thinknotes

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Applying Constrained Cubic Splines to Yield Curves

- As shown in a recent report, a smooth continuous risk-free yield curve can be fitted to the term structure of the GOVI benchmark bond yield by using a natural cubic spline interpolation method. Though this produces a smooth yield curve, it is unfortunately also prone to overshooting between node points. This decreases the accuracy of the interpolation and limit the curve's usefulness for practical purposes.
- In this note an improvement on the abovementioned method is suggested by using a constrained, as opposed to natural, cubic spline interpolation method. Interpolating a term structure of interest rate with this method produces a yield curve that scarifies some smoothness in return for a more stable shape that does not overshoot.

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Applying Constrained Cubic Splines to Yield Curves

What can go wrong when applying splines to yield curves?

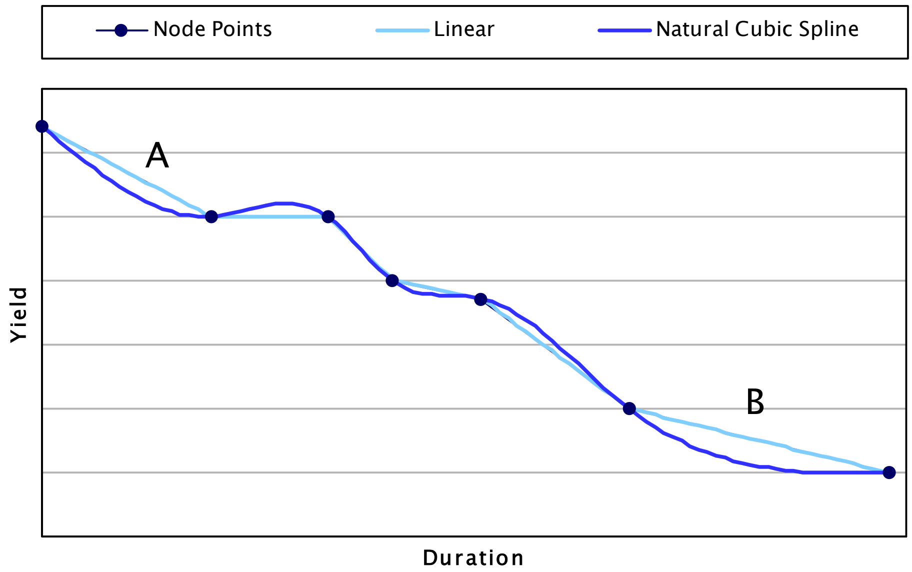
As shown in a recent report, a smooth continuous risk-free yield curve can be fitted to the term structure of the GOVI benchmark bond yields by using a natural cubic spline [1]. This method is preferred to ordinary linear interpolation since discontinuities occur at the node points associated with generating instruments. Therefore the natural cubic spline is a popular method for interpolating yield curves since it produces a very smooth yield curve [2]. However, these curves are also very prone to oscillating or overshooting between certain node points, which is an unfortunate result of the curve's smoothness requirement [3].

In Figure 1, a yield curve is shown, generated by interpolating the hypothetical term structure of interest-bearing instruments using a natural cubic spline. When comparing the curve to a linear interpolation, there are various regions where the curve overshoots to some extent. This phenomena might be acceptable for other applications, but a yield curve overshooting between node points implies that dissimilar yields are obtainable in the region surrounding a node point. For example, consider the region marked A on the graph where the instruments associated with the second and third node points both trade at the same yield. According to arbitrage theory, any similar instrument positioned in this region trading at a yield dissimilar to the ruling yield will quickly either be bid up or down, bringing it in line with the yields associated with the other two node points. Therefore, assuming a proper liquid market, a yield curve like the one represented by the natural cubic spline will supposedly not be observed in an efficient interest-bearing instrument market.

The presence of overshooting as associated with natural cubic splines is very dependant upon the exact distribution of the node points. For example, an evenly spaced set of continuously increasing or decreasing node points has a much lower probability of overshooting or oscillation than a similarly spaced set of random increasing or decreasing node points. The risk in using natural cubic splines to interpolate term structure lies in mispricing instruments using the yield curve, especially where short, mid and long-term yields are volatile relative to one another. For example, restrictive monetary policy might cause the short-term yields of the government yield curve to rise relative to mid and long-term yields. This situation may then lead to the yield curve overshooting in certain regions when the term structure of the government bond yield is interpolated.

In this short technical paper, a solution to the overshooting phenomena is proposed by utilising the constrained cubic spline, another member of the cubic spline family, to interpolate the term structure of yields. The principle behind a constrained cubic spline is that it sacrifices some degree of smoothness to prevent any overshooting. This is done by slightly modifying the mathematical constraints placed upon the natural cubic spline. A

Figure 1. A comparison between a linear and natural cubic spline interpolation of the term structure of hypothetical yields.



short review of the mathematical details underlying this curve will be given, followed by a comparison of the interpolation of a generic yield curve using both natural and constrained cubic splines.

Some theoretical background on the cubic spline family

Cubic spline interpolation is the most common piecewise polynomial method and is referred to as "piecewise" since a unique polynomial is fitted between each pair of data points. In turn, each of these polynomials are linked to adjacent polynomials by using a set of constraints. This collection of polynomials that form the curve is collectively referred to as "the spline".

Depending on the type of mathematical constraints placed upon the cubic spline, different members in the cubic spline family can be created [4]. For example, the traditional and constrained cubic splines are two different groups of the same family. Many more groups in this family can be identified by appropriately adjusting the constraints. The group of traditional cubic splines can furthermore be divided into subgroups, e.g. natural, parabolic run-out, cubic run-out and clamped cubic splines. The natural cubic spline is by far the most popular and widely-used version of the cubic spline family.

The essential theory underlying any spline is to fit a piecewise function of the following form:

$$S(x)=\begin{cases} s_1(x)\,;\; x_1 \le x < x_2 \\ s_2(x)\,;\; x_2 \le x < x_3 \\ \vdots \\ s_{n-1}(x)\,;\; x_{n-1} \le x < x_n \end{cases} \qquad (1)$$

on the interval $[x_1,x_n]$. In the case of cubic spline interpolation, $s_i(x)$ is a third order or cubic polynomial defined as follows:

$$s_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i \quad (2)$$

The first and second order derivatives of these polynomials are fundamentally important in this process and are given by:

$$s_i'(x) = 3a_i(x - x_i)^2 + 2b_i(x - x_i) + c_i \quad (3)$$

and

$$s_i''(x) = 6a_i(x - x_i) + 2b_i \quad (4)$$

for $i=1,2,\ldots,n-1$.

Based on Equations (1) to (4), four mathematical constraints that characterise the constrained cubic spline can be defined [3] as follows[1]:

1. The piecewise function $S(x_i)$ will interpolate all data points on the interval $[x_1,x_n]$ and therefore:

   $$S(x_i) = y_i \quad (5)$$

   for $i=1,2,\ldots,n$.

2. Since the curve $S(x)$ must be continuous on the interval $[x_1,x_n]$ each piecewise polynomial should join at shared node points:

   $$s_{i-1}(x_i) = s_i(x_i) \quad (6)$$

   for $i=2,\ldots,n-1$.

3. To ensure that the curve has smoothness on the interval $[x_1,x_n]$ the end and start slope of every pair of adjacent piecewise polynomials should be continuous at the shared node points. Furthermore, the slope of the curve at these node points

[1] The constraints listed originally by [3] were adapted slightly. Please refer to the original text for more information on the original constraints.

should reflect a weighted average of the slopes of the two adjacent straight lines. Instead of calculating the curve's slope by equal-weighting the adjacent slopes, a greater weighting is given to the second slope. This is done since, for this type of interpolation, the curve's overall shape is determined more by the latter. Also, the slope of the curve should be zero when one of the adjacent straight line slopes are zero or if one the slopes change in sign. The above requirements can be represented in the following expression:

$$s_i'(x_i) = s_{i-1}'(x_i) = \begin{cases} \dfrac{\frac{1}{3} m_{i-1,i} + \frac{2}{3} m_{i,i+1}}{2} & \text{if } m_{i-1,i} m_{i,i+1} > 0 \\ 0 & \text{if } m_{i-1,i} m_{i,i+1} \le 0 \end{cases} \tag{7}$$

where the slope between a pair of node points is defined as:

$$m_{i,i+1} = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} \tag{8}$$

for $i=2,\ldots,n-1$.

4. The slope at both endpoints of the curve are set to the slope of the adjacent straight lines:

$$s_1'(x_1) = \frac{y_2 - y_1}{x_2 - x_1} \tag{9}$$

and

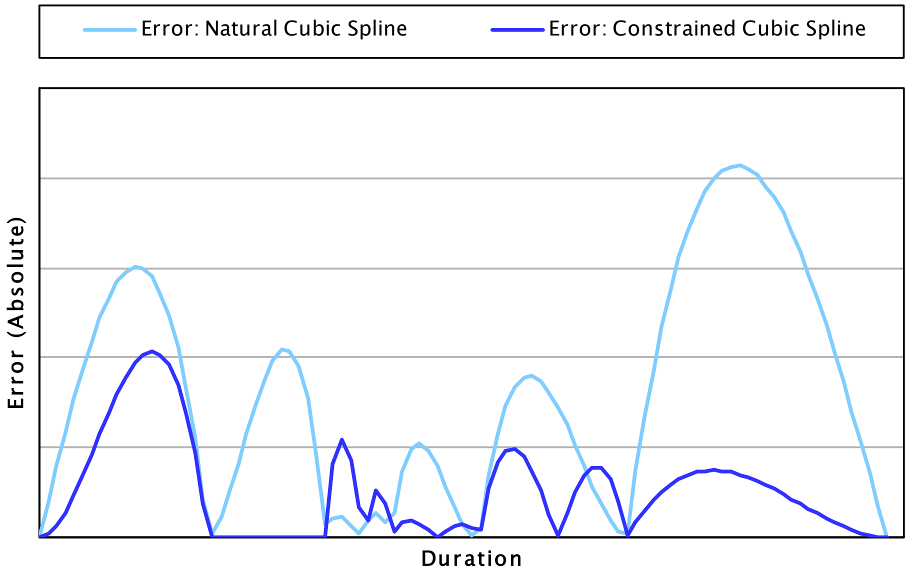
$$s'(x_n) = \frac{y_n - y_{n-1}}{x_n - x_{n-1}} \tag{10}$$

Using the definitions listed in Equations (5) to (10) above, the $4n-4$ unknown coefficients as defined in Equation (2) can be determined using an appropriate method. For example, to fit a constrained cubic spline through a set of three random node points, the system of equations to be solved will look as follows:

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ (x_2-x_1)^3 & (x_2-x_1)^2 & x_2-x_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (x_3-x_2)^3 & (x_3-x_2)^2 & x_3-x_2 & 1 \\ 3(x_2-x_1)^2 & 2(x_2-x_1) & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3(x_3-x_2)^2 & 2(x_3-x_2) & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \\ a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ s'(x_1) \\ s_2'(x_2) \\ y_2 \\ y_3 \\ s_2'(x_2) \\ s'(x_n) \end{bmatrix} \tag{11}$$

The above matrix equation is the familiar **Ax=b** format and can easily be solved using a suitable technique, like LU decomposition for example.

Figure 2. A comparison between the natural and constrained cubic spline interpolation error as compared to a linear interpolation.



Comparing the different cubic spline interpolations

In Figure 2 the hypothetical yield curve previously shown in Figure 1 is compared to the constrained cubic spline interpolation method. From the graph it can be seen that the constrained curve has noticeably less smoothness than the natural curve, but succeeds in its goal of not overshooting in any region between node points. Referring to the region marked A, notice that the constrained curve interpolates the region linearly instead of cubically like the natural curve.

The degree to which both cubic spline interpolation methods compare to ordinary linear interpolation can be quantified by taking the absolute difference between the two yields calculated using the cubic spline and linear interpolation methods. The interpolation errors for both the natural and constrained curves are compared in Figure 2. If the linear interpolated curve is taken to be the favoured "discontinuous" yield curve, the constrained cubic spline deviates considerably less from it compared to the natural cubic spline.

Finishing off...

Given the argument earlier regarding arbitrage theory and the shape of the yield curve between two equal yield node points, linear interpolation may actually be the best method to interpolate a given term structure of interest rates. The motivation behind moving from linear to cubic spline interpolation was to address the discontinuities in the linear interpolation and create a smooth curve. This in turn results in a curve that appears to be too smooth. Therefore the constrained cubic spline is a very suitable alternative to use, since it generates a sufficiently smooth yield curve that also adheres to the underlying principles of arbitrage theory.

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