characteristics function (see 1.4)
Let $Y_j = \frac{(X_j - \mu)}{\sigma}$ be a standardised random variable for each $j = 1, 2, \ldots, n$ then,

$$Z = \frac{T - n\mu}{\sigma\sqrt{n}} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_j \tag{1.7}$$

From (1.4) since $\mathbb{E}[X]^k < \infty$ then,

$$\mathbb{E}[X^k] = \frac{\varphi_X^{(k)}(0)}{i^k}\; (k = 1, 2)\; \mathbb{E}[X^1] = \mu \text{ and } \mathbb{E}[X^2] = \sigma^2 + \mu^2 \tag{1.8}$$

$$\begin{aligned}\varphi_Y(t) &= \mathbb{E}[e^{itY}] \\ &= \mathbb{E}\left[e^{it\left(\frac{X-\mu}{\sigma}\right)}\right] \\ &= e^{\frac{-it\mu}{\sigma}}\mathbb{E}\left[e^{\frac{it}{\sigma}X}\right] \\ &= \left[1 - \frac{it\mu}{\sigma} - \frac{t^2\mu^2}{2\sigma^2} + o(t^2)\right]\mathbb{E}\left[1 + \frac{itX}{\sigma} - \frac{t^2X^2}{2\sigma^2} + o(t^2)\right] \\ &= 1 - \frac{t^2}{2} + o(t^2) \end{aligned} \tag{1.9}$$

and hence,

$$\varphi_Y(t/\sqrt{n}) = 1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right) \tag{1.10}$$

As the X_i are independent, the Y_i are as well, and from (1.3) if Y_i are independent then,

$$\begin{aligned}\varphi_{Y_j}^n(t) &= \varphi_{Y_1}(t)\varphi_{Y_2}(t)\ldots\varphi_{Y_n}(t) \\ &= \left[1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right]^n \to e^{-\frac{1}{2}t^2} \text{ as } n \to \infty \end{aligned} \tag{1.11}$$

This limit is just the characteristics function of a standard normal distribution and the central limit theorem follows from the Levy continuity theorem which proves that the convergence of characteristic functions implies convergence in distribution.

1.4 Concepts in Mathematical Finance

Definition 1.13 : Asset

An *asset* is a resource that an individual, or cooperation, or country own or control that has economic values and is expected to provide future benefits.

Definition 1.14 : Options

These are financial instruments which give the holder the right, but not the obligation, to trade them at specified price at a specified date. A call option gives the holder the right to buy an asset, a put option gives the holder the right to sell an asset.

- The *American Option* is the one where you can exercise the option any time before some fixed time T i.e expiry date.
- The *European Option* is the one which you can exercise only at the expiry date T.
- The *Strike price* K is the price at which the future transaction will take place and is fixed in advance at time 0 'now'.

Note that since the holder has the *right* and not obligation to buy or sell he will only exercise it if it is profitable to him, otherwise he will not. In the case of a European call, he will exercise the option if the market price S_t is greater than the strike price K, while in the case of the European put; if the market price S_t is less than K. The difference between the two prices at the time of exercise gives the *p*ayoff of the option. Since the market price of the asset is generally unbounded the payoff of the call options is also an unbounded random variable. Suppose C_E and P_E denote the payoff of a European call and put option respectively, then:

$$C_E = \text{Max}(S_T - K, 0) = (S_T - K)^+ \tag{1.12}$$

$$P_E = \text{Max}(K - S_T, 0) = (K - S_T)^+ \tag{1.13}$$

Definition 1.15 : Derivative security

A derivative security (also called a contingent claim) is a financial contract whose value at expiry date T is fully determined by the prices at time T (or at a fixed range of times within $[0, T]$) of underlying assets $S_i, i = 0, 1, \dots, d$. Examples of derivative security are such as; *option*, *forward and futures* (binding agreement to buy or sell asset S at a future date T), and *swap.*

1.4.1 Market participants

Definition 1.16 : Hedgers

These are traders who aim to reduce the risk associated with price movement in the underlying assets by off-setting long and short positions. Since the future movement cannot be determined with certainty hedgers try to reduce some risk due to price movement.

Definition 1.17 : Speculators

These are traders who take risks with the intention of profiting from them. Speculators take large risks, especially with respect to anticipating future price movement. Speculators are not interested in holding the underlying asset for long period but seeking only to profit by trading in it.

Definition 1.18 : Arbitrageurs

This is the third group of traders who seek to profit from market deficiencies. They purchase securities in one market for immediate resale in another market in hope of profiting from price differences. They want to lock in a risk-less profit by simultaneous holding positions in many markets, and exploiting possible mis-pricing of assets.

All these groups of traders are important in making the financial market more efficient.

Chapter 2

Option Pricing in the Binomial model

Since the future values of risky assets are uncertain, a mathematical model of the market dynamics requires us to work with probability spaces. The Binomial model was first proposed by Cox, Ross and Rubinstein in a paper published in 1979 [6]. This solution to pricing an option is probably the most common model used for pricing derivative securities. The model divides the time to an option's expiry into a large number of intervals, or steps. At each interval it calculates that the stock price will move either up or down with a given probability. In this chapter we are going to analyse critically the binomial mode in relation to other market models.

2.1 A single-period binomial model

This is the market model with *one time step*, and the price of the single stock in the model takes one of just two possible values at the end of this step. So there are just two time points (0 and 1) and the price $S(0)$ changes to to $S(1,\omega_1)$ or $S(1,\omega_2)$ at time 1. In this model we have two assets, a bond and a stock. At time t a bond is denoted by B_t and the price of stock is denoted by S_t

- The bond price process is deterministic and is given by the following formula, $B_t = (1+r)^t$ such that

$$B_0 = 1$$
$$B_1 = 1 + r$$

  where r is the risk-less interest rate.

- The stock price process is stochastic process and is described by the following;

$$S_1(\omega) = \begin{cases} s_1(\omega_1) = S_0(1+u) & \text{with probability } p, \\ s_1(\omega_2) = S_0(1+d) & \text{with probability } 1-p. \end{cases} \tag{2.1}$$

Theorem 2.1 :
The market is viable (Arbitrage-free) if $-1 < d < r < u$

Proof-(By contradiction) Suppose $d < u < r$, consider a portfolio $\Theta = (1. - S_0)$. The value of portfolio at time $t = 0$ is $V_0(\Theta) = 0$. At time $t = 1$, we consider the two cases
If the stock prices move upward by the factor u;
Then

$$\begin{aligned} V_1(\Theta) &= S_1(\omega_1) - S_0(1+r) \\ &= S_0(1+u) - S_0(1+r) \\ &= S_0(u-r), \end{aligned}$$

$$\begin{aligned} V_1(\Theta) &= S_1(\omega_2) - S_0(1+r) \\ &= S_0(1+d) - S_0(1+r) \\ &= S_0(u-r), \end{aligned}$$

If there is no arbitrage then $V_1(\omega_1) < 0$ or $V_1(\omega_2) < 0$ and also either $V_1(\omega_1) > 0$ or $V_1(\omega_2) > 0$ this shows that either $u > r > d$ or $d > r > u$

Definition 2.1 :

A probability measure Q is called a *martingale measure* if the following conditions holds

$$S_0 = \frac{1}{R}\mathbb{E}_Q[S_1]$$

Proposition 2.1 : The market model is arbitrage free if and only if there exists a martingale measure Q.

A single-period binomial model has the following martingale probabilities;

$$\begin{cases} q = \frac{r-d}{u-d} \\ 1 - q = \frac{u-r}{u-d}. \end{cases} \tag{2.2}$$

2.1.1 Replicating Portfolios

Let $\Theta = (\theta_0, \theta_1)$ be a self-financing portfolio and r be the risk-less interest rate. The portfolio Θ replicates the derivative security C if its value at time 1 equals that of C for all scenarios. That is

$$V_1(\Theta) = C_1.$$

Theorem 2.2 : *In a single-period binomial model any derivative C has a replicating portfolio*

Proof. We need a portfolio $\Theta = (\theta_0, \theta_1)$ which satisfies the following

$$V_1(\Theta) = \begin{cases} \theta_1 S_u + \theta_0(1+r) = C_u \\ \theta_1 S_d + \theta_0(1+r) = C_d \end{cases} \tag{2.3}$$

where S_u and S_d denotes the prices of the stock when either moves up or down. C_u and C_d are their corresponding prices of derivative securities.
Solving (2.3) for θ_0, θ_1 gives the unique solutions.

$$\theta_1 = \frac{C_u - C_d}{S_u - S_d}$$
$$\theta_0 = \frac{1}{1+r}\left[\frac{C_d S_u - C_u S_d}{S_u - S_d}\right], \tag{2.4}$$

where θ_1 indicates shares in stock and θ_0 indicates units of cash or bond. By the **law of one price**; "*If two assets have the same terminal values, then they must have the same initial values; otherwise an arbitrage profit is feasible*"

$$C_0 = V_0(\Theta) = \theta_1 S_0 + \theta_0 = \frac{C_u - C_d}{S_u - S_d} S_0 + \frac{C_d S_U - C_u S_d}{(1+r)(S_u - S_d)}$$
$$= \frac{1}{1+r}\left[\left(\frac{r-d}{u-d}\right) C_u + \left(\frac{u-r}{u-d}\right) C_d\right]. \tag{2.5}$$

2.1.2 Risk-Neutral Probabilities

From the result (2.5), the coefficients of C_u and C_d add up to 1. They can be interpreted as probability. In fact, the equation can be simplified by defining a subjective probability measure $\mathbb{Q} = (q, 1-q)$ such that, $q = \frac{r-d}{u-d}$ and $1 - q = \frac{u-r}{u-d}$ Therefore

$$C_0 = \frac{1}{1+r}[qC_u + (1-q)C_d] = \frac{1}{1+r}\mathbb{E}_Q[C_1] \tag{2.6}$$

The probability measure $\mathbb{Q} = (q, 1-q)$ is called *risk-neutral probability* or *equivalent martingale measure*(EMM). It is a feature of every complete market. In general, the arbitrage price (fair) price of any derivative security X, at time $t = 0$ in a one step binomial setup is given by

$$X_0 = \frac{1}{1+r}\mathbb{E}_Q[X_1] = \mathbb{E}_Q[\bar{X}_1], \tag{2.7}$$

where $\bar{X}$ is the discounted price at time $t = 1$ and $\mathbb{E}[\ldots]$ denotes the expectation taken with respect to probabilities $(q, 1-q)$, that is "the present value of a derivative is equal to its discounted expected value under the risk neutral measure".

2.2 The Multi-period Binomial Model

A multi-period binomial model is a discrete model with time running from $t = 0$ to $t = T$. This is the extension of single step. In this model also we have two underlying assets, a bond with prices B_t and a stock with prices S_t.

- **The bond price**

The bond price is predictable and the risk-free interest rate r is known and constant over the period $0 \leq t \leq T$. The bond value increased by the factor of e^{rT}(compound interest). However we require that the interest rate over the period is known at the beginning of the interval time t_n, $n = 0, 1, \ldots, (N-1)$

- **The Stock price dynamics**

The possible trajectories of the stock price can be encoded in a tree such that over each mini-period the stocks follow a simple binomial model. The mini-periods are of equal length $h = \frac{T}{N}$ with time $t_n = nh$. Therefore, the stock prices can be given as a vector: $[S_0, S_1, S_2, \dots, S_n, \dots, S_N = S_T]$ for $0 \leq n \leq N$. We consider the notation $U = 1 + u$ and $D = 1 + d$ to denote the growth factors in prices, with the risk-neutral measure $\mathbb{Q} = (q, 1 - q)$ representing the probability of an up or downward movement in prices respectively, At any time step n,

- Each scenario (path) with exactly j upward moves and $n - j$ downward moves gives the same stock prices; $S_n = S_0 U^j D^{n-j}$
- There are $\binom{n}{j}$ such paths and the probability of each is $q^j(1 - q)^{(n-j)}$ and hence $S_n = S_0 U^j D^{n-j}$ with probability $\binom{n}{j} q^j (1 - q)^{n-j}$
- The stock price is a discrete random variable with $n + 1$ different values and at each n-step the stock has 2^n possible prices.
- The number of j move and $n - j$ are random variables with binomial distribution.

2.3 Cox-Ross-Rubinstein (CRR) Model

This is the most widely used market model in discrete time, which includes an important simplification of the multi-period binomial model in order to allow easier calculations. The two parameters, denoted by u (up) and d(down) which satisfy $d < u$ are fixed for all nodes ω at time t with prices $S(t, \omega)$, the prices at its two successors are $S(t, \omega)(1 + u)$ and $S(t, \omega)(1 + d)$ The only parameters needed to specify the *CRR* model are $T, r, S(0), u$ where r denotes risk-less and $r > 0$

2.3.1 Change of Parameters used in CRR Model

In this model there is a slight change of notation: we let $R = 1 + r, U = 1 + u, D = 1 + d$ in order to simplify the description of the prices at later time, e.g for $t = 3$ there are four nodes with with prices $S(0)U^3, S(0)U^2D, S(0)UD^2$ and $S(0)D^3$ respectively. In general in this model the 2^T different scenarios (paths) lead to only $T + 1$ different prices by time T, given by

$$(S(0)U^T, S(0)U^{T-1}D, S(0)^{T-2}D^2, \dots, S(0)D^{T-1}U, S(0)D^T).$$

2.3.2 CRR pricing formula

The pricing and hedging in a CRR model also in multi-step binomial model is determined using *backward Induction.* In particular, suppose that the price of stock is known at time $n - 1$. Using the risk neutral probability, $S(n - 1) = \Psi^{(n)} \mathbb{E}_{\mathbb{Q}}^{(n-1)}[S_n]$, where $\Psi^{(n)} = \frac{1}{R}$ is the discount factor over each time step, similarly the the value of derivative $f_{n-1} = \Psi^{(n)} \mathbb{E}_{\mathbb{Q}}^{(n-1)}[f_n]$. This notion of pricing is used to develop the CRR model price for any derivative whose price dynamics can be encoded in a tree structure as shown.

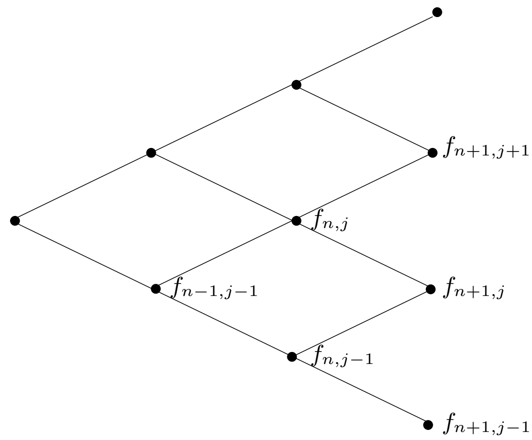

Figure 2.1: A CRR model

Let $f_{n,j}$ denote the value of the derivative at the n^{th} time step $(0 \leq n \leq N)$ in state j, as shown in the figure 2.1 where j represents the number of times that the price of the underlying stock has had an upward jump. Then using EMM Q , for all $(0 \leq n \leq N-1)$, $f_{n,j}$ is equal to the discounted expected payoff of the immediately succeeding time-step $n+1$, as shown in figure 2.1

$$\begin{aligned} f_{n,j} &= \frac{1}{R}\mathbb{E}_{\mathbb{Q}}(f_{n+1}) \\ &= \frac{1}{R}[qf_{n+1,j+1} + (1-q)f_{n+1,j}] \end{aligned}$$

and

$$f_{n,j-1} = \frac{1}{R}[qf_{n+1,j} + (1-q)f_{n+1,j-1}]$$

Also

$$\begin{aligned} f_{n-1,j-1} &= \frac{1}{R}[qf_{n,j} + (1-q)f_{n,j-1}] \\ &= \frac{1}{R^2}[q(qf_{n+1,j+1} + (1-q)f_{n+1,j}) + (1-q)(qf_{n+1,j} + (1-q)f_{n+1,j-1})] \\ &= \frac{1}{R^2}[q^2 f_{n+1,j+1} + 2q(1-q)f_{n+1,j} + (1-q)^2 f_{n+1,j-1}] \end{aligned}$$

By mathematical induction:

$$f_0 = \frac{1}{R^n}\sum_{j=0}^{n}\binom{n}{j}q^j(1-q)^{n-j}f_{n,j} \tag{2.8}$$

At the terminal nodes of the binomial tree, the value of an option $f_{N,j}$, determined by the price of the underlying asset $S_{N,j}$, replicates the value of the portfolio. Let $S_{N,j} = S_0U^jD^{N-j}$ denote the price of the underlying stock at the N^{th}-step in state j. Also let the payoff of the European call option at expiry be $C_{N,j}$ in state j $(0 \leq j \leq N+1)$ that is

$$C_{N,j} = \text{Max}\{S_{N,j} - X, 0\} = \text{Max}\{S_0U^jD^{N-j} - X, 0\} \tag{2.9}$$

where X is the strike price. Therefore equation above can be written as

$$C = \frac{1}{R^N} \sum_{j=0}^{N} \binom{N}{j} q^j (1-q)^{N-j} [S_0 U^j D^{N-j} - X]^+. \tag{2.10}$$

For the claim to be exercisable, we require $S_0 U^j D^{N-j} > X$. Let a denote the minimum number of up moves required for the option to end up in money. That is

$$S_0 U^a D^{N-a} > X.$$

Thus

$$a = \frac{\ln(X/S_0) - N \ln D}{\ln(U/D)} + \xi \text{ where } 0 < \xi < 1.$$

where ξ is a factor added to make a an integer. Thus equation (2.10) can be written as

$$C_0 = S_0 \left(\frac{1}{R^N} \sum_{j=a}^{N} \binom{N}{j} q^j (1-q)^{N-j} U^j D^{N-j} \right) - \frac{X}{R^N} \left(\sum_{j=a}^{N} \binom{N}{j} q^j (1-q)^{N-j} \right) \tag{2.11}$$

Note that;

$$\frac{q^j (1-q)^{N-j} U^j D^{N-j}}{R^N} = \left(\frac{qU}{R}\right)^j \left(\frac{D(1-q)}{R}\right)^{N-j}.$$

Thus (2.11) becomes

$$C_0 = S_0 \sum_{j=a}^{N} \binom{N}{j} \left(\frac{qU}{R}\right)^j \left(\frac{(1-q)D}{R}\right)^{N-j} - \frac{X}{R^N} \sum_{j=a}^{N} \binom{N}{j} q^j (1-q)^{N-j}. \tag{2.12}$$

Since $\mathbb{Q} = (q, 1-q)$ is a risk-neutral probability measure, and the model is viable then $R = qU + (1-q)D$ therefore we write,

$$1 = \frac{qU}{R} + \frac{(1-q)D}{R}$$

Let $q^* = \frac{qU}{R}$ and $1 - q^* = \frac{(1-q)D}{R}$ Then

$$\begin{aligned} C_0 &= S_0 \sum_{j=a}^{N} \binom{N}{j} q^{*j} (1-q^*)^{N-j} - \frac{X}{R^N} \sum_{j=a}^{N} \binom{N}{j} q^j (1-q)^{N-j} \\ &= S_0 \Psi(a; N, q^*) - X R^{-N} \Psi(a; N, q). \end{aligned} \tag{2.13}$$

The above equation(2.13) is called the **Cox-Ross-Rubinstein Formula** for pricing option, where,

$$\Psi(a; N, q) = \sum_{j=a}^{N} \binom{N}{j} q^j (1-q)^{N-j}$$

is the complementary binomial distribution function. Using the continuously compounded risk-free interest rate we can rewrite (2.13) as

$$C_0 = S_0 \Psi(a; N, q^*) - X e^{-rT} \Psi(a; N, q). \tag{2.14}$$

2.4 Modelling in Continuous Market Models

This section develops a continuous-variable, continuous-time stochastic process for stock prices. An understanding of this process is the first step to understanding the pricing option and other more complicated derivatives. We will analyse critically the Black-Scholes formula for pricing option with the applications of Ito's lemma. We suppose that the stock price follows a stochastic differential equation

$$dS = \mu S dt + \sigma S d\mathcal{W}$$

Or

$$\frac{dS}{S} = \mu dt + \sigma d\mathcal{W} \tag{2.15}$$

Where μ is the expected rate of return of the stock, μdt is the drift term which gives the deterministic component in the rate of return, σ is the volatility of the stock, σ d $\mathcal{W}$ is the stochastic component of the return and S is the stock price at time t.

2.4.1 Ito's Lemma

The price of stock option is a function of the underlying stock's price and time. More generally, we can say that the price of any derivative is the function of the stochastic variables underlying the derivative and time. From Taylor series expansion of $f(S,t)$ it follows that

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}dS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}dS^2 + O(dS^3)$$

Where by $dS^2 = \mu^2 S^2 dt^2 + 2\mu\sigma S^2 dt d\mathcal{W} + \sigma^2 S^2 d\mathcal{W}_t^2 +$ and $d\mathcal{W}^2 \rightarrow dt$, $dt d\mathcal{W} \rightarrow 0$, $dt^2 \rightarrow 0$ Therefore the above equation becomes

$$df = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial f}{\partial S}\sigma S d\mathcal{W} \tag{2.16}$$

Note that the above equation is made up of a random component proportional to the random variable $d\mathcal{W}$ and a deterministic component proportional to dt. This proves to be very important in the derivation of Black-Scholes results.

2.4.2 Black -Scholes Formula

This was first published in 1973. It has been extended so that it can be used to value an option on the foreign exchange. It is based on the following fundamental assumptions

- The prices of assets follow a log-normal random walk.i.e $\frac{dS}{S} = \mu dt + \sigma d\mathcal{W}$
- The risk free interest rate r and the asset volatility σ are functions of time over the life of security.
- There are no dividends during the life of derivative security.
- There are no arbitrage opportunities or possibilities.

- There are no transaction costs associated with hedging a portfolio.
- Trading of the underlying assets takes place continuously.
- Short selling is permitted, borrowing and lending at the risk-free rate is possible and assets are divisible.
- The market is liquid and there is no default risk.

Suppose S denotes the price of an underlying asset and $V(S,t)$ denotes the price of the derivative security, particularly an option. Then the Black- Scholes formula is given the following partial differential formula:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} + rS\frac{\partial V}{\partial s} = rV \tag{2.17}$$

Black-Scholes pricing formulas

The Black-Scholes formulas for the prices at time zero of European a call option on a non-dividend-payment are

$$C = S_0 N(d_1) - Xe^{-rT} N(d_2)$$

and

$$P = Xe^{-rT} N(-d_2) - S_0 N(-d_1)$$

where

$$d_1 = \frac{\ln(S_0/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$
$$d_2 = \frac{\ln(S_0/X) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}$$
$$= d_1 - \sigma\sqrt{T},$$

where $\mathcal{N}(d_i) i = 1, 2 =$ the cumulative probability function for a variable that is normally distributed with mean 0 and standard deviation 1, S_0 is the current stock price, X the strike price, C is the European call option, P is the European put option and T is the time to maturity.

Chapter 3

Convergence in distribution

In this chapter, we are going to analyse critically convergence in distributions of the stock and option prices. We will be able to show how the option price in the binomial model converges to the Black-Scholes formula as the length of the time steps decreases. In other words the binomial model provides discrete approximations to the continuous process underlying the Black-Scholes model. We will show also the convergence of stock prices in the binomial model to the geometric Wiener process. In particular we investigate the relationships between the parameters of the binomial model (the up and down returns) and the parameters of the log-normal distribution of the stock prices (drift and volatility)

3.1 Convergence of stock price to the geometric Wiener process.

Stock prices always follow the geometric Wiener process. In this section we are going to see how the stock prices in the binomial model converge to the geometric Wiener process. In particular, we will show that the stock prices follow the log-normal random walk. The geometric Wiener process with a constant drift term μt and scaled Brownian motion $\sigma \mathcal{W}_t$ is given by

$$\ln\left(\frac{S(t)}{S(0)}\right) = \mu t + \sigma \mathbb{W}_t. \tag{3.1}$$

So we are going to see that the stock price have the same equation in order to prove the convergence criteria.
Let S_0 denote the initial price of the stock and $S(t)$ its price at a future time t. We consider a discrete time model of a financial market such as the binomial model with the set dates $0, 1, \ldots, T$. The stock price process can be expressed for all $t \in [0, T-1]$ as

$$\chi_t = \frac{S(t+1)}{S(t)} \in \{u, d\}, \tag{3.2}$$

where χ_t is the return of the stock. We also require that the market is arbitrage free and therefore $-1 < d < r < u$ where $u, d \in \mathbb{R}$ and $S(0)$ is strictly non-negative. To provide a probability model, we assume that χ_t, for $t = 1, 2, 3, \ldots, T$ are mutually independent and identically distributed random variables on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with identical probability, That is

$$\mathbb{P}[\chi_t = u] = p = 1 - \mathbb{P}[\chi_t = d] \quad \text{for} \quad t = 0, 1, \ldots, T$$

Therefore, the stock price dynamics can be modelled for all $t \in [0, T]$ as

$$S(t) = S(0) \prod_{j=1}^{t} \chi_j \quad \text{for all} \quad t \leq T \tag{3.3}$$

Further we divide the time axis into units of length equal to $1/n$ and assume that the stock price at time $t_j = j/n$ increases by the factor u with probability p or decreases by a factor d with probability $1 - p$, That is

$$S_{j/n}(t) = \begin{cases} uS_{j-1/n}(t) & \text{with probability } p, \\ dS_{j-1/n}(t) & \text{with probability } 1-p. \end{cases} \tag{3.4}$$

Let us denote the rate of return over the time interval $t_j - t_{j-1} = 1/n$ by χ_j^n, where the superscript indicates the dependence on the choice of n. Then

$$\chi_j^n = \frac{S_{j/n}(t) - S_{j-1/n}(t)}{S_{j-1/n}(t)} \; j = 1, 2, \ldots, n \text{ and } t = 0, 1, \ldots, T. \tag{3.5}$$

As $n \to \infty$ we can write (3.5) as

$$\begin{aligned} \chi_j^n &= \frac{S_{j/n}(t) - S_{j-1/n}(t)}{S_{j-1/n}(t)} \\ &= \ln\left(1 + \frac{S_{j/n}(t) - S_{j-1/n}(t)}{S_{j-1/n}(t)}\right) \\ &= \ln\left(\frac{S_{j/n}(t)}{S_{j-1/n}(t)}\right) \; j = 1, 2, \ldots, n \text{ and } t = 0, 1, \ldots, T \end{aligned} \tag{3.6}$$

This follows from approximation of $\ln(1 + t) = t$ for all small $t \in \mathbb{R}, j = 1, 2, \ldots, n$ and $t = 0, 1, \ldots, T$ We are going to show, using the idea of central limit theorem that, $S(t)/S_0$ has a log-normal distribution. The basic idea is to present $\ln(S(t)/S_0)$ as a sum of independent, identically distributed random variable, that is

$$\frac{S(t)}{S_0} = \frac{S(1/n)}{S_0} \times \frac{S(2/n)}{S(1/n)} \cdots \frac{S(k/n)}{S((k-1)/n)} \times \frac{S(t)}{S(k/n)} \tag{3.7}$$

where $\frac{k}{n} \leq t \leq \frac{(k+1)}{n}$ that is $k = \lfloor nt \rfloor$ and therefore $\frac{\lfloor nt \rfloor}{n} \leq t < \frac{\lfloor nt \rfloor + 1}{n}$. Consequently,

$$\lim_{n\to\infty} \frac{\lfloor nt \rfloor}{n} = t$$

Since the rate of return over the a time interval of length $1/n$ goes to zero as $n \to \infty$ it follows that $\lim_{n\to\infty} \ln(S(t)/S(\lfloor nt \rfloor/n)) = 0$. Taking the logarithm on both sides of (3.7) we see that

$$\begin{aligned} \ln\left(\frac{S(t)}{S_0}\right) &= \sum_{j=1}^{\lfloor nt \rfloor} \ln\left(\frac{S(j/n)}{S((j-1)/n)}\right) + \ln\left(\frac{S(t)}{S(\lfloor nt \rfloor/n)}\right) \\ &= \sum_{j=1}^{\lfloor nt \rfloor} \ln\left(\frac{S(j/n)}{S((j-1)/n)}\right) \\ &= \sum_{j=1}^{\lfloor nt \rfloor} \chi_j^n \quad \text{see} \quad (3.6) \end{aligned} \tag{3.8}$$

The returns χ_j^n are independent, identically distributed random variable with $\mathbb{E}(\chi_j^n) = \mu_n$, $Var(\chi_j^n) = \sigma_n^2$, thus

$$\mathbb{E}\left[\ln\left(\frac{S(t)}{S_0}\right)\right] = \mathbb{E}\left[\sum_{j=1}^{\lfloor nt \rfloor} \chi_j^n\right] = \lfloor nt \rfloor \mu_n \tag{3.9}$$

$$\text{Var}\left[\ln\left(\frac{S(t)}{S_0}\right)\right] = \text{Var}\left[\sum_{j=1}^{\lfloor nt \rfloor} \chi_j^n\right] = \lfloor nt \rfloor \sigma_n^2 \tag{3.10}$$

Using the fact that $\lim_{n\to\infty} \lfloor nt \rfloor / n = t$ it follows that

$$\lim_{n\to\infty} \lfloor nt \rfloor \mu_n = \lim_{n\to\infty} \frac{\lfloor nt \rfloor}{n}(n\mu_n) = t \lim_{n\to\infty}(n\mu_n) \tag{3.11}$$

$$\lim_{n\to\infty} \lfloor nt \rfloor \sigma_n^2 = \lim_{n\to\infty} \frac{\lfloor nt \rfloor}{n}(n\sigma_n^2) = t \lim_{n\to\infty}(n\sigma_n^2) \tag{3.12}$$

But the above two equations exist if the limits $\lim_{n\to\infty} n\mu_n = \mu$ and $\lim_{n\to\infty} n\sigma_n^2 = \sigma^2$ exist. We make the assumption that the rate of return and its variance are proportional to the length of the time interval $1/n$; that is we assume

$$\mu_n = \frac{\mu}{n}, \sigma_n^2 = \frac{\sigma^2}{n}, \sigma_n = \frac{\sigma}{\sqrt{n}}$$

Therefore

$$\lim_{n\to\infty} \lfloor nt \rfloor \mu_n = \mu t$$
$$\lim_{n\to\infty} \lfloor nt \rfloor \sigma_n^2 = \sigma^2 t$$

We apply the central limit theorem (see section 1.3). *The sum of independent and identically distributed random variables with finite mean μ_n and variance σ_n^2 is normally distributed with mean μ and variance σ^2*. Therefore

$$T = \sum_{j=1}^{\lfloor nt \rfloor} \chi_j^n = \ln\left(\frac{S(t)}{S_0}\right) \text{ is } \mathcal{N}(\mu, \sigma^2)$$

with $\lfloor nt \rfloor$ instead of n, μ/n and $\sigma/\sqrt{n}$ instead of μ and σ, respectively. Therefore

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}\left(\frac{\sum_{j=1}^{\lfloor nt \rfloor} \chi_j^n - \lfloor nt \rfloor (\frac{\mu}{n})}{\sqrt{\lfloor nt \rfloor}(\sigma/\sqrt{n})} \le z\right) &= \lim_{n\to\infty} \mathbb{P}\left(\frac{\sum_{j=1}^{\lfloor nt \rfloor} \chi_j^n - t\mu}{\sigma\sqrt{t}} \le z\right) \\ &= \mathbb{P}(Z \le z)\end{aligned} \tag{3.13}$$

This means that as $n \to \infty$ we have

$$\sum_{j=1}^{\lfloor nt \rfloor} \chi_j^n - t\mu/(\sigma\sqrt{t}) \approx Z \tag{3.14}$$

This means that, the distribution of the random variable is approximately the same as a standard normal distribution,i.e $Z \sim \mathcal{N}(0,1)$. This is equivalent to

$$\ln\left(\frac{S(t)}{S_0}\right) \approx \sum_{j=1}^{\lfloor nt \rfloor} \chi_j^n \approx t\mu + \sigma\sqrt{t}z. \tag{3.15}$$

In particular, as $\lim_{n\to\infty}$, we have equality, that is the distribution of $\ln(S(t)/S_0)$ is equal to that of $\mu t + \sigma\sqrt{t}z$. But $z\sqrt{dt} = d\mathcal{W}_t$ Where z is a random sample drawn from normal distribution, therefore equation (3.1) can be written as

$$\ln\left(\frac{S(t)}{S_0}\right) \approx \sum_{j=1}^{\lfloor nt \rfloor} \chi_j^n \approx t\mu + \sigma\mathcal{W}_t. \tag{3.16}$$

Therefore this proves the fact that the stock price in the binomial model converges to the geometric Wiener process and in-fact $\ln(S(t)/S_0)$ is log-normally distributed. Compare (3.1) and (3.16).

3.2 Convergence of the binomial model to the Black-Scholes model

It is well-known that the binomial model converges to the Black-Scholes model when the number of time steps increases to infinity. The proof relies on a specific case of the central limit theorem. The formula for valuing a call option given by Black-Scholes is

$$C = S_0\mathcal{N}(d_1) - Xe^{-rT}(d_2) \tag{3.17}$$

$$d_1 = \frac{\ln(S_0/X) + (r_c + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

where S_0 is the current stock price, X is the strike price, r_c is the continuously compounded risk-less interest rate, T is the time to expiration and σ^2 is the variance of continuously compounded return of the stock. On the other hand, the option price in the binomial model is given by the following formula as discussed in the previous chapter.

$$C = \frac{1}{R^N}\sum_{j=a}^{N}\binom{N}{j}q^j(1-q)^{N-j}\text{Max}\{S_0U^jD^{N-j} - X, 0\}, \tag{3.18}$$

where $\binom{N}{j}$ represent the number of paths the stock take to reach a certain point in a binomial tree, q is the risk neutral probability of an up move. U and D are parameters of the binomial tree model, $R = 1 + r$ and r is the risk-free interest rate. This expression can be simplified. For some outcome $\text{Max}\{S_0U^jD^{N-j} - X, 0\}$ is zero. Let a represent the minimum number of upward moves for the call to finish in the money. That is, a is the smallest integer such that

$$S_0U^aD^{N-a} > X.$$

Then for all $j < a$, $\max\{S_0U^jD^{N-j} - X, 0\} = 0$, but for j and a, $\max\{S_0U^jD^{N-j} - X, 0\} = (S_0U^jD^{n-j} - X)^+$. We need only to count binomial paths from $j = a$ to N, hence the above model can be written as

$$C = \frac{1}{R^N}\sum_{j=a}^{N}\binom{N}{j}q^j(1-q)^{N-j}[S_0U^jD^{N-j} - X]. \tag{3.19}$$

If we split into two terms we obtain the following;

$$C = S_0\left(R^{-N}\sum_{j=a}^{N}\binom{N}{j}q^j(1-q)^{N-j}U^jD^{N-j}\right) - \frac{X}{R^N}\left(\sum_{j=a}^{N}\binom{N}{j}q^j(1-q)^{N-j}\right) \tag{3.20}$$

Let us call the two terms in the large parentheses $\Psi(a; N, q^*)$ and $\Psi(a; N, q)$. The latter is the probability density function for the binomial distribution.
Note that

$$\frac{q^j(1-q)^{N-j}U^jD^{N-j}}{R^N} = \left(\frac{qU}{R}\right)^j \left(\frac{D(1-q)}{R}\right)^{N-j}$$

Thus the equation above can be written as

$$q^{*j}(1-q^*)^{N-j}$$

where $q^* = \frac{qU}{R}$, and $(1-q^*) = \frac{D(1-q)}{R}$. Thus $\Psi(a; N, q^*)$ also is the binomial probability with probability of each trial being q^*
Therefore the binomial model can be written as

$$C = S_0\Psi(a; N, q^*) - XR^{-N}\Psi(a; N, q). \tag{3.21}$$

We need this to converge to the Black-Scholes formula as given above. We shall have to get $\Psi(a; N, q^*)$ and $\Psi(a; N, q)$ to converge to $\mathcal{N}(d_1)$ and $\mathcal{N}(d_2)$, respectively. However recall that R^{-N} is the present factor for N periods where the per capital rate is R. This present value factor is equivalent to $\exp(-rT)$ when the interest rate is continuously compounded. So the binomial formula is equivalent to

$$C = S_0\Psi(a; N, q^*) - Xe^{-rT}\Psi(a; N, q),$$

Which is the same as (2.13) in the previous chapter. We proceed to get this binomial formula to converge to the Black-Scholes formula.
Since we require $S_0U^aD^{n-a} > X$, it follows that

$$a\ln U + (N-a)\ln D + \ln S_0 > \ln X$$

$$a\ln U + N\ln D - a\ln D + \ln S_0 > \ln X$$

$$a(\ln U - \ln D) > \ln X - \ln S_0 - N\ln D$$

$$a > \frac{\ln(X/S_0D^N)}{\ln(U/D)}$$

But a should be an integer therefore,

$$a = \frac{\ln(X/S_0) - n\ln D}{\ln(U/D)} + \xi$$

where ξ is the number added to make a an integer.
From the famous De Moivre-La Place limit theorem, which says that a binomial model converges to normal if $np \to \infty$ as $n \to \infty$. That is, we need to show for example

$$\Psi(a; N, q^*) \to \int_a^\infty f(j)dj,$$

where $f(j)$ is the probability density function for a normal distribution. We need to convert a random variable j in a standard normal such that $z = \frac{j-\mathbb{E}(j)}{\sigma_j}$. Then we would have

$$\Psi(a; N, q^*) \to \int_a^\infty f(j)dj = \int_d^\infty f(z)dz.$$

where $d = (a - \mathbb{E}(j))/\sigma_j$.
By the symmetry of the normal distribution we can define $d = -(a - \mathbb{E}(j))/\sigma_j$. Hence

$$\Psi(a; N, q^*) \rightarrow \int_a^\infty f(j)dj = \int_{-\infty}^d f(z)dz = \mathcal{N}(d_1)$$

where $\mathcal{N}(d)$ is the cumulative probability distribution function for the standard normal distribution.
Similarly;

$$\Psi(a; N, q) \rightarrow \int_a^\infty f(j)dj = \int_{-\infty}^d f(z)dz = \mathcal{N}(d_2)$$

Now let S_T be the stock price at expiration, such that;

$$S_T = S_0 U^j D^{N-j}$$

$$S_T/S_0 = U^j D^{N-j}.$$

Thus the log return on the stock over the life of the option is

$$\begin{aligned} \ln(S_T/S_0) &= j \ln U + (N - j) \ln D \\ &= j \ln(U/D) + N \ln D. \end{aligned}$$

Then we take the expectation such that,

$$\mathbb{E} \ln(S_T/S_0) = \mathbb{E}(j) \ln(U/D) + N \ln D.$$

Hence

$$\mathbb{E}(j) = \frac{\mathbb{E}[\ln(S_T/S_0)] - N \ln D}{\ln(U/D)}. \tag{3.22}$$

The variance of the log return on the stock over the life of option is
$\mathrm{Var}[\ln(S_T/S_0)] = \mathrm{Var}(j)[\ln(U/D)]^2$.
Thus

$$\mathrm{Var}(j) = \frac{\mathrm{Var}[\ln(S_T/S_0)]}{[\ln(U/D)]^2}. \tag{3.23}$$

Since $d = (-a + \mathbb{E}(j))/\sigma_j$, $a = \frac{\ln(X/S_0) - N \ln D}{\ln(U/D)} + \xi$, and $\mathbb{E}(j)$ and $\mathrm{Var}(j)$ are as given above,

$$\begin{aligned} d &= \frac{\ln(S_0/X) + \mathbb{E}[\ln(S_T/S_0)] - \xi \ln(U/D)}{\sqrt{\mathrm{Var}[\ln(S_T/S_0)]}} \\ &= \frac{\ln(S_0/X) + \mathbb{E}[\ln(S_T/S_0)]}{\sqrt{\mathrm{Var}[\ln(S_T/S_0)]}} - \frac{\xi}{\frac{\sqrt{\mathrm{Var}[\ln(S_T/S_0)]}}{\ln(U/D)}}. \end{aligned} \tag{3.24}$$

From the properties of the binomial distribution, it is known that Var(j)=nq(1-q) where q is the probability per outcome. Therefore

$$d = \frac{\ln(S_0/X) + \mathbb{E}[\ln(S_T/S_0)]}{\sqrt{\mathrm{Var}[\ln(S_T/S_0)]}} - \frac{1}{\sqrt{q(1-q)}} \frac{\xi}{\sqrt{n}}. \tag{3.25}$$

As n goes to infinity, the last term can be ignored. The discrete binomial process is then converging to a continuous log-normal process, for which it is known that $Var[\ln(S_T/S_0)]=\sigma^2 T$. Thus we have

$$d = \frac{\ln(S_0/X) + \mathbb{E}[\ln(S_T/S_0)]}{\sigma\sqrt{T}}. \tag{3.26}$$

But we need this to be equal to d_1 and d_2 as defined by the Black-Scholes formula when the probabilities are q^* and q respectively. This means that we need

$$\mathbb{E}[\ln(S_T/S_0)] = (r + \sigma^2/2)T$$

if the probability is q^* and

$$\mathbb{E}[\ln(S_T/S_0)] = (r - \sigma^2/2)T$$

if the probability is q.
But

$$\begin{aligned} q^* &= \frac{qU}{R} \\ &= \frac{U}{R}\left(\frac{R-D}{U-D}\right) \end{aligned} \tag{3.27}$$

Rearrange to solve for R

$$R = [q^*(1/U) + (1-q^*)(1/d)]^{-1}.$$

Recall that $R^N = r^T = [q^*(1/U) + (1-q^*)(1/d)]^{-N}$ Note that S_0/S_T can be expressed as follow:

$$S_0/S_T = (S_0/S_1)(S_1/S_2)\ldots(S_{n-1}/S_n) = \prod_{i=1}^{N}(S_{i-1}/S_i). \tag{3.28}$$

The expectation of this would be

$$\mathbb{E}(S_0/S_T) = \left[\mathbb{E}\prod_{i=1}^{n}(S_{i-1}/S_i)\right] = \prod_{i-1}^{n}\mathbb{E}(S_{i-1}/S_i) \tag{3.29}$$

Now recall that the probability for $\Psi(a;N,q^*)$ is q^*. Since $S_i = S_{i-1}U$ with probability q^* and $S_i = S_{i-1}D$ with probability $(1-q^*)$, then it follows that

$$\mathbb{E}(S_{i-1}/S_i) = q^*(1/U) + (1-q^*)(1/D) \tag{3.30}$$

Thus

$$\begin{aligned} \mathbb{E}(S_0/S_T) &= \prod_{i=1}^{n}[q^*(1/U) + (1-q^*)(1/D)] \\ &= [q^*(1/U) + (1-q^*)(1/D)]^n. \end{aligned} \tag{3.31}$$

Inverting this gives

$$[\mathbb{E}(S_0/S_T)]^{-1} = [q^*(1/U) + (1-q^*)(1/D)]^{-n}. \tag{3.32}$$

Since $r^T = [q^*(1/U)+(1-q^*)(1/D)]^{-n}$, then $r^T = \mathbb{E}[S_0/S_T]^{-1}$ or $r^{-T} = \mathbb{E}[S_0/S_T]$. Since S_T/S_0 is log-normally distributed, it is also true that the inverse of log-normal distribution is also log-normal distributed. And for any random variable x that is log-normally distributed the following equation holds, i.e $\ln[\mathbb{E}(x)] = \mathbb{E}[\ln x] + \mathrm{Var}[\ln x]/2$. Therefore since S_0/S_T is a random variable which is log-normally distributed, it is true that

$$\begin{aligned} -T\ln r &= \ln[\mathbb{E}(S_0)/S_T] \\ &= \mathbb{E}[\ln(S_0/S_T)] + \mathrm{Var}[\ln(S_0/S_T)]/2 \\ &= \mathbb{E}[-\ln(S_T/S_0)] + \mathrm{Var}[-\ln(S_T/S_0)]/2 \\ &= -\mathbb{E}[\ln(S_T)/S_0] + \mathrm{Var}[\ln(S_T/S_0)]/2. \end{aligned}$$

What we have now is

$$\mathbb{E}[\ln(S_T/S_0)] = T\ln r + \mathrm{Var}[\ln(S_T/S_0)]/2.$$

But we know that $\mathrm{Var}[\ln(S_T/S_0)] = \sigma^2 T$, thus we have

$$\mathbb{E}[\ln(S_T/S_0)] = (\ln r + \sigma^2/2)T. \tag{3.33}$$

c But $\ln r = r_c$ Therefore by Combining (3.26) (3.33) and $\Psi(a; N, q^*)$ converges to $\mathcal{N}(d_1)$ For $\Psi(a; N, q)$ to converge to $\mathcal{N}(d_2)$, recall that $q = \frac{(R-D)}{(U-D)}$, then $R = qU + (1-q)D$. Since $S_i = S_{i-1}U$ with probability q and $S_i = S_{i-1}D$ probability $1-q$, then

$$\mathbb{E}(S_i)/S_{i-1} = qU + (1-q)D.$$

Since

$$\begin{aligned} \mathbb{E}(S_T/S_0) &= \mathbb{E}\prod_{i=1}^{N}(S_i/S_{i-1}) = \prod_{i=1}^{N}\mathbb{E}(S_i/S_{i-1}) \\ &= \prod_{i=1}^{n}[qU + (1-q)D] \\ &= [qU + (1-q)D]^N = R^N = r^T. \end{aligned} \tag{3.34}$$

By taking the logarithm

$$\ln[\mathbb{E}(S_T/S_0)] = \mathbb{E}[\ln(S_T/S_0)] + \mathrm{Var}[\ln(S_T/S_0)]/2 = T\ln r. \tag{3.35}$$

Thus we have

$$\mathbb{E}[\ln(S_T/S_0)] = T\ln r - \mathrm{Var}[\ln(S_T/S_0)]/2 \tag{3.36}$$

Recalling that $\mathrm{Var}[\ln(S_T/S_0)] = \sigma^2 T$, we have

$$\mathbb{E}[\ln(S_T/S_0)] = (\ln r - \sigma^2/2)T \tag{3.37}$$

recall $\ln r = r_c$ hence $\Psi(a; N, q)$ converges to $\mathcal{N}(d_2)$. Thus, the binomial model converges to Black-Scholes.

Chapter 4

Numerical implementations

In this chapter we are going to analyse the numerical methods used to explain the rate and order of convergence of the binomial model to the Black-Scholes model. We will explain three lattice approaches developed simultaneously by Cox, Ross, and Rubinstein CRR(1979), Jarrow and Rudd (1983), and Tian(1993)[2] to price derivative securities numerically. From the second chapter we have seen that the formula for pricing options in the binomial model is given by the equation below,

$$C(S_0, t) = S_0 \Psi(a; n, q^*) - X R^{-N} \Psi(a; n, q) \tag{4.1}$$

where $a = \left\lfloor \frac{\ln(X/S_0) - n \ln D}{\ln U - \ln D} \right\rfloor$, $q = \frac{R-D}{U-D}$ and $q^* = \frac{qU}{R}$ and n is the number of periods. These authors use different numerical approaches to improve the binomial option pricing model.

4.1 Parameters used in the CRR model

The numerical method developed by Cox, Ross and Rubinstein[79] used the following parameters;

$$U = \exp\left(\sigma\sqrt{T/n}\right), \quad D = \exp\left(-\sigma\sqrt{T/n}\right) \text{ and } R = \exp\left(rT/n\right)$$

where n is the number of steps used. T is the expiration date. The values of the parameters used in running the simulations are as indicated in figure 4.1. The algorithms used to simulate the oscillations are shown in the appendix. From figure 4.1 the oscillations and wave patterns indicate results from option price from the binomial model. The straight line indicates the Black-Scholes solution. At the beginning of the oscillations, the magnitude of the amplitude ranges from 9.78 to 10.29. It can be observed that as the number of steps increase the amplitudes of oscillations decrease which shows that the binomial model converges to the Black -Scholes model.

4.2 Parameters used in the JR model

There exist many extensions of the CRR model. Jarrow and Rudd (1983)[2] JR, adjusted the CRR model by adding the local drift term. They constructed a binomial model where the first two moments of the discrete and continuous time return processes match. As a consequence a probability measure equal to a half results. Therefore the CRR and JR models are sometimes attributed as

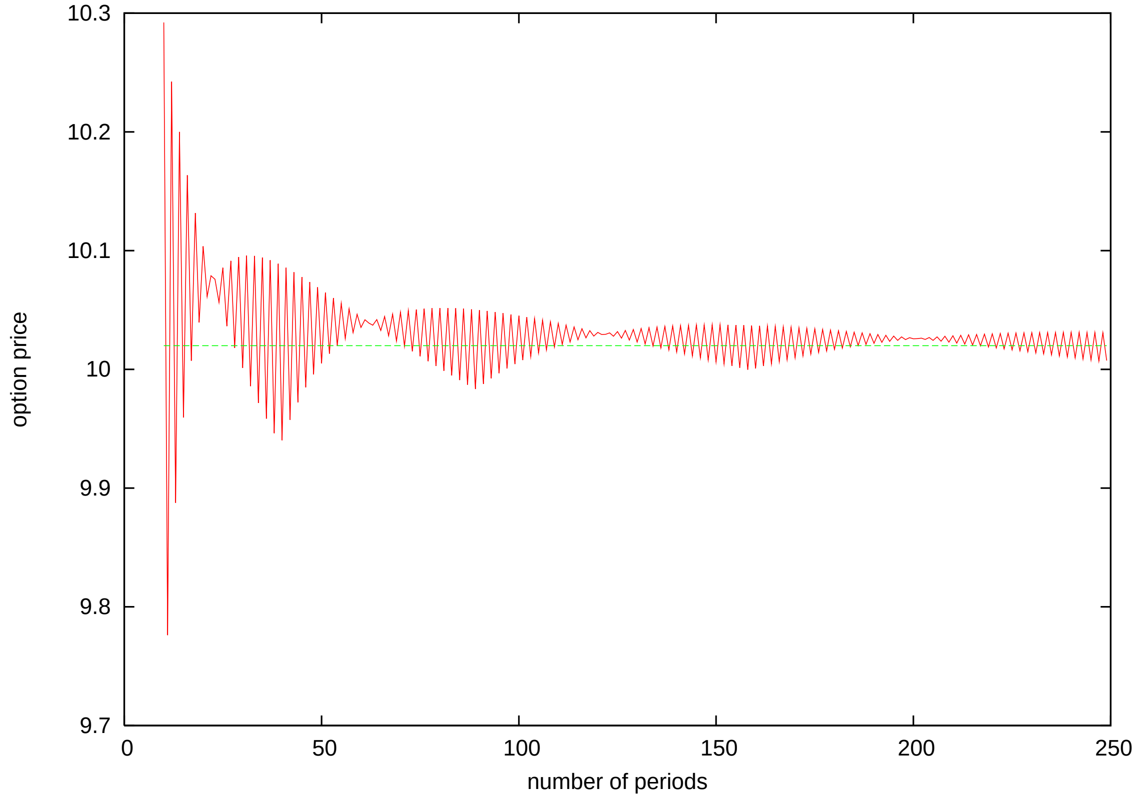

Figure 4.1: typical pattern resulting from option price for valuing call option calculations with binomial model: example with CRR-Model with parameters: $S = 100, X = 110, T = 1, r = 0.05, \sigma = 0.3, n = 10, \ldots, 250$

equal jump binomial trees and equal probabilities binomial trees [2]. Jarrow and Rudd(1983) used the following parameters in their numerical implementation to simulate the oscillations of option pricing in the binomial model.

$$U = \exp\left(\mu T/n + \sigma\sqrt{T/n}\right), \quad D = \exp\left(\mu T/n - \sigma\sqrt{T/n}\right), \quad \mu = r - 1/2\sigma^2$$

where n is the number of steps, μ is the expected return of the asset price, r is the risk-free interest rate. The algorithms used to simulate the oscillations are as indicated in the appendix. The figure resulted after running the simulation is shown in figure 4.2. At the beginning of the oscillations, the magnitude of the amplitudes ranges from 9.72 to 10.29 which is slightly bigger than the CRR model. Also it can be observed that as the number of steps increases the amplitudes of the oscillations decrease. The option prices oscillate unsymmetrically with changing amplitude around the Black-Scholes solution for the European call option.

4.3 Parameters used in TIAN model

Tian (1993) proposed binomial and trinomial models where the model parameters are derived as unique solutions to equation systems, established from sufficient conditions to acquire weak

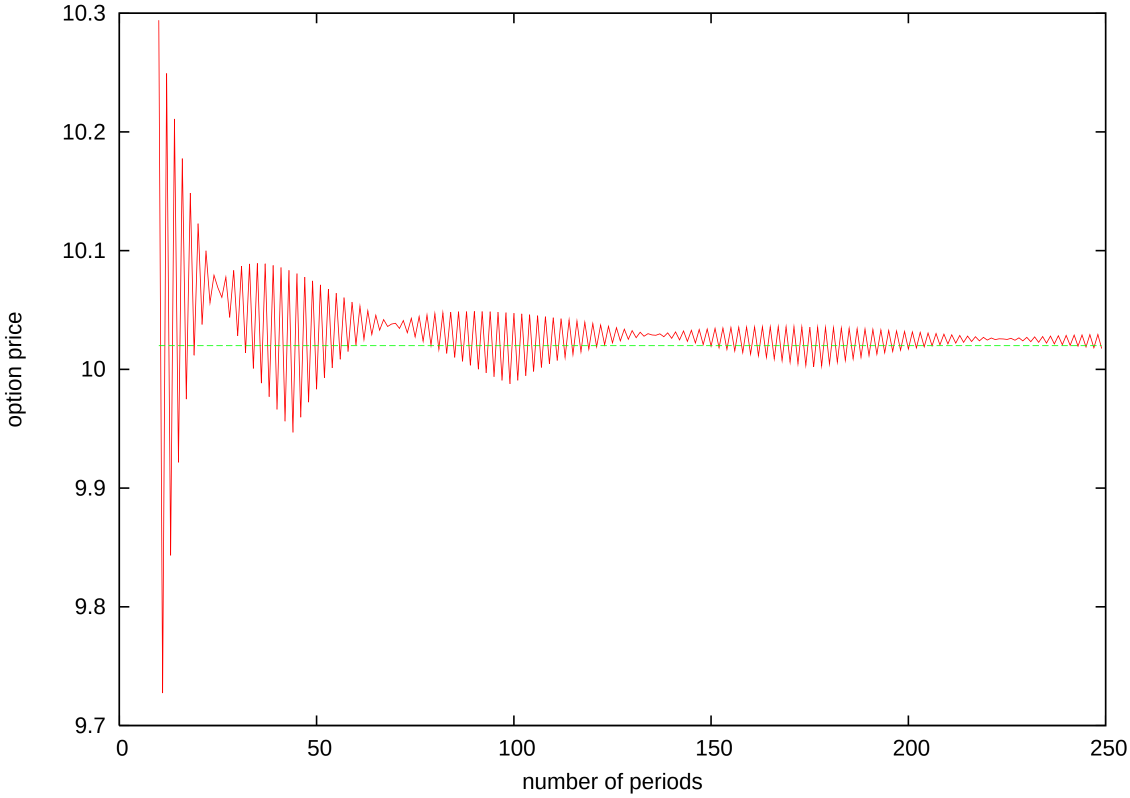

Figure 4.2: typical pattern resulting from option price for valuing call option calculations with binomial model: example with JR-Model with parameters: $S = 100, X = 110, T = 1, r = 0.05, \sigma = 0.3, n = 10, \ldots, 250$

convergence. The parameters are as indicated below,

$$U = \frac{R\nu}{2}\left(\nu + 1 + \sqrt{\nu^2 + 2\nu - 3}\right), D = \frac{R\nu}{2}\left(\nu + 1 - \sqrt{\nu^2 + 2\nu - 3}\right), \nu = \exp\left(\sigma^2 T/n\right), R = \exp\left(rT/n\right)$$

The results obtained after running the simulation is as shown in figure 4.3. The straight line is the value of the option calculated from the Black-Scholes formulae. At the beginning of the oscillations, the magnitude of the amplitude ranges from 9.95 to 10.15 which shows a slight difference compared to the CRR model and JR model. But as the number of steps increases, the magnitude of the amplitudes decrease which indicates that the option price in the binomial model converges to the Black-Scholes model.

Proposition 4.1 : The lattice-approaches proposed by CRR(1979), JR(1983), TIAN(1993) converge with order one. The proof is shown in [2]

4.4 Odd-even binomial model

In the sections above, we have been dealing with the homogeneous binomial model in which there is no distinction between the increments. In particular in the homogeneous binomial model, conver-

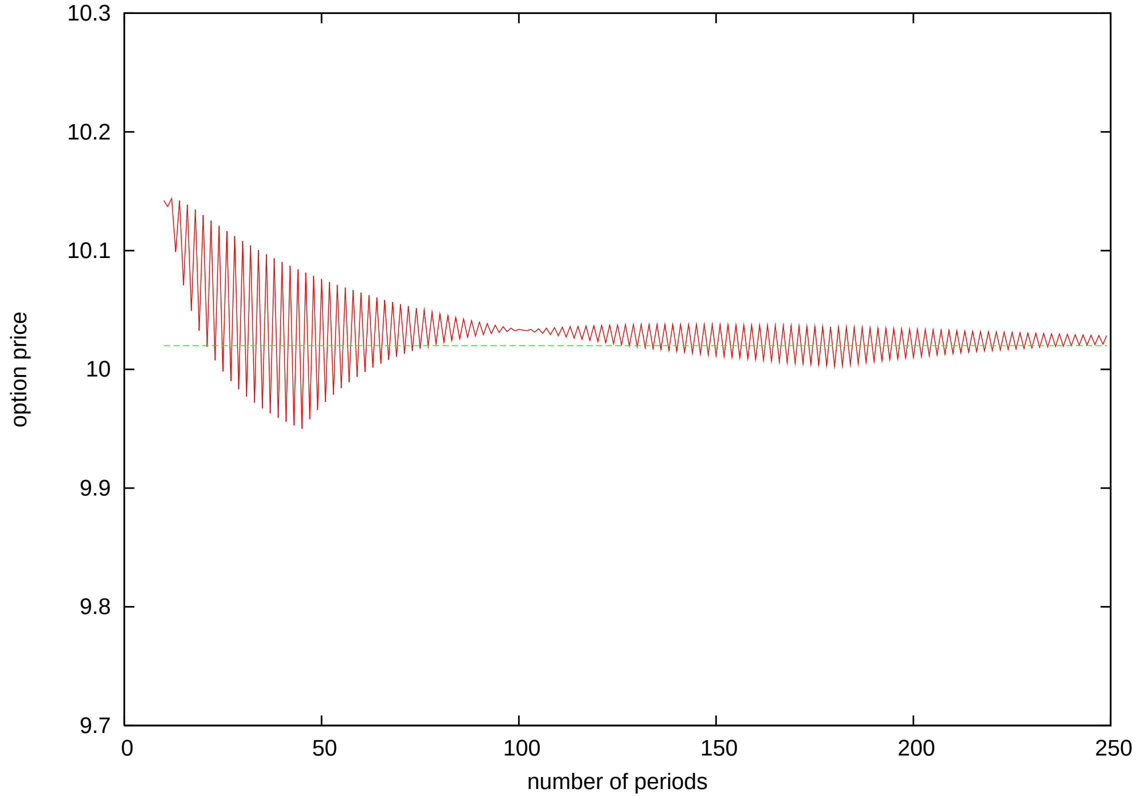

Figure 4.3: typical pattern resulting from option price for valuing call option calculations with binomial model: example with JTIAN-Model with parameters: $S = 100, X = 110, T = 1, r = 0.05, \sigma = 0.3, n = 10, \ldots, 250$

gence in stock price implies convergence in option price.[4] In this section we describe the odd-even binomial model in contrast to the homogeneous binomial model.

Definition 4.1 :

Generally a sequence of asset prices (S^n) is a binomial model if each discounted asset price process (S^n) evolves as follows: for $t \in [0, T]$

$$S_t^n = S_0 \exp\left(\sum_{k=1}^{\lfloor nt \rfloor} \xi_k^n\right) \tag{4.2}$$

where n is the number of periods, and k is the number of intervals, $S_0 > 0$ is constant and the increments (ξ_k^n) of logarithmic discounted returns $X_t^n = \sum_{k=1}^{\lfloor nt \rfloor} \xi_k^n$ form a row-wise independent triangular array. The random variables ξ_k^n assume two values U_k^n and D_k^n with positive probabilities p_k^n and $1 - p_k^n$, thus

$$\begin{cases} \xi_k^n = U_k^n & \text{with probabilities } \ p_k^n, \\ \xi_k^n = D_k^n & \text{with probabilities } \ 1 - p_k^n. \end{cases} \tag{4.3}$$

For $k = 1, \ldots, n$ and $D_k^n < 0$ and $U_k^n > 0$. The model is called homogeneous if (U_k^n, D_k^n, p_k^n) depend on n but not on k. It is called an **odd-even binomial model** if these parameters depend on n

and the parity of k.

In the odd-even binomial model, we find some convergence criteria. Since it has been difficult to investigate the nature of convergence for some odd-even parameters of the binomial tree, it is not easy to say numerically that the odd-even binomial model converges. This should depend on the choice of the parameters of the binomial tree. Analytically, the odd-even binomial model under risk-neutral probability converges to a geometric Wiener process[4]. See example 4.1. Thus, the nature of convergence is as indicated in figure 4.4.

Example 4.1 : *There is an odd-even model such that under the physical probability measures P^n the sequence (S^n) converges in distribution to geometric Brownian(Wiener) motion with parameters μ and σ^2, and under the risk-neutral probability measure (Q^n) the sequence of stock prices (S^n) converges to the geometric Brownian motion with parameters $-\sigma^2/2$ and σ.[4]*

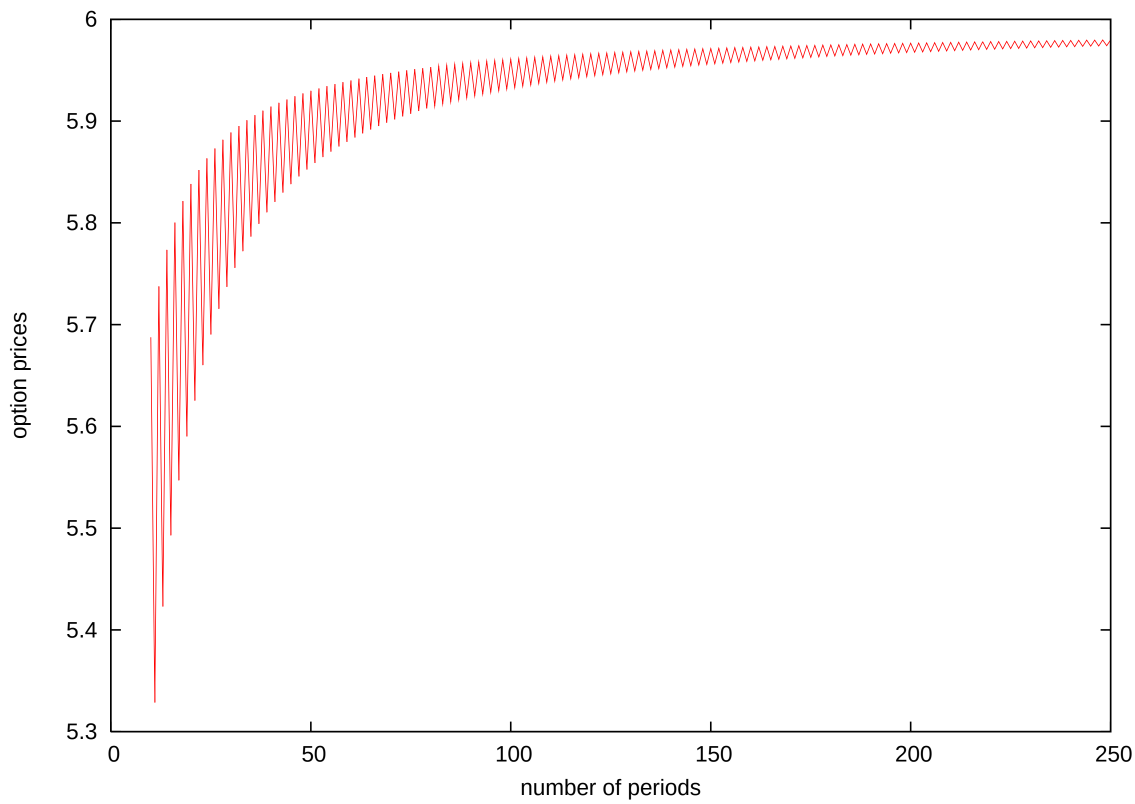

Figure 4.4: typical pattern resulting from option price for valuing call option calculations with odd-even binomial model: with parameters: $S = 100, X = 110, T = 1, r = 0.05, \sigma = 0.3, n = 10, \ldots, 250$

From the data obtained we can say that the stock price has the log-normal distribution which agreed with the assumptions of the Black-Scholes model. The random variable has a log-normal distribution if the natural logarithm of the random variable is normally distributed. The histogram shows that the logarithm of the stock price is normally distributed.

Theorem 4.1 : *Suppose a sequence of homogeneous binomial models (S^n) with $U_k^n \to 0$, $D_k^n \to 0$ convergences in distribution under P^n to Black-Scholes model with parameters μ, σ^2. Then under*

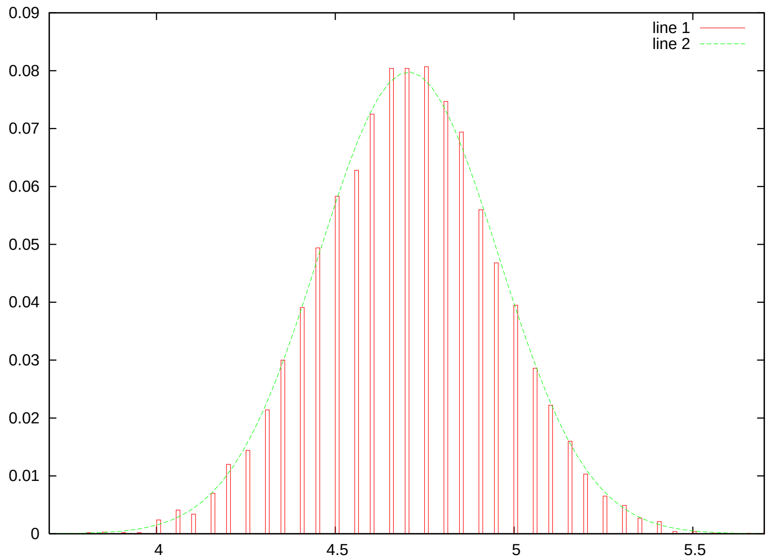

Figure 4.5: Histogram of the logarithmic of the stock price plotted against frequency

the corresponding martingale measures Q^n *the sequence* $\{S^n|Q^n\}$ *converges to the Black-Scholes model with parameters* $-\sigma^2/2, \sigma^2$

Proof

Given the following assumptions

$$n[U^n p^n + D^n(1-p^n)] \to \mu, n(U^n - D^n)^2 p^n(1-p^n) \to \sigma^2. \tag{4.4}$$

where n is the number of steps of the binomial tree model, U^n and D^n are tree parameters (binomial model parameters). This follows from the central limit theorem. We claim that

$$\sigma^2 = -\lim nU^n D^n. \tag{4.5}$$

By the assumption (4.4) we have, $U^n p^n + D^n(1-p^n) = O(1/n)$. Therefore we can write

$$(U^n - D^n)p^n = -D^n + O(1/n),\ (U^n - D^n)(1-p^n) = U^n + O(1/n) \tag{4.6}$$

Then by multiplying the equations in (4.6) you obtain the following;

$$n(U^n - D^n)p^n(1-p^n) = -nD^nU^n + O(U^n) + O(D^n) + O(1/n) \tag{4.7}$$

But from the Theorem (4.1)we consider models with $U^n \to 0$ and $D^n \to 0$. Therefore from (4.4) we have proved the claim that $\sigma^2 = -\lim nU^nD^n$

The risk-neutral probability measure is given by the familiar formula (Cox, Ross, and Rubinstein 1979[6]; Rachev, Rüschendorf 1994[11]; Shiryaev, Kramkov, and Mel'nikov 1994[14]; Pliska (1997)[10]

$$q_k^n = \frac{1 - e^{D_k^n}}{e^{U_k^n} - e^{D_k^n}}. \tag{4.8}$$

But the homogeneous binomial model depends on n and not on k, therefore we have

$$q^n = \frac{1 - e^{D^n}}{e^{U^n} - e^{D^n}} \tag{4.9}$$

By doing the Taylor expansion and calculate the asymptotic expansion of the risk neutral probabilities, we get the following expansion

$$q^n = \frac{-D^n}{U^n - D^n}\left[1 - \frac{D^n}{2} + O(U^n - D^n)^2\right]. \tag{4.10}$$

We find,

$$n(U^n - D^n)^2 q^n(1 - q^n) = -nU^n D^n + O(D^n U^n (U^n - D^n)) + O(\frac{U^n - D^n}{n}). \tag{4.11}$$

Finally equation (4.11) shows that $n(U^n - D^n)q^n(1 - q^n) \to \sigma^2$. Also we can see that,

$$U^n q^n + D^n(1 - q^n) = \frac{D^n U^n}{2} + O(U^n - D^n)^3. \tag{4.12}$$

Equation (4.12) shows that $n(U^n q^n + D^n(1 - q^n)) \to -\sigma^2/2$. Hence the theorem proved.

In the financial market this theorem is useful, it explains why is it necessary to price option under risk-neutral probability measure rather than the normal physical probability. This is because if the option is wrongly priced it can allow the arbitrage opportunity (see prop.2.1), and this is obvious when we price under physical probability measure.

Conclusion

In this essay we have seen how the option price in the binomial model converges to the option price in the Black-Scholes model. We have also seen that the stock price in the binomial model converges to the geometric Wiener process. These concepts have been possible through the application of the Central limit theorem which says that *the sum of independent, identically distributed random variables with finite mean* μ *and variance* σ^2 *is approximately normal as the sample size* n *becomes large.* For this case we have considered the stock price as log-normally distributed random variables which are independent and identically distributed. We have used both analytical and numerical approaches to prove the convergence criteria. In the numerical approaches we have seen the lattice model developed by Jarrow and Rudd; and Tian models. These authors made some modifications to the binomial model developed by Cox, Ross and Rubinstein by adding some parameters such as the local drift term. The binomial model is a powerful discrete market model that converges to the continuous market model as the length of the time step decreases. We have seen the homogeneous binomial model as well as the odd-even binomial model. In the homogeneous binomial model there is no difference between the increments of the parameters U_k^n and D_k^n, that is for every n-period model U and D are the constant for all the levels k of the tree, and in particular in the homogeneous binomial model convergence in the stock price implies the convergence in the option price. In the odd-even binomial model the parameters U_k^n and D_k^n depend on n and the parity of k. Some of the proof has been attached in the appendix as well as the programs for running the simulations which were written in the octave programing language.

Appendix A

Derivation of Black-Scholes pricing formula for European options

The Black-Scholes equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{A.1}$$

as shown in chapter two. This equation must be solved with final condition depending on the payoff, each contract will have different functional form prescribed at expiry $t = T$, depending on whether it is a call or put. Let us change from the present to the future since we valuing the option at the time t and payoff is received at time T. Then the value of an option in the future will be given as

$$V(S,t) = e^{-r(T-t)}U(S,t)$$

this takes our differential equation to

$$\frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S} = 0$$

Because we are solving backward let $\tau = T - t$ so that

$$\frac{\partial U}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S} = 0 \tag{A.2}$$

Let $\xi = \log S$ then we find $\frac{\partial}{\partial S} = e^{-\xi}\frac{\partial}{\partial \xi}$ and $\frac{\partial^2}{\partial S^2} = e^{-2\xi}\frac{\partial^2}{\partial \xi^2} - e^{-2\xi}\frac{\partial}{\partial 2\xi}$ thus equation (A.2) becomes

$$\frac{\partial U}{\partial \tau} = \frac{1}{2}\sigma^2 e^{2\xi}\left(e^{-2\xi}\frac{\partial^2 U}{\partial \xi^2} - e{-}2\xi\frac{\partial U}{\partial \xi}\right) + re^{\xi}\left(e^{-\xi}\frac{\partial U}{\partial \xi}\right)$$

$$\frac{\partial U}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 U}{\partial \xi^2} + (r - \frac{1}{2}\sigma^2)\frac{\partial U}{\partial \xi}$$

Note we change the problem from $0 \leq S \leq \infty$ to one defined as $-\infty \leq \xi \leq \infty$; and all coefficients are constant independent of the underlying. this is called the *log-normality* of the underlying asset.

Let's write $x = \xi + (r - \frac{1}{2}\sigma^2)\tau$ and $U = W(x,\tau)$. This is just translation of the coordinate system. Let $y = \tau$ so that for any arbitrary function $f(\xi,\tau)$

$$\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial^2 f}{\partial \xi^2} = \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi}\right) + \frac{\partial y}{\partial \xi}\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi}\right)$$

From these chain formula we have

$$\frac{\partial U}{\partial \tau} = (r - \frac{1}{2}\sigma^2)\frac{\partial W}{\partial x} + \frac{\partial W}{\partial \tau}$$

$\frac{\partial U}{\partial \xi} = \frac{\partial W}{\partial x}$ and $\frac{\partial^2 U}{\partial \xi^2} = \frac{\partial^2 W}{\partial x^2}$ so that we have:

$$\frac{\partial W}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 W}{\partial x^2} \tag{A.3}$$

Now we are going to derive an expression for the value of any option whose payoff is known function of the asset price at expiry. The fundamental solution of equation A.3 has the form $W(x,\tau) = \tau^\alpha f\left(\frac{x-x'}{\tau^\beta}\right)$, where x' is arbitrary constant. Need to find parameters α, β: Using the composite function differentiation $\frac{df[g(x)]}{dx} = f'[g(x)] \times g'(x)$ we have

$$\frac{\partial W}{\partial \tau} = \tau^{\alpha-1}\left(\alpha f - \beta\frac{(x-x')}{\tau^\beta}f'\left(\frac{x-x'}{\tau^\beta}\right)\right)$$

Let $\eta = \frac{x-x'}{\tau^\beta}$ then L.H.S

$$\frac{\partial W}{\partial \tau} = \tau^{\alpha-1}\left(\alpha f - \beta\eta\frac{\partial f}{\partial \eta}\right)$$

and $\frac{\partial W}{\partial x} = \frac{\tau^\alpha}{\tau^\beta}f'\left(\frac{x-x'}{\tau^\beta}\right)$, $\frac{\partial^2 W}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\tau^\alpha}{\tau^\beta}f'\left(\frac{x-x'}{\tau^\beta}\right)\right) = \frac{\tau^\alpha}{\tau^{2\beta}}f'\left(\frac{x-x'}{\tau^\beta}\right) = \tau^{\alpha-2\beta}\frac{\partial^2 f}{\partial \eta^2}$ Thus equation A.1 becomes

$$\tau^{\alpha-1}\left(\alpha f - \beta\eta\frac{\partial f}{\partial \eta}\right) = \frac{1}{2}\sigma^2\tau^{\alpha-2\beta}\frac{\partial^2 f}{\partial \eta^2} \tag{A.4}$$

Thus we can have solution if $\alpha - 1 = \alpha - 2\beta$ that is $\beta = \frac{1}{2}$

We want our 'special solution' to have the property that its integral over all ξ is independent of τ. To ensure this we require $\int_{-\infty}^{\infty} \tau^\alpha f\left(\frac{x-x'}{\tau^\beta}\right)dx$ to be constant. So we can write $\int_{-\infty}^{\infty} \tau^{\alpha+\beta} f(\eta)d\eta$ and we need $\alpha = -\beta = \frac{1}{2}$.

The function f now satisfies $-f - \eta\frac{\partial f}{\partial \eta} = \sigma^2\frac{\partial^2 f}{\partial \eta^2}$ this can be written

$$\sigma^2\frac{\partial^2 f}{\partial \eta^2} + \frac{\partial(\eta f)}{\partial \eta} = 0$$

Integrate once you get

$$\sigma^2\frac{\partial f}{\partial \eta} + \eta f = a$$

(a is constant). for special solution we require $a = 0$ and on integration we have;

$$f(\eta) = be^{-\frac{\eta^2}{2\sigma^2}}$$

We choose the constant b in such away that the integral of f from $-\infty$ to $+\infty$ is equal to one.

$$f(\eta) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{\eta^2}{2\sigma^2}}$$

This is the special function we were seeking then $W(x,\tau) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-x')^2}{2\sigma^2\tau}}$

This function can be approximated by Dirac delta function $\delta(x'-x)$ as $\tau \to 0$ we know

$$\int \delta(x'-x)g(x)dx' = g(x)$$

Thus in the limit as $\tau \to 0$ the function W becomes a delta function at $x = x'$. This means that;

$$\lim_{\tau\to 0} \frac{1}{\sigma\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{(x-x')^2}{2\sigma^2\tau}} g(x')dx' = g(x)$$

This property of special function and linearity of the Black-Scholes equation are all that we needed to find some explicit solutions.

Now the payoff(S) is the value of option at time $t = T$. It is the final condition for the function V, satisfying the Black-Scholes equation $V(S,T) = \text{payoff}(S)$ with our variable this function is $W(x,0) = \text{payoff}(e^{x'})$. But for $\tau > 0$ it claimed that [9] $W(x,\tau) = \int_{-\infty}^{\infty} W_f(x,\tau;x')\text{payoff}(e^{x'})dx'$.

Referring to our steps to write our solution in terms of the original variables, we get

$$V(S,t) = \frac{e^{-(T-t)}}{\sigma\sqrt{s\pi(T-t)}} \int_0^{\infty} e^{-\frac{\log(\frac{S}{S'})+(r-\frac{1}{2}\sigma^2)(T-t)^2}{2\sigma^2(T-t)}} \text{payoff}(S')\frac{dS'}{S'} \tag{A.5}$$

We have $x' = \log S'$. This is exact solution for the option value in terms of the arbitrary payoff function. c

A.0.1 Formula for Call option

payoff(S) $= \max(S-X,0)$, using equation A.5 we have

$$\frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_X^{\infty} e^{-\frac{\log(\frac{S}{S'})+(r-\frac{1}{2}\sigma^2)(T-t)^2}{2\sigma^2(T-t)}} (S'-X)\frac{dS'}{S'}$$

But $x' = \log S'$, then

$$\begin{aligned}
&\frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{\log X}^{\infty} e^{-\frac{-x+\log S+(r-\frac{1}{2})(T-t)^2}{2\sigma^2(T-t)}} (e^{x'}-X)dx' \\
&= \frac{e^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{\log X}^{\infty} e^{-\frac{-x+\log S+(r-\frac{1}{2})(T-t)^2}{2\sigma^2(T-t)}} e^{x'}dx' \\
&-\frac{Ee^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}} \int_{\log X}^{\infty} e^{-\frac{-x+\log S+(r-\frac{1}{2})(T-t)^2}{2\sigma^2(T-t)}} dx'
\end{aligned}$$

Both of this integral can written as $\int_d^{\infty} e^{-\frac{x'^2}{2}}$ for some d. Thus the call option can be written as

$$C = S\mathcal{N}(d_1) - Xe^{-r(T-t)}\mathcal{N}(d_2) \tag{A.6}$$

with

$$d_1 = \frac{\log(\frac{S}{X}) + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\log(\frac{S}{X}) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$$

When there is continuous dividend yield on the underlying,

$$C = Se^{-q(T-t)}\mathcal{N}(d_1) - Xe^{r(T-t)}\mathcal{N}(d_2) \tag{A.7}$$

where

$$d_1 = \frac{\log(\frac{S}{X}) + (r - q + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\log(\frac{S}{X}) + (r - q - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$$

A.0.2 Formula for Put option

The put option has payoff

$$\text{Payoff}(S) = \max(X - S, 0).$$

The value of put option can be found in the same way as in the case of call option.

$$P = -S\mathcal{N}(-d_1) + Xe^{-r(T-t)}\mathcal{N}(-d_2) \tag{A.8}$$

with the same d_1 and d_2.

When there is continuous dividend yield on the underlying,

$$P = -Se^{-q(T-t)}\mathcal{N}(-d_1) + Xe^{r(T-t)}\mathcal{N}(-d_2) \tag{A.9}$$

with the same d_1 and d_2 as equation A.6.

Appendix B

Numerical methods

```
%THE ALGORITHMS RUN BY OCTAVE PROGRAMING LANGUAGE TO SIMULATE
%CCR-MODEL,JR-MODEL AND TIAN-MODEL.
-------------------------------------------------------------------------------

%Simulation by using CRR model for valuing call option.
clear all;

s=100;                              %The current stock price
k=110;                              %Strike price
T=1;
r=0.05;
sigma=0.3;

out=fopen("converge.dat","w");
i=10;
while(i<100)
  n=i;
  u=exp(sigma*sqrt(T/n));
  d=1/u;
  R=exp(r*T/n);
  p=(R-d)/(u-d);                     %Risk-neutral probability measure.
  q=(u*p)/R;
  a=floor((log(k/s)-n*log(d))/log(u/d));
 %cn is the option price from the binomial pricing formula
  cn=s*(1-binomial_cdf(a, n, q))-(k*(1-binomial_cdf(a,n,p)))/R^n;
  fprintf(out,"%d\t%f\n",n,cn);
  i=i+1;
endwhile
fclose(out);


-------------------------------------------------------------------------------
%Simulation by using JR[83] model for valuing call option.
```

```
clear all;

s=100;                      %The current stock price
k=110;                       %Srike price
T=1;
r=0.05;
sigma=0.3;
mu=r-sigma^2/2;
out=fopen("JRmodel.dat","w");
i=10;
while(i<100)
  n=i;
% u and d are the binomial tree parameters proposed by Jarrow and Rudd in 1983
  u=exp(mu*T/n+sigma*sqrt(T/n));
  d=exp(mu*T/n-sigma*sqrt(T/n));
  R=exp(r*T/n);
  p=(R-d)/(u-d);
  q=(u*p)/R;
  a=floor((log(k/s)-n*log(d))/log(u/d));
  cn=s*(1-binomial_cdf(a, n, q))+(k*(1-binomial_cdf(a,n,p)))/R^n;
  fprintf(out,"%d\t%f\n",n,cn);
  i=i+1;
endwhile
fclose(out);

--------------------------------------------------------------------------------
%Simulation by using TIAN[93] model for valuing the call option.
clear all;

s=100;                      %The stock price at time zero
k=110;                      %The strike price
T=1;
r=0.05;
sigma=0.3;
out=fopen("TIANmodel.dat","w");
i=10;
while(i<120)
  n=i;
% u and d are the binomial tree parameters proposed by Tian in 1993
  R=exp(r*T/n);
  v=exp(sigma^2*T/n);
  u=R*v/2*(v+1+sqrt(v^2+2*v-3));
  d=R*v/2*(v+1-sqrt(v^2+2*v-3));
  p=(R-d)/(u-d);
  q=(u*p)/R;
  a=floor((log(k/s)-n*log(d))/log(u/d));
  cn=s*(1-binomial_cdf(a, n, q))-(k*(1-binomial_cdf(a,n,p)))/R^n;
  fprintf(out,"%d\t%f\n",n,cn);
```

```
  i=i+1;
endwhile
fclose(out);
```

```
%PROGRAM TO RUN BROWNIAN MOTION BY USING OCTAVE PROGRAMING LANGUAGE
clear all;

out=fopen("pendo.dat","w");
i=0;
x=0;
dt=0.001;
mu=0.1; % mu is given
sigma=0.2; % also sigma

while( i < 3)
  b=randn(1);                   %This generate random number from -1 to 1
%This is the log-normal random walk differential equation.
  x= x + mu*dt + sigma * (sqrt(dt)*b);
  fprintf(out,"%f\t%f\n",i,x);
  i=i+dt;
endwhile

fclose(out);
```

```
% THIS PROGRAM CONSTRUCTS THE ODD-EVEN BINOMIAL TREE
% as defined in [4], Definition 3.3
% References: [4] Hubalek and Schachermayer, "When does convergence of
% assest prices imply convergence of option prices?"
% Authors: Pendo Kiyiro, Diane Wilcox and Mike Pickles.
% Date: 23 May 2005
% Revision: 1.1

clear all;

% Initialise variables
s=100;
k=110;
T=1;
r=0.05;
sigma=0.3;
a=0.1 ;
out=fopen("ODDpendo2.dat","w");

for n=10:250
```

```
  disp(n);                 %how far has the program got?
   %Construct tree with n periods
   % determine u and d at ith level of n-period tree
   % (for each n-period tree the u and d values will depend on n
   % and on i, where i denotes the level in the n-period tree)
mu=0.1;
sigma1=0.3;
sigma2=0.2;

  uodd= exp(sigma1/sqrt(n)+mu/n);
  ueven=exp(sigma2/sqrt(n)+mu/n);
  dodd = exp(-sigma2/sqrt(n)+mu/n);
  deven= exp(-sigma1/sqrt(n)+mu/n);

  qodd  = (1-dodd)/(uodd-dodd);
  qeven = (1-deven)/(ueven-deven);

  if (qodd<0)
    disp('q negative!');
  endif
  if (qeven<0)
    disp('q negative!');
  endif
 % determine stock prices at ith level of n-period tree
  neven=floor((n-1)/2);
  nodd=floor(n/2);

  S(1,1,1)=s;

  S(2,1,1)=S(1,1,1)*dodd;
  S(2,2,1)=S(1,1,1)*uodd;

  for i=2:(n-1)

    ieven=floor((i-1)/2);
    iodd=floor(i/2);
    if (rem(i,2)==0)              %for even i the steps look like this
      for jodd=1:iodd+1
for jeven=1:ieven+1
  S(i+1,jodd,jeven+1)=S(i,jodd,jeven)*ueven;
endfor
S(i+1,jodd,1)=S(i,jodd,1)*deven;
      endfor

    endif

    if (rem(i,2)==1)              %for odd i the steps look like this
      for jeven=1:ieven+1
```

```
for jodd=1:iodd+1
  S(i+1,jodd+1,jeven)=S(i,jodd,jeven)*uodd;
endfor
S(i+1,1,jeven)=S(i,1,jeven)*dodd;
      endfor
    endif
   endfor                        %end of the for i=1..n loop


  % determine option payoffs at nth period (expiry)
  for jeven=1:neven+1
    for jodd=1:nodd+1
      C(n,jodd,jeven)=max(S(n,jodd,jeven)-k,0);
    endfor
  endfor

   % evaluate option values at each level by iterating backwards through
   % the tree
  for i=n-1:-1:1

    ieven=floor((i-1)/2)+1;
    iodd=floor(i/2)+1;

    if (rem(i,2)==1)
      for jeven=1:ieven
for jodd=1:iodd
  C(i,jodd,jeven)=qodd*C(i+1,jodd+1,jeven)+(1-qodd)*C(i+1,jodd,jeven);
endfor
      endfor

    endif

    if (rem(i,2)==0)
      for jodd=1:iodd
for jeven=1:ieven
 C(i,jodd,jeven)=qeven*C(i+1,jodd,jeven+1)+(1-qeven)*C(i+1,jodd,jeven);
endfor
      endfor
    endif
    cn=C(1,1,1);
    endfor
  fprintf(out,"%d\t %f\n ",n,cn);
endfor
fclose(out);

-------------------------------------------------------------------------------
%THE PROGRAM TO GENERATE THE LOG-NORMAL
%DISTRIBUTION OF THE STOCK PRICE
```

```
%Author:Mike Pickles, Pendo Kivyiro and Diane Wilcox.
clear all;

n=100;             %number of timesteps in one experiment
total=10000;       %run total experiments
S0=100;
nodd=floor((n+1)/2);
neven=floor(n/2);
storeS=[];

p=0.5;            %probability of moving down
mu=0.1;
sigma1=0.3;
sigma2=0.2;

  uodd= exp(sigma1/sqrt(n)+mu/n);
  ueven=exp(sigma2/sqrt(n)+mu/n);
  dodd = exp(-sigma2/sqrt(n)+mu/n);
  deven= exp(-sigma1/sqrt(n)+mu/n);
  qodd  = (1-dodd)/(uodd-dodd);
  qeven = (1-deven)/(ueven-deven);
  if (qodd<0)
    disp('q negative!');
  endif
  if (qeven<0)
    disp('q negative!');
  endif
  for j=1:total
  S=S0;
  for i=1:n             %do n timesteps
   test=rand(1);
    if (test>p)         %move up with probability 0.5
      if (rem(i,2)==0)  %i is even
S=S*ueven;
      endif
     if (rem(i,2)==1)   %i is odd
S=S*uodd;
      endif
    endif
   if (test<=p)         %move down with probability 0.5
   if (rem(i,2)==0)     %i is even
S=S*deven;
      endif
    if (rem(i,2)==1)  %i is odd
S=S*dodd;
      endif
    endif
  endfor
```

```
storeS=[storeS;S];
endfor
y=log(storeS)                              %changes to logarithm
sigma=sqrt(p*(1-p))*(sigma1+sigma2);       %theoretical standard deviation
z = linspace(min(y)-0.5,max(y)+0.5,100)  %an approximation to where
 the histogram is.
z2=1/(20*sqrt(2*pi*sigma^2)) *
  exp(-(z-(mu+log(S0))).*(z-(mu+log(S0)))/(2*sigma^2));
                %The theoretical mean of log(storeS/S0)
hist(y,175,1)   %use 175 bins, so each bin will be very narrow
hold on
plot(z,z2)
hold off
```

Refinement	CRRmodel	JRmodel	TIANmodel	CRRerror	JRerror	TIANerror
10	10.292187	10.294023	10.142217	-0.272187	-0.274023	-0.122217
11	9.776203	9.727338	10.137172	0.243797	0.292662	-0.117172
12	10.242369	10.249297	10.143919	-0.222369	-0.229297	-0.123919
13	9.887574	9.843215	10.098915	0.132426	0.176785	-0.078915
14	10.199981	10.210896	10.142120	-0.179981	-0.190896	-0.122120
15	9.959455	9.921630	10.070905	0.060545	0.098370	-0.050905
16	10.163487	10.177631	10.138674	-0.143487	-0.157631	-0.118674
17	10.007212	9.974899	10.049512	0.012788	0.045101	-0.029512
18	10.131694	10.148523	10.134468	-0.111694	-0.128523	-0.114468
19	10.039445	10.011855	10.032642	-0.019445	0.008145	-0.012642
20	10.103695	10.122802	10.129952	-0.083695	-0.102802	-0.109952
100	10.045145	10.047339	10.033662	-0.025145	-0.027339	-0.013662
101	10.007846	9.990731	10.033165	0.012154	0.029269	-0.013165
102	10.043951	10.046732	10.032530	-0.023951	-0.026732	-0.012530
103	10.010961	9.994590	10.033758	0.009039	0.025410	-0.013758
104	10.042695	10.046046	10.031425	-0.022695	-0.026046	-0.011425
105	10.013823	9.998175	10.034295	0.006177	0.021825	-0.014295
106	10.041384	10.045289	10.030347	-0.021384	-0.025289	-0.010347
107	10.016449	10.001505	10.034781	0.003551	0.018495	-0.014781
108	10.040024	10.044468	10.029294	-0.020024	-0.024468	-0.009294
109	10.018855	10.004597	10.035219	0.001145	0.015403	-0.015219
110	10.038619	10.043588	10.028265	-0.018619	-0.023588	-0.008265
240	10.031088	10.028554	10.019904	-0.011088	-0.008554	0.000096
241	10.009649	10.020173	10.029287	0.010351	-0.000173	-0.009287
242	10.031069	10.028799	10.020233	-0.011069	-0.008799	-0.000233
243	10.008782	10.019520	10.029061	0.011218	0.000480	-0.009061
244	10.031029	10.029022	10.020550	-0.011029	-0.009022	-0.000550
245	10.007913	10.018862	10.028834	0.012087	0.001138	-0.008834
246	10.030970	10.029221	10.020856	-0.010970	-0.009221	-0.000856
247	10.007042	10.018199	10.028607	0.012958	0.001801	-0.008607
248	10.030892	10.029399	10.021151	-0.010892	-0.009399	-0.001151
249	10.007439	10.017531	10.028381	0.012561	0.002469	-0.008381

Table B.1: Some of the results generated from lattice methods developed by CRR, JR and TIAN models. The exact values of the price of the option is 10.02 calculated from the Black-Scholes pricing formula

Bibliography

[1] A.V. Melnikov, “Financial Markets”, *Translation of AMS*, Volume.184, 1999.

[2] D. Leisen & M. Remer, “Binomial models for option valuation-Examining and improving convergence”, *JEL classification. G13*, journal of Mathematical Finance, 1995.

[3] D. Chance, “Convergence of the binomial to the Black-Scholes model”, *Financial Engineering news*, 1996.

[4] F.Hubalek & W.Schachermayer “When Does Convergence of Asset Price Processes Imply Convergence of Option Prices?”, *Blackwell*, journal of Mathematical Finance, pp.385–403, 1998.

[5] J. Jacob & A. Shiryaev “Limit Theorems for Stochastic Processes”, *Springer*, 1987.

[6] J. C. Cox & M. Rubinstein “Option pricing”, *Prentice-Hall*, 1985.

[7] J.C. Hull, “Options,Futures and Other Deriavatives”, *University of Toronto*, 2000.

[8] J.W. Nieuwenhuis, “Weak convergence of tree methods to price options on defaultable asset”, *Department of applied mathematics, University of Twente*, Journal of Economic Literature Classification:G13.

[9] P. Wilmott, “DERIVATIVES: The Theory and practice of Financial Engineering”, *John Wiley and Sons Ltd*, 1999.

[10] S.R. Pliska, “Introduction to Mathematical Finance: Discrete Time Models”, *Malden, Mass.: Blackwell*, 1997.

[11] S.T. Rachev & L.Rüschendorf, “Models for option Prices”, *Theory Prob.Appl*, 39(1)pp.120–152, 1994.

[12] R. Merton, “Continuous Time Finance”, *Blackwell*, 1993.

[13] R.T.H. Peters, “Nonconvergence in the variation of the hedging strategy of a European call option”, *Blackwell*, volume 13, Journal of Mathematical Finance, pp.469–480, 2003.

[14] A.N. Shiryaev, Y.M. Kabanov,D.O. Krankov & A.V. Mel’nikov “Toward the theory of Pricing of Options of Both European and American Types.I.Discrete Time”, *Theory Prob.Appl*, 39(1),pp 14–60, 1994.

[15] W.A. Rosenkrantz, “Why Stock prices Have a log-normal Distribution”, *University of Massachusetts at Amherst*, Class Notes, 2004.