MODELS FOR OPTION PRICES

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Abstract. Cox, Ross and Rubinstein introduced a simple binomial option price model and derived the seminal Black-Scholes pricing formula for a geometric Brownian motion as a limiting case of the binomial option pricing formula. In this paper we characterize all possible infinitely divisible stock price models which can be approximated by the binomial models and derive the corresponding approximations for the pricing formulas. We introduce two additional randomizations in the binomial price models seeking for more general and more realistic limiting models. The first type of models is based on a randomization of the number of price changes, the second one on a randomization of the up's and down's in the price process. As a result we obtain also price models with fat tails, higher peaks in the centre, nonsymmetric etc. which are observed in typical asset return data. Following similar ideas as in the Cox, Ross, Rubinstein model we also derive approximating option pricing formulas and discuss several examples.

Key words: option prices, Black-Scholes formula, stable distributions, binomial pricing process

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1. Introduction

The classical Black-Scholes option pricing formula deals with a continuous market model with two assets; one risky asset, the stocks with a stock price X_t^1 modelled by a geometric Brownian motion, and one non-risky asset - the bonds with price $X_t^2 = X_0^2 \exp(rt)$, r a discounting factor. A trading stategy (portfolio) of an investor is given by $\Phi_t = (\Phi_t^0, \Phi_t^1)$, Φ_t^0 describing the amount of bonds held at time t, Φ_t^1 the amount of stocks. The value of the portfolio at time t is given by $V_t = \Phi_t^0 X_t^0 + \Phi_t^1 X_t^1$.

The value of a (European) call option $B = (X_T^1 - K)_+$ (T is the expiration time, and K is the striking price) is derived in the Black-Scholes model in the following way. There exists a "hedging" strategy Φ with constant cost $C_t = \pi$, in other words, the corresponding value process satisfies $V_T = (X_T^1 - K)_+ = B$, and with $I_t := \int_0^t \Phi_u^0 dX_u^0 + \int_0^t \Phi_u^1 dX_u^1$, describing the investment until time t, the cost process $C_t = V_t - I_t = \pi$ is constant over time t. Then π is the Black-Scholes value of the call B. π can be explicitely calculated (using simple stochastic calculus),

(1.1) $$\pi = f(X_0^1, T),$$

where $f(x,t) := x\, \Phi(g(x,t)) - K\, e^{-rt}\, \Phi(g(x,t) - \sigma\sqrt{t})$ and $g(x,t) := [\log \frac{x}{K} + (r + \frac{1}{2}\sigma^2)t] / \sigma\sqrt{t}$, Φ is the standard normal distribution function, and σ^2 is the diffusion parameter in the geometric Brownian motion. For this derivation of the Black-Scholes formula we refer to Merton (1973), Cox and Ross (1976), Smith (1976) and for some extensions see Föllmer and Sondermann (1986).

Cox, Ross and Rubinstein (1979) introduced a simple discrete binomial pricing model for the price of the stock. The time interval [0,T] is divided into n steps, T = nh. In each step the price S moves up to u S with probability q down to dS with probability 1-q; here the up's, the down's and the interest rate r are constants, d < r < u. A suitable choice of the four parameters u, d, q, n leads in the limit to a log normal price model. The binomial pricing formula for a call $B = (S_n - K)_+$ is now based on the same hedging (arbitrage) idea as in the Black-Scholes model. For the Cox, Ross, Rubinstein derivation of this formula assume in the first step that n = 1, that is, there is only one time period. The value of the call C at the end of the period is either C_n or C_d,

$$(1.2)\qquad C \begin{cases} \nearrow C_u = (uS-K)_+ & \text{with probability } q \\ \searrow C_d = (dS-K)_+ & \text{with probability } 1-q \end{cases} .$$

Consider a hedging strategy (Δ, B), that is, the value of the portfolio changes as

$$(1.3)\qquad \Delta S + B \begin{cases} \nearrow \Delta u S + rB \overset{!}{=} C_u & \text{with probability } q \\ \searrow \Delta d S + rB \overset{!}{=} C_d & \text{with probability } 1-q \end{cases} ;$$

S is the initial price. This equivalent portfolio determines the value of the call

$$(1.4)\qquad C = \Delta S + B.$$

It is easy to see that any other call value leads to riskless arbitrage opportunities. From (1.3) we obtain

$$(1.5)\qquad \Delta = \frac{C_u - C_d}{(u-d)S}, \quad B = \frac{u C_d - d C_u}{(u-d)r}$$

and

$$(1.6)\qquad C = \frac{1}{r}(p C_u + (1-p) C_d) \text{ with } p := \frac{r-d}{n-d} .$$

Note that p describes a risk-neutral world

$$(1.7)\qquad p n S + (1-p) d S = r S$$

and (1.6) has the interpretation that the value of the call C is, in fact, the expectation of the call B w.r.t. the risk-neutral measure (a martingale measure) p, discounted by r,

$$(1.8)\qquad C = C_1 = E_*(S_1 - K)_+/r; \ E_* \text{ is the expectation w.r.t. p.}$$

This argument can be extended to n-periods of price-movements until T arguing for each step (starting from $(n-1)h$ to $nh = T$) as above. This results in *the binomial pricing formula*

$$(1.9)\qquad C_n = E_*(S_n - K)_+/r^n = S\Phi(a,n,p') - K r^{-n}\Phi(a,n,p),$$

$$\Phi(a,n,p) := P(X_{n.p} \geq a), \ X_{n,p} \overset{d}{=} \mathfrak{B}(n,p), \ p = \frac{r-d}{u-d}, \ p' = \frac{u}{r} p, \ a = \left[\frac{\log \frac{K}{S d^n}}{\log \frac{u}{d}} \right]_+ ,$$

($[\ ]_+$ stands for the positive Gauss bracket.)
From (1.9) - which is a useful pricing formula in its own right - Cox, Ross and Rubinstein (1979) derived (with their particular choice of parameters) the approximating Black-Scholes formula (1.1) in the limit. For a special second choice of the parameters they also obtained the Cox and Ross (1975) option pricing formula for the Poisson price process.

In the first part of this paper we characterize all possible limits of the Cox-Ross-Rubinstein model and the corresponding approximate pricing formulas. In the second part we consider an extension of the binomial model based on a randomization of the number n of price movements until the terminal time T. This leads to a richer class of possible price models including, in particular, the stable (Paretian) distributions with index $0 < \alpha < 2$ and the geometric stable laws. This is an important extension since it has become evident from empirical work of Mandelbrot (1963, 1977), Fama (1965), Teichmoeller (1971), Officer (1972), Hagerman (1978), du Mouchel (1971, 1983) and Mittnik and Rachev (1989) that stock return distributions with fatter (than the normal law) tails lead to better fits of the stock price data, see also the example at the end of this paper. For this extension we determine a pricing formula based on similar ideas as described above and we calculate explicitely several examples.

In the third part we propose a second modification of the binomial model which is based on a randomization of the up's and down's in the binomial model. This modification allows to deal e.g. with Weibull distributions providing a very satisfactory fit to return data - cf. Mittnik and Rachev (1989). A more detailed analysis of the pricing formula in this model is given in a subsequent paper by Rachev and Samorodnitsky (1992). We remark that some of the results of this paper have been announced (without proofs and examples) in Rachev and Rüschendorf (1991).

2. Limits of the Binomial Option Pricing Model

Let $S = S_o$ be the known stock price at moment $t_o = 0$, let t be the length of calender time representing the expiration of a call. In the binomial option pricing model t is divided into n periods of length h, $t = nh$; at the end of each period $(k, k+1)$ the value S_{k+1} is equal to uS_k with probability p and to dS_k with probability $q = 1 - p$, $0 < d \le 1 \le u$. Therefore, with $U = \log u$, $D = \log d$

(2.1)
$$\log (S_n / S) = \sum_{k=1}^{n} X_{n,k},$$

where

(2.2)
$$X_{n,k} = \zeta_{n,k} U + (1 - \zeta_{n,k}) D$$

and $\zeta_{n,k}$ are iid Bernoulli with success probability p.

We assume that u, d, p are functions of n (resp. h) and consider the class of possible limits of (2.1) in the class of all infinite divisible distributions (ID), assuming that

(2.3)
$$\lim_{n\to\infty} U = \lambda, \quad \lim_{n\to\infty} D = -\mu, \quad \lambda \ge 0,\ \mu \ge 0.$$

(We omit the index n in $U = U(n)$, $D = D(n)$, $p = p(n)$, if it is not ambigous.) Cox, Ross and Rubinstein (1979) considered the special case $U = \sigma\sqrt{t/n} = \sigma\sqrt{h}$, $D = -\sigma\sqrt{h}$ to obtain normal limits and $U = U(0)$, $D = -\sigma h$ to obtain Poisson limits for (2.1). The following different cases arise in this way:

Case (1): $\lambda = \mu = 0$;

Case (2): $0 < \lambda < \infty$, $\mu = 0$;

Case (3): $0 < \lambda < \infty$, $0 < \mu < \infty$;

Case (4): $\lambda = 0$, $0 < \mu < \infty$;

Case (5): $0 < \lambda < \infty$, $\mu = \infty$;

Case (6): $\lambda = \infty$, $0 < \mu < \infty$;

Case (7): $\lambda = 0$, $\mu = \infty$;

Case (8): $\lambda = \infty$, $\mu = 0$;

Case (9): $\lambda = \infty$, $\mu = \infty$.

To investigate the limit distributions of (2.1), we assume that the $X_{n,k}$'s satisfy the uniform asymptotic negligibility (UAN) condition: as $n \to \infty$ (i.e. $h = \frac{t}{n} \to 0$)

(C.1) $\max_{1\le k\le n} P(|X_{n,k}| \ge \varepsilon) \to 0$ for all $\varepsilon > 0$.

In Case (1) ($\lambda = \mu = 0$) (C.1) is satisfied. This follows from the following equivalent form of (C.1)

(2.4) $$\max_{1\le k\le n} \int \frac{x^2}{1+x^2}\, dF_{n,k}(x) \to 0,$$

where $F_{n,k}$ is the df of $X_{n,k}$.

In Cases (2) and (8) ($0 < \lambda \le \infty$, $\mu = 0$)

(2.5) $$(\text{C.1}) \Leftrightarrow p \to 0.$$

In Cases (4) and (7) ($\lambda = 0$, $0 < \mu \le \infty$)

(2.6) $$(\text{C.1}) \Leftrightarrow q \to 0.$$

In the other cases ($0 < \lambda \le \infty$, $0 < \mu \le \infty$) (C.1) does not hold. This can be seen from (2.4) or, equivalently, from

(2.7) $$p \frac{U^2}{1+U^2} + q \frac{D^2}{1+D^2} \to 0 \quad \text{as } n \to \infty.$$

So from now on we shall investigate only the cases 1, 2, 4, 7, 8.

Our main result of this section shows that the only possible infinitely divisible (ID) limits of the binomial model (2.1) are the normal, the Poisson, and the degenerate case. This "characterization" theorem confirms the completeness of the results in the classical paper of Cox, Ross, and Rubinstein (1975) who obtained these limits for very particular choices of the sequences U, D, p. Moreover, we additionally determine all possible choices of U, D, p leading to ID-limits.

Theorem 2.1. a) Case (1) ($\lambda = \mu = 0$). The only possible ID limit distribution of $\log S_n/S$ in this case is the normal distribution. We have:

(2.8) $$\mathfrak{L}(\log (S_n/S)) \xrightarrow{w} N(\alpha,\sigma^2)$$

if and only if

(2.9) $$n\left(qD + pU - \frac{pqR}{1+p^2R^2} + \frac{pqR}{1+q^2R^2}\right) \to \alpha$$

and

(2.10) $$n\left(q\frac{R^2p^2}{1+R^2p^2} + p\frac{R^2q^2}{1+R^2q^2}\right) \to \sigma^2.$$

b) <u>Case (2) (0 ≤ λ ≤ ∞, μ ≡ 0).</u> The only possible ID limit distribution of $\log S_n/S$ in this case is the scaled and shifted Poisson distribution. We have:

(2.11) $\quad \mathfrak{L}(\log S_n/S) \xrightarrow{w} b + \lambda$ Poisson (a)

if and only if

(2.12) $\quad a = \lim np, \quad b = \lim nD, \quad p \to 0.$

c) <u>Case (8) (λ ≡ ∞, μ ≡ 0).</u> The only possible ID limit distribution of $\log S_n/S$ in this case is the degenerate distribution. We have:

(2.13) $\quad \mathfrak{L}(\log S_n/S) \xrightarrow{w} \alpha$

if and only if

(2.14) $\quad \alpha = \lim (nD + \frac{np}{U}), \quad p \to 0.$

d) <u>Case (4) (λ ≡ 0, 0 ≤ μ ≤ ∞).</u> The only possible ID limit is the scaled and shifted Poisson distribution. We have:

(2.15) $\quad \mathfrak{L}(\log (S_n/S)) \xrightarrow{w} b - \mu$ Poisson (a)

if and only if

(2.16) $\quad a = \lim nq, \quad b = \lim nU, \quad q \to 0.$

e) <u>Case (7) (λ ≡ 0, μ ≡ ∞).</u> The only possible limit is the degenerate distribution. We have:

(2.17) $\quad \mathfrak{L}(\log (S_n/S)) \xrightarrow{w} \alpha$

if and only if

(2.18) $\quad \lim nU = \alpha, \quad \lim nq = 0, \quad q \to 0.$

<u>Proof.</u> The proof of Theorem 2.1 is based on the CLT for triangular arrays of independent r.v.'s subject to the UAN condition (cf. Loeve (1977), section 23) which formulation we invoke here only as a reference.

<u>Lemma</u> 2.1. Suppose that $(X_{n,k})$ is any independent triangular array of UAN r.v.'s.

a) The family of weak limits of $\mathfrak{L}(\sum_{k=1}^{n} X_{n,k})$, $n \in \mathbb{N}$, coincides with the family of infinitely divisible (ID) laws or, equivalently, with the family of laws of r.v.'s X with ch.f.

(2.19) $$\varphi_X(u) = E\, e^{iuX} = \exp\{iu\,\alpha + \int (e^{iux} - 1 - \frac{iux}{1+x^2}) \frac{1+x^2}{x^2}\, d\psi(x)\},$$

where $\alpha \in \mathbb{R}$ and ψ is a df up to a multiplicative constant.

b) $\sum_{k=1}^{n} X_{n,k}$ converges to X with ch.f. (2.19) iff

(C.2) $$\alpha_n \to \alpha$$

and

(C.3) $$\psi_n \xrightarrow{w} \psi \quad \text{(convergence in distribution)}$$

where

(2.20) $$\alpha_n := \sum_{k=1}^{n} \{a_{n,k} + \int \frac{x}{1+x^2}\, d\overline{F}_{n,k}(x)\},$$

(2.21) $$\psi_n(x) := \sum_{k=1}^{n} \int_{-\infty}^{x} \frac{y^2}{1+y^2}\, d\overline{F}_{n,k}(y),$$

$a_{n,k} := \int x\, I\{|x| < \tau\} dF_{n,k}(x)$, $F_{n,k}$ the df of $X_{n,k}$, $\overline{F}_{n,k}(x) = F_{n,k}(x + a_{n,k})$ and $\infty > \tau > 0$ arbitrary fixed. □

For the proof of Theorem 2.1 we have to check that the conditions stated in our theorem are equivalent to the conditions (C.1), (C.2), (C.3).

a) Case (1) ($\lambda = \mu = 0$): For n large enough

(2.22) $$a_{n,k} = p\, U\, I\{|U| < \tau\} + q\, D\, I\{|D| < \tau\} = p\, U + q\, D = E\, X_{n,k}$$

and

$$\int \frac{x}{1+x^2}\, d\overline{F}_{n,k}(x) = q \frac{-pR}{1+p^2R^2} + p \frac{qR}{1+q^2R^2},$$

which implies (2.9). Furthermore, by (2.21)

$$\psi_n(x) = n \int_{-\infty}^{(x/R)+p} \frac{R^2(y-p)^2}{1+R^2(y-p)^2}\, dF_{n,1}(y)$$

$$= \begin{cases} 0 & \text{if } x < -pR \\ n\,q \dfrac{R^2p^2}{1+R^2p^2} & \text{if } -pR \le x < qR \\ n\,q \dfrac{R^2p^2}{1+R^2p^2} + n\,p \dfrac{R^2q^2}{1+R^2q^2} & \text{if } x \ge qR \end{cases}$$

Since $\lambda = \mu = 0$ and, thus $R = U - D \to 0$, we obtain

$$(2.23) \qquad \psi_n(x) \to \psi(x) = \begin{cases} 0 & \text{if } x < 0 \\ \lim\limits_{n\to\infty} n\left(q \dfrac{R^2p^2}{1+R^2p^2} + p \dfrac{R^2q^2}{1+R^2q^2}\right) & \text{if } x \geq 0 \end{cases}$$

iff the limit in the R.H.S. exists.

b) Case (2) ($0 < \lambda < \infty$, $\mu = 0$): For $\tau < \lambda$ and n large enough $a_{n,k} = qD$ and

$$(2.24) \qquad \int \frac{x}{1+x^2}\, d\overline{F}_{nk} = q \frac{pD}{1+p^2D^2} + p \frac{U-qD}{1+(U-qD)^2}.$$

Therefore

$$(2.25) \qquad (C.2) \Leftrightarrow n\left(qD + \frac{pqD}{1+p^2D^2} + p \frac{U-qD}{1+(U-qD)^2}\right) \to \alpha.$$

Similarly,

$$(2.26) \qquad \psi_n(x) = n \int_{-\infty}^{\frac{x-pD}{R}} \frac{(yR+pD)^2}{1+(yR+pD)^2}\, dF_{n,1}(y)$$

and, thus

$$(2.27) \qquad (C.3) \Leftrightarrow \psi(x) = \begin{cases} 0 & \text{if } x < 0 \\ \lim\limits_{n\to\infty} nq \dfrac{p^2D^2}{1+p^2D^2} & \text{if } 0 \leq x < \lambda \\ \lim\limits_{n\to\infty} \left[nq \dfrac{p^2D^2}{1+p^2D^2} + np \dfrac{(U-qD)^2}{1+(U-qD)^2}\right] & \text{if } x > \lambda \end{cases}$$

Since $1 + p^2D^2 \to 1$ it follows that $\lim np^2qD^2$ exists. Furthermore, $\lim\limits_{n\to\infty} np \dfrac{(U-qD)^2}{1+(U-qD)^2} = \lim\limits_{n\to\infty} np \dfrac{\lambda^2}{1+\lambda^2}$ and, thus $\lim\limits_{n\to\infty} np^2qD^2 = \lim\limits_{n\to\infty} nq \dfrac{p^2D^2}{1+p^2D^2} = 0$. Together, we obtain

$$(2.28) \qquad \psi_n(x) \xrightarrow{w} \psi(x) = \begin{cases} 0 & \text{if } x < \lambda \\ \dfrac{\lambda^2}{1+\lambda^2} \lim\limits_{n\to\infty} (np) & \text{if } x \geq \lambda \end{cases}$$

c) Case (8) ($\lambda = \infty$, $\mu = 0$): In this case we obtain for the limiting function after some calculations

$$(2.29) \qquad \psi(x) = \lim_{n\to\infty} \psi_n(x) = \begin{cases} 0 & \text{if } x < 0 \\ \lim\limits_{n\to\infty} nqp^2D^2 & \text{if } x > 0 \end{cases}$$

d) Case (4) is quite similar to the Case (2). We obtain

$$(2.30) \qquad \psi_n(x) \xrightarrow{w} \psi(x) = \begin{cases} 0 & \text{if } x < -\mu \\ \lim\limits_{n\to\infty} nq \dfrac{(D-pU)^2}{1+(D-pU)^2} & \text{if } -\mu < x < 0 \\ \lim\limits_{n\to\infty} [nq \dfrac{(D-pU)^2}{1+(D-pU)^2} + np \dfrac{q^2U^2}{1+q^2U^2}] & \text{if } x > 0 \end{cases}$$

Since $\lim\limits_{n\to\infty} nq \dfrac{(D-pU)^2}{1+(D-pU)^2} = \lim\limits_{n\to\infty} nq \dfrac{\mu^2}{1+\mu^2}$, we obtain $\lim\limits_{n\to\infty} np \dfrac{q^2U^2}{1+q^2U^2} = 0$ and thus

$$(2.31) \qquad \psi_n(x) \xrightarrow{w} \psi(x) = \begin{cases} 0 & \text{if } x < -\mu \\ \lim\limits_{n\to\infty} nq \dfrac{\mu^2}{1+\mu^2} & \text{if } x \geq -\mu \end{cases}$$

e) In Case (7) ($\lambda = 0$, $\mu = \infty$) it is easily seen that

$$(2.32) \qquad (C.2) \Leftrightarrow n(pU + \frac{pqU}{1+q^2U^2} + q\frac{D-pU}{1+(D-pU)^2}) \to \alpha.$$

Furthermore, (C.3) is equivalent to $\psi_n(x) \xrightarrow{w} \psi(x) = \lim nq$ for all x. Since ψ is a df of a nonnegative measure, $\psi(x) = 0$. □

By Theorem 2.1 we can derive the limiting distribution of the binomial model. In the next example we shall examine the choice of appropriate parameters U, D, p as in Cox, Ross and Rubinstein (1979), resp. Cox and Ross (1975).

<u>Example</u> 2.1. a) Let $U = \tilde{\sigma}\sqrt{h}$, $D = -\tilde{\sigma}\sqrt{h} = -\tilde{\sigma}\sqrt{t/n}$, $p = p(n) = \frac{r(n)-d(n)}{n(n)-d(n)}$, $r(n) = r_o^{t/n}$, then

$$(2.33) \quad p \approx \frac{1}{2} + \frac{1}{2}\frac{\alpha}{\tilde{\sigma}}\sqrt{t/n} \quad \text{and } E(\log S_n/S) = n(qD + pU) \to \alpha = (\log r_o - \frac{1}{2}\tilde{\sigma}^2)t.$$

Since we are in Case (1) we must check conditions (2.9), (2.10).

$$\begin{aligned}(2.34) \qquad & \lim n(qD + pU - qp(q-p)\frac{R^3}{(1+p^2R^2)(1+q^2R^2)} \\ &= \lim n[(\tfrac{1}{2} - \tfrac{1}{2}\frac{\log r_o - \frac{1}{2}\tilde{\sigma}^2}{\tilde{\sigma}})\sqrt{n}(-\tilde{\sigma}\sqrt{h}) \\ &\quad + (\tfrac{1}{2} + \tfrac{1}{2}\frac{\log r_o - \frac{1}{2}\tilde{\sigma}^2}{\tilde{\sigma}})\sqrt{h}(\tilde{\sigma}\sqrt{h})] \\ &= t(\log r_o - \tfrac{1}{2}\tilde{\sigma}^2) = \alpha.\end{aligned}$$

Furthermore,

$$\begin{aligned}(2.35) \qquad & \lim (nq\frac{4\tilde{\sigma}^2hp^2}{1+4\tilde{\sigma}^2hp^2} + np\frac{4\tilde{\sigma}^2hq^2}{1+4\tilde{\sigma}^2hq^2}) \\ &= \tfrac{1}{8}\, 4\tilde{\sigma}^2t + \tfrac{1}{8}\, 4\tilde{\sigma}^2t = \tilde{\sigma}^2t = \sigma^2.\end{aligned}$$

So by (2.8) we obtain $\mathfrak{L}(\log S_n/S) \xrightarrow{w} N(\alpha,\sigma^2)$.

b) Let $U = \lambda$, $D = -\bar{\mu} h = -\bar{\mu}\, {}^{t}/_{n}$, $\lambda > 0$, $\bar{\mu} > 0$, and $p = vh$. Now we are in Case (2). By (2.12) $a = \lim np = vt$, $\alpha = \lim nD + a \frac{\lambda}{1+\lambda^2} = -\bar{\mu}t + vt \frac{\lambda}{1+\lambda^2}$. Thus by (2.11) we obtain the shifted and scaled Poissonian case

$$\mathfrak{L}(\log S_n/S) \xrightarrow{w} -\bar{\mu}t + \lambda \text{ Poisson } (vt).$$

So in Case (2) there are instantaneous up turns of the stock price (cf. Cox and Ross (1975), Cox and Rubinstein (1985)).

c) Let $U \to \lambda = \infty$, $D = -v\frac{t}{n}$, $v > 0$, $\frac{p}{U} = c \cdot \frac{t}{n}$, then by (2.14) $S_n \xrightarrow{w} S\, e^{(-v+c)t}$. □

In the next step we establish the functional CLT's corresponding to Theorem 2.1. The motivation for the functional convergence result follows the following general "continuity" approach. If the (discrete) price process $X_n = (X_n(u))_{0 \le u \le t}$ is approximated by a limiting price process $X = (X(u))_{0 \le u \le t}$, then under some regularity conditions also the pricing formulas for X_n should approximate the pricing formulas for the limiting process X. This "continuity" argument is underlined in e.g. the development in Cox, Ross and Rubinstein, where it is shown that the pricing formula for the binomial process converges to the Black Scholes pricing formula for the geometric Brownian motion. A general result of this type seems still to be missing in the literature. We shall not elaborate on this question in this paper, but concentrate on asymptotic approximations for the valuation formulas. We use the limiting price model to argue that our results might be relevant also to the limiting (continuous) models.

Recall that $X_{n,k} = \zeta_{n,k}U + (1 - \zeta_{n,k})D$, where $(\zeta_{n,k})$ is an iid sequence of binomials $\mathfrak{B}(1,p)$. Define the centered random variables

$$(2.36) \qquad \bar{X}_{n,k} = X_{n,k} - a_{n,k} \overset{d}{=} X_{n,1} - a_{n,1},$$

where $a_{n,k} = \int x\, I\{|x| < \tau\}\, dF_{n,k}(x)$ for $\tau > 0$ suitably chosen and define

$$(2.37) \qquad \bar{S}_{n,k} = \bar{X}_{n,1} + \ldots + \bar{X}_{n,k}.$$

Consider the D[0,t] valued random process

(2.38) $\overline{X}_n(u) = \overline{S}_{n,k}$ for $\frac{k-1}{n} \le \frac{u}{t} < \frac{k}{n}$, $1 \le k \le n$, $\overline{X}_n(t) = \overline{S}_{n,n}$,

where t is the fixed expiration time.

<u>Theorem</u> 2.2. <u>(Case (1) ($\lambda = \mu = 0$).</u> If $v = \lim_n n\,pq\,R^3 \frac{p-q}{(1+p^2R^2)(1+q^2R^2)}$ and $\sigma^2 = \lim_n \frac{n\,pq\,R^2}{(1+p^2R^2)(1+q^2R^2)}$ exist, then $\overline{X}_n \xrightarrow{w} \overline{X}$, a Wiener process with

$$E\,\overline{X}(u) = v\,\frac{u}{t}\,, \quad \mathrm{Var}\ \overline{X}(u) = \sigma^2\,\frac{u}{t}\,.$$

If additionally $n\,a_{n,1} \to a$, then the non-centered process $X_n(u) = \overline{X}_n(u) + [\frac{nu}{t}]\,a_{n,1} \xrightarrow{w} X$, a Wiener process with

(2.39) $EX(u) = \alpha\,\frac{u}{t}\,,\ \alpha = v + a,\ \mathrm{Var}\ X(u) = \tilde{\sigma}^2 u.$

<u>Proof.</u> We introduce for the general process $\overline{X}_n$ as in (2.38) the following notations

(2.40) $v_n := n \int_{-\infty}^{\infty} \frac{x}{1+x^2}\, d\overline{F}_{n,1}(x),\ \overline{F}_{n,1}(x) = F_{n,1}(x + a_{n,1})$

and

(2.41) $$\psi_n(x) = n \int_{-\infty}^{x} \frac{y^2}{1+y^2}\, d\overline{F}_{n,1}(y).$$

and consider the conditions:

($\overline{C.2}$) $v_n \to v$

and

(C.3) $\psi_n \xrightarrow{w} \psi.$

The proof is based on the following functional central limit theorem (cf. Gikhman and Skorohod (1969), Th. 2, p. 480) corresponding to Lemma 2.1 after centering.

<u>Lemma</u> 2.2 Under conditions (C.1), ($\overline{C.2}$), (C.3) the process $\overline{X}_n$ converges weakly in D[0,t] to a homogeneous process $\overline{X}$ with independent increments and characteristic function

(2.42) $$E\,e^{i\vartheta \overline{X}(u)} = \exp\{\frac{u}{t}\,[\,i\vartheta v + \int_{-\infty}^{\infty} (e^{i\vartheta x} - 1 - \frac{i\vartheta x}{1+x^2})\,\frac{1+x^2}{x^2}\, d\psi(x)]\},$$

$0 \le u \le t.$ □

Our next step is to check the conditions in the above lemma. By (2.22) $a_{n,k} = pU + qD = E\ X_{n,k}$ for n large enough. Furthermore, as in Example 2.1. a)

$$v_n = n \int \frac{x}{1+x^2}\, d\overline{F}_{n,1}(x) = n\,p\,q\,R^3 \frac{p-q}{(1+p^2R^2)(1+q^2R^2)} \to v,$$

and by (2.23)

$$\psi(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ \sigma^2 & \text{if} \quad x \geq 0 \end{cases},$$

where

$$\lim_n n\Big(q \frac{p^2R^2}{1+p^2R^2} + p \frac{q^2R^2}{1+q^2R^2}\Big) = \lim_n \frac{n\,p\,q\,R^2}{(1+p^2R^2)(1+q^2R^2)} = \sigma^2 .$$

Thus by the functional CLT, Lemma 2.2, $\overline{X}_n \xrightarrow{w} \overline{X}$, $\overline{X}$ a Wiener process with

$$E\,\overline{X}(u) = v\frac{u}{t}\,, \quad \text{Var}\;\overline{X}(u) = \sigma^2\,\frac{u}{t}$$

as desired. □

Note that X_n is a non-centered partial sum process

$$(2.43) \qquad X_n(u) = \begin{cases} \sum_{j=1}^{k} X_{n,j}, & \frac{k-1}{n} \leq \frac{u}{t} < \frac{k}{n} \\ \log S_n/S, & u = t \end{cases} .$$

In the Cox, Ross and Rubinstein model $U = \tilde{\sigma}\sqrt{t/n}$, $D = -\tilde{\sigma}\sqrt{t/n}$, $a_{n,k} = E\,X_{n,k} = q\,D + p\,U \approx \frac{\alpha}{n}$, $\alpha = (\log r_o - \frac{\tilde{\sigma}^2}{2})t$, we have $p \approx \frac{1}{2} + \frac{1}{2}\,\frac{\alpha}{\tilde{\sigma}}\,\sqrt{t/n}$, $v = 0$, $\sigma^2 = \tilde{\sigma}^2 t$. By Theorem 2.2, therefore,

$$(2.44) \qquad X_n \xrightarrow{w} X,$$

where X is a Wiener process with

$$(2.45) \qquad E\,X(u) = \frac{\alpha u}{t}\,, \quad \text{Var}\;X(u) = \tilde{\sigma}^2 u .$$

We next examine the cases (2) and (8) in the following remarks.

<u>Remarks.</u> a) <u>Case</u> (2) ($0 < \lambda < \infty$, $\mu = 0$) In this case (for $\tau < \lambda$) $a_{n,k} = q\,D$, $v_n = n\,q \frac{p\,D}{1+p^2D^2} + n\,p \frac{U-qD}{1+(U-qD)^2} \to \; a\frac{\lambda}{1+\lambda^2} = v$, where $a = \lim_{n\to\infty} n\,p$, and

$$\psi_n(x) \to \psi(x) = \begin{cases} 0, & \text{if} \;\; x < \lambda \\ \frac{\lambda^2}{1+\lambda^2}, & \text{if} \;\; x > \lambda \end{cases} . \text{ Thus,}$$

$$(2.46) \qquad \overline{X}_n \xrightarrow{w} \overline{X}$$

a homogeneous Poisson process with

$$(2.47) \qquad \log E\,e^{i\vartheta \overline{X}(u)} = \frac{u}{t}\,[\,i\vartheta\, a \frac{\lambda}{1+\lambda^2} + a(e^{iu\lambda} - 1)\,] .$$

In the Cox and Ross (1975) model $U = \lambda$, $D = -\overline{\mu}\,\frac{t}{n}$, $p = v\,\frac{t}{n}$, we have $a_{n,k} = qD \approx -\overline{\mu}\,\frac{t}{n}$ and $a = \lim(np) = vt$. Thus $\overline{X}$ has the characteristic function

(2.48) $$\log E\, e^{i\vartheta \overline{X}(u)} = vu\,[\,i\vartheta\,\frac{\lambda}{1+\lambda^2} + (e^{iu\lambda} - 1)\,].$$

b) <u>Case</u> (8) ($\lambda = \infty$, $\mu = 0$) $a_{n,k} = qD$ and

(2.49) $$\overline{X}_n(u) \to \overline{X}(u) \overset{d}{=} \alpha\,\frac{u}{t},$$

where $\alpha = \lim\,[\,nD + \frac{np}{U}\,]$ (see (2.14)). Consider again example 2.1.c). Then $U \to \lambda = \infty$, $\frac{p}{U} = c\,\frac{t}{n}$, $D = -v\,\frac{t}{n}$ and $\alpha = -vt + ct$ and so

(2.50) $$\overline{X}(u) = (c - v)\,u,\ 0 \le u \le t$$

(degenerate case).

c) Let $W = (W(u),\ 0 \le u \le t)$ be a standard Wiener process. Then in Case (1) ($\lambda = \mu = 0$) suppose that $\sum_{k=1}^{n} a_{n,k} = n\,a_{n,1} \to a\ (a_{n,1} = E X_{n,1})$. Then

(2.51) $$X_n(u) = \overline{X}_n(u) + \frac{[\frac{un}{t}]}{n}\; n\,a_{n,1} \xrightarrow{w} X(u) = a\,\frac{u}{t} + \sigma\sqrt{\frac{u}{t}}\,W(u)$$
$$=:\ \tilde{a}\,u + \tilde{\sigma}\,W(u),$$

where $\tilde{a} = \frac{a}{t}$, $\tilde{\sigma} = \sigma\sqrt{\frac{u}{t}}$. In other words the limiting stock price at time u is given by

(2.52) $$S_n(u) = S\,e^{X_n(u)} \xrightarrow{w} S(u) = S\,e^{X(u)} \overset{d}{=} S\,e^{\tilde{a}u + \tilde{\sigma}W(u)}.$$

By Ito's formula S(u) satisfies the stochastic differential equation

(2.53) $$dS(u) = S(u)\,[\,(\tilde{a} + \frac{\tilde{\sigma}^2}{2})du + \tilde{\sigma}\,dW(u)\,].$$

In the Cox, Ross and Rubinstein (1979) model $\tilde{a} = \log r_o - \frac{1}{2}\,\tilde{\sigma}^2$, $\tilde{\sigma} = \sigma$, and thus in this case

(2.54) $$dS(u) = S(u)\,[\,(\log r_o)du + \tilde{\sigma}\,dW(u)\,].$$ □

3. Convergence of the Binomial Pricing Formula

In the binomial option pricing model (2.1) the value C_n of a call $B = (S_n - K)_+$ is given by the option pricing formula (cf. [6] and (1.9))

(3.1) $$C_n = S\Phi(a_n, n, p') - K r^{-n} \Phi(a_n, n, p),$$

where $\Phi(a_n,n,p) = P(\sum_{k=1}^{n} \zeta_{n,k} \geq a_n)$, $(\zeta_{n,k})$ are iid $\mathfrak{B}(1,p)$, $a_n = [\frac{\log(K/d^n S)}{\log(u/d)}]_+$, $[\ ]_+$ the positive Gauss bracket, r is one plus the riskless interest rate over one period of the length h, riskless meaning that

(3.2) $$pu + (1-p)d = r, \quad \text{or "equivalently"}$$

(3.3) $$p = \frac{r-d}{u-d}, \quad \text{and} \quad p' = \frac{u}{r} p.$$

If $a_n > n$, $C = 0$.

If r_o denotes one plus the interest rate over the full time period t, then there is the relation

(3.4) $$r^n = r_o^t, \text{ equivalently, } r = r_o^h.$$

Formula (3.3) can be interpreted as transition to a new "riskless" measure P^* in the sense that $(r^{-k} S_k)$ becomes a martingale sequence. The option pricing formula is then given by

(3.5) $$C_n = E^*(B r_o^{-t}) = E^*(B r^{-n}),$$

with E^* being the expectation w.r.t. the measure P^*. The martingale property of $(r^{-k} S_k)$ is equivalent to (3.2), since, if for example k = 1, $E(\frac{S_1}{r} | S_o = s) = \frac{1}{r}(pus + (1-p)ds)$, which equals s if and only if $pu + (1-p)d = r$. In other words, given r, P^* is the unique measure such that (S_k/r^k) is a martingale and S_{k+1} is either uS_k or dS_k. This property has an obvious equilibrium interpretation and thus justifies the valuation formula in (3.1). Cox, Ross and Rubinstein (1979) show that for any pair (r,p) not satisfying (3.2) there are arbitrage possibilities. They also give a recursive algorithm (based on the knowledge of u,d,n and p resp. r) for riskless hedging strategies.

Cox, Ross and Rubinstein (1979) obtained the Black-Scholes formula for the normal case as limiting case of (3.1) (cf. our discussion before Theorem 2.2) choosing $U = \tilde{\sigma}\sqrt{h}$, $D = -\tilde{\sigma}\sqrt{h}$ as in Example 2.1, i.e.

(3.6) $$p = \frac{r-d}{u-d} = \frac{r_o^h - e^{-\tilde{\sigma}\sqrt{h}}}{e^{\tilde{\sigma}\sqrt{h}} - e^{-\tilde{\sigma}\sqrt{h}}} \approx \frac{1}{2} + \frac{1}{2}\,\frac{\log r_o - \frac{1}{2}\tilde{\sigma}^2}{\tilde{\sigma}}\sqrt{h}\,.$$

Then in the limit they obtained the Black-Scholes formula

$$C = S\,\Phi(x) - K\,r_o^{-t}\,\Phi(x - \tilde{\sigma}\sqrt{t}), \tag{3.7}$$

where

$$x = \frac{\log(S/K\,r_o^{-t})}{\tilde{\sigma}\sqrt{t}} + \frac{1}{2}\,\tilde{\sigma}\sqrt{t} \tag{3.8}$$

and $\Phi(x) = P(N_{0,1} \geq x)$, $N_{0,1}$ a standard normal rv. This limiting Black-Scholes formula is considered in our context as approximation to the binomial pricing formula.

Based on Section 2 we next determine all possible limiting cases for the binomial option pricing formula, assuming again generally that $U \to \lambda$, $D \to -\mu$. We replace (3.4) by the somewhat weaker assumption

$$\lim_{n\to\infty} r^n = r_o^t. \tag{3.9}$$

<u>Theorem</u> 3.1. (Case (1) $(\lambda = \mu = 0)$)

Suppose the existence of the following limits:

$$\alpha = \lim_{n\to\infty} n\,(q\,D + p\,U - \frac{pqR}{1+p^2R^2} + \frac{pqR}{1+q^2R^2}), \tag{3.10}$$

α' the similar limit with p, q in (3.10) replaced by

$$p' = p\,\frac{u}{r},\ q' = 1 - p' = q\,\frac{d}{r}, \tag{3.11}$$

$$\sigma^2 = \lim_{n\to\infty} n\,(q\,\frac{p^2R^2}{1+p^2R^2} + p\,\frac{q^2R^2}{1+q^2R^2}) \tag{3.12}$$

and σ'^2 the similar limit with the choice in (3.11). Then

$$C_n \to C := S\,\Phi(x') - K\,r_o^{-t}\,\Phi(x), \tag{3.13}$$

where

$$x = \frac{\log(S/K)+\alpha}{\sigma},\quad x' = \frac{\log(S/K)+\alpha'}{\sigma'}. \tag{3.14}$$

<u>Proof.</u> With $\log(S_n/S) = (\sum_{k=1}^{n} \zeta_{n,k})R + nD$ we have $\Phi(a,n,p) =$ $= P(\log(S_n/S) \geq a\,R + n\,D)$. From Theorem 2.1.a), we obtain

$$\mathfrak{L}(\log(S_n/S)) \xrightarrow{w} N(\alpha,\sigma^2).$$

For some $\varepsilon \in (0,1)$, $a_n R + n\,D = \log(K/S\,d^n) + \varepsilon R + n\,D = \log(K/S) + \varepsilon\,R \to \log(K/S)$. So by the uniform convergence above we arrive at $\Phi(a_n,n,p) \to P(N_{0,1} \geq$ $\geq \frac{\log(K/S)-\alpha}{\sigma}) = \Phi(x)$; $N_{0,1}$ a $N(0,1)$-distributed rv'e. The same arguments apply to get: $\Phi(a_n,n,p') \to \Phi(x')$. □

We note that by Theorem 2.1 conditions (3.10), (3.12) are also necessary to obtain normal limits.

Example 3.1. In the case $U = \tilde{\sigma}\sqrt{h}$, $D = -\tilde{\sigma}\sqrt{h}$, $p = \frac{r-d}{u-d}$ we have $\alpha = t(\log r_o - \frac{1}{2}\tilde{\sigma}^2)$ (cf. Example 2.1.a)), $\sigma^2 = \tilde{\sigma}^2 t$ and from (3.11) $\alpha' = t(\log r_o + \frac{1}{2}\tilde{\sigma}^2)$, $\sigma'^2 = \tilde{\sigma}^2 t$ and so (3.13) coincides with the Black-Scholes formula (3.7) in the limit. But note that Theorem 3.1 allows us to obtain also corresponding formulas for different choices of (U,D,p). □

We next describe the cases leading to a Poissonian Black-Scholes formula.

Theorem 3.2. (Case (2), $(0 < \lambda < \infty,\ \mu = 0)$)
Under the assumptions of Theorem 2.1.b) for U, D, p and $p' = p\frac{u}{r} < 1$, i.e. $a = \lim np$, $b = \lim nD$ and $a' = \lim np'$ holds:

(3.15) $$C_n \to C = S\,P'(x) - K\,r_o\ P(x),$$

where

(3.16) $$\begin{aligned} P(x) &= P\ (\text{Poisson }(a) \ge x), \\ P'(x) &= P\ (\text{Poisson }(a') \ge x), \\ x &:= [\tfrac{\log(K/S) - b}{\lambda}]_+, \end{aligned}$$

provided $\frac{\log(K/S)-b}{\lambda}$ is not an integer.

Proof. With $R = U - D$, $a_n = [\log(K/S\,d^n)/R]_+$, $\Phi(a_n,n,p) = P(\sum_k \zeta_{n,k} \ge a_n)$ $= P(\log(S_n/S) \ge a_n R + nD) = P(\log(S_n/S) \ge \log(K/S) + \varepsilon_n R)$, where $\varepsilon_n = a_n - \frac{1}{R}\log(K/S\,d^n) \ge 0$. From Theorem 2.1.b), $\log(S_n/S) \xrightarrow{w} b + \lambda$ Poisson (a), and thus $\Phi(a_n,n,p) = P(\frac{\log(S_n/S) - b}{\lambda} \ge \frac{\log(K/S) + \varepsilon_n R - b}{\lambda}) \to P(\text{Poisson }(a) \ge x)$ $= P(x)$ if $x = [\frac{\log(K/S)-b}{\lambda}]_+$ is not an integer. In a similar way one can consider $\Phi(a_n,n,p')$ and thus obtains (3.15) from (3.1). □

Remarks.

a) If x is an integer and $R > 0$, $\varepsilon_n > 0$ for all n, then in the limit one obtains

(3.17) $$C = S\,P'(x+1) - K\,r_o^{-t}\,P(x+1).$$

If $R < 0$, then (3.16) remains valid.

b) Suppose $U = \lambda$, $D = -\bar{\mu}h$, $p = vh$ (as in Example 2.1.b)) and suppose $v(e^{\lambda} - 1) - \bar{\mu} > 0$, then $r^n = (p e^U + q e^D)^n \to r_o^t$, where $\log r_o = v(e^{\lambda} - 1) - \bar{\mu} > 0$. Moreover,

(3.18) $$a = \lim n p = vt, \; b = \lim n D = -\bar{\mu} t, \; a' = vt e^{\lambda}.$$

So from (3.15)

(3.19) $$C = S P'(x) - K r_o^{-t} P(x)$$

with a, a', b as in (3.18). (3.19) is given in Cox and Rubinstein (1985), p. 366 (the assumption $x \notin \mathbb{N}$ is missing in this work, which seems to be a technical gap).

c) Note that in (3.1) $C = 0$ if $a_n > n$. Equivalently, $\frac{\log (K/S) - nD}{U - D} > n$ resp. $\log (K/S) > nU$ implies $C = 0$. Therefore, for the analogue of Theorem 3.2 in Case (4) ($\lambda = 0$, $0 < \mu < \infty$) one should require

(3.20) $$\log (K/S) < b := \lim n U$$

to avoid the degenerate case.

d) If f is a convex, nondecreasing function on $\mathbb{R}^1$, then f has a representation of the form

(3.21) $$f(x) = \int (x - K)_+ \, d\mu(K) \quad \text{with a measure } \mu \text{ on } \mathbb{R}^1.$$

This implies that a call of the form $B = f(S_t)$ has the options value

(3.22) $$C(f) = \int C(K) \, d\mu(K),$$

where $C(K)$ is the value of the call $(S_n - K)_+$. For μ bounded one can approximate $C_n(f) \to C(f)$ by the binomial pricing model. □

4. A Random Number of Price Changes

Several recent papers on the distribution of stock price changes show a large variety of alternative (to the normal) distributions which fit better the stock price changes data (see Hagerman (1978), Du Mouchel (1983), Mittnik and Rachev (1989)). The subject of this section is to give a simple extension of the binomial stock price model which gives a large class of alternative distributions for $\log (S^*/S)$, where

S^* is the price at time t. Let (N_n) be a sequence of integer valued random times independent of the underlying r.v.'s

(4.1) $$X_{nk} = \zeta_{n,k} U + (1-\zeta_{n,k}) D, \; \zeta_{n,k} \stackrel{d}{=} \mathfrak{B}(1,p)$$

with typically $E N_n = n$ and let

(4.2) $$S^* = S_{N_n} \quad \text{resp.} \quad \log(S^*/S) = \sum_{k=1}^{N_n} X_{n,k}.$$

So we consider a random number of jumps in [0,t], each jump being of the simple Bernoulli type. For the placement of the jumps we may think in a first step on a uniform placement in [0,t], so that we obtain random intervals of constant lenght $h = t/N_n$. In a second step we may model the placements (more realistically) by a point process on [0,t] with N_n points, such that we can identify our price change model with a marked point process (with independent marks). For the options pricing formula both models lead to the same result only in the case of r = 1. In this paper we shall concentrate on the simpler case with constant inter arrival times $h = t/N_n$. The idea of randomizing the time in evaluating stock price change distributions goes back to Clark (1973) and Mandelbrot and Taylor (1967).

In the same way as in Section 2 we consider the limit behaviour of $\log(S_{N_n}/S)$, where the constants $U = U(n) = \log u$, $D = D(n) = \log d$ are dependent on n and

(4.3) $$\lim_{n\to\infty} U(n) = \lambda, \quad \lim_{n\to\infty} D(n) = -\mu.$$

In each case we shall assume (4.2), (4.3).

<u>Theorem</u> 4.1. Suppose that $\frac{N_n}{n} \xrightarrow{w} Y$ and that $\sum_{k=1}^{n} X_{n,k} \xrightarrow{w} X$.

a) In Case (1) $(\lambda = \mu = 0)$ holds

(4.4) $$\log(S_{N_n}/S) \xrightarrow{w} Z,$$

where the distribution of Z is a mixture of normals

(4.5) $$\varphi_Z(u) = \int_0^\infty e^{i\alpha z u - \frac{\sigma^2 z}{2} u^2} \, dF_Y(z),$$

α, σ^2 as determined in (2.9), (2.10).

b) In Case (2) $(0 < \lambda < \infty, \mu = 0)$ holds (4.4), where Z is a mixture of Poisson distributions, (4.6) $$\varphi_Z(u) = \int_0^\infty \exp\{iubz + za(e^{iu\lambda} - 1)\} dF_Y(z)$$

and $a = \lim(np)$, $b = \lim(nD)$, $\alpha = b + a\frac{\lambda}{1+\lambda^2}$.

c) In Case (8) ($\lambda = \infty$, $\mu = 0$) (4.4) holds with

(4.7) $$\varphi_Z(u) = \int_0^\infty e^{i\alpha uz} dF_Y(z) \quad \text{and} \quad \alpha = \lim_{n\to\infty} nD.$$

Proof. The proof of Theorem 4.1 follows from Theorem 2.1 and the following simple lemma, the so-called "transfer theorem" which is well-known from the early works of Robbins (1948), Reyni (1967), Gnedenko (1970), (1983).

Lemma 4.1. Let $(X_{n,k})_{k\in\mathbb{N}}$ be an iid sequence of real rv's and let N_n be an integer valued rv independent of $X_{n,k}$, $n \in \mathbb{N}$. If as $n \to \infty$

(4.8) $$\sum_{k=1}^{n} X_{n,k} \xrightarrow{w} X, \quad \text{and}$$

(4.9) $$N_n/n \xrightarrow{w} Y,$$

then

(4.10) $$\sum_{k=1}^{N_n} X_{n,k} \xrightarrow{w} Z,$$

where the ch.f. of Z is given by

(4.11) $$\varphi_Z(u) = \int_0^\infty (\varphi_X(u))^z dF_Y(z),$$

φ_X the ch.f. of X and F_Y the d.f. of Y.

If Y is ID, then by a remark of Feller (1966) also Z is ID.

Example 4.1.

a) Geometric case. Let N_n be a geometric r.v. with mean n, i.e.

(4.12) $$P(N_n = k) = \frac{1}{n}\left(1 - \frac{1}{n}\right)^{k-1}, \; k = 1,2,\ldots \; .$$

Then Y is exponential with mean 1 and

(4.13) $$\varphi_Z(u) = \frac{1}{1 - \log \varphi_X(u)} \; .$$

Since by (4.2) X is ID, the distribution of Z is a geometric infinite divisible distribution (cf. Klebanov, Manija and Melamed (1984)).

In Case (1) ($\lambda = \mu = 0$) we have

(4.14) $$Z \stackrel{d}{=} Z_1 - Z_2,$$

where (Z_i) are independent exponentials with mean a_i

$$(4.15)\qquad a_1 = \frac{\alpha + \sqrt{\alpha^2+2\sigma^2}}{2}, \quad a_2 = \frac{-\alpha + \sqrt{\alpha^2+2\sigma^2}}{2},$$

α, σ^2 as in (2.9), (2.10).

For the proof note that by (4.13)

$$(4.16)\qquad \varphi_Z(u) = \frac{1}{1-\log \varphi_X(u)} = \frac{1}{1-i\alpha u+\frac{\sigma^2u^2}{2}} = \frac{1}{1-iu\,a_1}\;\frac{1}{1+iu\,a_2}$$

$$= \varphi_{Z_1}(u)\,\varphi_{Z_2}(-u).$$

The Laplace distribution plays the role of the normal distribution in the class of geometric stable distributions. Recall that X is stable with parameters $\alpha \in (0,2)$, $\sigma \ge 0$, $-1 \le \beta \le 1$, $\mu \in \mathbb{R}$, if

$$(4.17)\qquad \log \varphi_X(u) = \begin{cases} -\sigma^\alpha|u|^\alpha(1-i\beta(\operatorname{sign}\vartheta)\tan\frac{\pi\alpha}{2} + i\mu u & \text{if } \alpha \ne 1 \\ -\sigma|u|\,(1+i\beta\frac{2}{\pi}(\operatorname{sign} u)\,\ell n|u| + i\mu u & \text{if } \alpha = 1 \end{cases}.$$

Then Z with $\varphi_Z(u) = \frac{1}{1-\log\varphi_X(u)}$ is called geometric stable. The family of geometric stable distributions seems to fit well the empirical distribution of $\log(S^*/S)$ (see Mittnik and Rachev (1989)).

In Case (2) $(0 < \lambda < \infty,\ \mu = 0)$ the limiting distribution has a ch.f.

$$(4.18)\qquad \varphi_Z(u) = \frac{1}{1-iub-(e^{iu\lambda}-1)a}.$$

In Case (8) $(\lambda = \infty,\ \mu = 0)$ the limiting distribution is exponential with parameter α (cf. 4.7).

b) If N_n is uniformly distributed on $\{1,\dots,2n-1\}$

$$(4.19)\qquad P(N_n = k) = \frac{1}{2n-1},\quad 1 \le k \le 2n-1,$$

then

$$(4.20)\qquad \varphi_Z(u) = \frac{1}{2}\int_0^\infty (\varphi_Z(u))^z dz = \frac{1}{2\,\ell n\,\varphi_Z(u)}(\varphi_X(u)^2-1).$$

c) If N_n is Poisson distributed with mean n, then $N_n/n \to 1$ a.s. and, therefore, $\varphi_Z = \varphi_X$.

d) If X is normal distributed, then by a result of Gnedenko (1983), Z again is normal if and only if F_Y is a one-point measure. □

<u>Remarks.</u>

a) Note that (4.5) is equivalent to

$$(4.21)\qquad Z \stackrel{d}{=} \alpha Y + N_{0,\sigma^2}\sqrt{Y},$$

where N_{0,σ^2} is a normal rv with zero mean and variance σ^2 independent of Y. So the density of Z is given by

$$(4.22) \qquad f_Z(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_0^\infty e^{-\frac{(x-\alpha y)^2}{2y\sigma^2}} y^{-1/2}\, dF_Y(y).$$

b) Differentiating (4.11) one gets easily

$$(4.23) \qquad EZ = EX\, EY, \quad \frac{\mathrm{Var}\, Z}{EZ} = \frac{\mathrm{Var}\, X}{EX} + EX\, \frac{\mathrm{Var}\, Y}{EY},$$

provided $EX \neq 0$ and $EX^2 < \infty$, $EY^2 < \infty$ (see Keilson and Steutel (1979)). □

We next establish the functional CLT in the randomized model corresponding to Theorem 2.2 in the binomial model. Define

$$(4.24) \qquad \overline{X}_{n,k} = X_{n,k} - a_{n,k}, \quad X_{n,k} = \zeta_{n,k} U + (1-\zeta_{n,k}) D$$

and as in Section 2, assume that

$$(4.25) \qquad \nu_n := n \int \frac{x}{1+x^2}\, d\overline{F}_{n,1}(x) \to \nu$$

and

$$(4.26) \qquad \psi_n(x) = n \int_{-\infty}^{x} \frac{y^2}{1+y^2}\, d\overline{F}_{n,1}(y) \to \psi(x);$$

so by Theorem 2.2

$$(4.27) \qquad \overline{X}_n \xrightarrow{w} \overline{X}, \quad \text{where} \quad \overline{X}_n(u) = \sum_{j=1}^{[\frac{u}{t}n]} \overline{X}_{n,j}$$

and $\overline{X}(u)$ is a homogenous process with independent increments and ch.f.

$$(4.28) \qquad E e^{i\vartheta \overline{X}(u)} = \exp\{\frac{u}{t}[i\vartheta\nu + \int_{-\infty}^{\infty} (e^{i\vartheta x} - 1 - \frac{i\vartheta x}{1+x^2}) \frac{1+x^2}{x^2}\, d\psi(x)]\}, \quad 0 \le u < \infty$$

(For convergence in $D[0,\infty)$ cf. e.g. Resnick (1987), section 4.4.) Define for an integer valued random sequence (N_n) the randomized process

$$(4.29) \qquad \overline{Z}_n(u) := \sum_{j=1}^{[N_n \frac{u}{t}]} \overline{X}_{n,j}, \quad 0 \le u \le t.$$

<u>Theorem</u> 4.2. Assume (4.25), (4.26) and assume that $\frac{N_n}{n} \xrightarrow{w} Y$, then the sequence of processes $\overline{Z}_n$ converges weakly to

$$(4.30) \qquad \overline{Z}(u) = \overline{X}(Yu), \quad 0 \le u \le t.$$

If additionally $n\, a_{n,1} \to a$, then

(4.31) $$Z_n \xrightarrow{w} Z \quad \text{on} \quad D[0,\infty),$$

where $Z_n(u) = \sum_{j=1}^{[N_n \frac{u}{t}]} X_{n,j}$ and $Z(u) = \overline{X}(Y u) + a\frac{u}{t}\, Y$.

Proof. Note that $\overline{Z}_n(u) = \overline{X}_n(\frac{N_n}{n}\, u)$, $0 \le u \le t$, i.e. $\overline{Z}_n$ is a random time transformation of $\overline{X}_n$. Since (N_n) is independent of $(\overline{X}_n)$ we have the weak convergence of the joint process $(\frac{N_n}{n}, \overline{X}_n) \xrightarrow{w} (Y,\overline{X})$ (assuming w.l.g. that Y, $\overline{X}$ also are independent). By Skorohod's a.s. representation theorem, there exist (on a possibly different probability space) versions (K_n,U_n), (K,U) of the processes converging a.s., i.e. $(K_n,U_n) \overset{d}{=} (\frac{N_n}{n},\overline{X}_n)$, $(K,U) \overset{d}{=} (Y,\overline{X})$ and $(K_n,U_n) \to (K,U)$ a.s. Since $U_n(\omega) \in D[0,\infty)$ and $(K_n(\omega)u)_{u\ge 0} \in D[0,\infty)$ is non-decreasing for $n \ge 0$ and $(K(\omega)u)_{n\ge 0}$ is continuous, we obtain (cf. Resnick (1987), p. 221)

(4.32) $$d(U_n(K\cdot), U(K\cdot)) \to 0 \quad \text{a.s.},$$

where d is the Skorohod metric on $D[0,\infty)$, and thus

(4.33) $$\overline{Z}_n = \overline{X}_n(\frac{N_n}{n}\cdot) \xrightarrow{w} \overline{Z} = \overline{X}(Y\cdot).$$

The proof of the second part is similar. □

We apply Theorem 4.2 to the normal case. Then by (2.33) $X_n \xrightarrow{w} X$, where $X(u) \overset{d}{=} \alpha\frac{u}{t} + \frac{\sigma}{\sqrt{t}}\, W(u)$, $\alpha = \nu + a$, $W(u)$ a standard Wiener process. Therefore, $Z_n \xrightarrow{w} Z$, where

(4.34) $$Z(u) \overset{d}{=} \alpha/t\ Yu + \frac{\sigma}{\sqrt{t}}\sqrt{Y}\, W(u),$$

in the last line using that $(W(Yu))_{0\le u} \overset{d}{=} (\sqrt{Y}\, W(u))_{0\le u}$. So we obtain:

Theorem 4.3. (Case (1), $\lambda = \mu = 0$) Assume that $\frac{N_n}{n} \xrightarrow{w} Y$ and $n\, a_{n,1} = n(pU + qD) \to a$, $n(q\frac{R^2p^2}{1+R^2p^2} + p\frac{R^2q^2}{1+R^2q^2}) \to \sigma^2$, and $npq\,R^3 \frac{p-q}{(1+p^2R^2)(1+q^2R^2)} \to \nu$, then

(4.35) $$Z_n \xrightarrow{w} Z,$$

where $(Z(u))_{0\le u\le\infty} \overset{d}{=} (\frac{\alpha}{t}\, Yu + \frac{\sigma}{\sqrt{t}}\sqrt{Y}\, W(u))_{0\le u<\infty}$, W a Wiener process independent of Y. □

<u>Remark.</u> In the Cox, Ross, Rubinstein model $U = \tilde{\sigma}\sqrt{t/n}$, $D = -\tilde{\sigma}\sqrt{t/n}$, $r^n = r_o^t$, $\alpha/t = \log r_o - \frac{1}{2}\tilde{\sigma}^2$, $\frac{\sigma}{\sqrt{t}} = \tilde{\sigma}$, and thus

$$(Z(u)) \stackrel{d}{=} (\log r_o - \tfrac{1}{2}\tilde{\sigma}^2)uY + \tilde{\sigma}\sqrt{Y}\,W(u). \tag{4.36}$$

Applying Ito's formula it is seen that the price process satisfies the stochastic differential equation

$$dS(u) = S(u)[\tilde{\sigma}\sqrt{Y}\,dW(u) + (\log r_o)Y\,du], \tag{4.37}$$

where $Z(u) = \log (S(u)/S$ or

$$S(u) = S\,e^{\tilde{\sigma}\sqrt{Y}\,W(u) + aYu}, \quad a = \log r_o - \tfrac{1}{2}\tilde{\sigma}^2. \tag{4.38}$$

In fact, $S(u) = f(u,\zeta(u))$, where $f(u,x) = S\,e^x$ and $\zeta(u) = \tilde{\sigma}\sqrt{Y}\,W(u) + aYu = B\,W(u) + A(u)$. Thus $dS(u) = [f'_u(u,\zeta(u)) + f'_x(u,\zeta(u))A(u) + \frac{1}{2}f''_{xx}(u,\zeta(u))B^2(u)]du + f'_x(u,\zeta(u))B(u)dW(u) = (S(u)a\,Yu + \frac{1}{2}\,S(u)\tilde{\sigma}^2 Y)du + S(u)\tilde{\sigma}\,\sqrt{Y}\,dW(u) = S(u)(\tilde{\sigma}\sqrt{Y}\,dW(u) + \log r_o\,Y\,du$. □

5. <u>Valuation formulas for Models with a Random Number of Price Changes</u>

As in Section 4 we consider now a random number N_n of price changes in $[0,t]$, N_n independent of the jumps $(X_{n,k})$ of the price process $\log (S^*/S) = \sum_{k=1}^{N_n} X_{n,k}$. In order to determine a valuation formula for a call $B = (S^* - K)_+$ we use similar to Section 3 an equilibrium argument. The equilibrium measure P^* should satisfy the following conditions:

1. P^* is equivalent to P on the σ-algebra $\mathfrak{F}_k = \sigma(N_n, X_{n,1},\ldots,X_{n,k})$, $k \in \mathbb{N}_o$, where $\mathfrak{F}_o = \sigma(N_n)$.
2. W.r.t. P^*, N_n and the price process $(X_{n,k})$ are independent.
3. The distribution of N_n w.r.t. P^* is the same as w.r.t. P.
4. The price process $(r^{-k}S_k)$ is a martingale and S_{k+1} is either $u\,S_k$ or $d\,S_k$.

These conditions correspond to the situation that the random number N_n of price movements in the interval $[0,t]$ is known at the beginning of the period (in the classical case it is assumed to be known and equal to a fixed number n). So we allow some variation of the intrinsic time of the jumps in the market. The motivation for this time change goes back to Mandelbrot and Taylor (1967) and Clark (1973). An alternative and interesting assumption to the assumed independence of N_n, $(X_{n,k})$ would be to assume that N_n is a stopping time w.r.t. $\mathfrak{F}_k^S = \sigma(S_1,\dots,S_k)$, but we shall concentrate in this paper on the independence case.

From the consideration in Section 4 we can conclude that there exists a unique equilibrium measure P^* (unique on $\sigma(\bigcup_{k=0}^{\infty} \mathfrak{F}_k) = \sigma(N_n, X_{n,1}, X_{n,2},\dots))$ satisfying $(P^*)^{(N_n,(S_k))} = P^{*N_n} \otimes (P^*)^{(S_k)} = P^{N_n} \otimes (P^*)^{(S_k)}$, where $(P^*)^{(S_k)}$ is identical to the riskless martingale measure determined in Section 4. So if $p = p(n)$ denotes the probability of an up turn $q = 1-p$, then with the riskless discounting factor plus 1

$$r = pu + qd \tag{5.1}$$

(assuming $u > r > 1 > d > 0$) we have

$$r^{N_n} = r_o^t, \tag{5.2}$$

i.e. $r_o = r^{N_n/t}$ is random and the valuation formula in our model is the expected value of the discounted call w.r.t. the unique riskless measure P^*

$$C(N_n) = E_*(S_{N_n} - K)_+ = \sum_{k=1}^{\infty} E_*(S_k - K)_+ P^*(N_n = k) = \sum_{k=1}^{\infty} C_k P(N_n = k). \tag{5.3}$$

So $C(N_n)$ is the average of the valuations with fixed number of price steps in $[0,t]$.

<u>Remark.</u> An alternative derivation of a pricing formula is obtained by considering r_o to be a fixed discount rate per time unit, implying that $r = r_o^{t/N_n} = r_{N_n}$ is the random discount rate per unit period and, therefore, the risk neutral probabilities $p = \frac{r-d}{u-d} = p_{N_n}$ also are random. One obtains from (3.1) by conditioning on N_n

$$C(N_n) = E\, C_{N_n} = S\, E\Phi(a_{N_n}, N_n, p'_{N_n}) - K r_o^{-t} E\Phi(a_{N_n}, N_n, p_{N_n}) \tag{5.4}$$

with $p' = \frac{u}{r} p$. □

We next derive some asymptotic formulas for the pricing formula (5.3). With this in mind let

(5.5) $$p' := p\,\frac{u}{r}\,,\quad q' = 1 - p' = p\,\frac{d}{r}\,.$$

(5.6) $$\Omega_n := E\,r^{-N_n}$$

and define rv's N_n^* independent of $(\zeta_{n,i})$ with distributions

(5.7) $$P(N_n^* = k) = \frac{1}{\Omega_n}\, r^{-k}\, P(N_n = k)\,.$$

From (3.1) we obtain the representation

(5.8) $$C(N_n) = S\,E\,\Phi(a_{N_n}, N_n, p') - K\,\Omega_n\,E\,\Phi(a_{N_n^*}, N_n^*, p),$$
$$a_n = \left[\frac{\log(K/S) - nD}{R}\right],\quad n \in \mathbb{N}.$$

If $a_{N_n} > N_n$, i.e. if $U N_n < \log(K/S)$, then $C(N_n) = 0$. For the following we assume that

(5.9) $$\underline{\lim}_{n\to\infty}\; U(n) N_n > \log(K/S)\quad \text{a.s.}$$
$$\underline{\lim}_{n\to\infty}\; U(n) N_n^* > \log(K/S)\quad \text{a.s.},$$

and as before $U(n) \to \lambda$, $D(n) \to -\mu$.

<u>Lemma</u> 5.1. Assume that $\frac{N_n}{n} \xrightarrow{w} Y$ with Laplace-transform $\psi_Y(\vartheta) = E\,e^{-\vartheta Y}$ and $r^n \to \overline{r}_o^{\,t}$, then $\frac{N_n^*}{n} \xrightarrow{w} Y^*$ with Laplace-transform

(5.10) $$\psi_{Y^*}(\vartheta) = \frac{E\,\overline{r}_o^{\,-tY} e^{-Y\vartheta}}{E\,\overline{r}_o^{\,-tY}}\,.$$

<u>Proof.</u> $E\,e^{-\vartheta \frac{N_n^*}{n}} = \sum_{k=1}^{\infty} e^{-\vartheta \frac{k}{n}}\, P(N_n = k)\, r^{-k}\, \frac{1}{\Omega_n} =$

$= \frac{1}{E\,r^{-N_n}} \sum_{k=1}^{\infty} P(N_n = k)\, e^{-\frac{k}{n}(\vartheta + t\log \overline{r}_o + \varepsilon_n)}$ with $\varepsilon_n \to 0$. Since $E\,r^{-N_n} =$

$= E\,r^{-n\frac{N_n}{n}} \longrightarrow E\,\overline{r}_o^{\,-tY}$, we obtain from the continuity of Laplace-transforms $\psi_{\frac{N_n^*}{n}}(\vartheta) \to \frac{E\,e^{-Y(\vartheta + t\log \overline{r}_o)}}{E\,\overline{r}_o^{\,-tY}}$. □

<u>Theorem</u> 5.1. (Case (1), $\lambda = \mu = 0$)

Assume the existence of the limits α, α', σ^2, σ'^2 as in (3.10), (3.12) and assume that

(5.11) $$\frac{N_n}{n} \xrightarrow{w} Y \quad \text{and} \quad r^n \longrightarrow \overline{r}_o^{\,t}\,,$$

then

$$\text{(5.12)} \qquad C(N_n) \to C := S\,\Phi_{Z'}(x) - K(E\,\overline{r}_o^{\,-tY})\,\Phi_{Z*}(x),$$

where $x = \log(K/S)$, $\Phi_{Z*}(x) = P(Z^* \geq x)$, Z^* has ch.f. of a mixture of normals

$$\text{(5.13)} \qquad \varphi_{Z*}(u) = \int_0^\infty e^{i\alpha zu - \frac{\sigma^2}{2} zu^2}\, dF_{Y*}(z),$$

Y^* as in (5.10), and $\Phi_{Z'}$ is similarly defined with α', σ', Y instead of α, σ, Y^* in (5.13).

<u>Proof.</u> Recall that

$$\text{(5.14)} \qquad E\,\Phi(a_{N_n}, N_n, p') = P(\log(S_{N_n}/S) - a_{N_n}R - N_n D \geq 0),$$

where $p' = p\,\frac{u}{r}$. With $X_{ni} = \zeta_{ni}R + D$, the characteristic function of the random variable on the RHS of (5.14) equals

$$\text{(5.15)} \qquad \begin{aligned} &\sum_{k=1}^{\infty} P(N_n = k)\, e^{-(a_k R + kD)iu}\, E\exp\{iu \log(S_k/S)\} \\ &= \int_0^\infty e^{-(a_x R + xD)iu}\, (\varphi_{X_{n,1}}(u))^x\, dP^{N_n}(x) \\ &= \int_0^\infty e^{-(a_{nz}R + nzD)iu}\, (\varphi^n_{X_{n,1}}(u))^z\, dP^{N_n/n}(z) \\ &\longrightarrow \int_0^\infty e^{-\log(K/S)iu}\, (\varphi_X(u))^z\, dP^Y(z), \end{aligned}$$

where $X \stackrel{d}{=} N(\alpha', \sigma'^2)$. For the last step we use that by Theorem 2.1.a) $\varphi^n_{X_{n,1}}(u) \to \varphi_X(u)$ and, furthermore, since $R = R(n) \to 0$, $a_{nz}R + nzD \to \log(K/S)$. From (5.15) $\log(S_{N_n}/S) - a_{N_n}R - N_n D \xrightarrow{w} Z' - \log(K/S)$, which implies

$$\text{(5.16)} \qquad E\,\Phi(a_{N_n}, N_n, p') \to P(Z' \geq \log(K/S)).$$

By Lemma 5.1 and Theorem 4.1.a), a similar argument applies to $E\,\Phi(a_{N_n^*}, N_n^*, p)$; so from (5.8) we obtain (5.12), observing that by (5.11)

$$\text{(5.17)} \qquad \Omega_n \to E\,\overline{r}_o^{\,-tY}. \qquad \Box$$

<u>Example</u> 5.1. Let $U = \tilde{\sigma}\sqrt{t/n}$, $D = -\tilde{\sigma}\sqrt{t/n}$, $r = r_o^{t/n}$, $(r_o = \overline{r}_o)$, $p = \frac{r-d}{u-d} \approx \frac{1}{2} + \frac{1}{2}\,\frac{\log r_o - \frac{1}{2}\tilde{\sigma}^2}{\tilde{\sigma}}\sqrt{t/n}$ as in Example 3.1 and consider the geometric case (cf. Example 4.1.a) $P(N_n = k) = \frac{1}{n}(1 - \frac{1}{n})^{k-1}$, $k \geq 1$. The "random" Black- Scholes type formula (5.12) has the form:

$$\text{(5.18)} \qquad C = S\,\Phi_{Z'}(x) - \frac{K}{1 + t\log r_o}\,\Phi_{Z*}(x),$$

where $Z' \stackrel{d}{=} Z'_1 - Z'_2$, Z'_i are independent exponentials with means $a'_{1,2} = \frac{1}{2}(\pm\alpha' + \sqrt{\alpha'^2 + 2\sigma'^2})$ and $Z^* \stackrel{d}{=} Z^*_1 - Z^*_2$, where Z^*_i are independent exponentials with means

$a^*_{1,2} = \frac{1}{2}(\pm\alpha^* + \sqrt{\alpha^{*2}+2\sigma^{*2}})$, $\alpha^* = \frac{\alpha}{1+t\log r_o}$, $\sigma^{*2} = \frac{\sigma^2}{1+t\log r_o}$. In fact: $E\,r_o^{-tY} = \frac{1}{1+t\log r_o}$ and $E\,r_o^{-tY}e^{-Y\vartheta} = \frac{1}{1+\vartheta+t\log r_o}$ and thus by Lemma 5.1, $\psi_{Y^*}(\vartheta) = \frac{1+t\log r_o}{1+\vartheta+t\log r_o} = \frac{1}{1+\frac{\vartheta}{1+t\log r_o}}$. Furthermore, by (5.13), $\varphi_{Z^*}(t) = \int_0^\infty e^{i\alpha zu - \frac{\sigma^2}{2}zu^2}\,dF_{Y/1+t\log r_o}(u) = \int_0^\infty e^{i\alpha^* wu - \frac{\sigma^{*2}}{2}wu^2}\,dF_Y(w)$.

By Example 3.1, $\alpha = t(\log r_o - \frac{1}{2}\tilde\sigma^2)$, $\sigma^2 = \tilde\sigma^2 t$, $\alpha' = t(\log r_o + \frac{1}{2}\tilde\sigma^2)$, $\sigma'^2 = \tilde\sigma^2 t$. Further, $a'_{1,2} = \frac{1}{2}(\pm t(\log r_o + \frac{1}{2}\tilde\sigma^2) + \sqrt{t^2(\log r_o + \frac{1}{2}\tilde\sigma^2)^2 + 2\tilde\sigma^2 t})$,
$a^*_{1,2} = \frac{1}{2}\left(\pm\frac{t(\log r_o - \frac{1}{2}\tilde\sigma^2)}{1+t\log r_o} + \sqrt{\frac{t^2(\log r_o - \frac{1}{2}\tilde\sigma^2)^2}{(1+t\log r_o)^2} + 2\frac{\tilde\sigma^2 t}{(1+t\log r_o)}}\right)$ and thus $\Phi_{Z^*}(x) = P(Z^* \ge x) = P(Z^*_1 - Z^*_2 \ge x)$, $\Phi_{Z'}(x) = P(Z' \ge x) = P(Z'_1 - Z'_2 \ge x)$, where Z^*_i, Z'_i are independent exponentials with means a^*_i resp. a'_i. □

We next consider the case of mixtures of Poisson and degenerate distributions.

<u>Theorem</u> 5.2.

a) (Case (2), $0 < \lambda < \infty$, $\mu = 0$)

Assume the existence of the following limits

(5.19) $a = \lim np$, $b = \lim nD$, $a' = \lim np'$, $p' = \frac{pu}{r} < 1$

and suppose that

(5.20) $\frac{N_n}{n} \xrightarrow{w} Y$ and $r^n \to \bar r_o^{\,t}$,

then

(5.21) $C(N_n) \to C := S\Phi_{Z'}(0) - K(E\bar r_o^{\,-tY})\Phi_{Z^*}(0)$,

where $\mathfrak{L}(Z')$ is a Poisson mixture with ch.f.

(5.22) $\varphi_{Z'}(u) = \int_0^\infty e^{-[\frac{\log(K/S)-zb'}{\lambda}]_+ \lambda iu + a'z(e^{i\lambda u}-1)}\,dP^Y(z)$,

and where Z^* is similarly defined with a', b', Y in (5.22) replaced by a, b, Y^* from (5.19).

b) (Case (8), $\lambda = \infty$, $\mu = 0$)

Suppose that the following limits exist

(5.23) $\quad b = \lim nD,\ c = \lim \frac{np}{U},\ c' = \lim \frac{np'}{U},\ p' = \frac{pu}{r}$

and, furthermore, let

(5.24) $\quad \frac{N_n}{n} \xrightarrow{w} Y,\quad r^n \to \overline{r}_o^{\,t},$

then

(5.25) $\quad C(N_n) \to C := S\Phi_{Z'}(0) - K(E\,\overline{r}_o^{\,-tY})\Phi_{Z^*}(0),$

where

(5.26) $$\varphi_{Z'}(u) = \int_0^\infty e^{-[\frac{\log(K/S)-zb}{\lambda}]_+ \lambda iu + zc'iu}\, dP^Y(z)$$

and

(5.27) $$\varphi_{Z^*}(u) = \int_0^\infty e^{[\frac{\log(K/S)-zb}{\lambda}]_+ \lambda iu + zciu}\, dP^{Y^*}(z),$$

Y^* from (5.10), $[\ ]_+$ the positive Gauss-bracket.

<u>Proof.</u> a) By definition $E\Phi(a_{N_n}, N_n, p) = P(\log(S_{N_n}/S) - a_{N_n}R - N_nD \geq 0)$. The characteristic function of the RHS equals
$\int_0^\infty e^{-(a_{nz}R + nzD)iu}(\varphi^n_{X_{n,1}}(u))^z\, dP^{\frac{N_n}{n}}(z)$ and it converges to
$\varphi_Z(u) = \int_0^\infty e^{-[\frac{\log(K/S)-zb}{\lambda}]_+ \lambda iu + az(e^{i\lambda u}-1)}\, dP^Y(z)$. In fact, by Theorem 2.1.b), $\varphi^n_{X_{n,1}}(u) \to \varphi_X(u) = e^{iub + a(e^{i\lambda u}-1)}$. On the other hand,

since $R \to \lambda$, $nD \to b$, $np \to a$,

$a_{nz}R + nzD = [\frac{\log(K/Sd\ \ )}{R}]R + nzD \xrightarrow[n\to\infty]{} [\frac{\log(K/S)-zb}{\lambda}]\lambda + zb,$
for any z such that $\frac{1}{\lambda}(\log(K/S) - zb)$ is not an integer. Thus

$\log(S_N/S) - a_N R - N_nD \xrightarrow{w} Z$ and $\Phi(a_{N_n}, N_n, p) \to P(Z \geq 0)$. From this one can infer formula (5.21).

The proof of b) is similar. □

<u>Remarks.</u>

a) If the empirical data of the stock price changes suggest a mixture of the normals of the type $\alpha Y + \sigma\sqrt{Y}\,N$, where N is a standard normal then by Theorem 5.1 resp. Theorem 4.1 we can model these changes by a model with a random number N_n of changes, where $\frac{N_n}{n} \xrightarrow{w} Y$ and calculate the generalized Black-Scholes option price.

Since location and scale mixtures of normals are generally not identifiable, there are possibly different sequences $\frac{N'_n}{n} \to Y'$, but leading by (5.13) to the same approximative valuation formula.

b) In contrast to the "non random" case $N_n \equiv n$, where the Case (8), $\lambda = \infty$, $\mu = 0$, leads to degenerate convergence $\log(S_n/S) \to \alpha = \lim\limits_{n\to\infty} [nD + \frac{np}{U}]$ one obtains in the "random case" $\log(S_{N_n}/S) \to \alpha Y$.

c) Restriction to the case $r \equiv 1$?

The valuation formulas simplify if $r = r(n) = 1$. Recall from 5.1 that $r(n) = p(n)u(n) + q(n)d(n)$, $u(n) > r(n) > 1 > d(n)$. Defining $\tilde{d}(n) := \frac{d(n)}{r(n)} < 1$, $\tilde{u}(n) = \frac{u(n)}{r(n)} > 1$, the riskless rate is now $r(n) \equiv 1$. In several papers it is argued for this reason that one can assume w.l.g. that $r(n) \equiv 1$.

We demonstrate the effect of this substitution considering again Example 5.1. with $U(n) = \tilde{\sigma}\sqrt{t/n}$, $D(n) = -\tilde{\sigma}\sqrt{t/n}$, $r = r_o^{t/n}$, $p = \frac{r-d}{u-d} \approx \frac{1}{2} + \frac{1}{2}\frac{\log r_o - \frac{1}{2}\tilde{\sigma}^2}{\tilde{\sigma}}\sqrt{t/n}$. Define $\tilde{U} := \log\tilde{u} = \log\frac{u}{r} = U - \log r = \tilde{\sigma}\sqrt{t/n} - \frac{t}{n}\log r_o$, $\tilde{D} = \log\tilde{d} = \log d/r = D - \log r = -\tilde{\sigma}\sqrt{t/n} - \frac{t}{n}\log r_o$ and let $\tilde{\lambda} = \lim \tilde{U}(n)$, $\tilde{\mu} = -\lim \tilde{D}(n)$. In our case $\tilde{\lambda} = \tilde{\mu} = 0$ and the characteristics $\bar{\alpha}$, $\bar{\sigma}$ in this new model (cf. Example 3.1) can be seen by some calculations as

$$\text{(5.28)} \qquad \bar{\alpha} = t(\log r_o - \tfrac{1}{2}\tilde{\sigma}^2) - (\log r_o)t = -\tfrac{t}{2}\tilde{\sigma}^2, \quad \bar{\alpha}' = \tfrac{t}{2}\tilde{\sigma}^2,$$

$$\sigma = \bar{\sigma} = \tilde{\sigma}\sqrt{t} = \sigma' = \bar{\sigma}' \quad (p' = p\tfrac{u}{r} = p\tilde{u}).$$

For the proof note that $\tilde{R} = \tilde{U} - \tilde{D} = U - D = R$ and

$$\bar{\alpha} = \lim_n n[q\tilde{D} + p\tilde{U} + qp(q-p)\frac{\tilde{R}^3}{(1+p^2\tilde{R}^2)(1+q^2\tilde{R}^2)}] =$$

$$= \lim_n n[(\tfrac{1}{2} - \tfrac{1}{2}\frac{\log r_o - \frac{1}{2}\tilde{\sigma}^2}{\tilde{\sigma}})\sqrt{h}\,(-\tilde{\sigma}\sqrt{h} - h\log r_o) +$$

$$+ (\tfrac{1}{2} + \tfrac{1}{2}\frac{\log r_o - \frac{1}{2}\tilde{\sigma}^2}{\tilde{\sigma}})\sqrt{h}\,(\tilde{\sigma}\sqrt{h} - h\log r_o)]$$

$$= \lim_n n[(\tfrac{1}{2} - \tfrac{1}{2}\frac{\log r_o - \frac{1}{2}\tilde{\sigma}^2}{\tilde{\sigma}})\sqrt{h}\,(-\tilde{\sigma}\sqrt{h}) + (\tfrac{1}{2} + \tfrac{1}{2}\frac{\log r_o - \frac{1}{2}\tilde{\sigma}^2}{\tilde{\sigma}})\sqrt{h}\,\tilde{\sigma}\sqrt{h}]$$

$$- \lim_n (\log r_o)\, n[\tfrac{1}{2} - \tfrac{1}{2}\frac{\log r_o - \frac{1}{2}\tilde{\sigma}^2}{\tilde{\sigma}} + \tfrac{1}{2} + \tfrac{1}{2}\frac{\log r_o - \frac{1}{2}\tilde{\sigma}^2}{\tilde{\sigma}}]\, h$$

$$= t(\log r_o - \tfrac{1}{2}\tilde{\sigma}^2) - (\log r_o)t = -\tfrac{t}{2}\tilde{\sigma}^2.$$

The other cases are similar. It is interesting that r_o cancels in the formulas for $\overline{\alpha}$, $\overline{\alpha}'$, σ, $\overline{\sigma}$. Formula (3.13) becomes in this case

(5.29) $$C = S\Phi(x') - K\Phi(x), \text{ where } x = \frac{\log(S/K) - \frac{t}{2}\tilde{\sigma}^2}{\tilde{\sigma}\sqrt{t}},$$
$$x' = \frac{\log(S/K) + \frac{t}{2}\tilde{\sigma}^2}{\tilde{\sigma}\sqrt{t}},$$

which coincides with the Black-Scholes formula with $r \equiv 1$ but not with the pricing formula in the riskless model and it is not clear how to obtain by a transformation from the case $r = 1$ the valuation formula for the case $r \neq 1$. For this reason the suggestion to assume w.l.g. $r = 1$ seems not to be justified.

In the random binomial model with $r \equiv 1$ the calculations simplify. If $\frac{N_n}{n} \xrightarrow{w} Y$, then (5.12) becomes

(5.30) $$C = S\,\Phi_{\tilde{Z}'}(x) - K\,\Phi_{\tilde{Z}^*}(x), \quad \text{where} \quad x = \log(K/S),$$

(5.31) $$\varphi_{\tilde{Z}^*}(u) = \int_0^\infty e^{-\frac{\tilde{\sigma}^2}{2}tziu - \frac{\tilde{\sigma}^2}{2}tzu^2}\, dF_{Y^*}(z) = \int_0^\infty e^{-\frac{\sigma^2}{2}z(iu+u^2)}\, dF_{Y^*}(z)$$

and

(5.32) $$\varphi_{\tilde{Z}'}(u) = \int_0^\infty e^{\frac{\sigma^2}{2}z(iu-u^2)}\, dF_Y(u).$$

In the special case $P(N_n = k) = \frac{1}{n}(1 - \frac{1}{n})^{k-1}$, $k \in \mathbb{N}$, holds: $Y \overset{d}{=} \text{Exp}(1)$, $\tilde{Z}^* \overset{d}{=} \tilde{Z}_1 - \tilde{Z}_2$, $\tilde{Z}_i$ independent exponentials, $\tilde{a}_1 = E\tilde{Z}_1 = \frac{1}{2}(\tilde{\alpha} + \sqrt{\tilde{\alpha}^2 + 2\tilde{\sigma}^2})$ $= -\frac{t}{4}\tilde{\sigma}^2 + \frac{\tilde{\sigma}}{2}\sqrt{\frac{t^2\tilde{\sigma}^2}{4} + 2}$, $\tilde{a}_2 = E\tilde{Z}_2 = \frac{t\tilde{\sigma}^2}{4} + \frac{\tilde{\sigma}}{2}\sqrt{\frac{t^2\tilde{\sigma}^2}{4} + 2}$ and, similarly, $\tilde{Z}' \overset{d}{=} \tilde{Z}_1' - \tilde{Z}_2'$, $\tilde{Z}_i'$ independent exponentials with $E\tilde{Z}_1' = \tilde{a}_1' = \frac{t}{4}\tilde{\sigma}^2 +$ $+ \frac{\tilde{\sigma}}{2}\sqrt{\frac{t^2\tilde{\sigma}^2}{4} + 2t}$, $E\tilde{Z}_2' = \tilde{a}_2' = -\frac{t}{4}\tilde{\sigma}^2 + \frac{\tilde{\sigma}}{2}\sqrt{\frac{t^2\tilde{\sigma}^2}{4} + 2t}$. □

6. Examples

We shall consider examples for the random pricing model leading in particular to heavy tailed distributions in Theorem 4.1.a) resp. Theorem 5.1. (normal case).

6.1. Paretian Stable Y

Let Y be a positive Paretian stable rv with Laplace transform

(6.1) $$\psi_Y(\vartheta) = E\,e^{-\vartheta Y} = e^{-\vartheta^{\alpha/2}}, \quad \vartheta > 0, \; 0 < \alpha < 2.$$

If for example

(6.2) $$P(N_n = k) := P(k-1 < nY \le k),\ k \in \mathbb{N},\quad \text{then}$$

$$P(\frac{N_n}{n} > x) = P(N_n > xn) = P(N_n > [xn]) = P(nY > [xn])$$

$$= P(Y > \frac{[xn]}{n}) \to P(Y > x),\ \forall\, x \ge 0,\quad \text{i.e.}$$

(6.3) $$\frac{N_n}{n} \xrightarrow{w} Y.$$

Mandelbrot and Taylor (1967) suggested for the pricing process the continuous subordinated process $\log S(t) = X(T(t))$, where $(X(t))$ is a Wiener process and $(T(t))$ is a positive $\alpha/2$-stable, $0 < \alpha < 2$ process independent of X. So the resulting process is symmetric α-stable. Clark (1973) proposed for $(T(t))$ a lognormal process with independent increments implying that $\log S(t)$ has independent increments and thinner tails, e.g. finite variance.

In order to obtain by our mixing model the symmetric α-stable distribution proposed in Mandelbrot and Taylor (1967), we have to stick to the special situation with $p = \frac{1}{2}$ (implying $r_o = e^{-1/2\,\tilde{\sigma}^2}$).
If $U = -D = \tilde{\sigma}\sqrt{t/n}$, $\tilde{\sigma} > 0$, then $r = r(n) = \frac{1}{2}(e^{\tilde{\sigma}\sqrt{t/n}} + e^{-\tilde{\sigma}\sqrt{t/n}})$ and $r^n \to e^{\frac{1}{2}\tilde{\sigma}^2 t}$, i.e. $r_o = e^{\frac{1}{2}\tilde{\sigma}^2}$. By Theorem 4.1

(6.4) $$\log(S_{N_n}/S) \xrightarrow{w} Z = \log(S^*/S),$$

where

(6.5) $$Z \stackrel{d}{=} N_{o,\sigma^2}\sqrt{Y},\ \sigma^2 = \tilde{\sigma}^2 t,$$

and, therefore,

(6.6) $$E\, e^{i\vartheta Z} = \int_{-\infty}^{\infty} e^{-\frac{\vartheta^2 y \sigma^2}{2}}\, dP^Y(y) = e^{-(\frac{\vartheta\sigma}{\sqrt{2}})^{\alpha}}.$$

Z has a symmetric α-stable distribution with scale parameter $\frac{\sigma}{\sqrt{2}}$.

The valuation formula is given by Theorem 5.1

(6.7) $$C = S\,\Phi_{Z'}(x) - K(E\, r_o^{-tY})\,\Phi_{Z^*}(x),$$

where by (5.13)

(6.8) $$\varphi_{Z^*}(u) = \int_0^{\infty} e^{-\frac{\sigma^2}{2} z u^2}\, dP^{Y^*}(z) = E\, e^{-\frac{\sigma^2}{2} u^2 Y^*}.$$

By Lemma 5.1.

(6.9) $$\psi_{Y^*}(\vartheta) = \frac{E\, r_o^{-tY} e^{-Y\vartheta}}{E\, r_o^{-tY}} = \frac{E\, e^{-(\frac{t}{2}\tilde{\sigma}^2+\vartheta)Y}}{E\, e^{-\frac{t}{2}\tilde{\sigma}^2 Y}} = \frac{e^{-(\frac{t}{2}\tilde{\sigma}^2+\vartheta)^{\alpha/2}}}{e^{-(\frac{t}{2}\tilde{\sigma}^2)^{\alpha/2}}}.$$

Therefore,

(6.10) $$\varphi_{Z^*}(u) = e^{-(\frac{t}{2}\tilde{\sigma}^2+\frac{\tilde{\sigma}^2}{2}tu^2)^{\alpha/2}+(\frac{t}{2}\tilde{\sigma}^2)^{\alpha/2}}$$
$$= \exp\{-(\frac{\tilde{\sigma}}{\sqrt{2}})^\alpha t^{\alpha/2}[(1+u^2)^{\alpha/2}-1]\}.$$

It follows that $EZ^* = 0$; but Z^* does not have a second moment if $1 < a < 2$.

Concerning Z' in (6.7), it holds that

(6.11) $$Z' \stackrel{d}{=} \alpha' Y + N_{o,\sigma'^2}\sqrt{Y} \qquad \text{resp.}$$
$$\varphi_{Z'}(u) = \int_0^\infty e^{\alpha' ziu - \frac{\sigma'^2}{2}zu^2}\, dP^Y(z),$$

where by Theorem 3.1, $\alpha' = \lim_{n\to\infty} n(p'-q')\tilde{\sigma}\sqrt{t/n}$, $p' = p\frac{u}{r} = \frac{e^{\tilde{\sigma}\sqrt{t/n}}}{e^{\tilde{\sigma}\sqrt{t/n}}+e^{-\tilde{\sigma}\sqrt{t/n}}}$ and $q' = q\frac{d}{r} = \frac{e^{-\tilde{\sigma}\sqrt{t/n}}}{e^{\tilde{\sigma}\sqrt{t/n}}+e^{-\tilde{\sigma}\sqrt{t/n}}}$. Thus

$$\alpha' = \lim_n n\frac{1}{2}(e^{\tilde{\sigma}\sqrt{t/n}} - e^{-\tilde{\sigma}\sqrt{t/n}})\tilde{\sigma}\sqrt{t/n}$$
$$= \lim_n \frac{1}{2}\tilde{\sigma}\sqrt{tn}\,[1+\tilde{\sigma}\sqrt{t/n}+\frac{\tilde{\sigma}^2}{2}\frac{t}{n}+\frac{\tilde{\sigma}^3}{3!}(\frac{t}{n})^{3/2}$$
$$-1+\tilde{\sigma}\sqrt{t/n}-\frac{\tilde{\sigma}^2}{2}\frac{t}{n}+\frac{\tilde{\sigma}^3}{3!}(\frac{t}{n})^{3/2}+0(n^{-2})] = \tilde{\sigma}^2 t$$ and from Theorem 3.1, $\sigma'^2 = \sigma^2 = \tilde{\sigma}^2 t$, i.e.

(6.12) $$\alpha' = \tilde{\sigma}^2 t,\ \sigma'^2 = \sigma^2 = \tilde{\sigma}^2 t$$

and

(6.13) $$Z' \stackrel{d}{=} \tilde{\sigma}^2 tY + N_{o,\tilde{\sigma}^2 t}\sqrt{Y}.$$

Since $E\,r_o^{-tY} = e^{-(\frac{t}{2}\tilde{\sigma}^2)^{\alpha/2}}$, we finally obtain

(6.14) $$C = S\Phi_{Z'}(x) - Ke^{-(\frac{t}{2}\tilde{\sigma}^2)^{\alpha/2}}\Phi_{Z^*}(x),$$

with $x = \log(K/S)$ and Z', Z^* from (6.10), (6.13).

6.2. Finite Normal Mixtures

Boness et al. (1979) suggest that the observed kurtosis in the distribution of stock price changes may be caused by the fact that returns follow a finite mixture of normal distributions, i.e. for $Z = \log(S^*/S)$,

$$(6.15)\qquad \varphi_Z(u) = \sum_{j=1}^{k} \vartheta_j\, e^{ia_j u - \frac{\sigma_j^2}{2}u^2} = \int_0^\infty e^{\alpha z i u - \frac{\sigma^2 z}{2}u^2}\, dF_Y(z),$$

where $P(Y = z_j) = \vartheta_j$, $\alpha z_j = a_j$ and $\frac{\sigma^2 z_j}{2} = \sigma_j^2$, $1 \le j \le k$, assuming that $\frac{\alpha_j}{\alpha} = \frac{2\sigma_j^2}{\sigma^2}$, $j = 1,\dots,k$.

Consider k Poisson rv's, $N_n^{(j)}$ = Poisson (nz_j) and define N_n by the mixture $P(N_n = i) = \sum_{j=1}^{k} \vartheta_j P(N_n^{(j)} = i)$, then $\frac{N_n}{n} \xrightarrow{w} Y$ and by Theorem 4.1 $\log(S_{N_n}/S) \xrightarrow{w} Z$ with ch.f. (6.15), α, σ determined by (2.9), (2.10).

For the option pricing formula (5.12) we have

$$(6.16)\qquad E\, r_o^{-tY} = \sum_{j=1}^{k} \vartheta_j r_o^{-tz_j},$$

$$(6.17)\qquad \varphi_{Z'}(u) = \int_0^\infty e^{i\alpha' z u - \frac{\sigma'^2}{2} z u^2}\, dF_Y(z) = \sum_{j=1}^{k} \vartheta_j e^{i\alpha' z_j u - \frac{\sigma'^2}{2} z_j u^2},$$

α', σ' as in (3.10), (3.12), and

$$(6.18)\qquad \Phi_{Z'}(x) = \sum_{j=1}^{k} \vartheta_j\, \Phi_{N(\alpha' z_j, \sigma'^2 z_j)}(x).$$

Furthermore, by Lemma 5.1

$$(6.19)\qquad \psi_{Y^*}(u) = \frac{E r_o^{-tY} e^{-Yu}}{E\, r_o^{-tY}} = \sum_{j=1}^{n} \vartheta_j^{*-z_j u}, \quad \vartheta_j^* := \frac{\vartheta_j r_o^{-tz_j}}{\Sigma \vartheta_i r_o^{-tz_i}},$$

Y^* is a discrete distribution and by (5.13)

$$(6.20)\qquad \Phi_{Z^*}(x) = \sum_{j=1}^{k} \vartheta_j^* \Phi_{N(\alpha z_j, \sigma^2 z_j)}(x), \quad x = \log(K/S).$$

6.3. <u>Mixtures of Gamma Distributions</u>

Define for $m \in \mathbb{N}$ the generalized geometric distribution

$$(6.21)\qquad P(N_n^{(m)} = 1 + km) = \begin{cases} (\frac{1}{n})^{1/m} & \text{for } k = 0 \\ \frac{\prod_{j=0}^{k-1} (\frac{1}{m} + j)}{k!} (\frac{1}{n})^{1/m} (1 - \frac{1}{n})^k & \text{for } k = 1, 2, \dots \end{cases}$$

(cf. Melamed (1990)). For $m = 1$, $N_n^{(1)}$ is geometrically distributed with mean n. It holds

$$(6.22)\qquad \frac{N_n^{(m)}}{n} \xrightarrow{w} Y$$

a Gamma $(\frac{1}{m}, m)$ distributed rv with Laplace transform

$$(6.23)\qquad \psi_Y(\tau) = E\, e^{-\tau Y} = \left(\frac{1}{1+m\tau}\right)^{1/m}$$

and density

$$(6.24)\qquad f_Y(x) = \begin{cases} 0 & \text{for } x < 0 \\ (\Gamma(\frac{1}{m})\, m^{1/m})^{-1}\, x^{1/m-1}\, e^{-x/m} & \text{for } x > 0 \end{cases} .$$

Given $N_n := N_n^m$ and $P(N_n^* = k) = \frac{1}{\Omega_n} r^{-k} P(N_n = k)$, where $r = r_o^{t/n}$, by Lemma 5.1, $\Omega_n \to E\, r_o^{-tY'} = (\frac{1}{1+mt \log r_o})^{1/m}$, and $\frac{N_n^*}{n} \xrightarrow{w} Y^*$ with

$$(6.25)\qquad \psi_{Y^*}(\vartheta) = \frac{E\, e^{-(\vartheta + t \log r_o) Y}}{E\, r_o^{-tY}} = (1 + \frac{m}{1+mt \log r_o}\, \vartheta)^{-1/m} .$$

That is, Y^* has again a Gamma distribution with density

$$(6.26)\qquad f_{Y^*}(x) = \begin{cases} \frac{1}{\Gamma(1/m)(m\Delta)^{(1/m)+1}}\, x^{(1/m)-1} e^{-x/m\Delta} & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases} ,$$

where $\Delta = (1 + mt \log r_o)^{-1}$, i.e. $Y^* \stackrel{d}{=} \Delta Y \stackrel{d}{=}$ Gamma $(\frac{1}{m}, m\Delta)$. Therefore, the ch.f. of Z^* (cf. (5.13)) is given by

$$(6.27)\qquad \varphi_{Z^*}(u) = \int_0^\infty e^{\alpha z i u - \frac{\sigma^2}{2} z u^2}\, dF_{Y^*}(z) = \int_0^\infty e^{\alpha^* z i u - \frac{\sigma^{*2}}{2} z u^2}\, dF_Y(z),$$

where $\alpha^* = \Delta\alpha$, $\sigma^{*2} = \Delta\sigma^2$ and α, σ^2 are given by (3.10), (3.12). (In the classical case $U(n) = \tilde{\sigma}\sqrt{t/n}$, $D(n) = -\tilde{\sigma}\sqrt{t/n}$, $r = r_o^{t/n}$, $\alpha = t(\log r_o - \frac{1}{2}\tilde{\sigma}^2)$, $\sigma^2 = \tilde{\sigma}^2 t$).

Observe that for $\sigma^2 = 0$ (as in Case (8), $\lambda = \infty$, $\mu = 0$)

$$(6.28)\qquad \int_0^\infty e^{\alpha^* z i u}\, dF_Y(z) = (1 - \alpha^* i u m)^{-1/m}$$

and in case $\alpha^* = 0$,

$$(6.29)\qquad \int_0^\infty e^{-\frac{\sigma^{*2}}{2} z u^2}\, dF_Y(z) = (1 + m\frac{\sigma^{*2}}{2} u^2)^{-1/m}$$

$$= (1 - i\sqrt{m}\, \frac{\sigma^*}{\sqrt{2}}\, u)^{-1/m} (1 + i\sqrt{m}\, \frac{\sigma^*}{\sqrt{2}}\, u)^{-1/m},$$

i.e. the law of Z^* is in the first case given by

$$(6.30)\qquad Z^* \stackrel{d}{=} \text{Gamma}\,(\tfrac{1}{m}, \alpha^* m),$$

while in the second case

$$(6.31)\qquad Z^* \stackrel{d}{=} \text{Gamma}\,(\tfrac{1}{m}, \sqrt{m}\,\tfrac{\sigma^*}{\sqrt{2}}) * \text{Gamma}(\tfrac{1}{m}, -\sqrt{m}\,\tfrac{\sigma^*}{\sqrt{2}}).$$

In the same way

$$(6.32)\qquad \varphi_{Z'}(u) = \int_0^\infty e^{\alpha' ziu - \frac{\sigma'^2}{2} zu^2}\, dF_Y(z),$$

where α', σ' are determined by Theorem 5.1. (In the classical case $U(n) = \tilde{\sigma}\sqrt{t/n}$, $D(u) = -\tilde{\sigma}\sqrt{t/n}$, $\alpha' = t(\log r_o + \frac{1}{2}\tilde{\sigma}^2)$, $\sigma'^2 = \sigma^2 = \tilde{\sigma}^2 t$). The "randomized" valuation formula corresponds by Theorem 4.1.a) to a limiting Gamma mixture of the stock price, $\log(S_{N_n(m)}/S) \xrightarrow{w} Z$ with $\varphi_Z(u) = \int_0^\infty e^{\alpha ziu - \frac{\sigma^2}{2}} dF_Y(z)$, α, σ as determined by Theorem 3.1.

6.4. $\tilde{\alpha}$-Stable Limit Laws

Let $\frac{N_n}{n} \xrightarrow{w} Y$, where Y is a positive $\tilde{\alpha}/2$-(Paretian) stable random variable with Laplace transform

$$(6.33)\qquad E\, e^{-\vartheta Y} = e^{-|\vartheta|^{\tilde{\alpha}/2}}.$$

By Theorem 4.1.a) the limiting price S^* has ch.f.

$$(6.34)\qquad \varphi_{\log(S^*/S)}(u) = \int_0^\infty e^{\alpha ziu - \frac{\sigma^2}{2} zu^2}\, dF_Y(z),$$

where α, σ are determined by (2.9), (2.10).

From the definition of Y,

$$(6.35)\qquad \varphi_{\log(S^*/S)}(u) = \int e^{-z(\frac{\sigma^2}{2}u^2 - iu\alpha)}\, dF_Y(z) = e^{-|\frac{\sigma^2}{2}u^2 - iu\alpha|^{\tilde{\alpha}/2}}$$
$$= e^{-(\frac{\sigma^4}{4}u^4 + u^2\alpha^2)^{\tilde{\alpha}/4}}.$$

If $\alpha = 0$, then

$$(6.36)\qquad \varphi_{\log(S^*/S)}(u) = e^{-|\frac{\sigma u}{\sqrt{2}}|^{\tilde{\alpha}}},$$

i.e. $\log(S^*/S)$ has a symmetric $\tilde{\alpha}$-stable distribution. Note that for $\alpha \neq 0$, $1 < \tilde{\alpha} < 2$, $E\log(S^*/S) = \frac{1}{i}\frac{d}{du}(-(\frac{\sigma^4}{4}u^4 + u^2\alpha^2)^{\tilde{\alpha}/4})|_{u=0} = \infty$. Moreover, since $Z = \log(S^*/S) \overset{d}{=} \alpha Y + N_{o,\sigma^2}\sqrt{Y}$ it follows that for independent copies Y_i of Y, N_i of N_{o,σ^2}

$$(6.37)\qquad \frac{Z_1 + \ldots + Z_n}{n^{2/\tilde{\alpha}}} \overset{d}{=} \alpha\, \frac{Y_1 + \ldots + Y_n}{n^{2/\tilde{\alpha}}} + \frac{N_1\sqrt{Y_1} + \ldots + N_n\sqrt{Y_n}}{n^{2/\tilde{\alpha}}} \to Y;$$

so Z is in the domain of attraction of a $\tilde{\alpha}/2$-stable law $\mathfrak{L}(Y)$.

In the Black-Scholes formula (5.12) we have $E\, r_o^{-tY} = e^{-|t \log r_o|^{\tilde{\alpha}/2}}$, $E\, r_o^{-tY} e^{-Y\vartheta} = e^{-|\vartheta + t \log r_o|^{\tilde{\alpha}/2}}$ and, therefore, by Lemma 5.1

$$\psi_{Y^*}(\vartheta) = e^{-|\vartheta + t \log r_o|^{\tilde{\alpha}/2} + |t \log r_o|^{\tilde{\alpha}/2}} \quad (6.38)$$

6.5. Stock Price Changes With Student's Distribution

If $1/Y$ is χ_n^2-distributed, then $Z \stackrel{d}{=} N_{o,\sigma^2}\sqrt{Y}$ is t_n-distributed modulo a factor $\sqrt{\frac{n}{\sigma}}$. Blattbery and Gonedes (1979) argue that security returns follow a Student t-distribution. Again we can apply Theorem 5.1 to obtain a valuation formula in the case that $\log (S^*/S)$ is Student distributed.

Note that by Keilson and Steutel (1974) the class $\mathfrak{L}$ of all rv's Z with a representation of the form $N_{o,1} Y^{1/2}$ coincides with the class of all Z with ch.f. φ_Z, such that $\varphi_Z(|t|^{1/2})$ is a d.f. on $[0,\infty)$. $\mathfrak{L}$ contains all rv's with symmetric densities that are d.f.s in x^2. $\mathfrak{L}$ is closed under mixing and convolution.

7. A Different Kind of Randomization and Continuous Trading

While in Section 4 we modified the original binomial option model by introducing a random number of price changes we shall now introduce a randomization on the up's and down's of the price changes. We shall consider this idea in the example of a modified version of the standard example of Cox, Ross and Rubinstein (1979).

Let us consider the binomial model with

$$U = \tilde{\sigma}\sqrt{t/n}\,,\quad D = -\tilde{\tau}\sqrt{t/n}\,,\quad p = p(n) = \frac{r(n)-d(n)}{u(n)-d(n)},\quad r(n) = r_o^{t/n}. \quad (7.1)$$

Then by some calculations one obtains

$$p(n) = \frac{\tilde{\tau}}{\tilde{\sigma}+\tilde{\tau}} + \frac{\log r_o - \tilde{\tau}\tilde{\sigma}/2}{\tilde{\sigma}+\tilde{\tau}}\sqrt{t/n} + 0\,(t/n), \quad (7.2)$$

$$\alpha = \lim_{n\to\infty} n(qD + pU) + n\frac{pq(q-p)R^3}{(1+q^2R^2)(1+p^2R^2)} = (\log r_o - \frac{\tilde{\tau}\tilde{\sigma}}{2})t, \quad (7.3)$$

$$R = U - D,$$

$$\sigma^2 = \lim_{n\to\infty} n\left((1-p)\frac{R^2p^2}{1+R^2p^2} + p\frac{R^2q^2}{1+R^2q^2}\right) = \tilde{\sigma}\tilde{\tau}t, \quad (7.4)$$

so condition C.2, C.3 of Lemma 2.1 are satisfied, and

(7.5) $\log S_n/S \xrightarrow{w} N(\alpha,\sigma^2)$.

Furthermore,

(7.6) $p' = p\,\frac{u}{r} = \frac{\tilde{\tau}}{\tilde{\sigma}+\tilde{\tau}} + \frac{\log r_o + \frac{\tilde{\tau}\tilde{\sigma}}{2}}{\tilde{\sigma}+\tilde{\tau}}\sqrt{t/n} + O(t/n)$,

(7.7) $\alpha' = \lim n(q'D + p'D) = (\log r_o + \frac{\tilde{\sigma}\tilde{\tau}}{2})t$,

and

(7.8) $\sigma' = \sigma$.

So by the binomial pricing formula (3.1)

(7.9) $C_n = C_n(\tilde{\sigma},\tilde{\tau}) = S\Phi(a_n,n,p') - Kr^{-n}\Phi(a_n,n,p)$,

where $\Phi(a_n,n,p) = P(\Sigma\,\zeta_{n,k} \ge a_n)$, $a_n = [\frac{\log(K/S) - nD}{R}]$. By Theorem 3.1

(7.10) $C_n \to C = C(\tilde{\sigma},\tilde{\tau}) = S\,\Phi(x') - Kr_o^{-t}\,\Phi(x)$,

$x = \frac{\log(S/K)+\alpha}{\sigma}$, $x' = \frac{\log(S/K)+\alpha'}{\sigma'}$.

The limiting process of X_n is by Theorem 2.2

(7.11) $X(u) = (\log r_o - \frac{\tilde{\tau}\tilde{\sigma}}{2})u + \sqrt{\tilde{\sigma}\tilde{\tau}}\,W(u)$.

Let Σ, T be nonnegative random variables independent of $(\zeta_{n,i})$ and consider the randomized version of the binomial model with random up's and down's:

(7.12) $U = \Sigma\sqrt{t/n}$, $D = -T\sqrt{t/n}$,

(7.13) $\log S_n/S = \sum_{k=1}^{n} X_{n,k}$, $X_{n,k} = \zeta_{n,k}U + (1-\zeta_{n,k})D$.

Let P^X denote the law of X, $P^{X|Y=y}$ the conditional law given $Y = y$.

By (7.5): $P^{\log(S_n/S)|\Sigma=\tilde{\sigma},T=\tilde{\tau}} \xrightarrow{w} N(\alpha,\sigma^2)$, $\alpha = \alpha(\tilde{\sigma},\tilde{\tau})$, $\sigma^2 = \tilde{\sigma}\tilde{\tau}t$ (cf. (7.3), (7.4)). Therefore,

(7.14) $P^{\log(S_n/S)} \xrightarrow{w} \int N(\alpha(\tilde{\sigma},\tilde{\tau}),\ \tilde{\sigma}\tilde{\tau}t)\,dP^{\Sigma,T}(\tilde{\sigma},\tilde{\tau})$.

The equilibrium measure P^* in this model should satisfy that:

1. P^* is equivalent to P on $\mathfrak{F}_k = \sigma(\Sigma,T,X_{n,1},\dots,X_{n,k})$, $k \in \mathbb{N}$;
2. w.r.t. P^*, (Σ,T) and $(X_{n,k})$ are independent;
3. $(P^*)^{(\Sigma,T)} = P^{(\Sigma,T)}$;
4. $(r^{-k}S_k,\mathfrak{F}_k)$ isa martingale and S_{k+1} is either $n\,S_k$ or $d\,S_k$ conditional on $\Sigma = u$, $T = d$.

These conditions correspond to the situation, that the up and down prices may vary but should be known at the beginning of the period. Similarly to the introduction in Section 5 it follows that P^* is uniquely determined on $\sigma(\Sigma, T, (X_{n,k}))$ and is given by the mixture $P^* = \int P^*_{\tilde{\sigma},\tilde{\tau}}\, dP^{\Sigma,T}(\tilde{\sigma},\tilde{\tau})$, where $P^*_{\tilde{\sigma},\tilde{\tau}}$ are the riskless measures in the conditional model. This implies that the pricing formula of a call $B = (S_n - K)_+$ is given by

$$(7.15) \qquad C_n = E_* B = \int C_n(\tilde{\sigma},\tilde{\tau})\, dP^{(\Sigma,T)}(\tilde{\sigma},\tilde{\tau}).$$

From (7.10) we obtain by the dominated convergence theorem the approximative valuation formula

$$(7.16) \qquad C = S \int \Phi\left(\frac{\log(K/S) + \log r_o + \frac{\tilde{\sigma}\tilde{\tau}}{2} t}{\tilde{\sigma}\tilde{\tau} t}\right) dP^{\Sigma,T}(\tilde{\sigma},\tilde{\tau}) - K r_o^{-t} \int \Phi\left(\frac{\log(K/S) + \log r_o - \frac{\tilde{\tau}\tilde{\sigma}}{2} t}{\tilde{\sigma}\tilde{\tau} t}\right) dP^{\Sigma,T}(\tilde{\sigma},\tilde{\tau}).$$

The expression simplifies if $\Sigma = T$. The limiting process in the randomized model is given by

$$(7.17) \qquad X^{(2)}(u) = \left(\log r_o - \frac{T\Sigma}{2}\right) u + \sqrt{T\Sigma}\, W(u)$$

(cf. (7.11)). In the model with a random number of price changes we obtain in comparison as limiting (riskless) process

$$(7.18) \qquad X^{(1)}(u) = \left(\log r_o - \frac{\tilde{\tau}\tilde{\sigma}}{2}\right) Y u + \sqrt{\tilde{\tau}\tilde{\sigma}}\,\sqrt{Y}\, W(u).$$

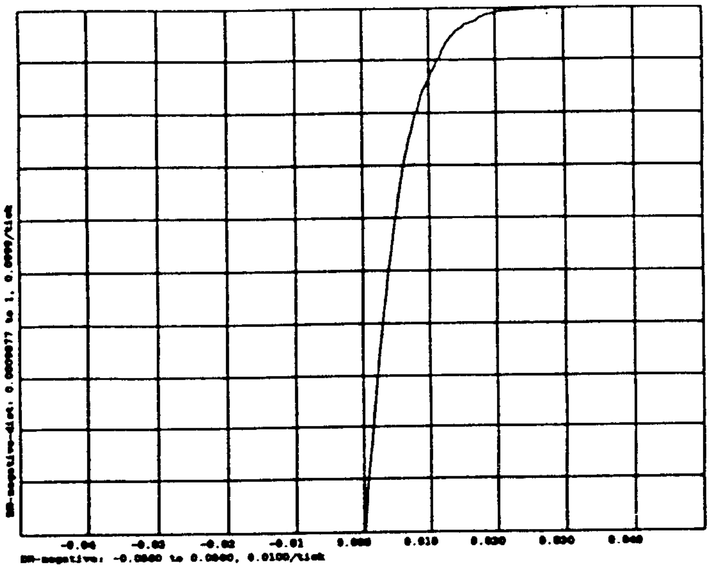
The differential equation corresponding to (7.11) is

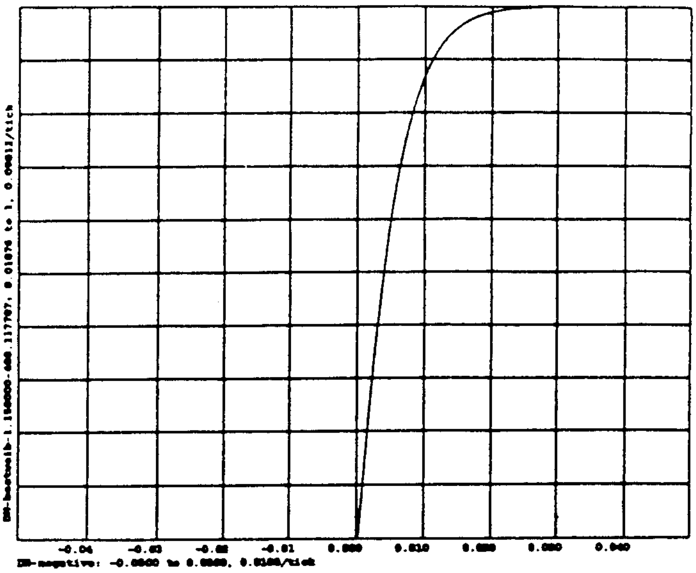
$$(7.19) \qquad d S^{(2)}(u) = S^{(2)}(u)\left((\log r_o)\, du + \sqrt{T\Sigma}\, dW(u)\right)$$

(cf. (3.37)). It was shown in an empirical study by Mittnik and Rachev (1989) that the best fit to the (logarithmic) stock price changes for the S & P 500 index is given by a fit for the positive and negative jumps, with two different Weibull-distributions

$$(7.20) \qquad P(\Sigma > x) = e^{-\lambda_+ x^{\alpha_+}}, \; x \geq 0, \qquad P(T > x) = e^{-\lambda_- x^{\alpha_-}}, \; x \geq 0.$$

For the estimation of the four parameters of this model cf. [9]. By our second randomized model (7.13) we can directly model these type of jumps and determine the corresponding pricing formulas. A different example where the Weibull distributions give a very good fit is the exchange rate of DM versus US Dollar. The following picture shows the negative log daily changes of a sample of more than 2 000 observations and the best Weibull fit with parameters $\lambda_- = 408{,}117$, $\alpha_- = 1.15$. (The data and the fit was kindly provided by Professor A. Berlekamp, Axcom, Berkeley.)





sample distribution　　　　　　　　Weibull fit

<u>Remark.</u> The idea of randomizing the parameters can also be carried out in continuous time option models. If we take e.g. the classical diffusion model (in its riskless version, cf. Harrison and Pliska (1981) or Karatzas (1989))

$$dS(u) = S(u)((\log r_o)\,du + \sigma\, dW(u)) \tag{7.21}$$

and consider a random volatility σ, then we can carry out directly the analysis in this continuous model leading to a Black-Scholes formula as in (7.16).

The idea of random time changes of Gaussian models as better fits to the stock price changes has been suggested by Mandelbrot and Taylor (1967) and Clark (1973). This idea is closely related to our first randomization model, leading in the limit to the random time transformed Wiener process

$$Z(u) = \log \frac{S(u)}{S} = \frac{\alpha}{t} Yu + \frac{\sigma}{\sqrt{t}} W(Yu). \tag{7.22}$$

As remarked in the introduction of Section 3 it is of interest to use more realistic transformations for the development of the price process. An interesting problem seems to be the modeling of the price process by a point process in [0,t]. Our method applies so far only to the case r = 1 since then the pricing formulas only depend on the (random) number of points.

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