Pricing American Call Options

with

Dividend and Stochastic Interest Rates

Shu-Ing Liu *

Yu-Chung Liu

Professor Dr. Shu-Ing Liu
Department of Finance,
Shih Hsin University
Taipei, Taiwan
Phone: 886-2-2236-8225-63433
Fax: 886-2-2378-4805
Email: siliu@cc.shu.edu.tw

Master Yu-Chung Liu

*Corresponder: Shu-Ing Liu

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0. Introduction

Option pricing models were first presented by Black and Scholes (1973), and Merton (1973). It is well-known that an American call option on a stock that pays no dividends during the life of the option will not be exercised early, and hence can be evaluated as a European stock option with the standard Black-Scholes formula. If the underlying stock does pay a dividend during the life of the option, however, early exercise could possibly be an optimal choice and further investigation becomes interesting. Moreover, the pricing of American options becomes more complicated as the randomness of the interest rates becomes further involved. Therefore, most pricing models for such American options are either constructed by numerical methods or by approximation solutions.

American options were analytically evaluated as early 1977 by Roll (1977) who constructed a replicating portfolio, and presented a pricing model for American stock call options with given dividends; later, Geske (1979) modified the Roll (1977) model to evaluate compound options for an underlying stock paying a single dividend during the life time of the option. This model is called the Roll-Geske-Whaley model, referring to Roll (1977), Geske (1979, 1981), and Whaley (1981).

The randomness of the influence of interest rates on option pricing, on the other hand, has been discussed by Ho and Lee (1986). Assuming the log-normal distribution property of the stock price process and combining this with an appropriate discount factor process, Wilhelm (2001) derived a closed-form solution to the modeling of European call stock options under stochastic interest rates.

The purpose of this article is to developed closed form pricing formula for American stock call options with one given dividend, subject to the Ho-Lee stochastic interest rates model. The correlation between the underlying stock price and the discount factor is explicitly expressed. And numerical analyses illustrate that the correlation between the underlying stock price and the discount factor imposes a discernible influence on both the dynamics of the option price and the delta hedge ratio. Furthermore, the impacts of the dynamic for distinct initial stock prices, are inspected as well. These offer profitable information that can be applied in the real financial market.

The remainder of this article is organized as follows. In the next section the valuation model is established, and the conditions and closed-form formulae for finding the option pricing and delta hedge ratio for the discussed American stock call option are derived. Numerical illustrations for comparative analyses, for both pricing and delta hedging, are then conducted. In the final section some conclusion are discussed. Parts of the detailed proof of the closed-form pricing formula are relegated to the Appendix.

1. Review of Some Related Models

In this article, an analytical pricing formula for an American call option with one known dividend D to be disposed of at time t, before the maturity date T, under the discussed stochastic interest rate considerations is derived. Before introducing the two above mentioned models separately, some notations employed throughout this article are stated as follows:

S: the underlying stock price
K: the constant strike price
D: the fixed amount of the cash dividend
t: time to the ex-dividend date
$c_u(S_u,K,v)$: the European stock call option price with constant interest rate at time u, with underlying stock price S_u, strike price K, and time to maturity v
$C_u(S_u,B_{u,v},K,v)$: the European stock call option price with stochastic interest rate at time u, with underlying stock price S_u, strike price K, time to maturity v, and $B_{u,v}$ being present price of a zero-coupon bond at time u, with time expiration v
$\varphi_1(\cdot)$: the density function of a uni-variate standard normal random variable
$\Phi_1(\cdot)$: the cumulative distribution function of a uni-variate standard normal random variable
$N_1(\mu,\sigma^2)$: the cumulative distribution function of a uni-variate normal random variable, with mean μ and variance σ^2
$\Phi_2(\cdot,\cdot;\rho)$: the cumulative distribution function of a bi-variate standard normal random vector with correlation coefficient ρ
$N_2(\mu_1,\mu_2.\sigma_1^2,\sigma_2^2;\rho)$: the cumulative distribution function of a bi-variate normal random vector, with means μ_i, variances σ_i^2, and correlation coefficient ρ

1.1 Stochastic Interest Rate model

Wilhelm (2001) considered option prices with stochastic interest rates. They unified Black and Scholes (1973) and the term structure model by Ho and Lee (1986). The randomness in the model is based on a well-defined probability space (Ω, A, μ) and a suitable filtration, $\{A_u\}$. An economy where all the securities evolve is assumed to be governed by a stochastic discount factor, so a positive stochastic process, denoted by $\{Q_v\}$, is adapted to the given filtration, $\{A_u\}$. For any security which pays the A_u for the measurable random amount p_u at time u, the price at time $\tau < u$, is determined by

$$p_\tau = E\left(\frac{Q_u}{Q_\tau} \cdot p_u \mid A_\tau\right).$$

Suppose that the discussed security does not pay out any cash during the period (τ, u). For simplicity, the dynamic behavior of the stochastic discount factor can be specified as

$$Q_u = \exp\left(-m_u - \sigma_Q Z_{Q_u}\right),$$

where $\{Z_{Q_u}\}$ represents Brownian motion with mean zero and variance u. Then $E(Q_u) = \exp\left(-m_u + \frac{1}{2}\sigma_Q^2 u\right)$. It is obvious that if $Q_0 = 1$, then $m_0 = 0$. Moreover, assume that the initial term structure of the interest rates $r_{0,u}$, and the stochastic discount factor to zero-bond-prices $B_{0,u}$ are related by

$$E(Q_u) = B_{0,u} = \exp(-r_{0,u} u).$$

Then, after substitution, the following result is obtained:

$$\frac{Q_u}{Q_0} = B_{0,u} \exp\left(-\frac{1}{2}\sigma_Q^2 u - \sigma_Q Z_{Q_u}\right).$$

Let $W_u = \frac{Q_u}{Q_0}$, and $\mu_{W_u} = \ln B_{0,u} - \frac{1}{2}\sigma_Q^2 u$, hence W_u follows the log-normal distribution, say

$$\ln W_u \sim N_1\left(\mu_{W_u}, \sigma_Q^2 u\right). \tag{1}$$

Since there is a fixed cash dividend of D dollars at time t, the underlying stock price prior to t will involve the effect of paying the dividend. To evaluate the option at time 0, the associated stock price at time 0 must be adjusted so that the dynamics of the stock price throughout its life time can be modeled. Let the stock price containing dividends be denoted by S, and the stock price without dividends be S'. Since the value of S_0' is equal to S_0 then after subtracting the present value of the dividend, that is, $S_0' = S_0 - E(W_t D)$, it follows that $S_0' = S_0 - DB_{0,t}$.

In this article, the dynamics of Q and S at the expiration date T, are expressed by

$$\begin{cases} \dfrac{Q_T}{Q_0} = B_{0,T}\exp\left(-\dfrac{1}{2}\sigma_Q^2 T - \sigma_Q Z_{Q_T}\right), \\ \dfrac{S_T}{S_0'} = \dfrac{S_T}{S_0 - DB_{0,t}} = \dfrac{1}{B_{0,T}}\exp\left\{\dfrac{1}{2}\left(2\rho\sigma_Q\sigma_S - \sigma_S^2\right)T + \sigma_S Z_{S_T}\right\}, \end{cases}$$

where $\left(Z_{Q_T}, Z_{S_T}\right) \overset{d}{\sim} N_2\left(0,0,T,T;\rho\right)$.

Adopting the aforementioned notation, define $W_T = \dfrac{Q_T}{Q_0}$, $\mu_{W_T} = \ln B_{0,T} - \dfrac{1}{2}\sigma_Q^2 T$, and $\mu_{S_T} = \ln\left(S_0 - DB_{0,t}\right) - \ln B_{0,T} + \rho\sigma_Q\sigma_S T - \dfrac{1}{2}\sigma_S^2 T$. It follows that

$$\left(\ln S_T, \ln W_T\right) \overset{d}{\sim} N_2\left(\mu_{S_T}, \mu_{W_T}, \sigma_S^2 T, \sigma_Q^2 T; \rho\right). \qquad (2)$$

Therefore, the joint relation between S and Q, after the setting of stochastic interest rates, can be established by expression (2). Here, ρ is the correlation coefficient between the two processes, $\{\ln S_T\}$ and $\{\ln W_T\}$.

1.2 Replicating portfolio for the American stock call option with one known dividend

Following the Roll-Geske-Whaley development, but with a constant interest rate, usually an American stock call option with dividend disposed at time t, can be replicated by the following combination of European stock call options:

(a) A long position in a European stock call option with time to maturity T and exercise price K, where the underlying asset is the same stock as the replicated American call targets.

(b) A short position in a European compound stock call option with time to maturity t and exercise price $\widetilde{S}_t + D - K$, where the underlying asset is the same as the European stock call option defined in portfolio (a). Here $\widetilde{S}_t$ is the unique solution to the equation, $c_t\left(\widetilde{S}_t, K, T-t\right) = \widetilde{S}_t + D - K$.

(c) A long position in a European stock call option with time to maturity t and exercise price $\widetilde{S}_t$, and again the underlying asset is the same stock as the replicated American call targets.

It is easy to determine that the synthetic portfolio of the stated European options (a), (b) and (c) is a replicating portfolio when used for the American stock call option, with known dividends and constant interest rate. And the closed form pricing formula for each investment can be explicitly obtained.

2. Proposed Model

2.1 Replicating portfolio for the discussed American stock call option

For simplicity, the current time is assumed to be zero. For the underlying stock, with maturity date T, a fixed cash dividend of D dollars is paid at time t, where $t \in (0, T)$. That is, there is only one known dividend paid; the amount of the dividend is D, and the time scale in years until the ex-dividend date is t. Moreover, in this article, the interest rate is supposed to be stochastic and governed by a suitable discount factor process, instead of a constant interest rate. Some modifications of the portfolios mentioned in subsection 1.1 are made in order to provide a synthetic portfolio with a combination of the following stated options:

(A). A long position in a European stock call option with time to maturity T and exercise price K, where the underlying asset is the same stock as the replicated American call option.

(B). A short position in a European compound stock call option with time to maturity t and exercise price $S_t^* + D - K$, where the underlying asset is the same as the European call option defined in portfolio (A), and S_t^* is the "unique" solution to the equation, $C_t(S_t^*, B_{t,T}, K, T-t) = S_t^* + D - K$.

(C). A long position in a European stock call option with time to maturity t and exercise price S_t^*. Again the underlying asset is the same stock as the replicated American call option.

2.2 Some analytical results

In this subsection, the equivalence of the aforementioned synthetic portfolio, combining (A), (B) and (C), and the discussed American stock call option, will be proved analytically under some conditions. First, at time zero, prices for each of the European options, (A), (B) and (C), denoted by $C_0^{(A)}$, $C_0^{(B)}$ and $C_0^{(C)}$, respectively, are obtained as follows:

Theorem 1:

$$C_0^{(A)} = (S_0 - DB_{0,t})\exp(2\rho\sigma_Q\sigma_S T)\Phi_1(a) - KB_{0,T}\Phi_1(c) \tag{3}$$

$$C_0^{(B)} = (S_0 - DB_{0,t})\exp(2\rho\sigma_Q\sigma_S T)\Phi_2(a,b;\nu) - KB_{0,T}\Phi_2(c,d;\upsilon) - (S_t^* + D - K)B_{0,t}\Phi_1(d) \tag{4}$$

$$C_0^{(C)} = (S_0 - DB_{0,t})\exp(2\rho\sigma_Q\sigma_S t)\Phi_1(b) - S_t^* B_{0,t}\Phi_1(d) \tag{5}$$

where $a = \dfrac{\ln\left(\dfrac{S_0 - DB_{0,t}}{KB_{0,T}}\right) + \left(2\rho\sigma_Q\sigma_S + \dfrac{1}{2}\sigma_S^2\right)T}{\sigma_S\sqrt{T}}$, $b = \dfrac{\ln\left(\dfrac{S_0 - DB_{0,t}}{S_t^* B_{0,t}}\right) + \left(2\rho\sigma_Q\sigma_S + \dfrac{1}{2}\sigma_S^2\right)t}{\sigma_S\sqrt{t}}$,

$c = a - \sigma_S\sqrt{T}$, $d = b - \sigma_S\sqrt{t}$, and $\nu = \sqrt{t/T}$, a fixed number.

Proof. Cf. Appendix 1.

Corollary 1: At time t, the ex-dividend date, the price of the European option (A) is

$C_t^{(A)} = C_t(S_t, B_{t,T}, K, T-t) = \exp[2\rho\sigma_Q\sigma_S(T-t)]S_t\Phi_1(d_1) - KB_{t,T}\Phi_1(d_2)$, where

$$d_1 = \frac{1}{\sigma_S\sqrt{T-t}}\left\{\ln\left(\frac{S_t}{KB_{t,T}}\right) + \left(2\rho\sigma_Q\sigma_S + \frac{1}{2}\sigma_S^2\right)(T-t)\right\}, \text{ and } d_2 = d_1 - \sigma_S\sqrt{T-t}.$$

Proof. Adopted from Theorem 1 immediately.

To prove that the synthetic portfolio of the defined European stock call options (A), (B) and (C), is a replicating portfolio for the discussed American stock call option, it is sufficient to show that $f(S_t)$ is an increasing function of S_t, where $f(S_t) = (S_t - X + D) - C_t(S_t, B_{t,T}, K, T-t)$.

Since $\dfrac{\partial\Phi_1(d_1)}{\partial S_t} = \varphi_1(d_1)\dfrac{\partial d_1}{\partial S_1}$, $\dfrac{\partial d_2}{\partial S_t} = \dfrac{\partial d_1}{\partial S_t} = \dfrac{1}{S_t\sigma_S\sqrt{T-t}}$,

$\varphi_1(d_2) = \varphi_1(d_1)\exp\left(d_1\sigma_S\sqrt{T-t} - \dfrac{1}{2}\sigma_S^2\sqrt{T-t}\right)$, $KB_{t,T}\varphi_1(d_2) = S_t\varphi_1(d_1)\exp[2\rho\sigma_Q\sigma_S(T-t)]$,

after algebra, the following partial derivative is obtained,

$$\frac{\partial f(S_t)}{\partial S_t} = 1 - \exp\{2\rho\sigma_Q\sigma_S(T-t)\}\Phi_1(d_1).$$

If we define $g(\rho) = 1 - \exp\{2\rho\sigma_Q\sigma_S(T-t)\}\Phi_1(d_1)$, then

$$\frac{\partial g(\rho)}{\partial\rho} = -(2\sigma_Q\sigma_S(T-t))\exp\{2\rho\sigma_Q\sigma_S(T-t)\}\Phi_1(d_1) - \exp\{2\rho\sigma_Q\sigma_S(T-t)\}\cdot\varphi_1(d_1)\frac{\partial d_1}{\partial\rho}$$

$$= -2\sigma_Q\sqrt{(T-t)}\exp\{2\rho\sigma_Q\sigma_S(T-t)\}\cdot\left\{\sigma_S\sqrt{(T-t)}\Phi_1(d_1) + \varphi_1(d_1)\right\}.$$

It is obvious that $\dfrac{\partial g(\rho)}{\partial\rho} < 0$, and since $g(0) \geq 0$, therefore $\dfrac{\partial f(S_t)}{\partial S_t} \geq 0$, when $\rho \leq 0$. That is, $f(S_t)$ is an increasing function of S_t, as $\rho \leq 0$. When $\rho > 0$, the sign of $\dfrac{\partial f(S_t)}{\partial S_t}$ is uncertain. However, since $\dfrac{\partial f(S_t)}{\partial S_t}$ is a continuous function of ρ, thus there exists $\rho_* > 0$, which is function of σ_S, σ_Q, $S_t / KB_{t,T}$ and $T-t$, such that $\frac{\partial f(S_t)}{\partial S_t}|_{\rho=\rho_*} = 0$. Let ρ_* satisfy the equation $\exp\{2\rho_* \sigma_Q \sigma_S (T-t)\}\Phi_1(d_1(\rho_*)) = 1$, then $\rho_* > 0$. The results can be summarized as follows:

Corollary 2: If we set ρ_* to satisfy the equation $\exp\{2\rho_* \sigma_Q \sigma_S (T-t)\}\Phi_1(d_1(\rho_*)) = 1$, then $f(S_t)$ is an increasing function of S_t, when the value of ρ satisfies the inequality $\rho < \rho_*$.

For simplicity, we suppose that $\rho < \rho_*$; therefore $f(S_t)$ is an increasing function of S_t in the following discussion. Furthermore let S_t^* be the unique solution satisfying $f(S_t^*) = 0$, that is, $S_t^* - X + D = C_t(S_t^*, B_{t,T}, K, T-t)$. The discussed American stock call option shall be exercised if $S_t - X + D > C_t(S_t, B_{t,T}, K, T)$, that is $S_t > S_t^*$; otherwise it will be held to maturity, if $S_t - X + D \le C_t(S_t, B_{t,T}, K, T)$, that is $S_t \le S_t^*$.

To prove that the synthetic portfolio of the defined European stock call options (A), (B) and (C) is a replicating portfolio for the discussed American stock call option, each payoff function can be reduced to two exclusive cases expressed by cash flows.

(I). Direct Cash flows of the discussed American stock call option

At time t if $S_t > S_t^*$, then $f(S_t) > 0$, and the option will be exercised immediately. The value of the discussed American stock call option is thus $S_t - X + D$. However, if $S_t \le S_t^*$, then $f(S_t) \le 0$, and consequently, the option will be held to maturity, with the value $C_t(S_t, B_{t,T}, K, T-t)$.

(II). Cash flows of the discussed replicating portfolio

At time t, if $S_t > S_t^*$, then $C_t(S_t, B_{t,T}, K, T-t) > C_t(S_t^*, B_{t,T}, K, T-t) = S_t^* - X + D$. Therefore, the cash flow of the European compound option (B) is

$$\max\left\{C_t(S_t, B_{t,T}, K, T-t) - \left(S_t^* - X + D\right), 0\right\} = C_t(S_t, B_{t,T}, K, T-t) - \left(S_t^* - X + D\right).$$

Similarly, since $\max\left\{S_t - S_t^*, 0\right\} > 0$, the cash flow of the European stock call option (C) at time t is $S_t - S_t^*$. In summary, the total cash flows of the replicating portfolio are

$$C_t(S_t, B_{t,T}, K, T-t) - \left\{C_t(S_t, B_{t,T}, K, T-t) - \left(S_t^* - X + D\right)\right\} + S_t - S_t^* = S_t - D + X .$$

This value is exactly the same as that computed by directly analyzing the cash flows for the discussed American stock call option.

On the other hand, if $S_t \leq S_t^*$, then

$$C_t(S_t, B_{t,T}, K, T-t) \leq C_t(S_t^*, B_{t,T}, K, T-t) = S_t^* - X + D .$$

Therefore, both options in (B) and (C) are zero and the total cash flows of the replicating portfolio are $C_t(S_t, B_{t,T}, K, T-t)$. Again, this value is exactly the same as that computed by directly analyzing the cash. The stated cash flows at time t are summarized in Table 1.

Therefore, when the condition that $\rho < \rho_*$ holds, irregardless of whether $S_t > S_t^*$ or $S_t \leq S_t^*$, the value of the replicating portfolio is always equal to that of the discussed American stock call option. Therefore the synthetic portfolio defining by European stock call options (A), (B) and (C) can be regarded as a replicating portfolio for the discussed American stock call option. To evaluate the price of the discussed American stock call option, it is sufficient to evaluate the discussed European stock call options (A), (B), (C), respectively. The results are summarized as follows:

Theorem 2: Suppose that $\rho < \rho_*$, then the synthetic portfolio combining by, the European options (A), (B) and (C), is a replicating portfolio for the discussed American stock call option, with price C_0, which is $C_0 = C_0^{(A)} - C_0^{(B)} + C_0^{(C)}$.

Theoretically, the existence of a replicating portfolio for the discussed American stock call option is proved. Unfortunately, the value of ρ_* depends upon S_t, the stock price at time t, which is unknown when the current time is at time zero. Nevertheless, Theorem 2 is definitely applicable when $\rho \leq 0$. Usually, in most situations, the value of the correlation coefficient ρ will be negative, thus the price of the discussed American stock call option is $C_0 = C_0^{(A)} - C_0^{(B)} + C_0^{(C)}$. The result can be stated as follows:

Corollary 3: When $\rho \leq 0$, the correlation between the stock price process and the discount factor process, is negatively correlated. The time zero price of the discussed American stock call option, with a fixed cash dividend of D dollars paid at time t, before the maturity date T, under the discussed stochastic discount factor process, is then given by

$$C_0 = \left(S_0 - DB_{0,t}\right)\left[\exp\left(2\rho\sigma_S\sigma_Q t\right)\Phi_1(b) + \exp\left(2\rho\sigma_S\sigma_Q T\right)\Phi_2(a,-b;-\upsilon)\right]$$

$$- KB_{0,T}\Phi_2(c,-d;-\upsilon) - (K-D)B_{0,t}\Phi_1(d) \quad . \tag{6}$$

Proof. Adopted from Theorem 2 immediately, where a, b, c and d are defined in Theorem 1.

When interest rate is uncertain, hedging becomes rather complicated and cannot done on stock price sensitivity basis only. However, a closed formula for the latter is easily to obtain. For simplicity, in this article only hedging on stock price is considered. Differentiating C_0 with respect to S_0 results in the delta hedge ratio, expressed by the following analytical formula:

Theorem 4. (Delta hedge ratio) When $\rho \le 0$, the time zero delta hedge ratio for the discussed American stock call option is given by

$$\Delta_S = \frac{\partial C_0}{\partial S_0} = \exp\left(2\rho\sigma_Q\sigma_S t\right)\Phi_1(b) + \exp\left(2\rho\sigma_Q\sigma_S T\right)\Phi_2(a,-b;-\upsilon)$$

$$+\left(S_0 - DB_{0,t}\right)\left[\exp\left(2\rho\sigma_Q\sigma_S t\right)\frac{\varphi_1(b)}{S_0\sigma_S\sqrt{t}} + \exp\left(2\rho\sigma_Q\sigma_S T\right)\left\{\frac{h(a,-b)}{S_0\sigma_S\sqrt{T}} - \frac{h(-b,a)}{S_0\sigma_S\sqrt{t}}\right\}\right]$$

$$- KB_{0,T}\left\{\frac{h(c,-d)}{S_0\sigma_S\sqrt{T}} - \frac{h(-d,c)}{S_0\sigma_S\sqrt{t}}\right\} - (K-D)B_{0,t}\frac{\varphi_1(d)}{S_0\sigma_S\sqrt{t}}, \tag{7}$$

where $\varphi_1(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)$, $\Phi_1(x) = \int_{-\infty}^{x}\varphi_1(y)dy$, and $h(x,y) = \Phi_1\left(\frac{\nu x + y}{\sqrt{1-\nu^2}}\right)\varphi_1(x)$.

The most significant difference between the Roll-Geske-Whaley model and our proposed model lies in the assumption of random interest rates. The randomness of the interest rates is carried out by a discount factor process motivated by Wilhelm (2001). In addition the correlation between the discount factor and the underlying stock price is established via the correlation coefficient ρ. Without the stochastic interest rate assumption, we may set $\sigma_Q = 0$, (or $\rho = 0$), and the price of the zero-coupon bond to be $B_{0,u} = e^{-ru}$. Then $S_t^* = \widetilde{S}_t$ and the result of Corollary 3 will be simplified. In fact, the reduced formula is exactly equal to that of the Roll-Geske-Whaley model, as follows:

Proposition 1. The time zero price of the American stock call option in the Roll-Geske-Whaley model is given by

$$\widetilde{C}_0 = \left(S_0 - De^{-rt}\right)\cdot\left[\Phi_1\left(\widetilde{b}\right) + \Phi_2\left(\widetilde{a}, -\widetilde{b}; -\upsilon\right)\right] - Ke^{-rT}\Phi_2\left(\widetilde{c}, -\widetilde{d}; -\upsilon\right) - (K - D)e^{-rt}\Phi_1\left(\widetilde{d}\right), \quad (8)$$

where $\widetilde{a} = \dfrac{\ln\left(\dfrac{S_0 - De^{-rt}}{Ke^{-rT}}\right) + \dfrac{1}{2}\sigma_S^2 T}{\sigma_S\sqrt{T}}$, $\widetilde{b} = \dfrac{\ln\left(\dfrac{S_0 - De^{-rt}}{\widetilde{S}_t e^{-rt}}\right) + \dfrac{1}{2}\sigma_S^2 t}{\sigma_S\sqrt{t}}$, $\widetilde{c} = \widetilde{a} - \sigma_S\sqrt{T}$, $\widetilde{d} = \widetilde{b} - \sigma_S\sqrt{t}$, and $\widetilde{S}_t$ satisfying $\widetilde{S}_t - X + D = c_t(\widetilde{S}_t, K, T - t)$.

Proposition 2. The time zero delta hedge ratio for the American stock call option in the Roll-Geske-Whaley model is given by

$$\widetilde{\Delta}_S = \frac{\partial \widetilde{C}_0}{\partial S_0} = \Phi_1\left(\widetilde{b}\right) + \Phi_2\left(\widetilde{a}, -\widetilde{b}; -\upsilon\right) + \left(S_0 - De^{-rt}\right)\left[\frac{\varphi_1\left(\widetilde{b}\right)}{S_0\sigma_S\sqrt{t}} + \left\{\frac{h\left(\widetilde{a}, -\widetilde{b}\right)}{S_0\sigma_S\sqrt{T}} - \frac{h\left(-\widetilde{b}, \widetilde{a}\right)}{S_0\sigma_S\sqrt{t}}\right\}\right]$$

$$- Ke^{-rT}\left\{\frac{h\left(\widetilde{c}, -\widetilde{d}\right)}{S_0\sigma_S\sqrt{T}} - \frac{h\left(-\widetilde{d}, \widetilde{c}\right)}{S_0\sigma_S\sqrt{t}}\right\} - (K - D)e^{-rt}\frac{\varphi_1\left(\widetilde{d}\right)}{S_0\sigma_S\sqrt{t}}. \quad (9)$$

3. Numerical Illustrations

Static analyses of the closed-form solutions for both option pricing and delta hedge ratio are offered in this section with the goal of offering detailed insight into their sensitivity to the some various parameter settings. A set of parameters, called the *base case*, is: $D = 1.5$, $K = 25$, $t = 0.25$, $r = 0.15$, $B_{0,u} = \exp(-ru)$, $T = 0.5$, $\sigma_S = 0.5$, and $\sigma_Q = 0.01$. Three distinct initial stock values, $S_0 = \{30, 25, 20\}$, are set for call options whose statue stays in-the-money, at-the-money, and out-of-the-money, respectively.

It is worthy to note that the sufficient condition, $\rho < \rho_*$, stated in Theorem 2, is useful to theoretically verify that $f(S_t)$ is an increasing function of S_t, and furthermore to guarantee that the synthetic portfolio of the defined options (A), (B) and (C), is a replicating portfolio for the discussed American stock call option. Actually, $\dfrac{\partial f(S_t)}{\partial S_t}$ is a decreasing function of either S_t/K or ρ. For numerical demonstrations, instead of simply computing values of ρ_* in terms of S_t/K, under the discussed *base case*, the values of $\dfrac{\partial f(S_t)}{\partial S_t}$ are tabulated, according to different values of S_t/K and ρ, to examine the increasing phenomena of $f(S_t)$. The results listed in Table 2 show that as long as the stock price at time t is a moderate in-the money situation, say, $S_t/K \le 1.8$, also $\rho \le 0.8$, the increasing property of $f(S_t)$ in

S_t, is ensured; furthermore, the price of the discussed American stock call option defined by equation (6), could be straightforwardly applied.

The following numerical illustrations are presumed to be under the condition that the stock price at time t satisfies $S_t / K \leq 1.8$. The variation in the American stock call option prices under various amounts of cash dividends and correlation coefficients with different stock values is illustrated in Figure 1. At all option-value statues, the option price decreases as the cash dividend increases; for a fixed correlation the difference is about 1 dollar when the amount of the cash dividend ranges from 0.5 to 5.5. This is because a larger cash dividend causes a lower post-dividend stock price, thereby resulting in a diminishing call option price. On the other hand, the option price increases as the correlation coefficient increases. This agrees with the commonly accepted economic principle that a boom in the stock market goes along with a sag in the short-term interest rate market. The impact of the correlation, however, is not as dominant as the cash dividend; for a fixed amount of cash dividends, the extreme variation is around 0.1 dollar when the correlation ranges from -0.8 to 0.8.

In the following numerical analyses we focus on the difference between the Roll-Geske-Whaley model and the results derived from our model. The ρ parameter, the correlation coefficient between the stock price and the discount factor, plays an important role in the comparisons. The percentage difference between equation (6) and (8), called the *price difference*, is defined by

$$dC = \left[\left(C_0 - \widetilde{C}_0 \right) \Big/ \widetilde{C}_0 \right] \times 100\% ,$$

and the percentage difference between equations (7) and (9), called the *delta hedge ratio difference*, is defined by

$$d\Delta = \left[\left(\Delta_S - \widetilde{\Delta}_S \right) \Big/ \widetilde{\Delta}_S \right] \times 100\% .$$

The variation of the option price difference under various correlation coefficients and values is illustrated in Figure 2. For all option-value statues, the option price difference increases as the correlation coefficient increases; in fact, the price difference is negative under a negative correlation coefficient, and it turns positive whenever the stock price and the discount factor are positively correlated. This means that when the stock price and discount factor are negatively correlated, the theoretical option price should be lower than the one computed from Roll-Geske-Whaley model; on the other hand, when they are positively correlated, the theoretical option price may be underestimated by Roll-Geske-Whaley model. In addition, the largest price difference occurs when the stock price and the discount factor are highly correlated, reaching an absolute amount of 3%.

The sensitivity of the option price difference with respect to the correlation coefficient, as illustrated in Figure 2 and Figure 3, also relies on the state of the initial stock price. When the initial stock price is low, the price difference of the out-of-the-money American stock call option is more sensitive to the variation of the correlation coefficient; in contrast, the sensitivity of the option price difference with respect to the change of correlation coefficient is smaller when the option is at-the-money or in-the-money. This is because the option is of the American type and so would be exercised early if the stock price is high enough. Nonetheless, when the American option is out-of-the-money at present, it should be held, to wait for a more benign evolution of the stock price. Since the out-of-the-money option depends on the uncertainty of future market behavior, the correlation coefficient between the stock price and the discount factor imposes a rather heavier impact on it.

The dynamics of the delta hedge ratio difference under various correlation coefficients at distinct stock values are demonstrated in Figure 4 and Figure 5. Similar to the price difference pattern, the delta hedge ratio difference increases as the correlation coefficient enhanced, for all option-value statues. Although the largest delta hedge ratio difference still occurs when the stock price and the discount factor are highly correlated, the extreme absolute amount, reaching around 2.5%, is smaller than that aforementioned the price difference.

In addition, the sensitivity of the delta hedge ratio difference with respect to the correlation coefficient is also reliant on the state of the initial stock price. The delta hedge ratio difference of the out-of-the-money option is more sensitive to a variation in the correlation when the initial stock price is low; when the option is at-the-money or in-the-money, on the other hand, the sensitivity of the delta hedge ratio difference with respect to changes in the correlation coefficient is smaller. The analysis of this pattern is analogous to that for the price difference. Furthermore, the dynamics of the delta hedge ratio difference are rather stable; in particular, the delta hedge ratio difference remains steady when the option is in-the-money, and the initial stock price is 35.

4. Conclusions

In this article, an analytical formula for evaluating American stock call options with a given dividend under stochastic interest rates is derived. By fabricating the proper correlation between the underlying stock price process and the discount factor process, we can construct an equivalent replicating portfolio under suitable conditions, and the closed-form solutions for both option price and delta hedge ratio can then be analytically derived.

As indicated by the numerical illustrations, the discussed American stock call option is sensitive to variations in the cash dividend amount and the introduced correlation coefficient. The dynamics of the call option value varies for different stock prices, and is most influenced by changes in the cash dividend amounts. Numerical analyses show that a positive correlation brings a higher option price and delta hedge ratio, and a negative correlation leads to a lower one. In particular, absolute values of both price difference and delta hedge ratio difference reach a maximum when the stock price and the discount factor are highly correlated. Moreover, despite the fact that both the option price and delta hedge ratio dynamics are affected by variations in the correlation coefficients, the numerical results reveal that it is the option price rather than the delta hedge ratio that is more sensitive to a change in the correlation coefficient, in all cases. These results should help researchers and participants be better informed and make accurate decisions for dealing with this specific American stock call option in the real financial market.

Appendix

Before developing the proofs, for convenience, some preliminary notations and facts are stated without proof as follows:

Fact 1: Suppose $(\ln S_u, \ln W_u) \overset{d}{\sim} N_2(\mu_{S_u}, \mu_{W_u}, \sigma_S^2 u, \sigma_Q^2 u; \rho)$, then

(1) $\ln S_u | \ln W_u = \ln w_u \overset{d}{\sim} N_1[m_{S_u}, \sigma_S^2 u(1-\rho^2)]$,

(2) $\ln W_u | \ln S_u = \ln s_u \overset{d}{\sim} N_1[m_{W_u}, \sigma_Q^2 u(1-\rho^2)]$, where

$m_{S_u} = \mu_{S_u} + \alpha(\ln w_u - \mu_{W_u})$, $\mu_{S_u} = \ln(S_0 - DB_{0,t}) - \ln B_{0,u} + \rho\sigma_S\sigma_Q u - \frac{1}{2}\sigma_S^2 u$, $\alpha = \frac{\rho\sigma_S}{\sigma_Q}$,

$\mu_{W_u} = \ln B_{0,u} - \frac{1}{2}\sigma_Q^2 u$, $m_{W_u} = \mu_{W_u} + \delta(\ln s_u - \mu_{S_u})$, and $\delta = \frac{\rho\sigma_Q}{\sigma_S}$, for $t < u < T$.

Fact 2: Let $\Phi_2(x, y; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{x} \int_{-\infty}^{y} \exp\left[-\frac{1}{2(1-\rho^2)}(u^2 - 2\rho uv + v^2)\right] du dv$, then

(1) $\int_{-\infty}^{x} \Phi_1\left(\frac{y - z\rho}{\sqrt{1-\rho^2}}\right) \varphi_1(z) dz = \Phi_2(x, y; \rho)$,

(2) $\Phi_2(x, y; \rho) = \Phi_2(y, x; \rho)$,

(3) $\Phi_1(x) = \Phi_2(x, \infty; \rho) = \Phi_2(\infty, x; \rho)$, and

(4) $\Phi_1(x) - \Phi_2(x, y; \rho) = \Phi_2(x, -y; -\rho)$

Fact 3**:** Let Y be a random variable and distributed as $\ln Y \sim N_1(\mu_Y, \sigma_Y^2)$, then

(1) The density function of Y is $f_Y(y) = \dfrac{1}{\sigma_Y y}\varphi_1\left(\dfrac{\ln y - \mu_Y}{\sigma_Y}\right)$.

(2) $E(Y) = \exp\left(\mu_Y + \dfrac{\sigma_Y^2}{2}\right)$.

(3) $E[\max(Y - K, 0)] = E(Y)\Phi_1\left\{\sigma_Y + \dfrac{(\mu_Y - \ln K)}{\sigma_Y}\right\} - \Phi_1\left\{\dfrac{(\mu_Y - \ln K)}{\sigma_Y}\right\}$.

Lemma 1: Let Y be log-normal distributed, say $\ln Y \sim N_1(\mu_Y, \sigma_Y^2)$. Define

$\zeta = \mu_Y + p\sigma_Y^2/2$, and $\psi = \mu_Y + p\sigma_Y^2$, with $p > 0$, then for any $q \in R$,

$$E\left[Y^p I_{\{s<Y<\infty\}}\Phi_1(q\ln Y + r)\right] = \exp(p\zeta)\Phi_2\left(\frac{\psi - \ln s}{\sigma_Y}, \frac{q\psi + r}{\sqrt{1+q^2\sigma_Y^2}}; \frac{q\sigma_Y}{\sqrt{1+q^2\sigma_Y^2}}\right),$$

where $I_{\{\cdot\}}$ is an indicator function.

Pf): $$E\left[Y^p I_{\{s<Y<\infty\}}\Phi_1(q\ln Y + r)\right] = \int_s^\infty y^p \Phi_1(q\ln y + r)\cdot\frac{1}{\sigma_Y y}\varphi_1\left(\frac{\ln y - \mu_Y}{\sigma_Y}\right)dy$$

$$= \int_{\frac{\ln s - \mu_Y}{\sigma_Y}}^\infty \exp\{p(\sigma_Y z + \mu_Y)\}\Phi_1\{q(\sigma_Y z + \mu_Y) + r\}\varphi_1(z)dz$$

$$= \exp(p\zeta)\int_{\frac{\ln s - \mu_Y}{\sigma_Y}}^\infty \Phi_1\{q(\sigma_Y z + \mu_Y) + r\}\varphi_1(z - p\sigma_Y)dz$$

$$= \exp(p\zeta)\int_{\frac{\ln s - \psi}{\sigma_Y}}^\infty \Phi_1(q\sigma_Y x + q\psi + r)\varphi_1(x)dx$$

$$= \exp(p\zeta) \int_{-\infty}^{\frac{\psi - \ln s}{\sigma_Y}} \Phi_1(-q\sigma_Y x + q\psi + r)\varphi_1(x)dx$$

After some algebra, together with result (1) of Fact 2, Lemma 1 is obtained. □

Lemma 2: Under the same assumption and notations stated in Lemma 1, the following special results are reduced and stated as follows:

(1) $E\left[Y^p \Phi_1(q \ln Y + r)\right] = \exp(p\zeta)\Phi_1\left(\frac{q\psi + r}{\sqrt{1 + q^2\sigma_Y^2}}\right)$

(2) $E\left[Y^p I_{\{s<Y<\infty\}}\right] = \exp(p\zeta)\Phi_1\left(\frac{\psi - \ln s}{\sigma_Y}\right)$

Appendix A

Proof of equation (1) of Theorem 1:

$$C_0^{(A)} = (S_0 - DB_{0,t})\exp(2\rho\sigma_Q\sigma_S T)\Phi_1(a) - KB_{0,T}\Phi_1(c),$$

where $a = \dfrac{\ln\left(\dfrac{S_0 - DB_{0,t}}{KB_{0,T}}\right) + \left(2\rho\sigma_Q\sigma_S + \dfrac{1}{2}\sigma_S^2\right)T}{\sigma_S\sqrt{T}}$ and $c = a - \sigma_S\sqrt{T}$.

Pf): By assumption of the European stock call option (A),

$$C_0^{(A)} = E\left[W_T \max(S_T - K, 0)\right] = E_{W_T}\left\{W_T E_{S_T|W_T}\left[\max(S_T - K, 0) \mid W_T\right]\right\}.$$

Let $\lambda_{S_T} = \sigma_S\sqrt{(1-\rho^2)T}$, then by results of Fact 1,

$$E_{S_T|W_T}\left[\max(S_T - K, 0) \mid W_T = w_T\right] = \exp\left(m_{S_T} + \frac{\lambda_{S_T}^2}{2}\right)\Phi_1\left(\lambda_{S_T} + \frac{(m_{S_T} - \ln K)}{\lambda_{S_T}}\right) - K\Phi_1\left(\frac{(m_{S_T} - \ln K)}{\lambda_{S_T}}\right).$$

Therefore, $C_0^{(A)} = E_{W_T}\left\{W_T\left[\exp\left(m_{S_T} + \dfrac{\lambda_{S_T}^2}{2}\right)\Phi_1\left(\lambda_{S_T} + \dfrac{(m_{S_T} - \ln K)}{\lambda_{S_T}}\right) - K\Phi_1\left(\dfrac{(m_{S_T} - \ln K)}{\lambda_{S_T}}\right)\right]\right\}$

$$= E_{W_T}\left[W_T \exp(\alpha \ln W_T + \beta)\Phi_1(\gamma \ln W_T + \theta)\right] - KE_{W_T}\left[W_T\Phi_1(\gamma \ln W_T + \eta)\right],$$

where $\gamma = \alpha \big/ \lambda_{S_T}$, $\beta = \mu_{S_T} - \alpha\mu_{W_T} + \lambda_{S_T}^2 \big/ 2$, $\eta = \left(\mu_{S_T} - \alpha\mu_{W_T} - \ln K\right) \big/ \lambda_{S_T}$, and $\theta = \lambda_{S_T} + \eta$.

Define $p = \alpha + 1$, $q = \gamma$, and $r = \theta$, then by result (1) of Lemma 2, we have

$$E_{W_T}\left[W_T \exp(\alpha \ln W_T + \beta)\Phi_1(\gamma \ln W_T + \theta)\right] = \exp(\beta) E_{W_T}\left[W_T^{\alpha+1}\Phi_1(\gamma \ln W_T + \theta)\right]$$

$$= \exp\left[\beta + (\alpha+1)\mu_{W_T} + \frac{(\alpha+1)^2\sigma_Q^2 T}{2}\right]\Phi_1\left(\frac{\gamma\left[\mu_{W_T} + (\alpha+1)\sigma_Q^2 T\right] + \theta}{\sqrt{1+\gamma^2\sigma_Q^2 T}}\right)$$

$$= S_0' \exp\left(2\rho\sigma_S\sigma_Q T\right)\Phi_1(a) = \left(S_0 - DB_{0,t}\right)\exp\left(2\rho\sigma_S\sigma_Q T\right)\Phi_1(a) .$$

Similarly, set $p = 1$, $q = \gamma$, $r = \eta$, then

$$E_{W_T}\left[W_T\Phi_1(\gamma \ln W_T + \eta)\right] = \exp\left(\mu_{W_T} + \frac{\sigma_Q^2 T}{2}\right)\Phi_1\left(\frac{\gamma\left(\mu_{W_T} + \sigma_Q^2 T\right) + \eta}{\sqrt{1+\gamma^2\sigma_Q^2 T}}\right) = B_{0.T}\Phi_1(c) .$$

Therefore, $C_0^{(A)} = \left(S_0 - DB_{0,t}\right)\exp\left(2\rho\sigma_Q\sigma_S T\right)\Phi_1(a) - KB_{0,T}\Phi_1(c)$. □

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Appendix B

Proof of equation (2) of Theorem 1:

$$C_0^{(B)} = \left(S_0 - DB_{0,t}\right)\exp\left(2\rho\sigma_Q\sigma_S T\right)\Phi_2(a,b;\nu) - KB_{0,T}\Phi_2(c,d;\upsilon) - \left(S_t^* + D - K\right)B_{0,t}\Phi_1(d) ,$$

where $b = \dfrac{\ln\left(\dfrac{S_0 - DB_{0,t}}{S_t^* B_{0,t}}\right) + \left(2\rho\sigma_Q\sigma_S + \dfrac{1}{2}\sigma_S^2\right)t}{\sigma_S\sqrt{t}}$, $d = b - \sigma_S\sqrt{t}$, and $\nu = \sqrt{t \big/ T}$.

Pf): Let $X = S_t^* + D - K$, the exercise price of the European compound option price, defined in (B), then the option price is

$$C_0^{(B)} = E_{S_t}\left[\max\left(C_t^{(A)} - X, 0\right)E_{W_t|S_t}\left(W_t | S_t\right)\right]$$

$$= E_{S_t}\left\{\max\left(C_t^{(A)} - X, 0\right)\exp\left[\mu_{W_t} + \delta\left(\ln S_t - \mu_{S_t}\right) + \frac{\lambda_{W_t}^2}{2}\right]\right\}$$

$$= \exp\left(\mu_{W_t} - \delta\mu_{S_t} + \frac{\lambda_{W_t}^2}{2}\right)\cdot E_{S_t}\left\{S_t^\delta \max\left[C_t^{(A)} - X, 0\right]\right\}.$$

From the expression of $C_t^{(A)}$ in Corollary 1, the second term is re-written as:

$$E_{S_t}\left\{S_t^{\delta}\max\left[C_t^{(A)}-X,0\right]\right\}=E_{S_t}\left\{S_t^{\delta}I_{\{S_t^*<S_t<\infty\}}\left[C_t^{(A)}-X\right]\right\}$$

$$=E_{S_t}\left\{S_t^{\delta}I_{\{S_t^*<S_t<\infty\}}\left[\exp\left(2\rho\sigma_S\sigma_Q(T-t)\right)\cdot S_t\Phi_1(d_1)-KB_{t,T}\Phi_1(d_2)-X\right]\right\}$$

$$=\exp\left[2\rho\sigma_S\sigma_Q(T-t)\right]\cdot E_{S_t}\left[S_t^{\delta+1}I_{\{S_t^*<S_t<\infty\}}\Phi_1(d_1)\right]-KB_{t,T}E_{S_t}\left[S_t^{\delta}I_{\{S_t^*<S_t<\infty\}}\Phi_1(d_2)\right]-XE_{S_t}\left[S_t^{\delta}I_{\{S_t^*<S_t<\infty\}}\right].$$

The remaining work is to respectively compute the three conditional expectations:

Let $p=\delta+1$, $s=S_t^*$, $r_1=q\left[-\ln\left(KB_{t,T}\right)+2\rho\sigma_S\sigma_Q(T-t)+\frac{\sigma_S^2(T-t)}{2}\right]$,

$q=\frac{1}{\sigma_S\sqrt{T-t}}$, then $d_1=q\ln S_t+r_1$. Applying Lemma 1,

$$E_{S_t}\left[S_t^{\delta+1}I_{\{S_t^*<S_t<\infty\}}\Phi_1(d_1)\right]=E_{S_t}\left[S_t^{\delta+1}I_{\{S_t^*<S_t<\infty\}}\Phi_1(q\ln S_t+r)\right]$$

$$=\exp\left\{(\delta+1)\mu_{S_t}+\frac{\left[(\delta+1)\sigma_S\sqrt{t}\right]^2}{2}\right\}\Phi_2(b,a;\upsilon)$$

$$=\exp\left\{(\delta+1)\mu_{S_t}+\frac{\left[(\delta+1)\sigma_S\sqrt{t}\right]^2}{2}\right\}\Phi_2(a,b;\upsilon)$$

Again, let $p=\delta$, $r_2=r_1-1/q$, then $d_2=q\ln S_t+r_2$, and by Lemma 1,

$$E_{S_t}\left[S_t^{\delta}I_{\{S_t^*<S_t<\infty\}}\Phi_1(d_2)\right]=E_{S_t}\left[S_t^{\delta}I_{\{S_t^*<S_t<\infty\}}\Phi_1(q\ln S_t+r_1)\right]$$

$$=\exp\left(\delta\mu_{S_t}+\frac{\delta^2\sigma_S^2t}{2}\right)\Phi_2(d,c;\nu).$$

Finally, by result (2) of Lemma 2, the last term becomes

$$E_{S_t}\left[S_t^{\delta}I_{\{S_t^*<S_t<\infty\}}\right]=\exp\left(\delta\mu_{S_t}+\frac{\delta^2\sigma_S^2t}{2}\right)\Phi_1(d).$$

Moreover, after some algebra, the following results are obtained:

- $\exp\left(\mu_{W_t}-\delta\mu_{S_t}+\frac{\lambda_{W_t}^2}{2}\right)\cdot\exp\left\{2\rho\sigma_S\sigma_Q(T-t)\right\}\cdot\exp\left\{(\delta+1)\mu_{S_t}+\frac{\left[(\delta+1)\sigma_S\sqrt{t}\right]^2}{2}\right\}$

$$=\left(S_0-DB_{0,t}\right)\exp\left(2\rho\sigma_S\sigma_QT\right),$$

- $\exp\left(\mu_{W_t} - \delta\mu_{S_t} + \frac{\lambda_{W_t}^2}{2}\right) \cdot \exp\left(\delta\mu_{S_t} + \frac{\delta^2\sigma_S^2 t}{2}\right) = B_{0,t}$.

Since $B_{0,t}B_{t,T} = B_{0,T}$, therefore

$$C_0^{(B)} = (S_0 - DB_{0,t})\exp(2\rho\sigma_Q\sigma_S T)\Phi_2(a,b;\nu) - KB_{0,T}\Phi_2(c,d;\upsilon) - (S_t^* + D - K)B_{0,t}\Phi_1(d).$$

This completes the proof. □

Appendix C

Proof of equation (3) of Theorem 1:

$$C_0^{(C)} = (S_0 - DB_{0,t})\exp(2\rho\sigma_Q\sigma_S t)\Phi_1(b) - S_t^* B_{0,t}\Phi_1(d),$$

where b and d are defined in the proof of equation (2) of Theorem 1,

Pf): Similar to the proof of equation (2), the result follows. □

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Table 1. Cash flows under the assumption that $\rho < \rho_*$

Value at time t	$S_t > S_t^*$	$S_t \le S_t^*$
The discussed American call option	$S_t - X + D$	$C_t(S_t, B_{t,T}, K, T-t)$
Option (A)	$C_t(S_t, B_{t,T}, K, T-t)$	$C_t(S_t, B_{t,T}, K, T-t)$
Option (B)	$-\{C_t(S_t, B_{t,T}, K, T-t) - (S_t^* - X + D)\}$	0
Option (C)	$S_t - S_t^*$	0
Replicating portfolio =(A)+(B)+(C)	$S_t - X + D$	$C_t(S_t, B_{t,T}, K, T-t)$

Table 2. Increasing property of $f(S_t)$: values of $\frac{\partial f(S_t)}{\partial S_t}$

ρ	S_t/K				
	0.6	0.8	1.0	1.4	1.8
-0.8	0.982	0.825	0.518	0.100	0.014
-0.6	0.981	0.824	0.516	0.098	0.013
-0.4	0.981	0.823	0.514	0.096	0.012
-0.2	0.981	0.822	0.512	0.095	0.011
0	0.981	0.821	0.510	0.093	0.010
0.2	0.981	0.819	0.508	0.092	0.009
0.4	0.980	0.818	0.506	0.090	0.008
0.6	0.980	0.817	0.504	0.089	0.007
0.8	0.980	0.816	0.502	0.087	0.006

Notes: 1. All the entries are positive, that means under the conditions, $S_t/K \le 1.8$ and $\rho \le 0.8$, $f(S_t)$ is an increasing function of S_t.

2. $\frac{\partial f(S_t)}{\partial S_t} = 1 - \exp\{2\rho\sigma_Q\sigma_S(T-t)\}\Phi_1(d_1)$ and

$$d_1 = \frac{1}{\sigma_S\sqrt{T-t}}\left\{\ln\left(\frac{S_t}{KB_{t,T}}\right) + \left(2\rho\sigma_Q\sigma_S + \frac{1}{2}\sigma_S^2\right)(T-t)\right\}$$

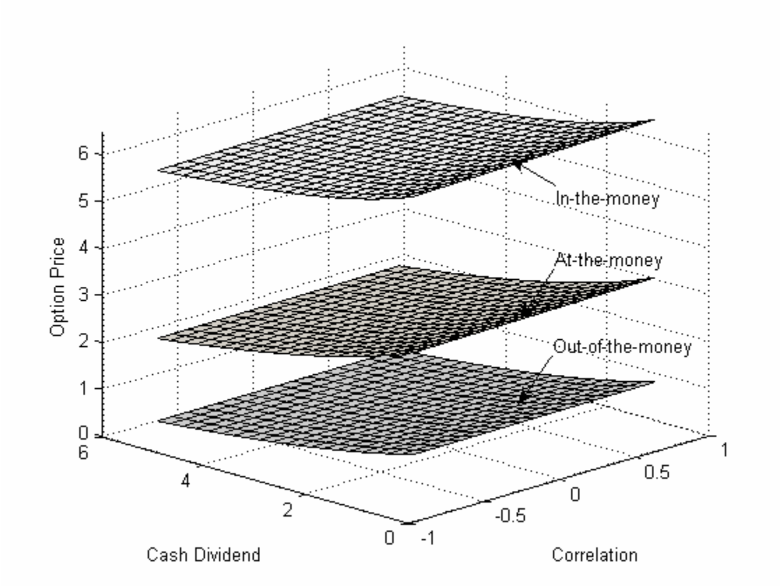

Figure 1.
Price dynamics for different cash dividend and correlation at distinct stock values

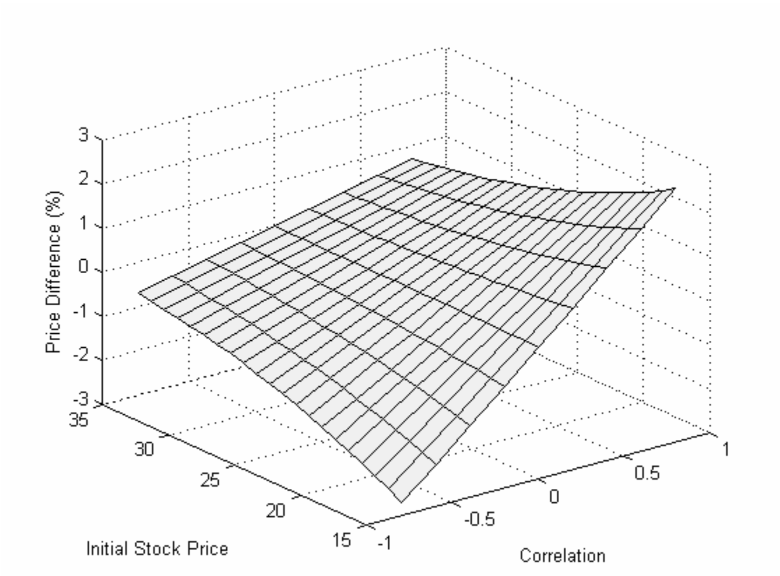

Figure 2.
Price difference analysis

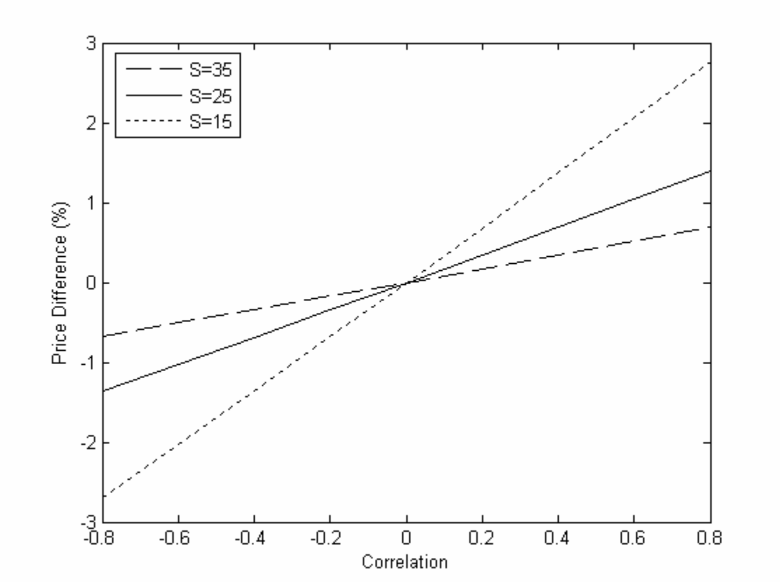

Figure 3.
Price difference at distinct initial stock values

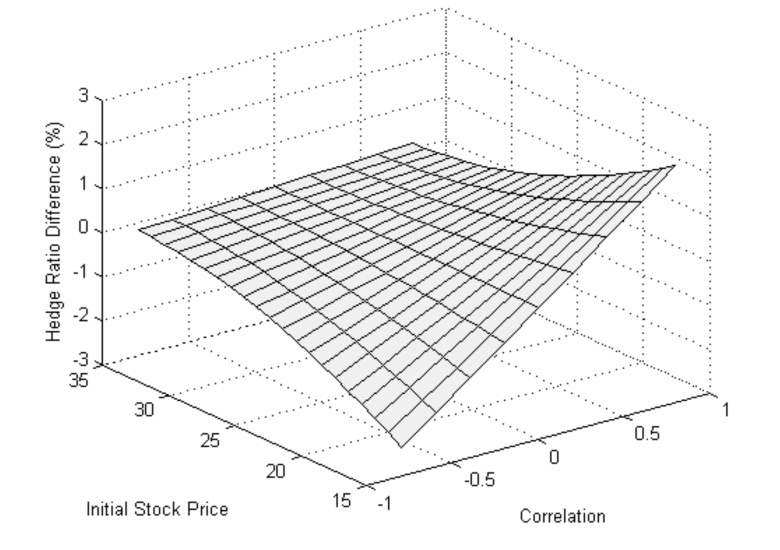

Figure 4.
Delta hedge ratio difference analysis

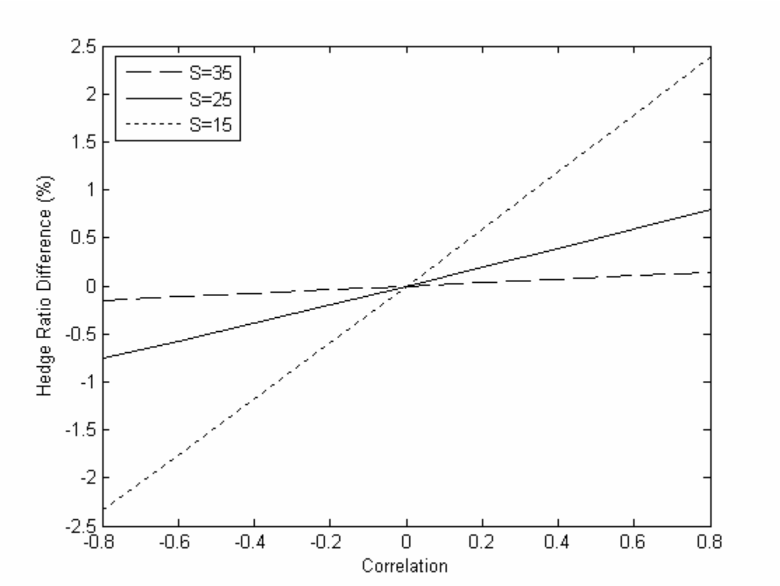

Figure 5.
Delta hedge ratio difference at distinct initial stock values