Valuation of American Options

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Abstract

The valuation of American options is a difficult problem. The basic reason is that the asset price at which early exercise is optimal isn't known in advance and has to be found as part of the solution of the problem. In mathematical terms, a partial differential equation known as *Black–Scholes equation* has to be solved with a moving boundary condition. This is known in general as a *moving boundary problem.* Analytic solutions of this kind of problems can be found only in very special cases (e.g. for the American call on an asset paying a single discrete dividend during the lifetime of the option). However, because of the practical importance of American options, their efficient and accurate pricing is vital for option market participants. Finite difference methods can be used to solve the differential equation numerically, but in order to obtain an accurate solution, considerable computational effort is necessary. Therefore other, more efficient methods have been developed.

In this thesis a comparison of numerical and approximative methods to solve this problem for equity (representing options on assets with discrete known payments) and FX options (representing options on assets with a continuous dividend yield or holding costs) is presented. The numerical methods are based on a binomial tree. A finite difference method the solution of which is considered exact is used as a benchmark all the other methods are compared to. The result is an assessment when these methods can be successfully applied.

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Chapter 1

Introduction

A (so called plain vanilla) financial option gives the holder the right to buy or sell the underlying asset for a *strike* or *exercise price* X (that is fixed when the option is written) at a later time. If the holder has the right to buy the asset, the option is a *call*, while a *put* gives the holder the right to sell the asset. The use of this right by the holder is called *exercise* of the option. The option is called *European* if it can be exercised only at a certain date (the *expiration* or *maturity date* T). An *American* option can be exercised at any time before the expiration date. If an American option is exercised before the expiration date, it is *early exercised*.

Becaue the option gives the holder the *right*, but not the obligation to buy or sell the asset, he will only exercise the option if it is profitable for him, i.e. he will only exercise a call if the market price of the asset is above the strike price or he will only exercise a put if the market price of the asset is below the market price. The option provides its holder with the possibility of unlimited profit at the risk of a limited loss. Therefore it has a value. Acquiring an option costs a *premium*.

The value of the option at expiry is the *payoff* of the option. It is zero if exercising the option doesn't provide a profit and positive otherwise. For plain vanilla options it can be written as

$$P_{\text{call}}(S) = \max(0, S - X) \tag{1.1a}$$

$$P_{\text{put}}(S) = \max(0, X - S). \tag{1.1b}$$

At any time before expiry, the value of the option will be different from the payoff. For a European option it will be the expectation value of the payoff at expiry discounted to the date the valuation is done for (given the interest rate, the price and the volatility of the underlying asset).

Because an American option can be exercised at any time, its value can never be less than the payoff. (Otherwise, it would be exercised immediately.) Furthermore, because the early exercise is an additional right to the exercise at expiry, an American option is at least as valuable as a European option. The exact determination of the value of the American option, however, is in general a difficult task. Analytic closed–form solutions can be found only in rare special cases. A numerical solution of this problem can be found (in principle) with arbitrary accuracy limited only by the properties of the numerical scheme, but involving a significant computational effort.

Most of the options traded at exchanges are American. Therefore, an accurate *and* efficient valuation of American options ist very important. Thus other ways must be found to find the option value. All known methods are a compromise between computational efficiency and accuracy. In this thesis the results of several popular methods are compared to the numerical solution. These methods fall in one of the two categories: Binomial tree models (and variants) and analytic approximations.

Chapter 2

Foundations

2.1 Pricing of Financial Derivatives

In this chapter the notation used in the following chapters shall be introduced. The derivation of the differential equation which is the basis of the discussion in the later chapters is outlined. The purpose of this chapter is not a rigorous mathematical treatment of stochastic calculus. A good introduction can be found e.g. in [1].

2.1.1 Wiener Processes

If the value of a variable changes over time in an uncertain way it is said to follow a *stochastic process*. This process can be discrete or continuous in time (discrete time or continuous time process) and in "space" (discrete or continuous variable). Although trading in financial markets isn't continuous in time (there is no trading outside business hours at exchanges) and asset price (e.g. stock prices are quoted in fixed ticks), the continuous-time, continuous-variable process is a useful model of financial asset prices for many purposes.

A *Markov process* is a stochastic process where only the present value of a stochastic variable is relevant for the next value. The next value is independent of the path the present value is obtained. A *Wiener process* is a particular Markov process with a mean change of 0 and variance 1. In physics the process is often referred to as *Brownian motion*. If a random variable X follows a Wiener process, its changes ΔX in discrete time steps Δt can be written as

$$\Delta X = \varepsilon\sqrt{\Delta t} \tag{2.1}$$

where ε is a random drawing from a standardised normal distribution[1]. In the limit $\Delta t \to 0$ this can formally be written as

$$\mathrm{d}X = \varepsilon\sqrt{\mathrm{d}t} \tag{2.2}$$

If the development of a stochastic variable S with time t can be described as a *generalised Wiener process*, its differential equation can be written as

$$\mathrm{d}S = M\mathrm{d}t + \Sigma\mathrm{d}X \tag{2.3}$$

[1] A normal distribution with mean 0 and variance 1

where the parameters M and Σ are constant. M describes the drift of the process, and Σ is a measure of its variation. The differential $\mathrm{d}X$ is a random variable drawn from a normal distribution with mean 0 and variance $\mathrm{d}t$ (i.e. a Wiener process as in equation 2.2). The values of $\mathrm{d}X$ for different times are independent.

The prices of financial assets are usually assumed to follow more general processes where the parameters can depend on S and t.

$$\mathrm{d}S = M(S,t)\mathrm{d}t + \Sigma(S,t)\mathrm{d}X \tag{2.4}$$

These processes are called *Itô processes.* For many financial assets (e.g. equities, FX rates) a log-normal random walk[2] of the asset price is assumed. In this case $M(S,t) = \mu S$ and $\Sigma(S,t) = \sigma S$ where σ and μ are constant. Equation 2.4 then can be written as

$$\mathrm{d}S = \mu S\mathrm{d}t + \sigma S\mathrm{d}X. \tag{2.5}$$

The parameter σ is called *volatility*, μ is called *drift.*

2.1.2 Risk-neutral Valuation

A world in which investors don't require a compensation for risk is called risk-neutral. In such a world the expected return on any security is the risk-free rate. The Girsanov theorem can be used to transform the random walk with drift μ in equation 2.5 to the risk–neutral random walk

$$\mathrm{d}S = rS\mathrm{d}t + \sigma S\mathrm{d}X. \tag{2.6}$$

where the drift is equal to the risk–free rate r. If the asset pays a continuous dividend yield q, the growth rate of the asset price has to be reduced by this amount in order to provide the same overall return (dividend + capital gain) as an asset without dividends:

$$\mathrm{d}S = (r-q)S\mathrm{d}t + \sigma S\mathrm{d}X. \tag{2.7}$$

The following derivation will be based on this random walk.

2.1.3 Itô's Lemma

Suppose a function $V(X_t,t)$ is given where the subscript t indicates that X is itself a function of time. In "normal"[3] calculus there is the chain rule how to calculate the *total* derivative of V with respect to t:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\partial V}{\partial X}\frac{\mathrm{d}X}{\mathrm{d}t} + \frac{\partial V}{\partial t}. \tag{2.8}$$

An equivalent formulation in terms of the differential is

$$\mathrm{d}V = \frac{\partial V}{\partial X}\mathrm{d}X + \frac{\partial V}{\partial t}\mathrm{d}t. \tag{2.9}$$

[2] The reason why it is called log-normal is explained in section 2.1.3.

[3] i.e. non-stochastic

Itô's lemma is the equivalent if the variable X_t is a stochastic variable following a Wiener process at time t. Using a Taylor expansion of V to order dt neglecting all terms of higher order as d$t \to 0$, we obtain

$$\begin{aligned} \mathrm{d}V &= \frac{\partial V}{\partial t}\mathrm{d}t + \frac{\partial V}{\partial X}\mathrm{d}X + \frac{1}{2}\frac{\partial^2 V}{\partial X^2}\mathrm{d}X^2 \\ &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial X^2}\right)\mathrm{d}t + \frac{\partial V}{\partial X}\mathrm{d}X \end{aligned} \tag{2.10}$$

because the increment dX is drawn from a normal distribution with variance dt (implied by the fact that X follows a Wiener process) and therefore dX^2 is of the order of dt. If V is contingent on a variable S following an Itô process 2.4, this can be generalised to

$$\mathrm{d}V = \left(M(S,t)\frac{\partial V}{\partial t} + \frac{1}{2}\Sigma(S,t)^2\frac{\partial^2 V}{\partial S^2}\right)\mathrm{d}t + \frac{\partial V}{\partial S}\mathrm{d}S \tag{2.11}$$

An application of Itô's lemma that will later be useful is to consider the process in equation 2.7 and $V(S) = \ln S$. Then

$$\begin{aligned} \mathrm{d}V &= \frac{\partial V}{\partial S}\mathrm{d}S + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\mathrm{d}t \\ &= \frac{1}{S}\left((r-q)S\mathrm{d}t + \sigma S\mathrm{d}X\right) - \frac{1}{2}\sigma^2 S^2\frac{1}{S^2}\mathrm{d}t \\ &= \left(r - q - \frac{\sigma^2}{2}\right)\mathrm{d}t + \sigma\mathrm{d}X \\ \mathrm{d}\ln S &= \left(r - q - \frac{\sigma^2}{2}\right)\mathrm{d}t + \sigma\mathrm{d}X \end{aligned} \tag{2.12}$$

The random variable $\ln S$ follows a normal generalised Wiener process[4], but the drift has an additional volatility–dependent component.

2.1.4 Derivation of the Black Scholes Equation

The following assumptions have to be made for the derivation:

- The asset price is a continuous variable that can change continuously in time and follows a random walk as in equation 2.7.
- Short selling of securities is permitted (with full use of the proceeds).
- There are no transaction costs, bid-ask spreads, or tax considerations.
- It is possible to adjust a portfolio continuously in time and asset quantity (i.e. the asset is perfectly divisible).
- The asset doesn't pay discrete dividends during the lifetime of the derivative.

[4] Therefore the process for S is called log-normal.

- There is no arbitrage possibility.
- Borrowing and lending at the risk-free rate is possible.
- The risk-free rate is constant in time and independent of the maturity (i.e. the rate curve is flat).[5]
- The liquidity of the market is unlimited.
- There is no counterparty/default risk.

Let S be the price of an asset with dividend yield q. Let $V(S,t)$ be the price of a derivative on this asset. The change of the derivative value can be expressed using Itô's lemma 2.11 and based on the random walk in equation 2.7 as

$$\mathrm{d}V = \frac{\partial V}{\partial t}\mathrm{d}t + \frac{\partial V}{\partial S}\mathrm{d}S + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\mathrm{d}t \tag{2.13}$$

A hedged portfolio with value Π is constructed containing the option on one unit of the asset and the fraction $-\Delta$ of the asset itself:

$$\Pi = V - \Delta S \tag{2.14}$$

In an infinitesimal time step $\mathrm{d}t$, the value of the portfolio changes by $\mathrm{d}\Pi$. The asset value changes by the (random) increment $\mathrm{d}S$. Using equation 2.13 and assuming that Δ is constant during $\mathrm{d}t$ we obtain utilising Itô's lemma (equation 2.11)

$$\begin{aligned}\mathrm{d}\Pi &= \mathrm{d}V - \Delta \mathrm{d}S \\ &= \frac{\partial V}{\partial t}\mathrm{d}t + \frac{\partial V}{\partial S}\mathrm{d}S + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\mathrm{d}t - \Delta \mathrm{d}S.\end{aligned}$$

The portfolio is risk-free if $\mathrm{d}\Pi$ is independent of $\mathrm{d}S$. This is accomplished by letting $\Delta = \frac{\partial V}{\partial S}$. Then the return on the portfolio reduced by the dividend the portfolio holder has to pay on the asset he is short should be equal to the risk-free rate.

$$\begin{aligned}\mathrm{d}\Pi - qS\frac{\partial V}{\partial S}\mathrm{d}t &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - qS\frac{\partial V}{\partial S}\right)\mathrm{d}t \\ &= r\Pi \mathrm{d}t \\ &= r(V - \Delta S)\mathrm{d}t \\ &= r\left(V - \frac{\partial V}{\partial S}S\right)\mathrm{d}t\end{aligned}$$

Rearranging this result yields the Black-Scholes equation

$$\boxed{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0.} \tag{2.15}$$

[5] This assumption can be relaxed as long as interest rates are deterministic. It only makes the formalism simpler.

The payoff at maturity determines the option price at any time prior to expiry. For vanilla options as described in chapter 1 it can be written as

$$P_{\text{call}}(S) = \max(0, S - X) \tag{2.16a}$$
$$P_{\text{put}}(S) = \max(0, X - S). \tag{2.16b}$$

Thus, the value of the option is known at expiry. Let t be the instant of time at which the fair option price $V(S,t)$ is to be determined. This option price is the solution of the Black-Scholes equation. The Black-Scholes equation is a parabolic equation that can be solved only for decreasing t. The payoff is a ”final condition”. Using $\tau = T - t$ as the time variable, the Black-Scholes equation can be solved as an initial value problem with the option payoff as the initial condition.

2.2 Solution of the Black Scholes Equation for European Options

2.2.1 Analytic solution

For European options, the Black-Scholes equation can be solved analytically by transforming it to the heat equation. The algebra of this solution can be found e.g. in [4]. In the original Black-Scholes equation the dividend yield $q = 0$, but conceptually this doesn't make any difference. Therefore here only the more general case is considered here.

2.2.1.1 Option on an Asset with a Continuous Dividend Yield

This case is valid for options on equity indices with a continuous dividend yield q, futures with $q = r$, FX rates with $q = r_f$ (where r_f is the foreign interest rate) and assets without dividends $(q = 0)$.

The price of a European Call is

$$V_{\text{call}} = e^{-q(T-t)} S N(d_1) - e^{-r(T-t)} X N(d_2) \tag{2.17}$$

with

$$d_1 = \frac{\ln(S/X) + (r - q + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}} \tag{2.18a}$$
$$d_2 = d_1 - \sigma\sqrt{T - t}. \tag{2.18b}$$

$N(x)$ is the cumulative normal distribution as defined in equation D.1. The price of a European put is

$$V_{\text{put}} = e^{-r(T-t)} X N(-d_2) - e^{-q(T-t)} S N(-d_1). \tag{2.19}$$

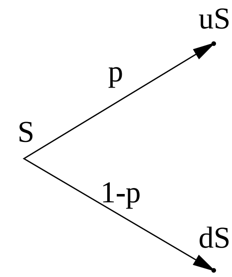

Figure 2.1: *Binomial model with a single time step*

2.2.1.2 Option on Equity with Discrete Dividends

If the asset underlying the options pays discrete dividends (known in advance), the process governing the asset price S can be divided in a deterministic part D (the present value of all future dividends falling into the lifetime of the option) and a stochastic process followed by S' according to

$$\mathrm{d}S' = rS'\mathrm{d}t + \sigma' S'\mathrm{d}X \tag{2.20}$$

The asset price is then $S = S' + D$ The volatility of the modified process has in principle to be scaled from the volatility of the process followed by S to $\sigma' = \sigma\frac{S}{S-D}$. However for sake of simplicity in all examples with discrete dividends it is assumed that the volatility of S' is given as σ instead of the volatility of S.

2.2.2 Numerical Solution of the Black-Scholes Equation

Because the Black-Scholes equation 2.15 is a parabolic equation, it can be solved by the usual implicit or explicit finite difference methods [12]. A Crank-Nicolson scheme has been used that is described in chapter 3. The numerical solution has the advantage over the analytic solution that it is applicable to a much greater variety of options. For European options the analytic solution can be used to assess the accuracy of the numerical solution.

2.2.3 Binomial Trees

The idea of the binomial tree is a simplified model of asset prices: after one time step δ, the asset with price S at time t can only take one of two different values uS and dS with probability p and $1-p$, respectively, at time $t+\delta$ (see figure 2.1). The drift and the volatility of the continuous process has to be matched by adjusting u, d and p. The valuation is done in a risk-neutral world, i.e. the drift is r.[6] The expected value of S at time $t+\delta$ given $S(t)$ is then (with $E[X|Y]$ denoting the expected value of X given Y)

$$\begin{aligned} E[S(t+\delta)|S(t)] &= pS(t)u + (1-p)S(t)d \\ &= e^{(r-q)\delta}S(t) \end{aligned} \tag{2.21a}$$

[6] Therefore the probability p is a risk-neutral probability. It must not be confused with the "true" probability in the real world.

Having defined $\hat{S}_t^\delta = E[S(t+\delta)|S(t)]$, the variance of S is

$$\begin{aligned} E[(S(t+\delta)-\hat{S}_t^\delta)^2|S(t)] &= E[S(t+\delta)^2|S(t)] - \hat{S}_t^{\delta 2} \\ &= pS(t)^2u^2 + (1-p)S(T)^2d^2 - (pS(t)u + (1-p)S(t)d)^2 \\ &= S(t)^2\sigma^2\delta \end{aligned} \tag{2.21b}$$

The two equations 2.21a and 2.21b can be simplified to obtain

$$e^{(r-q)\delta} = pu + (1-p)d \tag{2.22a}$$

$$\sigma^2\delta = pu^2 + (1-p)d^2 - (pu + (1-p)d)^2 \tag{2.22b}$$

One degree of freedom remains, leaving us with a choice. The choice made here is $u = 1/d$ as proposed by Cox, Ross and Rubinstein [2]. The solution of the nonlinear equations for u, d and p can be approximated using the Taylor expansion of the exponential function to first order in δ. The result is

$$u = e^{\sigma\sqrt{\delta}} \tag{2.23a}$$

$$d = e^{-\sigma\sqrt{\delta}} \tag{2.23b}$$

$$p = \frac{e^{(r-q)\delta} - d}{u-d}. \tag{2.23c}$$

This model is too simple to be realistic with only one time step. Generally the time to expiry T of an option is chopped into a number M of time steps such that $\delta = T/M$. For 3 time steps we get for the evolution of asset prices the picture in figure 2.2. In every time step k the nodes of the tree represent asset prices $S_i^k = Su^id^{k-i}$. The tree of asset prices is built using u and d until the expiry date of the option. Then the option price v_i^M for every asset price S_i^M is calculated at expiry as the payoff. Going back through the tree, the option price is calculated as

$$v_i^{k-1} = e^{-r\delta}\left(pv_{i+1}^k + (1-p)v_i^k\right). \tag{2.24}$$

Since p is interpreted as a probability, the condition $0 \le p \le 1$ must be satisfied. This imposes a stability condition of the method:

$$\delta < \frac{\sigma^2}{(r-q)^2} \tag{2.25}$$

If δ and therefore also the size of the step in the asset price is small enough, the continuous process can be modeled accurately. The stability condition 2.25 has to be obeyed.

2.2.4 Treatment of Discrete Dividends with Binomial Trees

If discrete dividends are payed during the lifetime of the option, an adjusted process is used according to equation 2.20 is used. Therefore the picture in figure 2.2 refers then to the process $\check{S}$. For every time step the present value of the future dividends until the expiry of the option is calculated

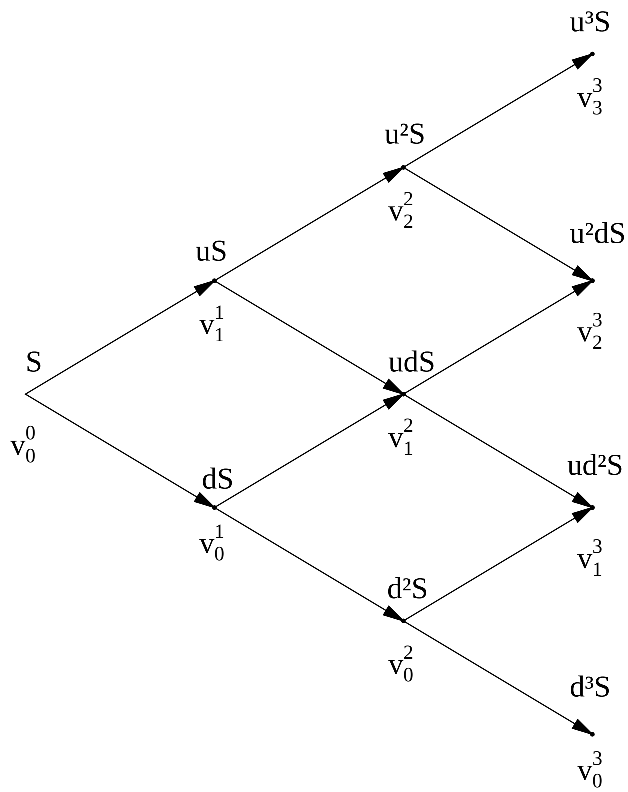

Figure 2.2: *Binomial tree with 3 time steps*

and added to the asset prices during the time step. Therefore if a single dividend D is payed at time τ with $0 < \tau < T$, the real stock price at node (i,k) is

$$S_i^k = {S'}_i^k + \Theta(\tau - k\delta) D e^{-r(\tau - k\delta)} \tag{2.26}$$

where $\Theta(.)$ is the Heaviside-function which is 1 when its argument is greater than 0 and zero elsewhere. This formula can be generalised easily to assets with more than one dividend.

2.3 American Options

For American options a solution of the Black Scholes equation in general can't be found analytically. The reason is that the point at which early exercise of the option at any instant of time is optimal is a priori unknown. In the framework of the PDE it can be treated as a free boundary problem. The equality in equation 2.15 becomes an inequality: For the Delta–hedged portfolio Π the change in value over time dt is dΠ. Arbitrage consideration show that it is impossible to have $d\Pi > r\Pi dt$ because an investor could borrow money at the risk–free rate and invest in the risk–free portfolio Π to make a risk–free profit. However, if $d\Pi < r\Pi dt$, the equivalent strategy (short the portfolio and earn the risk–free rate on the money) is not always possible in the presence of early exercise. While in the case of a European option the return on the option in a risk-neutral world must be equal to the risk-free rate, in the case of the American option the following inequality holds:

$$\begin{aligned} d\Pi &= \frac{\partial V}{\partial t}dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt \\ &\leq r\Pi dt \\ &= r(V - \frac{\partial V}{\partial S}S)dt \end{aligned}$$

$$\Leftrightarrow \quad \boxed{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV \leq 0.} \tag{2.27}$$

An additional condition is that the option value is always greater or equal to the payoff $p(S)$

$$V \geq p(S). \tag{2.28}$$

If the situation $V < p(S)$ occured, the option would be exercised immediately, and its payoff (and therfore also its value) would be $p(S)$. If the option value $V > p$ then early exercise doesn't occur, and therefore the portfolio can be shorted. Defining the operator

$$L = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r-q)S\frac{\partial}{\partial S} - r, \tag{2.29}$$

in this case $LV = 0$ must be satisfied. If $LV < 0$, the option should be exercised early, and therefore $V = p$. This can be formulated as the *linear complementarity problem*:

$$LV \leq 0 \tag{2.30a}$$

$$V - p(S) \geq 0 \tag{2.30b}$$

$$(V - p(S))\,LV = 0 \tag{2.30c}$$

For American calls on an asset without dividends it can be shown that early exercise is never optimal because the option value is always higher than the payoff. In this special case the value of the American option is equal to the value of the equivalent European option. If the asset pays discrete dividends, early exercise of a call can be optimal only immediately before a dividend payment in order to cash in the dividend. If there is only one dividend payment during the lifetime of the option, an analytic solution does exist.

2.3.1 Analytic Solution for American Calls with one Discrete Dividend

For the case that the asset pays exactly one known discrete dividend during the lifetime of the option, an exact solution of the Black-Scholes equation for an American call has been found by Roll, Geske and Whaley [6, 7, 8]. This is possible because early exercise is optimal only at one instance of time (namely at the dividend payment date). With the dividend D at time t' the solution is

$$V_{\text{call}} = \left(S - De^{-r(t'-t)}\right)\left(N(b_1) + M\left(a_1, -b_1, -\sqrt{\frac{t'-t}{T-t}}\right)\right) - Xe^{-r(T-t)}M\left(a_2, -b_2, -\sqrt{\frac{t'-t}{T-t}}\right) - (X-D)e^{-r(t'-t)}N(b_2) \quad (2.31a)$$

where

$$a_1 = \frac{\ln\frac{S-De^{r(t'-t)}}{X} + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \quad (2.31b)$$

$$a_2 = a_1 - \sigma\sqrt{T-t} \quad (2.31c)$$

$$b_1 = \frac{\ln\frac{S-De^{r(t'-t)}}{I} + (r+\sigma^2/2)(t'-t)}{\sigma\sqrt{t'-t}} \quad (2.31d)$$

$$b_2 = b_1 - \sigma\sqrt{t'-t}, \quad (2.31e)$$

$N(x)$ is the cumulative normal distribution as defined in equation D.1, and $M(x,y,\rho)$ is the cumulative bivariate normal distribution as defined in equation D.4. The variable I is the critical ex-dividend stock price I that solves

$$V^{\text{BS}}_{\text{call}}(I, X, T-t') = I + D - X \quad (2.32)$$

where $V^{\text{BS}}_{\text{call}}$ is the value of a European call with stock price I and time to maturity $T-t'$. If $D \leq X(1-e^{-r(T-t')})$ or $I = \infty$, it won't be optimal to exercise the option early. The price of the option is then equal to the price of the equivalent European option.

2.3.2 Numerical Solution

The linear complementarity problem can be solved numerically in principal with arbitrary accuracy limited only by computation time, machine accuracy, and the stability of the numerical scheme. The details of the Crank-Nicolson scheme used for the work presented in this thesis are described in chapter 3.

Because the numerical solution is (in principle) exact, it can be used as a benchmark for other (faster) methods approximating the solution of the Black-Scholes equation.

2.3.3 Binomial Tree

The binomial tree can be used almost unchanged to value American option. Only equation 2.24 has to be modified in the following way:

$$v_i^{k-1} = e^{-r\delta}\max\left(pv_{i+1}^k + (1-p)v_i^k, P(S_i^{k-1})\right), \quad (2.33)$$

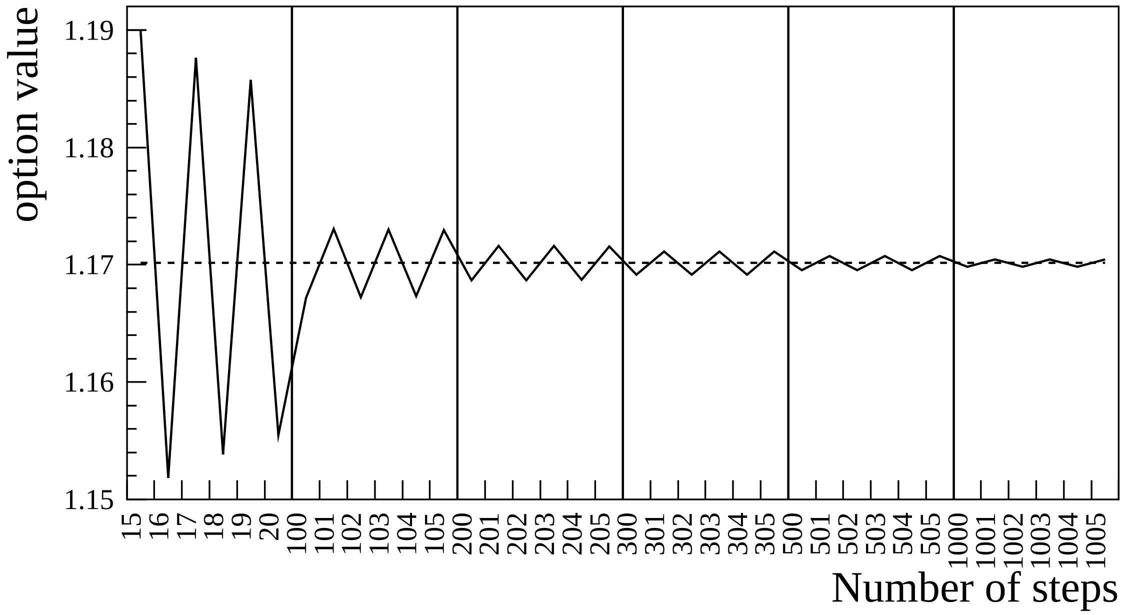

Figure 2.3: *Price of an American call option with one dividend calculated with binomial trees with different numbers of time steps (solid line). The dashed line indicates the true price calculated with the formula of Roll, Geske and Whaley. The parameters are* $S = X = 100$, $r = 0.04$, $\sigma = 0.4$, $D = 4$, *the time to maturity is 0.915 years and the time to the ex–dividend date is 0.61 years.*

where $P(S)$ is the payoff of the option at the asset price S. Figure 2.3 shows how the price of an American call option on an asset paying one discrete dividend varies with the number of steps in the binomial tree. The price oscillates about the true price (indicated by the horizontal dashed line) calculated with the analytic formula of section 2.3.1. The amplitude of these oscillations decays only very slowly.

A better option price can generally by obtained by calculating it with two binomial trees with step numbers N and $N+1$ and taking the average. In some cases this average is close to the true option value.

2.3.4 Control Variate Technique

The error of the binomial tree can be reduced by using it only to calculate the difference between the price of the American and the equivalent European option with the same strike and the same time to maturity. The analytical solution of the Black-Scholes equation for European option is then corrected with this difference, or in other words, the price of the European option is used as a control variate for the price of the American option. Letting v_e the Black-Scholes price of the European option, v_E the price of the European option calculated with the binomial tree, v_A the price of the American option calculated with the binomial tree, the improved price of the American option

v_a is

$$v_a = v_e + v_A - v_E \tag{2.34}$$

2.3.5 Analytic Approximations

There are two important analytic approximations for the value of American options without discrete dividends [5]. They both allow for a continuous dividend yield q. If $q \leq 0$, early exercise of an American call is never optimal, and its value is equal to the value of a European call.

One of these approximations has been developed by Barone-Adesi and Whaley [9]. The formula is shown in appendix A.1. The other later developed formula of Bjerksund and Stensland [10] is supposed to be computationally more efficient and more accurate for long–dated options. The formula can be found in appendix A.2.

Chapter 3

The Finite Difference Scheme

The differential equation to be solved is the Black Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0 \tag{3.1}$$

The coefficients σ, r and q are assumed to be constant. The Black–Scholes equation is a parabolic equation, similar to the *heat* or *diffusion equation* $\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial u}{\partial x}\right)$.[1] Thus it falls in the category of the *initial value problems* which describe the evolution of an initial state with time. For the numerical solution an important issue is the choice of a *stable* method (opposed to boundary value problems where efficiency is the greater issue).

Various methods are discussed in reference [12]. For this kind of problems one could generally choose any simple explicit or implicit method. While the implicit scheme is unconditionally stable, for the explicit scheme the size of the grid in “space” and time must be made small enough to obtain a numerically stable result. Both methods have a local truncation error of the order $O(\delta t)$. For the purpose of this project, the finite difference scheme used is the Crank–Nicolson scheme. It is basically a combination of an implicit and an explicit scheme that effectively calculates a central derivative in time for points in the middle of two time grid points. The local truncation error is of the order of $O(\delta t^2)$. At the boundaries $S = 0$ and $S = \infty$, the second derivative $\frac{\partial^2 V}{\partial S^2}$ is set to 0.

The finite difference scheme should operate on an equally spaced grid in order to keep the convergence properties. However the error is not only dependent on the grid spacing δS, but also on the second derivative $\frac{\partial^2 V}{\partial S^2}$. To improve the overall error without increasing the computational effort it would make sense to make the grid finer where the second derivative is large, i.e. where the asset price is close to the strike price. Both goals can be accomplished by transforming the asset price with a non–linear invertible function to the new variable

$$\tilde{s} = \operatorname{asinh}(S-k) - \operatorname{asinh}(-k) \tag{3.2}$$

(based on an idea from [11]) and using a homogeneous grid in $\tilde{s}$. The grid in S is then compressed near $S = k$ and stretched if S is far away from k (see figure 3.1). For the purpose of this calculation,

[1] Actually by some variable transforms, the Black–Scholes equation can be reduced to the heat equation.

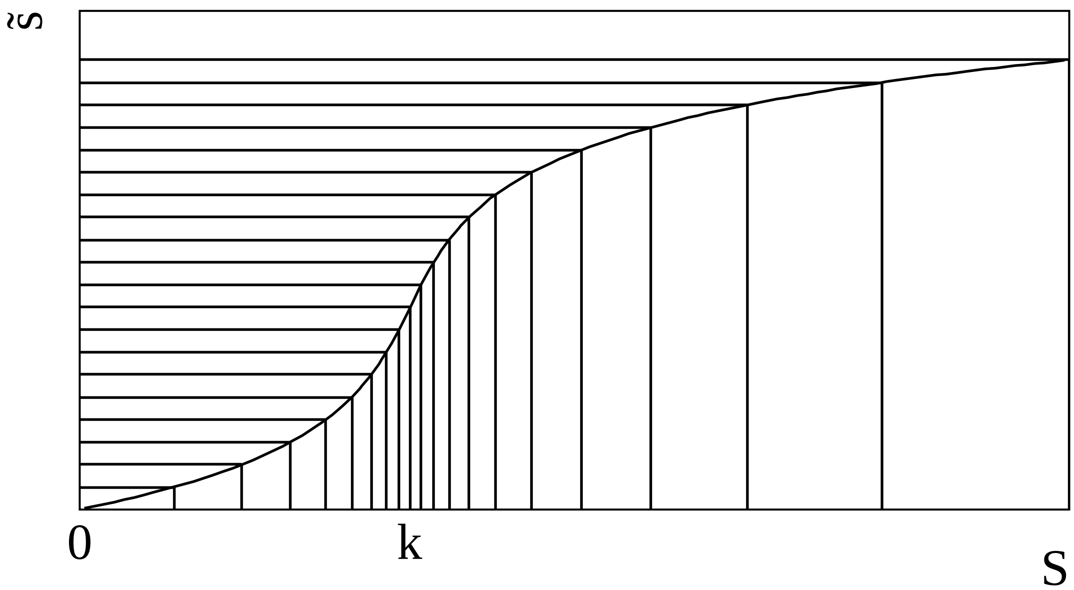

Figure 3.1: *Non-linear transform of S (equation 3.2). It can be seen how a constant spacing in $\tilde{s}$ leads to a compression of the grid in S about k and a stretching at large S.*

k is set equal to the strike X. The Black Scholes equation 3.1 can be written in terms of this new variable as

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 T(s')^2 \frac{\partial^2 V}{\partial \tilde{s}^2} + \underbrace{\left(r - q - \frac{1}{2}\sigma^2 T(s') \tanh s'\right)}_{=: f(s')} T(s') \frac{\partial V}{\partial \tilde{s}} - rV = 0 \tag{3.3}$$

with the definitions

$$c = \operatorname{asinh}(-k) \tag{3.4a}$$

$$s' = \tilde{s} + c \tag{3.4b}$$

$$T(x) = \frac{\sinh x + k}{\cosh x} \tag{3.4c}$$

3.1 Discretisation

The function V is mapped to a grid in "space" S[2] and time t. The number N of steps in S determines together with the lower boundary $S_{\min}$ and the upper boundary $S_{\max}$ the size of a step in S $\delta S = (S_{\max} - S_{\min})/N$, and the number M of time steps determines with the lifetime T of the option to be valued the size of a time step $\delta t = T/M$. The values $S_i = S_{\min} + i\delta S$ with $0 \leq i \leq N$ mark the grid points in S, the values $t^k = k\delta t$ with $0 \leq k \leq M$ mark the time steps. The function values on the grid are V_i^k where i represents the space dimension and k represents the time dimension. The following

[2] At this point it shall be left open how this space variable is related to the asset price.

discrete approximations for the partial derivatives of V are used:

$$\frac{\partial V}{\partial t} = \frac{V_i^{k+1} - V_i^k}{\delta t} + O(\delta t) \tag{3.5a}$$

$$\frac{\partial V}{\partial S} = \frac{V_{i+1}^k - V_{i-1}^k}{2\delta S} + O(\delta S^2) \tag{3.5b}$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{V_{i+1}^k - 2V_i^k + V_{i-1}^k}{\delta S^2} + O(\delta S^2). \tag{3.5c}$$

The discretisation of equation 3.3 with the Crank Nicolson scheme yields

$$\frac{V_i^{k+1} - V_i^k}{\delta t} + \frac{1}{2}\sigma^2 T(s_i')^2 \frac{V_{i+1}^k - 2V_i^k + V_{i-1}^k + V_{i+1}^{k+1} - 2V_i^{k+1} + V_{i-1}^{k+1}}{2\delta\tilde{s}^2} + f(s_i')T(s_i')\frac{V_{i+1}^k - V_{i-1}^k + V_{i+1}^{k+1} - V_{i-1}^{k+1}}{4\delta\tilde{s}} - \frac{r}{2}(V_i^k + V_i^{k+1}) = 0 \tag{3.6}$$

with a local truncation error of the order $O(\delta t^2, \delta\tilde{s}^2)$ Collecting the terms, multiplying by δt and rearranging the equation to have all variables of a time step on the same side one gets for the points inside the grid $(0 < i < N)$

$$A_i V_{i-1}^{k+1} + (1 + B_i)V_i^{k+1} + C_i V_{i+1}^{k+1} = -A_i V_{i-1}^k + (1 - B_i)V_i^k - C_i V_{i+1}^k \tag{3.7}$$

The coefficients A_i, B_i and C_i are independent of the time step k because the coefficients of equation 3.1 are constant in time. Their definition can be found as equation B.2 in appendix B.1.

The method is well–suited for diffusion equations. If, however, the coefficient of $\frac{\partial V}{\partial S}$ is very large compared to the coefficient of $\frac{\partial^2 V}{\partial S^2}$, the equation describes more a transport than a diffusion problem, and the discretisation scheme will not find the right solution because the central difference for $\frac{\partial V}{\partial S}$ can't describe the flow of information correctly. Therefore the condition

$$\left| \frac{\frac{1}{2}\sigma^2 T(s')^2 \delta\tilde{s}}{2f(s')T(s')} \right| < 1 \tag{3.8}$$

is checked during the calculation of the coefficients of the difference equation. If this condition is violated at a point S_i, an adaptive–upwind discretisation [13] is used making the following modification to the discretisation scheme 3.5:

$$\frac{\partial V}{\partial S} \approx \begin{cases} \frac{V_i^k - V_{i-1}^k}{\delta\tilde{s}} & f(s_i')T(s_i') < 0 \\ \frac{V_{i+1}^k - V_i^k}{\delta\tilde{s}} & f(s_i')T(s_i') > 0 \end{cases} . \tag{3.9}$$

This approximation is only accurate to order $O(\delta\tilde{s})$, but it has to be used only in corners of the parameter space that aren't too important for the valuation of real–world options (very low volatility and high interest rates). The coefficients in this case can be found in equations B.4 and B.5 in appendix B.2.

3.1.1 Boundary Conditions

At the boundary of the grid one off-grid point V_{N+1} and V_{-1}, respectively, is required at any time step. This off-grid point is expressed in terms of the points at the boundary by assuming $\frac{\partial^2 V}{\partial S^2} = 0$ at the upper bound $S_{\max}$ and the lower bound $S_{\min}$. Because the grid isn't uniform in S but in $\tilde{s}$, the condition reads

$$\frac{\partial^2 V}{\partial S^2} = \frac{1}{\cosh^2 s'} \frac{\partial^2 V}{\partial \tilde{s}^2} - \frac{\tanh s'}{\cosh^2 s'} \frac{\partial V}{\partial \tilde{s}} = 0. \tag{3.10}$$

From equation 3.10 follows after some arithmetic the difference equation

$$V_{i-1} - 2V_i + V_{i+1} = \frac{\delta\tilde{s} \tanh s_i'}{2} \left(V_{i+1} - V_{i-1}\right). \tag{3.11}$$

At the upper boundary,

$$V_{N+1} = \frac{1}{\beta_N} \left(2V_N - \alpha_N V_{N-1}\right) \tag{3.12}$$

with

$$\alpha_i = 1 + \frac{\delta\tilde{s} \tanh s_i'}{2}$$
$$\beta_i = 1 - \frac{\delta\tilde{s} \tanh s_i'}{2}.$$

The calculation of the coefficients at the boundary can be found in appendix B.3. Their definition is written down in equation B.7.

Analogously one can for the lower boundary deduce from equation 3.11

$$V_{-1} = \frac{1}{\alpha_0} \left(2V_0 - \beta_0 V_1\right) \tag{3.13}$$

and the result for the coefficients is shown in appendix B.3 in equation B.8.

3.1.2 Time steps

Equation 3.7 for $i = 0, \ldots, N$ can be summarised as the matrix equation

$$M_1 V^{k+1} = M_0 V^k \tag{3.14}$$

with the matrices

$$M_1 = \begin{pmatrix} 1+B_0 & C_0 & 0 & & \cdots & & 0 \\ A_1 & 1+B_1 & C_1 & 0 & \cdots & & 0 \\ 0 & A_2 & 1+B_2 & C_2 & 0 & & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & & \\ & \vdots & & & & & 0 \\ 0 & & \cdots & 0 & A_{N-1} & 1+B_{N-1} & C_{N-1} \\ 0 & & \cdots & & 0 & A_N & 1+B_N \end{pmatrix}$$

$$M_0 = \begin{pmatrix} 1-B_0 & -C_0 & 0 & & \cdots & & 0 \\ -A_1 & 1-B_1 & -C_1 & 0 & \cdots & & 0 \\ 0 & -A_2 & 1-B_2 & -C_2 & 0 & & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & & \\ & \vdots & & & & & 0 \\ 0 & & \cdots & 0 & -A_{N-1} & 1-B_{N-1} & -C_{N-1} \\ 0 & & \cdots & & 0 & -A_N & 1-B_N \end{pmatrix}$$

We are progressing backwards in time, i.e. given is V^{k+1} and we have to solve equation 3.14 for V^k. This equation is actually a linear equation system with a tridiagonal matrix. Such an equation system can be solved efficiently with iterative procedures of which the Jacobi method is the simplest. More efficient is the SOR method. It is shortly outlined in the following paragraphs.

We want to solve the equation $Ax = y$ where A is a quadratic $n \times n$ matrix and x, y are vectors of length n. The matrix A can be decomposed into three matrices $A = L + D + R$ with the lower left triangular matrix L[3], the diagonal matrix D and the upper right triangular matrix R. An initial estimate x_0 for the solution has to be guessed. (For the subsequent solutions of the equation for every time step, the vector that solved the linear equation in the time step before will be a good guess, e.g. in order to solve for V^k, V^{k+1} is a good initial vector.) Let x_n be the last calculated estimate of the solution of the linear equation system. Then the vector x_{n+1} of the next iteration is calculated as

$$x_{n+1} = \omega D^{-1}(y - Lx_{n+1} - Rx_n) + (1-\omega)x_n \tag{3.15}$$

where ω is a parameter. It can be shown that $0 < \omega < 2$ must be satisfied for the method to converge. A useful range is $1 \leq \omega < 2$.

Within this range there is an optimal value ω_0 with fastest convergence. Because the matrix A is constant in time and the vector y changes only little from one time step to the other, the optimisation is done by starting with $\omega = 1$[4] in the first time step and increasing ω in every time step until the number of iterations necessary to solve the equation system starts increasing again. The value ω_0 is chosen as the ω with the smallest number of iterations and kept constant for the rest of the time steps.

A stopping criterion for the iteration defines when the solution of the linear equation system is considered to be accurate enough. To measure the accuracy, the difference between two subsequent iterations is calculated using the vector norm

$$||x||_2 = \frac{1}{n}\sqrt{\sum_{i=0}^{n} x_i} \tag{3.16}$$

for the vector $x = (x_0, \ldots, x_n)$. With the given tolerance ε, the iterations therefore stop when

$$||x_{n+1} - x_n||_2 < \varepsilon. \tag{3.17}$$

[3] All matrix elements above the main diagonal of the matrix are 0.
[4] This choice reduces the method to the Gauss-Seidel method

For the desired accuracy of the solution of the PDE in the order of $< 10^{-3}$ with up to ≈ 1000 time steps, in every time step an accuracy of at least $\varepsilon = 10^{-6}$ is required (assuming that errors roughly add up linearly).

For American options, a variant of the SOR method, the *projected* SOR method is used. In every step of the SOR method the result of equation 3.15 is for every component of x_{n+1} compared to the option payoff. If the payoff is greater, the component of the vector is set equal to the payoff, i.e. for every component i of the estimated solution x_{n+1},

$$\hat{x}^i_{n+1} = \left(\omega D^{-1}(y - Lx_{n+1} - Rx_n) + (1-\omega)x_n\right)^i \tag{3.18a}$$

$$x^i_{n+1} = \max\left(\hat{x}^i_{n+1}, P(S_i)\right). \tag{3.18b}$$

Thus, the complementarity problem is solved consistently within every time step.

3.1.3 Choice of the Grid

To value an option with a defined accuracy, the spot price of the underlying asset at valuation time has to lie on a grid point, since the error estimates are valid only on the grid point. The cutoff point $S_{\max}$ should be far enough away from the spot that the probability of the asset price arriving outside the cutoff region is small. Here this probability was chosen to be less than 0.27%. This can be accomplished by choosing

$$S_{\max} = Se^{(r-\sigma^2/2)(T-t)+3\sigma\sqrt{T-t}} \tag{3.19}$$

where S asset price of the valuation day. The lower cutoff value for the asset price is set to $S_{\min} = 0$. Therefore the uniform grid is adjusted according to the following algorithm:

1. Given are a number of steps in the space dimension n and a cutoff value $S_{\max}$.

2. Calculate the grid size $\delta S = S_{\max}/n$.

3. Calculate the number of steps between 0 and the spot price of the asset and round to the nearest integer number $n_1 = \text{round}(S/\delta S)$.

4. Calculate a new grid size $\delta S' = S/n_1$.

5. Calculate a new upper cutoff value $S'_{\max} = \delta S' \cdot n$.

If the option value is to be calculated for a number of asset prices at the same time, this procedure can be skipped. For performance reasons the option price is calculated on the given grid in S. For the option price at a point in between, linear interpolation is used. The cutoff parameter $S_{\max}$ is chosen according to equation 3.19 where S is the highest asset price for which an option price shall be calculated.

For the comparisons to be done here, the interpolation error is avoided by first solving the PDE on the grid and then calculating the values from the other methods to be studied on the given grid points.

3.2 Optimisation of the Grid

Because the accuracy of the Crank-Nicolson scheme depends on the grid size in $\tilde{s}$ and t, it can't be improved alone by refining the grid only in one direction. For every spot grid size $\delta\tilde{s}$ there is an optimal time step size δt_{opt}. The accuracy of the numerical solution of the PDE can't be improved by reducing δt further without reducing $\delta\tilde{s}$. Therefore δt_{opt} can be determined as a function of $\delta\tilde{s}$.

When $\delta\tilde{s}$ and δt are decreased, at some point the accuracy of the solution is limited by the accuracy of the solution of the linear equation system 3.14 that has to be solved iteratively in every time step. A further improvement would then require tightening the stopping criterion (equation 3.17).

To measure the quality Q of the solution of the PDE, the average deviation relative to the option price from the analytically known solution for the case of a European call and put (without dividends) and for American calls with a single discrete dividend is calculated. The parameters $\sigma = 0.4$, $r = 0.05$, $T = 0.91$ and $X = 100$ are kept constant at typical values while the asset price S varies between 0 and 200. The average is calculated ignoring asset prices at which the option value is $< 10^{-2}$. With $V(S)$ being the option price as a function of the asset price S and $G = \{S_i | i = 0, \ldots, N\}$, Q can be written as

$$Q = \frac{1}{\#\{S \in G | V(S) > 10^{-2}\}} \sqrt{\sum_{\substack{S \in G \\ V(S) > 10^{-2}}} \left(\frac{V^{\text{num}}(S) - V^{\text{ana}}(S)}{V^{\text{ana}}(S)} \right)^2} \tag{3.20}$$

For various δS, the time step size δt is decreased and Q is determined for every combination of δS and δt until Q doesn't improve any more. Equivalently to δS and δt we use here the number of steps in S n_S and the number of steps in t n_t (while $S_{\max}$ and T are fixed). To the optimal time step δt_{opt} corresponds an optimal number of time steps n_t^{opt}. The result for European and American calls is shown in table 3.1. It has turned out that for European puts with the given financial parameters the quality Q is almost insensitive to the grid size at the optimal accuracy around $0.1 \cdot 10^{-3}$. For the further discussion only one relation between the number of steps in t and $\tilde{s}$ shall be used. Therefore

	EC		AC			
n_S	n_t^{opt}	$Q \cdot 10^3$	n_t^{opt}	$Q \cdot 10^3$	$n_t^{\text{opt,EC}}$	$Q \cdot 10^3$
100	100	7.7	100	7.8		
200	100	1.7	150	1.7	100	1.9
300	100	0.74	150	0.73	100	1.1
400	150	0.41	150	0.41		
500	200	0.28	300	0.31	200	0.41
600	250	0.25	300	0.27	250	0.35
700	300	0.27	300	0.31		

Table 3.1: *Optimisation of the grid for European and American calls. The last two columns show how Q changes for the American call if the optimal n_t of the European case is used.*

Parameter	Values
S	50, 70, 80, 90, 100, 110, 120, 140, 160, 200
σ	5%, 10%, 20%, 40%, 60%, 80%
r	1%, 2%, 4%, 8%, 16%, 25%
T	2D, 2W, 1M, 2M, 6M, 1Y, 2Y, 5Y

Table 3.2: *Parameters for validation scan. The total number of combinations is 2880.*

the last two columns show how Q changes for the American call if the optimal n_t of the European case is used. The deterioration is acceptable for the gain in simplicity to use only one relation

$$n_t^{\text{opt}} = \max\left(100, \left(n_{\tilde{s}} - 100\right)/2\right) \tag{3.21}$$

with an appropriate scaling for the time to maturity of the option. There isn't much to gain beyond 500 steps in $\tilde{s}$ because the accuracy of the solution of the linear equation system 3.14 is the limiting factor. Therefore $n_{\tilde{s}} = 500$ is chosen for all further studies.

3.3 Validation of the Numerical Solution

A large scale validation of the scheme has been done for equity options. There, analytical solutions exist for European calls and puts and for American calls with a single discrete dividend. The parameters shown in table 3.2 were scanned for the validation. Like in the definition of the quality Q in equation 3.20, only parameter combinations at which the analytical price is above 10^{-2} are considered to be not too sensitive to numerical artefacts. The following sections give a summary of the results. The numbers are summarised in table 3.3. The average deviation is the average of the relative deviations defined as

$$\left|\frac{V_i^{\text{num}} - V_i^{\text{ana}}}{V_i^{\text{ana}}}\right|$$

for all parameter combinations i with $V_i^{\text{ana}} > 0.01$.

For other option underlyings (FX rates, futures), several parameter combinations have been probed without performing such an extensive study. The tests showed for European options good agreement so that we can assume that the method is correctly implemented also including continuous dividend yields.

3.3.1 European Calls

For the European calls, 2390 of the 2880 parameter combinations were accepted with option values above 10^{-2}. The average relative deviation from the analytical solution was $0.28 \cdot 10^{-3}$, the largest deviation 16% at an option value of 0.025. Deviations larger than 1% occur only at $\sigma = 5\%$ and high interest rates $r \geq 16\%$ at the money forward.

	AC	EC	EP
samples total	1440	2880	2880
used	1263	2390	2129
average deviation ($\cdot 10^{-3}$)	0.53	0.28	0.70
RMS ($\cdot 10^{-3}$)	3.4	3.5	10
maximal deviation (%)	9.1	16	43

Table 3.3: *Deviation of the finite difference solution from the analytical solution.*

3.3.2 European Puts

For European puts, 2129 parameter combinations were accepted. The statistical quantities (especially RMS and maximal deviation) in table 3.3 are dominated by a single large deviation of 43% at an option value of 0.40 with the parameters $\sigma = 5\%$, $r = 16\%$, $T = 5Y$ and $S = 50.5$. The average deviation with $0.7 \cdot 10^{-3}$ is still well below the goal of 10^{-3}. Deviations above 1% occur only at the money forward at $\sigma = 5\%$ with high interest rates $r \geq 16\%$ and at $r = 8\%$ with $T \geq 2Y$.

3.3.3 American Calls on Equity with one Discrete Dividend

A dividend of 4 is payed 6 weeks after the valuation date for the American case. (For European options, the dividend makes conceptually no difference.) The last and the first three maturities were skipped for the American call because for the short maturities the dividend wouldn't fall in the option lifetime, and for the long maturity the early dividend is of little interest. Therefore only 1440 parameter combinations remain of which 1263 were accepted with option values above 10^{-2}.

The average deviation is with $0.53 \cdot 10^{-3}$ slightly higher than for the European calls, but the RMS and the maximal deviation are comparable. From the total parameter set probed only 7 lead to deviations $> 1\%$, all at the money forward with $\sigma = 5\%$ or $\sigma = 10\%$ and $r \geq 8\%$.

3.3.4 Conclusion

The numerical scheme for the valuation of options has been validated by comparing the results with the analytical solutions for European puts and calls and American calls on equity. An overall (average) accuracy better than 10^{-3} has been achieved over a wide variety of parameter combinations.

Most of the numerical scheme is independent of the presence of continuous dividend yields. There for it is sufficient to check that the yields are correctly implemented with some isolated parameter combinations. This test has been performed successfully.

With the achieved accuracy this finite difference scheme can serve as a benchmark the other methods to value American options can be compared to.

Chapter 4

Application of the Methods

Several methods outlined in chapter 2 and defined in the following paragraphs separately for equity and FX options have been compared to the finite difference scheme described in chapter 3 and selected as a benchmark pricing tool. In the first step the overall performance of the methods was compared for all parameters and in domains of parameter space (section 4.1). As a second step those domains were identified in which the methods yield the largest error and where they should not be applied (section 4.2).

The parameters have been set to all combinations of the values shown in table 4.1. The results of all models viable for the option/underlying combinations were compared to the option value calculated with the finite differences.

The particular role of equity options in this study is that they can have discrete dividends. The following methods for pricing American equity options have been compared to the benchmark:

- Binomial models with a moderate (200) and a very high (1000) number of time steps in the binomial tree. They will be called BIN200 and BIN1000, respectively.
- The average of binomial trees with 200 and 201 steps as described in section 2.3.3. This method will be called AVERAGE.
- The control variate technique described in section 2.3.4 with a binomial tree with 200 time steps. The symbol for this method will be CONVAR.

Parameter	Values
S	50, 70, 80, 90, 100, 110, 120, 140, 160, 200
σ	5%, 10%, 20%, 40%, 60%, 80%
r	1%, 2%, 4%, 8%, 16%, 25%
T	2D, 2W, 1M, 2M, 6M, 1Y, 2Y, 5Y
b	-10%, 0, 10%

Table 4.1: *Parameters for the comparison of the methods. The parameter b is varied only for FX options, and the foreign interest rate is set to $q = \max(r - b, 0)$.*

low domain	medium domain	high domain
$T < 0.5y$	$0.5y \leq T < 2y$	$T \geq 2y$
$S < 90$	$90 \leq S \leq 110$	$S > 110$
$r < 4\%$	$4\% \leq r \leq 8\%$	$r > 8\%$
$\sigma < 20\%$	$20\% \leq \sigma \leq 40\%$	$\sigma > 40\%$
$b < 0$	$b = 0$	$b > 0$

Table 4.2: *Parameter domains*

The number of time steps of the binomial tree was chosen such that the binomial tree was stable for all studied parameter combinations (i.e. the up–probability p was in the range $0 \leq p \leq 1$).

The models considered for FX options are:

- Binomial models BIN200 and BIN1000.
- The average of binomial trees with 200 and 201 steps AVERAGE
- The control variate technique CONVAR.
- The analytic approximation of Barone–Adesi and Whaley (BAW)
- The analytic approximation of Bjerksund and Stensland (BJST)

4.1 Comparison of the Methods

The performance of the methods in terms of accuracy and computational efficiency is compared over a broad range of parameters. Also the edges of the realistic parameter space are explored in order to find the limitations of the different methods. The “true” option value is determined with the finite difference scheme described in chapter 3.

A first comparison of the methods is done on global quantities: The mean of the relative deviation from the benchmark over all parameter sets, the RMS of the relative deviation, and the maximal relative deviation. In order to study the dependence of the accuracy of the methods on the parameters, each parameter set is divided in three domains. These domains are shown in table 4.2.

4.1.1 Equity Options

In figure 4.1 the result for American calls is shown as a histogram. The abscissa shows the relative deviation

$$\frac{v_{\text{model}} - v_{\text{FD}}}{v_{\text{FD}}} \tag{4.1}$$

where v_{FD} denotes the value of the option calculated with the finite difference scheme and v_{model} the value calculated with the model indicated in the histogram titles. Each histogram has 2366 entries, and a range of deviations of $\pm 10\%$ is shown. The statistical quantities can be found in the right upper corner.

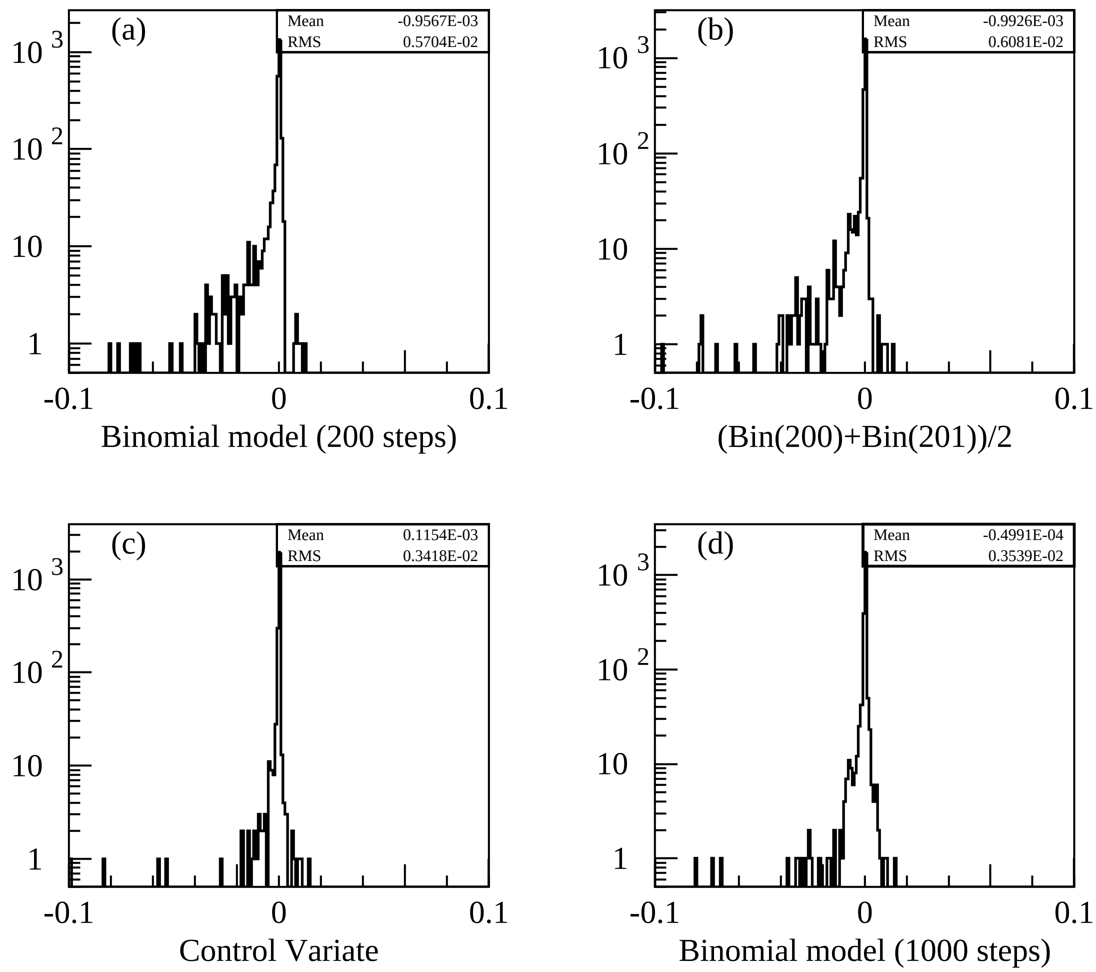

Figure 4.1: *Comparison of the call prices calculated with the method indicated in the histogram titles with the finite difference solution with a logarithmic scale for the ordinate. Each histogram has 2366 entries.*

Some more statistical data can be found in table 4.3. The average deviation is the average of quantity defined in equation 4.1 over all parameter combinations *i* with $V_i^{FD} > 0.01$. Obviously, the binomial models BIN200 and AVERAGE tend to yield too low values. The methods CONVAR and BIN1000 are on average very close to the finite difference solution. The RMS shows that these two have about the same accuracy globally. The means don't differ significantly. However, in terms of computation time[1], the method CONVAR is about 28 times more efficient.

The analysis of the parameter domain shows that BIN1000 is superior to CONVAR only for short–term at–the–money calls at high rates and low volatility, for medium term at–the–money and in–the–money calls at low rates and medium volatility, and for long–term in-the-money calls at low rates and low volatility.

[1]All computation times are measured on a K6-III/400 system running Linux kernel 2.2.5 and the GNU compiler suite egcs-2.91.66.

Method	BIN200	AVERAGE	CONVAR	BIN1000
average deviation $(\cdot 10^{-3})$	-0.96	-0.99	0.11	-0.05
RMS of deviations $(\cdot 10^{-2})$	0.57	0.61	0.34	0.35
number of deviations $> 1\%$	108	96	15	23
largest deviation	33%	33%	10%	14%
computation time (s)	29	60	28	777

Table 4.3: *Statistical data for calls on equity*

Method	BIN200	AVERAGE	CONVAR	BIN1000
average deviation $(\cdot 10^{-3})$	-1.27	-1.28	-0.49	0.28
RMS of deviations $(\cdot 10^{-2})$	0.68	0.67	0.58	0.27
number of deviations $> 1\%$	130	126	68	18
largest deviation	47%	47%	47%	7%
computation time (s)	29	61	29	795

Table 4.4: *Statistical data for puts on equity*

The methods BIN200 and AVERAGE deviate at 108 and 96 parameter combinations, respectively, from the benchmark by more than 1%. These parameter combinations are scattered over a wide domain of parameter space; therefore their distribution hasn't been studied further.

The 15 parameter combinations for CONVAR with deviations above 1% are shown in appendix C.1.1. Almost all these combinations have a low volatility $\leq 10\%$ and a high interest rate $r \geq 16\%$. These combinations are unrealistic for most practical purposes. Only three combinations occur at lower rates, but also at the low volatility $\sigma = 5\%$ out of the money.

In Appendix C.1.2 the parameter combinations are summarised for which method BIN1000 yields a deviation greater than 1%. The overall distribution of these parameter combinations is similar to that of CONVAR. All the additional critical parameter combinations occur far out of the money at low volatility and high rate or long time to maturity.

Figure 4.2 shows the equivalent results for puts on equity. The summarising statistical data are displayed in table 4.4. All methods are significantly less accurate than for calls. Especially method CONVAR has a maximal deviation of the same size as BIN200 and AVERAGE. As can be seen in the table of appendix C.2.1, the prices of in-the-money options with long times to maturity are not very good, in addition to the critical regions of the calls. The reason is that for equity puts, early exercise of the American option is much more important than for calls. Therefore, the difference between the price of an American put and a European put is larger than between the price of an American call and a European call. As this difference gets large, the correction introduced by the binomial tree calculation of American and European put gets larger, and the numerical error (of the same magnitude as in BIN200 and AVERAGE) becomes more important. However, the average deviation and the RMS are still significantly better than for those two methods, and the number of parameter combinations with deviations $> 1\%$ is only half as big.

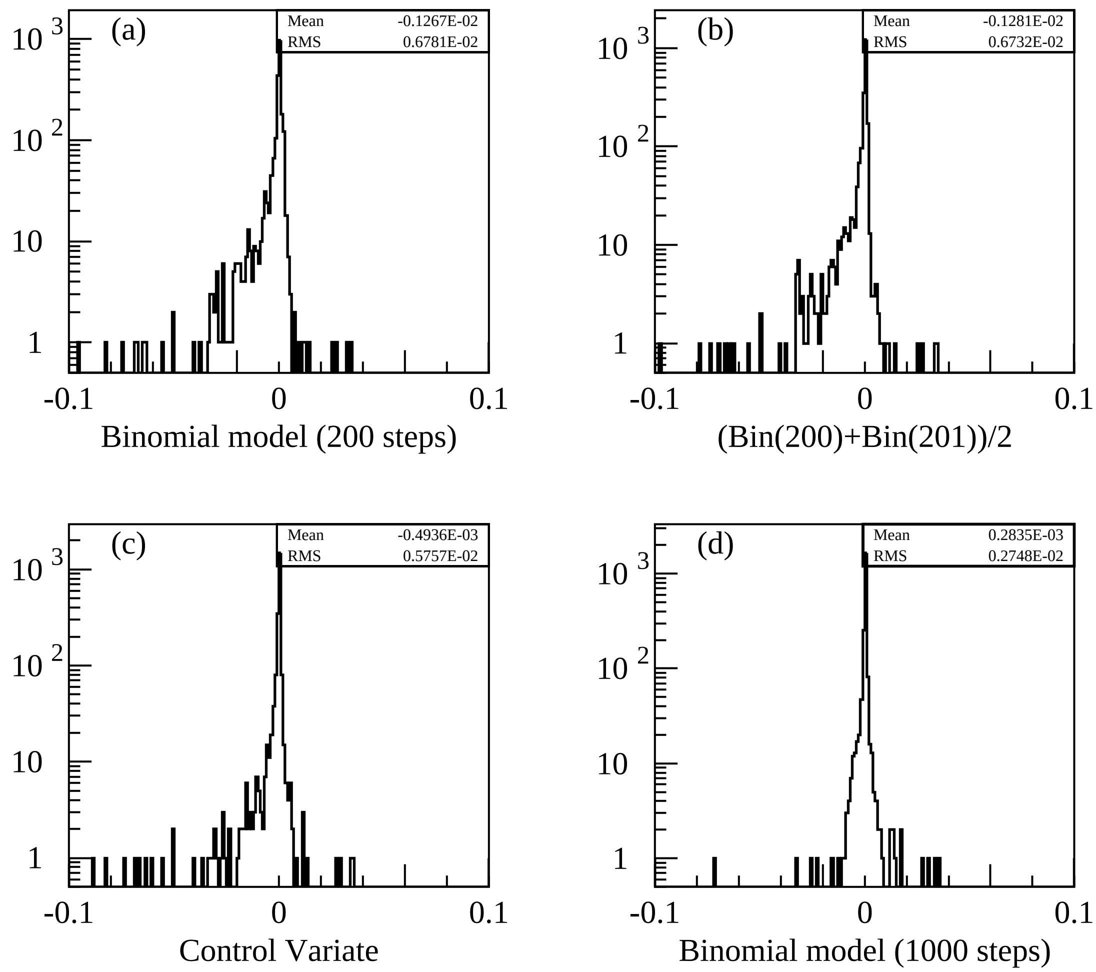

Figure 4.2: *Comparison of the put prices with the finite difference solution with a logarithmic scale for the ordinate. Each histogram has 2189 entries.*

Compared with BIN1000, CONVAR performs much worse in many parameter domains. These domains are scattered all over the parameter space. Clearly the best method for puts on equity is BIN1000 that yields even less deviations than for calls, at the cost of a high computational effort.

4.1.2 FX Options

The methods considered here have already been summarised in the beginning of this chapter. There is yet another parameter to be varied, the foreign interest rate (or dividend yield for index options) q. This parameter has been chosen to be $q = \max(r - b, 0)$ where the cost of carry b was chosen as -10%, 0 (corresponding to futures options) and +10%. Therefore there are in total 8640 parameter combinations.

For 7022 combinations the call price is > 0.01. The overall result for calls is shown in table 4.5. The distributions of deviations are shown in figure 4.3. All models based on binomial trees

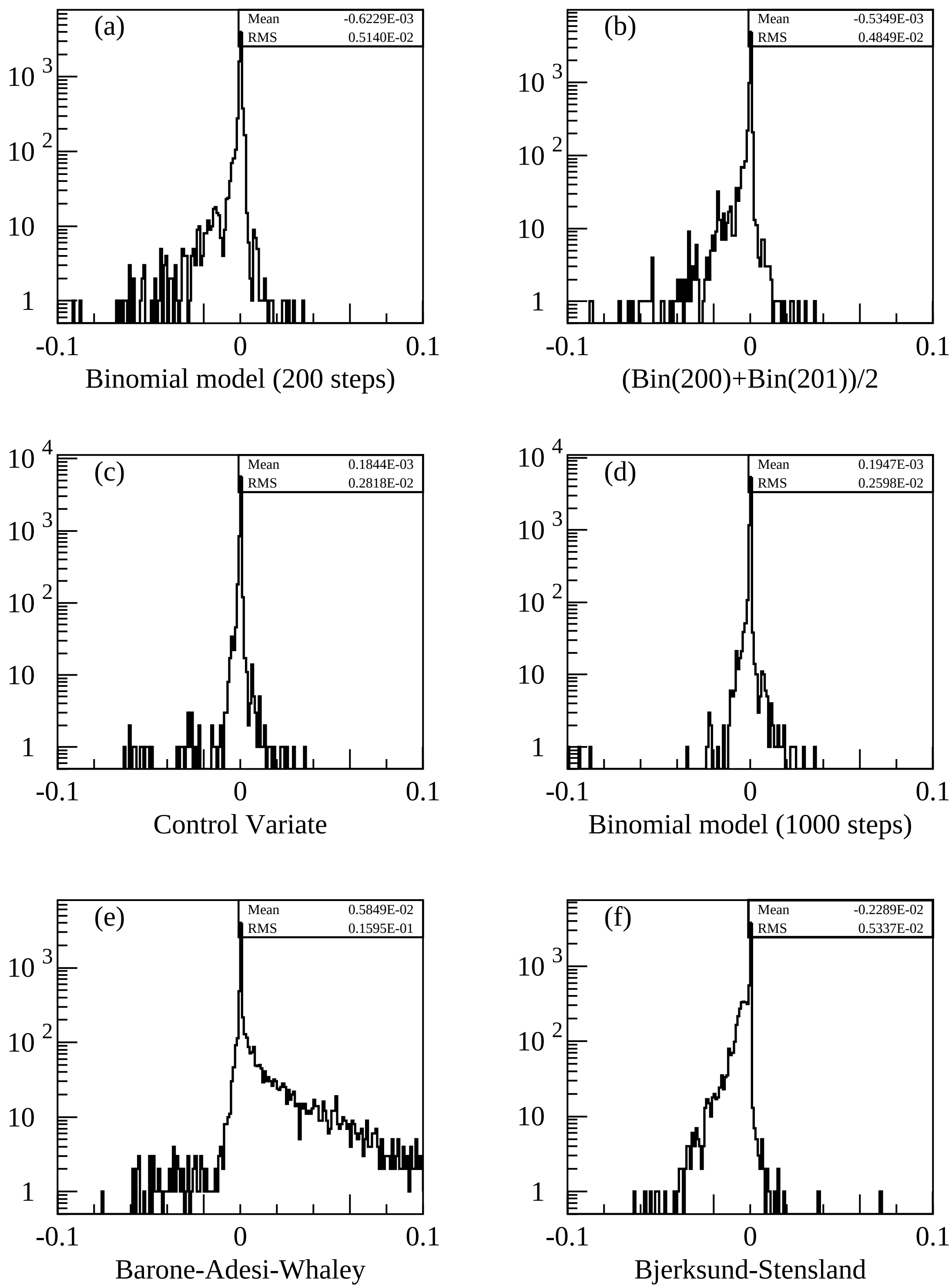

Figure 4.3: *Distribution of deviations for American calls on FX rates rates with a logarithmic scale for the ordinate. Histograms (a)-(e) have 7022 entries, histogram (f) has 6996 entries.*

Method	BIN200	AVERAGE	CONVAR	BIN1000	BAW	BJST
average deviation $(\cdot 10^{-3})$	-0.81	-0.75	0.09	0.07	5.61	-2.52
RMS of deviations $(\cdot 10^{-2})$	0.51	0.48	0.28	0.26	1.60	0.52
# deviations $> 1\%$	231	233	49	44	1338	523
largest deviation	49%	49%	13%	22%	554%	98%
computation time (s)	85	172	82	2307	0.5	0.27

Table 4.5: *Statistical data for the call on an FX rate*

have a similar global performance for calls on FX rates as for calls on equity. The mean deviation is about 30% smaller, the fraction of parameter combinations for which the deviation is $> 1\%$ is 30% or more smaller for the low-accuracy methods BIN200 and AVERAGE. The higher deviations are scattered all over the parameter space, but with higher density at the money with $b = -10\%$ and out of the money with $b \geq 0$. The highest deviations occur for long–term out–of–the–money calls. For the higher accuracy binomial models CONVAR and BIN1000, the parameter sets with deviations $> 1\%$ are shown in appendix C.3. BIN1000 is significantly more accurate than CONVAR at low volatility for short–dated in–the–money calls at high rates and for long–dated out–of–the–money calls at low rates.

BAW doesn't work well for out–of–the–money calls for all values of q, r and σ. In 198 cases errors above 10% are observed. The problem gets bigger with increasing time to maturity. If the time to maturity is longer than 1 year, even in–the–money calls get more and more affected. Because of the large number of high deviations, no table is shown in appendix C.3. BJST shows generally a reasonable behaviour, but with negative option prices for at–the–money and somewhat out–of–the–money calls with $b = -10\%$ and time to maturity 1 year or more. These obviously wrong results were omitted for the calculation of the quantities in table 4.5. Therefore the number of entries in the histogram and contributing to the statistics is only 6996. Neglecting those parameter sets, it is obvious from the means in table 4.5 and figure 4.3 (e) and (f) that BAW tends to too high option values while BJST tends to too low option values. Generally, BAW is slightly better than BJST only for short term calls at the money.

The strength of the analytic approximations is clearly the high computational efficiency. However, the accuracy is generally only of the order of 1%. The table in appendix C.3.3 show only parameter sets with deviations $> 10\%$. This table contains mainly those parameter combinations where a negative call price is obtained.

The put price is > 0.01 for 6700 parameter sets. The distribution of the deviation for the six methods can be found in figure 4.4, and the summary in table 4.6. Again the model BJST produces negative option values for six parameter combinations. Therfore only the remaining 6694 parameter sets are considered in the further study.

For the lower accuracy binomial models the picture is very similar to the calls. The average deviation is about twice the average for the calls, and the number of samples with deviations $> 1\%$ is about 50% higher. These cluster again out of the money scattered over wide ranges in the remaining

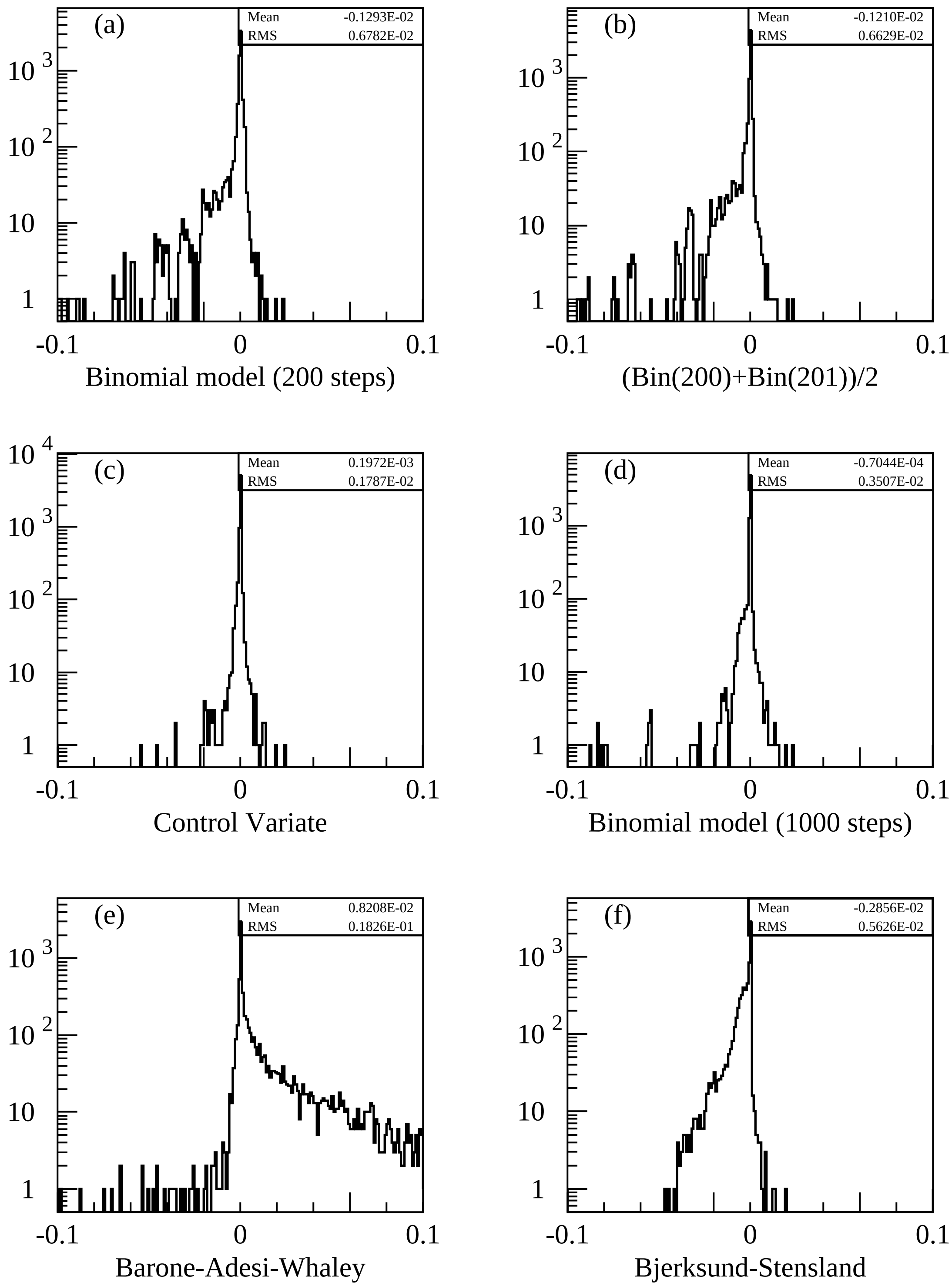

Figure 4.4: *Distribution of deviations for American puts on FX rates with a logarithmic scale for the ordinate. Histograms (a)-(e) have 6700 entries, histogram (f) has 6694 entries.*

Method	BIN200	AVERAGE	CONVAR	BIN1000	BAW	BJST
average deviation $(\cdot 10^{-3})$	-1.45	-1.39	0.05	0.28	8.04	-3.02
RMS of deviations $(\cdot 10^{-2})$	0.67	0.66	0.17	0.35	1.83	0.56
# deviations $> 1\%$	355	353	33	52	1649	559
largest deviation	33%	33%	5%	9%	680%	99%
computation time (s)	86	176	83	2324	25	0.25

Table 4.6: *Statistical data for the put on an FX rate*

variables. The method CONVAR deviates from the benchmark by more than 1% only at the money and out of the money for low volatilities $\sigma \leq 10\%$ and with long time to maturity and higher interest rate at $b = 10\%$. There's only one single case for $b = 0$. These parameter sets are shown in appendix C.4.1. For BIN1000 most of the cases occur at $b = -10\%$ and lower volatilities $\sigma \leq 20\%$ out of the money. The parameter sets for which BIN1000 deviates by more than 1% from the benchmark are shown in appendix C.4.2. A more detailed analysis shows that BIN1000 is only better than CONVAR for long–dated options at high interest rates and low volatility.

For BAW, deviations above 10% occur mainly for long–term at– and out–of–the–money puts, at interest rates of 16% or above and $b \leq 0$ also for in–the–money puts. Method BJST performs generally much better, with real problems mainly at long time to maturity out of the money when $b = 10\%$ and $\sigma \leq 20\%$, for $b \approx 0$, long time to maturity and $\sigma = 80\%$ in the money. The table in appendix C.4.3 shows the 19 parameter sets at which the deviation of BJST is greater than 5%. The six combinations with negative option price can be found in the table; they all occur at 5 years to maturity, volatility $\sigma \leq 10\%$ and $r \geq 16\%$. A direct comparison of the two methods shows that BAW is superior to BJST for short–dated at–the–money and out–of–the–money puts at high volatility and low rate, but nowhere else.

The computational efficiency of BJST for the puts is about the same as for the calls, but it's amazing that BAW takes 50 times longer for the puts than for the calls, and is only 3 times faster than the much more accurate method CONVAR. The reason is that the solution of equation A.6 takes far more iterations than its equivalent for calls.

4.2 Problematic Domains

The parameter domains defined in table 4.2 have been investigated separately for all applicable methods[2] and the four classes of options (call and put on equity and FX). For every combination of method and option class, the most problematic parameter domains defined by the RMS of the deviation of the result of the method from the finite difference result have been identified and are summarised in table 4.7.

[2]except BIN200 and AVERAGE having a much worse accuracy than CONVAR with the same computational effort

	Call on equity				Put on equity				Call on FX					Put on FX				
	T	*S*	*r*	σ	*T*	*S*	*r*	σ	*T*	*S*	*r*	σ	*b*	*T*	*S*	*r*	σ	*b*
BIN1000	o+	-	+	-	-+	=	+	-	+	=	o+	-	+	+	=	o+	-	+
	+	=	+	-	+	=	o	-	+	=	-o+	-	-	o	=	+	-	+
									o	=	-	-	-	+	-	o	-	+
									+	=	+	-	+	o	-	-	-	+
									o	-	+	-	+	o	=	o	-	+
CONVAR	o+	-	+	-	-+	=	+	-	+	=	o+	-	+	+	-=	+	-	+
	-	=	+	-	+	=	+	-	+	=	-o+	-	-	+	-	-o+	-	-
					+	=	o	-	o	-	+	-	+	-	=	+	-	+
					+	=	+	o	+	-	-	-	-	o	-	-o+	-	-
					+	+	+	-										
BAW									+	-	+	-	o+	+	-	+	-	-o
									+	-	+	o	o	+	-	o	-	-
									o	-	+	o	-	o	-	+	o	+
									o+	-	-o	-o	-	+	-	+	o	o+
														+	-	o	-o	+
BJST									+	=	o	-	+	o+	-	-o+	-	-
									o+	-	-o+	-	+	-+	=	+	-	+
									-+	=	-o+	-	-					

Table 4.7: *Problematic regions of the different methods. Legend: for all columns except S:* - *low,* o *medium,* + *high domain; for columns S:* - *out of the money,* = *at the money,* + *in the money.*

4.2.1 Options on Equity

For calls on equity, both methods CONVAR and BIN1000 are generally good, and there are only two domains in parameter space where both methods yield less accurate results than on average. In these parameter domains, rates are high and volatility is low. Both methods are less accurate than on average for out–of–the–money calls with medium to long time to maturity. Method BIN1000 is worse for long–term at–the–money calls, method CONVAR for short–term calls.

Both methods yield for at–the–money puts on equity at high rates and low volatility and for long–term at–the–money puts at medium rates and low volatility less accurate results than on average. There are a few clustered domains where the method CONVAR shows less accurate results than the method BIN1000, namely for puts with long time to maturity at the money and in the money at high rates and low volatility.

Therefore (and considering the results from section 4.1.1) for reasonable accuracy at high computational efficiency the method CONVAR can generally be used for American options on equity paying discrete dividends, except for short–term at–the–money calls, at–the–money and in–the–money puts with long time to maturity at high rates and low volatility.

4.2.2 Options on FX rates

In the following discussion it should be well understood that the methods CONVAR and BIN1000 are both much more accurate (in general) than the methods BAW and BJST. Comparisons should be read as relative to the average of the respective method.

Both binomial methods CONVAR and BIN1000 have problems in domains clustered at low volatility for at– and out–of–the–money calls with medium to long time to maturity. BAW shows rather poor results for out of the money calls while the method BJST is weaker than on average only for at and out of the money calls at low volatility.

For applying BIN1000 to puts the problematic regions are limited to $q < r$ ($b > 0$) and low volatility at and out of the money. Method CONVAR is less accurate for out–of–the–money puts at low volatility and $b < 0$ and for at–the–money puts at high rates, low volatility and $b > 0$. The main problem of BAW is the valuation of long–term out–of–the–money puts at medium to high rates and low to medium volatility. BJST is problematic at the valuation of at– and out–of–the–money puts at low volatility.

In the case of FX options, no cluster of regions can be found where one of the methods CONVAR and BIN1000 is better than the other. BAW generally performs rather bad for out of the money options. Therefore, if for computational efficiency an analytic approximation for the pricing of American FX options is desired, BJST should be generally preferred in this case.

It should be noted that for both, calls and puts, the case $b = 0$ doesn't occur amongst the problematic regions of the methods CONVAR, BIN1000 and BJST (and only rarely amongst the problematic regions of method BAW). Therefore all methods work rather well for futures options, while still CONVAR and BIN1000 are more accurate than BJST.

Chapter 5

Conclusions

In this thesis, a comparison of various method for the pricing of American options is discussed. Because an analytic solution to this problem isn't known, a Crank–Nicolon finite difference scheme was used to compute a benchmark option price considered as exact.

The finite difference scheme uses a non–uniform grid in direction of the underlying price in order to reduce the truncation error close to the strike price of the option without increasing the computational complexity of the problem by making the grid finer. In order to assess the accuracy of the finite difference scheme, its solutions were compared to option prices obtained analytically in cases where this is possible, i.e. for European options and American calls on an underlying paying a single discrete dividend.

In total, up to six methods to approximate the price of American options on equity and FX rates have been compared to the numerical solution of the Black–Scholes equation over a large number of combinations of parameter values covering the part of parameter space relevant for practical purposes. Two of the methods are plain Binomial Trees (with 200 and 1000 time steps), two are based on the Binomial Tree approach (the average of Binomial Trees with 200 and 201 steps, and a control variate technique), and two methods are popular analytic approximations which are not applicable to options on assets paying discrete dividends.

A general result is that the averaging of two subsequent, lower accuracy Binomial Trees doesn't improve as much over the plain Binomial Tree with the same number of time steps as the control variate technique with the same computational effort. Therefore more detailed studies have been limited to the high–resolution Binomial Tree, the control variate technique and the analytical approximations.

As one would expect, the high resolution Binomial Tree yields indeed the highest accuracy of the proposed methods. However, the control variate technique gets in most cases rather close to this accuracy at much lower computational effort. The analytic approximations yield significantly less accurate results and should only be used if computation time is much more critical than pricing accuracy.

Appendix A

Analytic Approximations

A.1 Approximation of Barone-Adesi and Whaley

According to [9], the value of an American call is approximated by

$$V_{\text{call}} = \begin{cases} V_{\text{call}}^{\text{BS}}(S,X,T) + A_1(S/S^*)^{q_1} & S < S^* \\ S - X & S \geq S^* \end{cases} \tag{A.1}$$

with

$$A_1 = \frac{S^*}{q_1}\left(1 - e^{-q(T-t)}N\left(d_1(S^*)\right)\right) \tag{A.2a}$$

$$d_1(S) = \frac{\ln(S/X) + (r - q + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \tag{A.2b}$$

$$q_1 = \frac{-(2(r-q)/\sigma^2 - 1) + \sqrt{(2(r-q)/\sigma^2 - 1)^2 + \frac{8r}{\sigma^2(1-e^{-r(T-t)})}}}{2} \tag{A.2c}$$

S^* is the asset price for which the call price satisfies

$$S^* - X = V_{\text{call}}^{\text{BS}}(S^*,X,T) + \frac{1 - e^{-q(T-t)}N\left(d_1(S^*)\right)S^*}{q_1} \tag{A.3}$$

The value of an American put is

$$V_{\text{put}} = \begin{cases} V_{\text{put}}^{\text{BS}}(S,X,T) + A_2(S/S^{**})^{q_2}, & S > S^{**} \\ S - X, & S \leq S^{**} \end{cases} \tag{A.4}$$

with

$$A_2 = -\frac{S^{**}}{q_2}\left(1 - e^{-q(T-t)}N\left(-d_1(S^{**})\right)\right) \tag{A.5a}$$

$$q_2 = \frac{-(2(r-q)/\sigma^2 - 1) - \sqrt{(2(r-q)/\sigma^2 - 1)^2 + \frac{8r}{\sigma^2(1-e^{-r(T-t)})}}}{2} \tag{A.5b}$$

S^{**} is the asset price for which the call price satisfies

$$S^{**} - X = V_{\text{put}}^{\text{BS}}(S^{**},X,T) - \frac{1 - e^{-q(T-t)}N\left(d_1(S^{**})\right)S^{**}}{q_2} \tag{A.6}$$

A.2 Approximation of Bjerksund and Stensland

The Approximation of Bjerksund and Stensland [10] can be written as

$$\begin{aligned} V_{\text{call}} &= \alpha S^{\beta} - \alpha\Phi(S, T-t, \beta, I, I) + \Phi(S, T-t, 1, I, I) - \Phi(S, T-t, 1, X, I) \\ &\quad - X\Phi(S, T-t, 0, I, I) + X\Phi(S, T-t, 0, X, I) \end{aligned} \tag{A.7}$$

with

$$\alpha = (I - X)I^{-\beta} \tag{A.8a}$$

$$\beta = \left(\frac{1}{2} - \frac{b}{\sigma^2}\right) + \sqrt{\left(\frac{b}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r}{\sigma^2}} \tag{A.8b}$$

$$b = r - q \tag{A.8c}$$

The function Φ is given by

$$\Phi(S, T, \gamma, H, I) = e^{\lambda} S^{\gamma} \left(N(d) - \left(\frac{I}{S}\right)^{\kappa} N\left(d - \frac{2\ln(I/S)}{\sigma\sqrt{T}}\right)\right) \tag{A.8d}$$

$$\lambda = \left(-r + \gamma b + \frac{1}{2}\gamma(\gamma - 1)\sigma^2\right) T \tag{A.8e}$$

$$d = -\frac{\ln(S/H) + \left(b + (\gamma - \frac{1}{2})\sigma^2\right) T}{\sigma\sqrt{T}} \tag{A.8f}$$

$$\kappa = \frac{2b}{\sigma^2} + (2\gamma - 1) \tag{A.8g}$$

$$I = B_0 + (B_\infty - B_0)\left(1 - e^{h(T)}\right) \tag{A.8h}$$

$$h(T) = -(bT + 2\sigma\sqrt{T})\left(\frac{B_0}{B_\infty - B_0}\right) \tag{A.8i}$$

$$B_\infty = \frac{\beta}{\beta - 1} X \tag{A.8j}$$

$$B_0 = \max\left(X, \frac{r}{q} X\right) \tag{A.8k}$$

The value of an American put in this approximation can be written as a transformation of the formula for a call:

$$V_{\text{put}}(S, X, T, r, q, \sigma) = V_{\text{call}}(X, S, T, q, q - r, \sigma) \tag{A.9}$$

Appendix B

Coefficients for the Finite Difference Scheme

B.1 Inside the Grid

Rearranging equation 3.6 to have all terms belonging to one time step on the same side yields

$$\begin{aligned}
&\left(\frac{1}{4}\frac{\sigma^2 T(s_i')^2}{\delta\tilde{s}^2} - \frac{f(s_i')T(s_i')}{4\delta\tilde{s}}\right)\delta t V_{i-1}^{k+1} + \left(1 - \left(\frac{1}{2}\frac{\sigma^2 T(s_i')^2}{\delta\tilde{s}^2} + \frac{r}{2}\right)\delta t\right) V_i^{k+1} \\
&\quad + \left(\frac{1}{4}\frac{\sigma^2 T(s_i')^2}{\delta\tilde{s}^2} + \frac{f(s_i')T(s_i')}{4\delta\tilde{s}}\right)\delta t V_{i+1}^{k+1} \\
&\quad = -\left(\frac{1}{4}\frac{\sigma^2 T(s_i')^2}{\delta\tilde{s}^2} - \frac{f(s_i')T(s_i')}{4\delta\tilde{s}}\right)\delta t V_{i-1}^{k} + \left(1 + \left(\frac{1}{2}\frac{\sigma^2 T(s_i')^2}{\delta\tilde{s}^2} + \frac{r}{2}\right)\delta t\right) V_i^{k} \\
&\qquad - \left(\frac{1}{4}\frac{\sigma^2 T(s_i')^2}{\delta\tilde{s}^2} + \frac{f(s_i')T(s_i')}{4\delta\tilde{s}}\right)\delta t V_{i+1}^{k}. \quad \text{(B.1)}
\end{aligned}$$

The definitions

$$A_i = \left(\frac{1}{4}\frac{\sigma^2 T(s_i')^2}{\delta\tilde{s}^2} - \frac{f(s_i')T(s_i')}{4\delta\tilde{s}}\right)\delta t \quad \text{(B.2a)}$$

$$B_i = \left(-\frac{1}{2}\frac{\sigma^2 T(s_i')^2}{\delta\tilde{s}^2} - \frac{r}{2}\right)\delta t \quad \text{(B.2b)}$$

$$C_i = \left(\frac{1}{4}\frac{\sigma^2 T(s_i')^2}{\delta\tilde{s}^2} + \frac{f(s_i')T(s_i')}{4\delta\tilde{s}}\right)\delta t \quad \text{(B.2c)}$$

allow to write equation B.1 as concise as in equation 3.7.

B.2 Adaptive Upwind Differencing

If the relation

$$\left|\frac{\frac{1}{2}\sigma^2 T(s')^2\delta\tilde{s}}{2f(s')T(s')}\right| < 1 \quad \text{(B.3)}$$

is violated, B.2 is modified as follows if $f(s_i')T(s_i') < 0$:

$$A_i = \left(\frac{1}{4}\frac{\sigma^2 T(s')^2}{\delta\tilde{s}^2} - \frac{f(s_i')T(s_i')}{2\delta\tilde{s}}\right)\delta t \tag{B.4a}$$

$$B_i = \left(-\frac{1}{2}\frac{\sigma^2 T(s_i')^2}{\delta\tilde{s}^2} + \frac{f(s_i')T(s_i')}{2\delta\tilde{s}} - \frac{r}{2}\right)\delta t \tag{B.4b}$$

$$C_i = \frac{1}{4}\frac{\sigma^2 T(s_i')^2}{\delta\tilde{s}^2}\delta t. \tag{B.4c}$$

If $f(s_i')T(s_i') > 0$, the coefficients are calculated according to

$$A_i = \frac{1}{4}\frac{\sigma^2 T(s_i')^2}{\delta\tilde{s}^2}\delta t \tag{B.5a}$$

$$B_i = \left(-\frac{1}{2}\frac{\sigma^2 T(s_i')^2}{\delta\tilde{s}^2} - \frac{f(s_i')T(s_i')}{2\delta\tilde{s}} - \frac{r}{2}\right)\delta t \tag{B.5b}$$

$$C_i = \left(\frac{1}{4}\frac{\sigma^2 T(s')^2}{\delta\tilde{s}^2} + \frac{f(s_i')T(s_i')}{2\delta\tilde{s}}\right)\delta t. \tag{B.5c}$$

B.3 Boundary Conditions

Inserting equation 3.12 into B.1 yields

$$\begin{aligned}
&\left(\frac{1}{4}\frac{\sigma^2 T(s_N')^2}{\delta\tilde{s}^2}\left(1-\frac{\alpha_N}{\beta_N}\right) - \frac{f(s_N')T(s_N')}{4\delta\tilde{s}}\left(1+\frac{\alpha_N}{\beta_N}\right)\right)\delta t V_{N-1}^{k+1} \\
&\quad + \left(1 + \left(\frac{1}{2}\frac{\sigma^2 T(s_N')^2}{\delta\tilde{s}^2}\left(\frac{1}{\beta_N}-1\right) + \frac{1}{2}\frac{f(s_N')T(s_N')}{\beta_N\delta\tilde{s}} - \frac{r}{2}\right)\delta t\right)V_N^{k+1} \\
&\quad = -\left(\frac{1}{4}\frac{\sigma^2 T(s_N')^2}{\delta\tilde{s}^2}\left(1-\frac{\alpha_N}{\beta_N}\right) - \frac{f(s_N')T(s_N')}{4\delta\tilde{s}}\left(1+\frac{\alpha_N}{\beta_N}\right)\right)\delta t V_{N-1}^{k} \\
&\qquad + \left(1 - \left(\frac{1}{2}\frac{\sigma^2 T(s_N')^2}{\delta\tilde{s}^2}\left(\frac{1}{\beta_N}-1\right) + \frac{1}{2}\frac{f(s_N')T(s_N')}{\beta_N\delta\tilde{s}} - \frac{r}{2}\right)\delta t\right)V_N^{k}.
\end{aligned} \tag{B.6}$$

Extending the scheme 3.7 to the boundary $i = N$,

$$A_N = \left(\frac{1}{4}\frac{\sigma^2 T(s_N')^2}{\delta\tilde{s}^2}\left(1-\frac{\alpha_N}{\beta_N}\right) - \frac{f(s_N')T(s_N')}{4\delta\tilde{s}}\left(1+\frac{\alpha_N}{\beta_N}\right)\right)\delta t \tag{B.7a}$$

$$B_N = \left(\frac{1}{2}\frac{\sigma^2 T(s_N')^2}{\delta\tilde{s}^2}\left(\frac{1}{\beta_N}-1\right) + \frac{1}{2}\frac{f(s_N')T(s_N')}{\beta_N\delta\tilde{s}} - \frac{r}{2}\right)\delta t \tag{B.7b}$$

$$C_N = 0. \tag{B.7c}$$

At the lower boundary, equation 3.13 is inserted in equation B.1 to obtain

$$A_0 = 0 \tag{B.8a}$$

$$B_0 = \left(\frac{1}{2}\frac{\sigma^2 T(s_0')^2}{\delta\tilde{s}^2}\left(\frac{1}{\alpha_0}-1\right) - \frac{1}{2}\frac{f(s_0')T(s_0')}{\alpha_0\delta\tilde{s}} - \frac{r}{2}\right)\delta t \tag{B.8b}$$

$$C_0 = \left(\frac{1}{4}\frac{\sigma^2 T(s_0')^2}{\delta\tilde{s}^2}\left(1-\frac{\beta_0}{\alpha_0}\right) + \frac{f(s_0')T(s_0')}{4\delta\tilde{s}}\left(1+\frac{\beta_0}{\alpha_0}\right)\right)\delta t. \tag{B.8c}$$

Appendix C

Tables of Large Deviations

This appendix summarises the parameter combinations of calls on equity for which the deviation from the benchmark is greater than 1% (for the high accuracy binomial methods) and 10% (for the analytic approximations), respectively. The asset price is rounded to the next integer for better readability of the tables. The binomial tree methods with 200 steps are not listed here because they show about 100 deviations above 1% at parameter combinations scattered widely over the entire parameter space.

In all tables, the first four columns show the parameters, the next two columns the option price calculated with the method indicated in the heading, and the last column shows the absolute value of the relative deviation

$$\left| \frac{V_i^{\text{num}} - V_i^{\text{FD}}}{V_i^{\text{FD}}} \right|$$

C.1 Calls on equity

C.1.1 Control Variate

With the control variate technique with 200 steps there are 15 parameter combinations with deviations greater than 1%.

T	S	r	σ	Control Variate	FD	Deviation
1.003	90	0.04	0.05	0.024	0.024	0.01758
5.005	70	0.04	0.05	0.096	0.098	0.01284
2.003	80	0.08	0.05	0.126	0.128	0.01409
0.005	100	0.16	0.05	0.161	0.159	0.01050
0.501	90	0.16	0.05	0.025	0.026	0.05325
1.003	80	0.16	0.05	0.016	0.017	0.08333
1.003	70	0.16	0.10	0.014	0.014	0.01108
2.003	70	0.16	0.05	0.211	0.216	0.02707
5.005	50	0.16	0.05	2.737	3.039	0.09944
0.005	100	0.25	0.05	0.194	0.191	0.01403
0.167	90	0.25	0.10	0.018	0.019	0.05789
2.003	50	0.25	0.10	0.074	0.076	0.01708
5.005	140	0.25	0.05	107.640	108.810	0.01075
5.005	160	0.25	0.05	127.160	128.700	0.01197
5.005	200	0.25	0.05	167.490	169.880	0.01407

For a discussion of the result, see section 4.1.1.

C.1.2 Binomial Tree with 1000 steps

All parameter combinations for which the deviation from the benchmark is greater than 1% are shown here. Their total number is 23.

T	S	r	σ	Binomial (1000)	FD	Deviation
1.003	90	0.04	0.05	0.024	0.024	0.02748
5.005	70	0.04	0.05	0.095	0.098	0.03148
5.005	50	0.04	0.10	0.023	0.023	0.01690
1.003	90	0.08	0.05	0.161	0.164	0.01758
2.003	80	0.08	0.05	0.124	0.128	0.03202
5.005	50	0.08	0.10	0.247	0.251	0.01456
0.005	100	0.16	0.05	0.161	0.159	0.01068
0.501	90	0.16	0.05	0.024	0.026	0.08003
0.501	70	0.16	0.20	0.035	0.036	0.01180
1.003	80	0.16	0.05	0.015	0.017	0.13712
1.003	70	0.16	0.10	0.013	0.014	0.03694
2.003	70	0.16	0.05	0.201	0.216	0.07243
5.005	50	0.16	0.05	2.687	3.039	0.11579
0.005	100	0.25	0.05	0.194	0.191	0.01408
0.501	90	0.25	0.05	0.484	0.495	0.02114
0.501	80	0.25	0.10	0.036	0.037	0.02938
1.003	80	0.25	0.05	0.801	0.822	0.02591
1.003	70	0.25	0.10	0.151	0.155	0.02673
1.003	50	0.25	0.20	0.018	0.019	0.02686
2.003	50	0.25	0.10	0.070	0.076	0.06830
5.005	140	0.25	0.05	107.640	108.810	0.01075
5.005	160	0.25	0.05	127.160	128.700	0.01197
5.005	200	0.25	0.05	167.490	169.880	0.01407

C.2 Puts on equity

C.2.1 Control Variate

With the control variate technique with 200 steps there are 68 parameter combinations with deviations greater than 1%.

T	S	r	σ	Control Variate	FD	Deviation
5.005	100	0.04	0.05	3.418	3.473	0.01583
0.005	100	0.08	0.05	0.166	0.164	0.01379
1.003	110	0.08	0.05	0.010	0.010	0.01139
2.003	110	0.08	0.05	0.014	0.014	0.01452
5.005	90	0.08	0.05	12.515	12.691	0.01387
5.005	100	0.08	0.05	2.962	3.137	0.05575
5.005	110	0.08	0.05	0.013	0.014	0.06811
5.005	90	0.08	0.10	12.695	12.862	0.01298
5.005	110	0.08	0.10	0.848	0.858	0.01133
5.005	140	0.08	0.10	0.012	0.012	0.01380
0.005	100	0.16	0.05	0.146	0.142	0.02767
0.038	100	0.16	0.05	0.247	0.244	0.01123
0.088	100	0.16	0.05	0.297	0.294	0.01102

T	S	r	σ	Control Variate	FD	Deviation
1.003	100	0.16	0.05	2.110	2.165	0.02550
2.003	100	0.16	0.05	2.111	2.181	0.03228
2.003	100	0.16	0.10	2.788	2.817	0.01008
2.003	110	0.16	0.10	0.167	0.170	0.01535
5.005	70	0.16	0.05	31.601	31.957	0.01114
5.005	80	0.16	0.05	21.524	21.880	0.01627
5.005	90	0.16	0.05	11.483	11.839	0.03007
5.005	100	0.16	0.05	1.605	1.961	0.18142
5.005	70	0.16	0.10	31.740	32.071	0.01032
5.005	80	0.16	0.10	21.494	21.844	0.01602
5.005	90	0.16	0.10	11.633	11.989	0.02969
5.005	100	0.16	0.10	2.747	2.932	0.06306
5.005	110	0.16	0.10	0.153	0.164	0.06679
5.005	70	0.16	0.20	31.752	32.078	0.01016
5.005	80	0.16	0.20	21.372	21.707	0.01543
5.005	90	0.16	0.20	12.335	12.557	0.01768
5.005	100	0.16	0.20	5.718	5.797	0.01359
5.005	120	0.16	0.20	1.268	1.284	0.01246
5.005	140	0.16	0.20	0.340	0.345	0.01509
5.005	160	0.16	0.20	0.107	0.108	0.01542
5.005	200	0.16	0.20	0.014	0.014	0.02382
0.005	100	0.25	0.05	0.127	0.122	0.03429
0.038	100	0.25	0.05	0.172	0.166	0.03587
0.088	100	0.25	0.05	0.179	0.173	0.02963
0.501	100	0.25	0.05	1.077	1.098	0.01913
1.003	100	0.25	0.05	0.977	1.065	0.08250
1.003	100	0.25	0.10	1.835	1.879	0.02342
1.003	110	0.25	0.10	0.036	0.037	0.02707
2.003	100	0.25	0.05	0.925	1.064	0.13085
2.003	100	0.25	0.10	1.899	1.965	0.03344
2.003	110	0.25	0.10	0.032	0.035	0.07337
2.003	120	0.25	0.20	0.407	0.412	0.01042
2.003	140	0.25	0.20	0.049	0.049	0.01523
5.005	51	0.25	0.05	49.649	50.212	0.01121
5.005	70	0.25	0.05	29.974	30.541	0.01857
5.005	80	0.25	0.05	20.199	20.749	0.02651
5.005	90	0.25	0.05	10.286	10.835	0.05067
5.005	100	0.25	0.05	0.630	1.197	0.47407
5.005	50	0.25	0.10	50.243	50.775	0.01048
5.005	70	0.25	0.10	29.946	30.496	0.01804
5.005	80	0.25	0.10	20.293	20.842	0.02634
5.005	90	0.25	0.10	10.343	10.893	0.05049
5.005	100	0.25	0.10	1.626	1.951	0.16656
5.005	110	0.25	0.10	0.030	0.038	0.20433
5.005	70	0.25	0.20	29.931	30.467	0.01759
5.005	80	0.25	0.20	20.258	20.805	0.02629
5.005	90	0.25	0.20	10.682	11.130	0.04025
5.005	100	0.25	0.20	4.115	4.249	0.03151
5.005	110	0.25	0.20	1.283	1.324	0.03089
5.005	120	0.25	0.20	0.420	0.436	0.03625
5.005	140	0.25	0.20	0.055	0.058	0.06070
5.005	160	0.25	0.20	0.010	0.011	0.08881
5.005	50	0.25	0.40	50.134	50.641	0.01001

T	S	r	σ	Control Variate	FD	Deviation
5.005	70	0.25	0.40	30.700	31.146	0.01432
5.005	80	0.25	0.40	22.283	22.519	0.01048

C.2.2 Binomial Tree with 1000 Steps

All parameter combinations for which the deviation from the benchmark is greater than 1% are shown here. Their total number is 18.

T	S	r	σ	Binomial (1000)	FD	Deviation
0.005	100	0.08	0.05	0.166	0.164	0.01355
1.003	110	0.08	0.05	0.010	0.010	0.01734
2.003	110	0.08	0.05	0.014	0.014	0.01775
5.005	110	0.08	0.05	0.014	0.014	0.01317
0.005	100	0.16	0.05	0.146	0.142	0.02753
0.038	100	0.16	0.05	0.247	0.244	0.01201
0.088	100	0.16	0.05	0.298	0.294	0.01214
5.005	100	0.16	0.05	1.912	1.961	0.02514
5.005	100	0.16	0.10	2.901	2.932	0.01061
0.005	100	0.25	0.05	0.127	0.122	0.03331
0.038	100	0.25	0.05	0.172	0.166	0.03581
0.088	100	0.25	0.05	0.179	0.173	0.03061
0.501	100	0.25	0.05	1.114	1.098	0.01494
5.005	100	0.25	0.05	1.112	1.197	0.07157
5.005	100	0.25	0.10	1.906	1.951	0.02292
5.005	110	0.25	0.10	0.037	0.038	0.03276
5.005	140	0.25	0.20	0.058	0.058	0.01279
5.005	160	0.25	0.20	0.011	0.011	0.01524

C.3 Calls on FX Rates

C.3.1 Control Variate

With the control variate technique with 200 steps there are 49 parameter combinations with deviations greater than 1%.

T	S	r	q	σ	Control Variate	FD	Deviation
0.005	100	0.01	0.11	0.05	0.156	0.153	0.01665
0.088	100	0.01	0.11	0.05	0.331	0.327	0.01017
5.005	100	0.01	0.11	0.05	0.466	0.490	0.04876
5.005	80	0.01	0.11	0.10	0.015	0.015	0.02898
5.005	90	0.01	0.11	0.10	0.181	0.186	0.02231
0.005	100	0.02	0.12	0.05	0.094	0.092	0.02216
0.088	100	0.02	0.12	0.05	0.335	0.332	0.01128
5.005	100	0.02	0.12	0.05	0.417	0.443	0.06053
5.005	80	0.02	0.12	0.10	0.015	0.015	0.02888
5.005	90	0.02	0.12	0.10	0.186	0.191	0.02613
0.005	100	0.04	0.14	0.05	0.095	0.093	0.02310
0.088	100	0.04	0.14	0.05	0.345	0.341	0.01050
5.005	100	0.04	0.14	0.05	0.483	0.500	0.03277
5.005	80	0.04	0.14	0.10	0.014	0.014	0.02707
5.005	90	0.04	0.14	0.10	0.180	0.185	0.02628
0.005	100	0.08	0.18	0.05	0.096	0.094	0.02510

T	S	r	q	σ	Control Variate	FD	Deviation
0.038	100	0.08	0.18	0.05	0.218	0.216	0.01024
2.003	100	0.08	0.18	0.05	0.432	0.441	0.02236
5.005	100	0.08	0.18	0.05	0.416	0.443	0.06042
5.005	80	0.08	0.18	0.10	0.013	0.013	0.02847
5.005	90	0.08	0.18	0.10	0.167	0.171	0.02654
0.005	100	0.16	0.26	0.05	0.100	0.097	0.02966
0.038	100	0.16	0.26	0.05	0.235	0.233	0.01091
2.003	100	0.16	0.26	0.05	0.403	0.408	0.01154
5.005	100	0.16	0.26	0.05	0.406	0.429	0.05393
5.005	80	0.16	0.26	0.10	0.010	0.011	0.03133
5.005	90	0.16	0.26	0.10	0.147	0.152	0.02943
0.005	100	0.25	0.35	0.05	0.104	0.100	0.03595
0.038	100	0.25	0.35	0.05	0.256	0.253	0.01365
0.088	100	0.25	0.35	0.05	0.334	0.330	0.01201
2.003	100	0.25	0.35	0.05	0.398	0.405	0.01587
5.005	100	0.25	0.35	0.05	0.411	0.436	0.05828
5.005	90	0.25	0.35	0.10	0.130	0.135	0.03436
5.005	70	0.25	0.35	0.20	0.314	0.317	0.01097
0.005	100	0.16	0.16	0.05	0.119	0.117	0.01376
0.005	100	0.25	0.25	0.05	0.123	0.121	0.01883
5.005	70	0.02	0.00	0.05	0.035	0.035	0.01023
2.003	80	0.04	0.00	0.05	0.052	0.052	0.01341
0.501	90	0.08	0.00	0.05	0.037	0.038	0.02464
5.005	50	0.08	0.00	0.05	0.009	0.010	0.13493
0.005	100	0.16	0.06	0.05	0.144	0.142	0.01095
0.501	90	0.16	0.06	0.05	0.091	0.092	0.01477
1.003	80	0.16	0.06	0.05	0.011	0.012	0.05768
2.003	70	0.16	0.06	0.05	0.022	0.023	0.05466
5.005	51	0.16	0.06	0.05	0.099	0.104	0.05088
0.005	100	0.25	0.15	0.05	0.149	0.147	0.01522
0.501	90	0.25	0.15	0.05	0.088	0.089	0.01588
2.003	70	0.25	0.15	0.05	0.022	0.023	0.05170
5.005	50	0.25	0.15	0.05	0.052	0.055	0.06377

C.3.2 Binomial Tree with 1000 Steps

All parameter combinations for which the deviation from the benchmark is greater than 1% are shown here. Their total number is 55.

T	S	r	q	σ	Binomial (1000)	FD	Deviation
0.005	100	0.01	0.11	0.05	0.156	0.153	0.01646
1.003	100	0.01	0.11	0.05	0.469	0.463	0.01203
5.005	100	0.01	0.11	0.05	0.495	0.490	0.01127
0.005	100	0.02	0.12	0.05	0.094	0.092	0.02240
2.003	100	0.02	0.12	0.05	0.423	0.415	0.01755
0.005	100	0.04	0.14	0.05	0.095	0.093	0.02324
5.005	100	0.04	0.14	0.05	0.509	0.500	0.01856
0.005	100	0.08	0.18	0.05	0.096	0.094	0.02493
0.005	100	0.16	0.26	0.05	0.100	0.097	0.02986
0.038	100	0.16	0.26	0.05	0.235	0.233	0.01143
2.003	100	0.16	0.26	0.05	0.413	0.408	0.01157
0.005	100	0.25	0.35	0.05	0.104	0.100	0.03525
0.038	100	0.25	0.35	0.05	0.256	0.253	0.01357

T	S	r	q	σ	Binomial (1000)	FD	Deviation
0.088	100	0.25	0.35	0.05	0.333	0.330	0.01050
0.501	100	0.25	0.35	0.05	0.453	0.448	0.01251
5.005	100	0.25	0.35	0.05	0.443	0.436	0.01577
0.167	50	0.01	0.01	0.60	0.011	0.011	0.01001
0.005	100	0.16	0.16	0.05	0.119	0.117	0.01401
0.005	100	0.25	0.25	0.05	0.123	0.121	0.01891
0.167	50	0.01	0.00	0.60	0.011	0.012	0.01003
5.005	70	0.02	0.00	0.05	0.034	0.035	0.02142
5.005	50	0.02	0.00	0.10	0.017	0.018	0.01146
2.003	80	0.04	0.00	0.05	0.051	0.052	0.02270
5.005	70	0.04	0.00	0.05	0.301	0.306	0.01459
5.005	50	0.04	0.00	0.10	0.064	0.065	0.01130
0.501	90	0.08	0.00	0.05	0.036	0.038	0.03439
2.003	80	0.08	0.00	0.05	0.612	0.618	0.01022
5.005	50	0.08	0.00	0.05	0.008	0.010	0.21508
5.005	50	0.08	0.00	0.10	0.560	0.566	0.01040
0.005	100	0.16	0.06	0.05	0.144	0.142	0.01116
0.167	90	0.16	0.06	0.10	0.021	0.021	0.01040
0.501	90	0.16	0.06	0.05	0.090	0.092	0.02248
0.501	80	0.16	0.06	0.10	0.015	0.015	0.01472
1.003	80	0.16	0.06	0.05	0.011	0.012	0.08738
1.003	70	0.16	0.06	0.10	0.012	0.012	0.02183
2.003	70	0.16	0.06	0.05	0.021	0.023	0.09995
5.005	51	0.16	0.06	0.05	0.093	0.104	0.10793
0.005	100	0.25	0.15	0.05	0.149	0.147	0.01536
0.167	80	0.25	0.15	0.20	0.013	0.014	0.01093
0.501	90	0.25	0.15	0.05	0.087	0.089	0.02387
0.501	80	0.25	0.15	0.10	0.013	0.014	0.01759
1.003	70	0.25	0.15	0.10	0.011	0.012	0.02295
2.003	70	0.25	0.15	0.05	0.021	0.023	0.09367
5.005	50	0.25	0.15	0.05	0.049	0.055	0.12468

C.3.3 Bjerksund and Stensland

Here only parameter sets for which the deviation is $> 10\%$ are shown. Most of the 36 table entries are those where the option price becomes negative.

T	S	r	q	σ	Bjerksund and Stensland	FD	Deviation
1.003	100	0.01	0.11	0.05	0.043	0.463	0.90769
2.003	100	0.01	0.11	0.05	0.150	0.511	0.70688
5.005	100	0.01	0.11	0.05	0.110	0.490	0.77498
5.005	80	0.01	0.11	0.10	-0.698	0.015	47.21344
5.005	90	0.01	0.11	0.10	-8.212	0.186	45.25338
5.005	100	0.01	0.11	0.10	0.061	1.805	0.96631
1.003	100	0.02	0.12	0.05	0.124	0.498	0.75110
2.003	100	0.02	0.12	0.05	-0.089	0.415	1.21473
5.005	100	0.02	0.12	0.05	-0.020	0.443	1.04483
5.005	80	0.02	0.12	0.10	-0.706	0.015	48.17877
5.005	90	0.02	0.12	0.10	-8.640	0.191	46.24323
5.005	100	0.02	0.12	0.10	-0.112	1.732	1.06447
1.003	100	0.04	0.14	0.05	-0.013	0.426	1.03113
2.003	100	0.04	0.14	0.05	-0.190	0.384	1.49443
5.005	100	0.04	0.14	0.05	0.146	0.500	0.70871

T	S	r	q	σ	Bjerksund and Stensland	FD	Deviation
5.005	80	0.04	0.14	0.10	-0.702	0.014	50.34107
5.005	90	0.04	0.14	0.10	-8.778	0.185	48.36765
5.005	100	0.04	0.14	0.10	-0.092	1.724	1.05330
1.003	100	0.08	0.18	0.05	-0.013	0.415	1.03126
2.003	100	0.08	0.18	0.05	-0.017	0.441	1.03909
5.005	100	0.08	0.18	0.05	-0.007	0.443	1.01589
5.005	80	0.08	0.18	0.10	-0.692	0.013	54.66592
5.005	90	0.08	0.18	0.10	-8.875	0.171	52.76134
5.005	100	0.08	0.18	0.10	0.063	1.753	0.96430
1.003	100	0.16	0.26	0.05	-0.012	0.389	1.03151
2.003	100	0.16	0.26	0.05	-0.096	0.408	1.23536
5.005	100	0.16	0.26	0.05	-0.027	0.429	1.06358
5.005	80	0.16	0.26	0.10	-0.685	0.011	64.29237
5.005	90	0.16	0.26	0.10	-9.325	0.152	62.54095
5.005	100	0.16	0.26	0.10	0.030	1.687	0.98224
1.003	100	0.25	0.35	0.05	-0.012	0.380	1.03186
2.003	100	0.25	0.35	0.05	-0.096	0.405	1.23827
5.005	100	0.25	0.35	0.05	0.013	0.436	0.97095
5.005	90	0.25	0.35	0.10	-10.039	0.135	75.34644
5.005	100	0.25	0.35	0.10	-0.066	1.597	1.04135
5.005	50	0.08	0.00	0.05	0.009	0.010	0.13493

C.4 Puts on FX Rates

C.4.1 Control Variate

With the control variate technique with 200 steps there are 33 parameter combinations with deviations greater than 1%.

T	S	r	q	σ	Control Variate	FD	Deviation
1.003	120	0.01	0.11	0.05	0.109	0.111	0.01263
2.003	140	0.01	0.11	0.05	0.072	0.073	0.01966
5.005	199	0.01	0.11	0.05	0.223	0.226	0.01599
1.003	120	0.02	0.12	0.05	0.102	0.104	0.01452
2.003	140	0.02	0.12	0.05	0.076	0.078	0.01847
5.005	200	0.02	0.12	0.05	0.196	0.200	0.01652
1.003	120	0.04	0.14	0.05	0.107	0.109	0.01317
2.003	140	0.04	0.14	0.05	0.071	0.072	0.01959
5.005	199	0.04	0.14	0.05	0.187	0.190	0.01697
1.003	120	0.08	0.18	0.05	0.099	0.100	0.01420
2.003	140	0.08	0.18	0.05	0.061	0.063	0.01885
5.005	200	0.08	0.18	0.05	0.144	0.146	0.01769
1.003	120	0.16	0.26	0.05	0.100	0.102	0.01423
2.003	140	0.16	0.26	0.05	0.058	0.059	0.01981
5.005	200	0.16	0.26	0.05	0.098	0.100	0.01923
0.005	100	0.25	0.35	0.05	0.207	0.204	0.01228
1.003	120	0.25	0.35	0.05	0.081	0.083	0.01581
2.003	140	0.25	0.35	0.05	0.050	0.051	0.02024
5.005	200	0.25	0.35	0.05	0.060	0.062	0.02134
0.005	100	0.25	0.25	0.05	0.175	0.173	0.01314
0.005	100	0.08	0.00	0.05	0.166	0.164	0.01379
5.005	120	0.08	0.00	0.10	0.104	0.105	0.01133

T	*S*	*r*	*q*	σ	Control Variate	FD	Deviation
0.005	100	0.16	0.06	0.05	0.157	0.154	0.01971
5.005	100	0.16	0.06	0.05	0.426	0.451	0.05453
5.005	110	0.16	0.06	0.10	0.245	0.249	0.01689
5.005	120	0.16	0.06	0.10	0.038	0.039	0.03528
0.005	100	0.25	0.15	0.05	0.152	0.149	0.02437
0.038	100	0.25	0.15	0.05	0.273	0.270	0.01187
0.088	100	0.25	0.15	0.05	0.336	0.332	0.01221
5.005	100	0.25	0.15	0.05	0.410	0.430	0.04597
5.005	110	0.25	0.15	0.10	0.214	0.218	0.01851
5.005	120	0.25	0.15	0.10	0.032	0.033	0.03529
5.005	160	0.25	0.15	0.20	0.257	0.259	0.01003

C.4.2 Binomial Tree with 1000 Steps

T	*S*	*r*	*q*	σ	Binomial (1000)	FD	Deviation
0.088	140	0.01	0.11	0.40	0.011	0.012	0.01059
0.501	110	0.01	0.11	0.05	0.179	0.182	0.01493
1.003	120	0.01	0.11	0.05	0.108	0.111	0.02761
1.003	140	0.01	0.11	0.10	0.035	0.035	0.01802
1.003	200	0.01	0.11	0.20	0.011	0.012	0.01506
2.003	140	0.01	0.11	0.05	0.069	0.073	0.05507
2.003	160	0.01	0.11	0.10	0.169	0.171	0.01252
5.005	199	0.01	0.11	0.05	0.209	0.226	0.07803
0.501	110	0.02	0.12	0.05	0.173	0.175	0.01351
1.003	120	0.02	0.12	0.05	0.101	0.104	0.03173
1.003	140	0.02	0.12	0.10	0.033	0.034	0.01577
1.003	200	0.02	0.12	0.20	0.012	0.012	0.01381
2.003	140	0.02	0.12	0.05	0.073	0.078	0.05470
2.003	160	0.02	0.12	0.10	0.167	0.169	0.01246
5.005	200	0.02	0.12	0.05	0.184	0.200	0.08125
0.501	110	0.04	0.14	0.05	0.160	0.163	0.01589
1.003	120	0.04	0.14	0.05	0.106	0.109	0.02800
1.003	140	0.04	0.14	0.10	0.033	0.034	0.01706
1.003	200	0.04	0.14	0.20	0.011	0.011	0.01735
2.003	140	0.04	0.14	0.05	0.068	0.072	0.05485
2.003	160	0.04	0.14	0.10	0.156	0.158	0.01306
5.005	199	0.04	0.14	0.05	0.175	0.190	0.07930
0.501	110	0.08	0.18	0.05	0.167	0.169	0.01473
1.003	120	0.08	0.18	0.05	0.097	0.100	0.03019
1.003	140	0.08	0.18	0.10	0.031	0.031	0.01557
1.003	200	0.08	0.18	0.20	0.011	0.011	0.01486
2.003	140	0.08	0.18	0.05	0.059	0.063	0.05486
2.003	160	0.08	0.18	0.10	0.152	0.154	0.01276
5.005	200	0.08	0.18	0.05	0.134	0.146	0.08317
0.501	110	0.16	0.26	0.05	0.149	0.152	0.01577
1.003	120	0.16	0.26	0.05	0.099	0.102	0.02989
1.003	140	0.16	0.26	0.10	0.029	0.029	0.01634
2.003	140	0.16	0.26	0.05	0.056	0.059	0.05630
2.003	160	0.16	0.26	0.10	0.123	0.124	0.01327
5.005	200	0.16	0.26	0.05	0.092	0.100	0.08333
0.005	100	0.25	0.35	0.05	0.207	0.204	0.01238
0.088	140	0.25	0.35	0.40	0.011	0.011	0.01028

T	S	r	q	σ	Binomial (1000)	FD	Deviation
0.501	110	0.25	0.35	0.05	0.156	0.159	0.01412
1.003	120	0.25	0.35	0.05	0.080	0.083	0.03290
1.003	140	0.25	0.35	0.10	0.026	0.027	0.01647
2.003	140	0.25	0.35	0.05	0.049	0.051	0.05527
2.003	160	0.25	0.35	0.10	0.102	0.104	0.01351
5.005	200	0.25	0.35	0.05	0.056	0.062	0.08724
2.003	120	0.01	0.01	0.05	0.011	0.011	0.01331
0.005	100	0.25	0.25	0.05	0.175	0.173	0.01320
0.005	100	0.08	0.00	0.05	0.166	0.164	0.01355
0.005	100	0.16	0.06	0.05	0.157	0.154	0.01958
5.005	100	0.16	0.06	0.05	0.458	0.451	0.01456
0.005	100	0.25	0.15	0.05	0.152	0.149	0.02383
0.038	100	0.25	0.15	0.05	0.273	0.270	0.01153
0.088	100	0.25	0.15	0.05	0.335	0.332	0.01067
5.005	100	0.25	0.15	0.05	0.437	0.430	0.01593

C.4.3 Bjerksund and Stensland

Here only parameter sets for which the deviation is $> 10\%$ are shown. There are 19 entries, 6 of which the Bjerksund and Stensland method yields a negative result.

T	S	r	q	σ	Bjerksund and Stensland	FD	Deviation
5.005	100	0.04	0.00	0.05	0.914	1.067	0.14392
5.005	110	0.04	0.00	0.05	0.039	0.045	0.12373
2.003	100	0.08	0.00	0.05	0.017	0.572	0.96984
5.005	100	0.08	0.00	0.05	0.007	0.565	0.98755
5.005	100	0.08	0.00	0.10	1.875	2.187	0.14296
5.005	110	0.08	0.00	0.10	0.408	0.472	0.13616
5.005	120	0.08	0.00	0.10	0.092	0.105	0.12684
1.003	100	0.16	0.06	0.05	0.156	0.509	0.69385
2.003	100	0.16	0.06	0.05	0.096	0.486	0.80223
5.005	100	0.16	0.06	0.05	0.027	0.451	0.93954
5.005	100	0.16	0.06	0.10	-0.030	1.733	1.01729
5.005	110	0.16	0.06	0.10	-9.877	0.249	40.70652
5.005	120	0.16	0.06	0.10	-1.980	0.039	51.68875
1.003	100	0.25	0.15	0.05	0.168	0.509	0.67112
2.003	100	0.25	0.15	0.05	0.096	0.480	0.79909
5.005	100	0.25	0.15	0.05	-0.013	0.430	1.02946
5.005	100	0.25	0.15	0.10	0.066	1.706	0.96129
5.005	110	0.25	0.15	0.10	-10.010	0.218	46.97226
5.005	120	0.25	0.15	0.10	-2.046	0.033	62.34101

Appendix D

Basic Functions

In this appendix some basic statistical functions are summarised.

D.1 Cumulative Normal Distribution

$N(x)$ is the cumulative normal distribution:

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2} dy \tag{D.1}$$

The numerical approximation from [14] with an accuracy of 10^{-6} is

$$N(x) = \begin{cases} 1 - N'(x)\left(a_1 k + a_2 k^2 + a_3 k^3 + a_4 k^4 + a_5 k^5\right) & x \geq 0 \\ 1 - N(-x) & x < 0 \end{cases} \tag{D.2}$$

where

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tag{D.3a}$$

and

$$\begin{aligned} k &= \frac{1}{1+\gamma x} \\ \gamma &= 0.2316419 \\ a_1 &= 0.319381530 \\ a_2 &= -0.356563782 \\ a_3 &= 1.781477937 \\ a_4 &= -1.821255978 \\ a_5 &= 1.330274429 \end{aligned} \tag{D.3b}$$

D.2 Cumulative probability in Bivariate Normal Distribution

$M(x,y,\rho)$ is the cumulative bivariate normal distribution with correlation coefficient ρ:

$$M(x,y,\rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{x} \int_{-\infty}^{y} e^{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}} \mathrm{d}x\mathrm{d}y. \tag{D.4}$$

It can be approximated according to [15] by

$$M(a,b,\rho) = \frac{\sqrt{1-\rho^2}}{\pi} \sum_{i,j=1}^{4} A_i A_j f(B_i, B_j) \tag{D.5}$$

where

$$f(x,y) = \exp\left(a'(2x-a') + b'(2y-b') + 2\rho(x-a')(y-b')\right) \tag{D.6a}$$

$$a' = \frac{a}{\sqrt{2(1-\rho^2)}} \tag{D.6b}$$

$$b' = \frac{b}{\sqrt{2(1-\rho^2)}} \tag{D.6c}$$

$$A_1 = 0.3253030$$
$$A_2 = 0.4211071$$
$$A_3 = 0.1334425$$
$$A_4 = 0.006374323$$
$$B_1 = 0.1337764$$
$$B_2 = 0.6243247$$
$$B_3 = 1.3425378$$
$$B_4 = 2.2626645$$

for $a \leq 0$, $b \leq 0$ and $\rho \leq 0$. In other cases the following identities can be used:

$$M(a,b,\rho) = N(a) - M(a,-b,-\rho) \tag{D.6d}$$

$$M(a,b,\rho) = N(b) - M(-a,b,-\rho) \tag{D.6e}$$

$$M(a,b,\rho) = N(a) + N(b) - 1 + M(-a,-b,\rho). \tag{D.6f}$$

If $ab\rho > 0$, the identity

$$M(a,b,\rho) = M(a,0,\rho_1) + M(b,0,\rho_2) - \delta \tag{D.6g}$$

with

$$\rho_1 = \frac{(\rho a - b)\,\mathrm{sgn}(a)}{\sqrt{a^2 - 2\rho ab + b^2}} \tag{D.6h}$$

$$\rho_2 = \frac{(\rho b - a)\,\mathrm{sgn}(b)}{\sqrt{a^2 - 2\rho ab + b^2}} \tag{D.6i}$$

$$\delta = \frac{1 - sgn(a)sgn(b)}{4} \tag{D.6j}$$

can be used.

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