

# Randomization and the American Put

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# Randomization and the American Put

## **Abstract**

While American calls on non-dividend paying stocks may be valued as European, there is no completely explicit exact solution for the values of American puts. We use a technique called randomization to value American puts and calls on dividend-paying stocks. This technique yields a new semi-explicit approximation for American option values in the Black Scholes model. Numerical results indicate that the approximation is both accurate and computationally efficient.

Closed-form solutions for the value of European-style options have been known since the seminal papers of Black-Scholes (1973) and Merton (1973). Since American calls on non-dividend paying stocks are not rationally exercised early, they can be valued in closed form. Unfortunately, the vast majority of listed options are American-style and subject to early exercise. Despite a profusion of research on the subject, no completely satisfactory analytic solution for the value of such options has been found.

The principal difficulty in obtaining an analytic solution arises from the absence of a simple expression for the optimal exercise boundary. An exercise boundary is a time path of critical stock prices at which early exercise occurs. The optimal exercise boundary of an American option is not known *ex ante*, and must be determined as part of the solution to the valuation problem. Furthermore, it is difficult to analytically approximate American option values using boundary approximations which are consistent with the known short and long time behavior of the exercise boundary.

The purpose of this paper is to develop a new approach for determining American option values and exercise boundaries based on a technique called randomization. In general, randomization describes a three step procedure which can be used to solve a host of problems. The first step is to randomize a parameter by assuming a plausible distribution for it. The second step is to somehow calculate the expected value of the dependent variable in this random parameter setting. This is the difficult step since one does not know the dependent variable in the fixed parameter setting. The final step is to let the variance of the distribution governing the parameter approach zero, holding the mean of the distribution constant at the fixed parameter value.

For standard options, one can randomize the initial stock price, the strike price, the initial time, or the maturity date. In this paper, we randomize the maturity date of an American option and determine the exact solution for its value. The owner of this random maturity American option can exercise at any time up to and including some random maturity date. Thus, a random maturity American put gives its owner the right to sell an underlying security for a fixed price at any time up to and including its random maturity, while the call gives the corresponding right to buy. In this paper, the maturity date is

determined by the waiting time to a pre-specified number of jumps of a standard Poisson process, which is assumed to be independent of the underlying stock price process. We note that the only role of the Poisson process is to determine maturity; the stock price process used is continuous.

A random maturity contract has a value which approximates the value of its fixed maturity counterpart. In order to distinguish between these values, we refer to the former values as randomized. Our formulas for randomized values are generally simpler than the formulas for fixed maturity contracts. The simplest expression arises when the randomized American option matures at the first jump time of a Poisson process, in which case the maturity date is exponentially distributed. This random horizon problem is equivalent to an infinite horizon problem with an adjusted discount rate, as shown in a portfolio optimization setting by Merton (1971) and Cass and Yaari (1967). In the option pricing context, American options with infinite horizons were valued long ago by Samuelson (1965) and McKean (1965). So it is somewhat natural<sup>1</sup> that randomizing the maturity will lead to simpler option valuation formulas.

For American options, the simplicity of the solution arising from randomization is mainly due to the taming of the behavior of the exercise boundary. When the option matures with the first jump, the memoryless property of the exponential distribution implies that the exercise boundary is independent of time. As calendar time elapses, the option gets no closer to its random maturity, and thus its value suffers no time decay. The stationarity in value implies that the exercise boundary is also independent of time. When the underlying security has either no dividends or a constant continuous dividend flow, we can solve explicitly for the critical stock price. In contrast, if the underlying pays continuous proportional dividends, then a fairly simple algebraic equation must be solved numerically. As a result, the general formulation leads to semi-explicit valuation formulas.

While the assumption of an exponentially distributed maturity leads to simple approximations for American options, the approximation has too much error to be used in practice. To improve the approximation, we instead assume that the time to maturity may be subdivided into  $n$  independent exponential sub-periods. Thus, the randomized American option matures at the  $n$ -th jump time of a standard Pois-

son process. The maturity time is thereby Erlang distributed with a mean equal to the fixed maturity date of the true American option. In this case, the exercise boundary takes the form of a staircase, with the levels being determined by optimizing within each sub-period. The resulting expression for the randomized option value is a triple sum, involving no special functions other than the natural log.

As the number of random sub-periods becomes large, the variance of the random maturity approaches zero, so that the probability density function governing maturity approaches a Dirac delta function centered at the American option's fixed maturity. Thus, increasing the number of periods increases the accuracy of the solution at the expense of greater computational cost. However, when Richardson extrapolation is used, our numerical results indicate that our randomized option value converges to the true American option value in a computationally efficient manner.

The randomization approach taken in this paper is to exactly value a contract which approximates the nature of an American option. An alternative approach is to approximate the valuation operator rather than the contract. This is the approach taken when finite differences (see eg. Brennan and Schwartz (1977)) are used to numerically solve the partial differential equation (p.d.e.) governing the value of an American option. As is well-known, the standard finite difference approach replaces all of the partial derivatives in a p.d.e. with finite differences. When only the time derivative is discretized, the approach is termed the (horizontal) method of lines or *Rothe's method* (see Rothe (1930) and Rektorys (1982)). The application of the method of lines to free boundary problems has been promulgated in Meyer (1970), Meyer (1979) and in Meyer & van der Hoek (1994), who use it to numerically value American options. Goldenberg and Schmidt (1995) test this numerical scheme against other approaches and find that it is highly accurate, although slightly slower<sup>2</sup> than some other approaches. Carr and Faguet (1994) gave a semi-explicit solution to the sequence of ordinary differential equations which arise when the method of lines is applied to the Black Scholes p.d.e. In fact, the solution obtained via randomization in this paper is mathematically equivalent to the solution in Carr and Faguet.

The structure of this paper is as follows. The next section reviews standard results on the pricing of

American puts in the Black-Scholes model. The following section presents the randomization technique in the context of valuing an American put on a non-dividend-paying stock with an exponential maturity. The subsequent section discusses the more general case of a Erlang distributed maturity. The following section discusses the implementation of our formula and compares this implementation with extant approaches in terms of both speed and accuracy. The penultimate section extends the analysis to dividends and American calls. The final section summarizes, while the appendix collects all the formulas needed to implement the randomization approach.

## 1 American Put Valuation in the Black-Scholes Model

In this section, we focus on the valuation of American puts in the Black-Scholes model. We defer the corresponding development for American calls until dividends have been introduced. The Black-Scholes model assumes that over the option's life  $[0, T]$ , the economy is described by frictionless markets, no arbitrage, a constant riskless rate  $r > 0$ , no dividends from the underlying stock, and that the underlying spot price process  $\{S_t, t \in (0, T)\}$  is a geometric Brownian motion with a constant volatility rate  $\sigma > 0$ . Let  $P(t, S; T)$  denote the value of an American put as a function of the current time  $t$ , the current stock price  $S$ , and the maturity date  $T$ . The critical stock price  $\underline{S}(t; T), t \in [0, T]$  is defined as the largest price  $S$  at which the American put value  $P(t, S; T)$  equals its exercise value  $K - S$ , where  $K$  is the strike price. As the maturity is shortened, the alive American put value falls, while the exercise value remains constant. A reduction in time to maturity therefore raises the critical stock price at which exercise occurs. When graphed against time, the critical stock price is a smoothly increasing function termed the *exercise boundary*.

For quite general stochastic processes, the American put's initial value is given by the solution to an *optimal stopping problem*:

$$P(0, S; T) = \sup_{\tau_s \in [0, T]} E_{0, S} \{ e^{-r\tau_s} [K - S_{\tau_s}]^+ \}, \quad (1)$$

where  $\tau_s$  is a stopping time and the expectation is calculated under a risk-neutral probability measure.

In the Black-Scholes model, this optimal stopping time is the earlier of maturity and the first passage time to the exercise boundary. Consequently, the alive American put may alternatively be valued as:

$$P(0, S; T) = \sup_{B(t); t \in [0, T]} E_{0, S} \{ e^{-r(\tau_B \wedge T)} [K - S_{\tau_B \wedge T}]^+ \}, \quad S > \underline{S}(0; T), \quad (2)$$

where  $\tau_B$  is the first passage time<sup>3</sup> from  $S$  to an exercise boundary  $B(t), t \in [0, T]$ .

McKean (1965) showed that an application of Itô's lemma to (1) implies that the alive American put value and exercise boundary jointly solve a *free boundary problem*, consisting of the Black-Scholes partial differential equation (p.d.e.):

$$\frac{\sigma^2}{2} S^2 P_{ss}(t, S; T) + r S P_s(t, S; T) - r P(t, S; T) = P_T(t, S; T), \quad S \in (\underline{S}(t; T), \infty), t \in (0, T), \quad (3)$$

and the following boundary conditions:

$$P(T, S; T) = (K - S)^+, \quad S \in (\underline{S}(T; T), \infty), \quad \text{and} \quad \underline{S}(T; T) = K,$$

$$\lim_{S \uparrow \infty} P(t, S; T) = 0, \quad \lim_{S \downarrow \underline{S}(t; T)} P(t, S; T) = K - \underline{S}(t; T), \quad \lim_{S \downarrow \underline{S}(t; T)} P_s(t, S; T) = -1, \quad t \in (0, T).$$

Unfortunately, there is no known exact and completely explicit solution to either the optimal stopping problem (1) or to the free boundary problem (3). The next section presents a new approach for obtaining approximate solutions to these problems.

## 2 Exponential Maturity Valuation

In order to obtain an approximate solution for the value of an American put and its exercise boundary, we now suppose that the maturity date is random. Let  $\tau$  denote the random maturity time. In this section, we assume that  $\tau$  is exponentially distributed with scale parameter  $\lambda$ :

$$\text{Prob}\{\tau \in dt\} = \lambda e^{-\lambda t} dt.$$

Since the mean of  $\tau$  is the reciprocal of  $\lambda$ , we set  $\lambda = \frac{1}{T}$ , so that the mean maturity of the randomized American put is  $T$ , the maturity of the true American put. Let  $P^{(1)}(S)$  denote the randomized value of an

American put, which matures at the first jump time of a standard Poisson process with intensity  $\lambda = \frac{1}{T}$ . We assume that the Poisson process is independent of the stock price process. Furthermore, we assume that the Poisson process is also uncorrelated with any market factor. It follows that the risk associated with the randomness of maturity can be diversified away by holding a large portfolio of random maturity options on different stocks. Thus, the randomized value can be calculated in a “risk-neutral” fashion.

The analog to (2) for randomized American option values is:

$$P^{(1)}(S) = \sup_B E_{0,S} \{ e^{-r(\tau_B \wedge \tau)} [K - S_{\tau_B \wedge \tau}]^+ \}, \quad S > \underline{S}_1, \quad (4)$$

where  $\underline{S}_1$  is the unknown optimal exercise boundary. Note that the supremum is taken only over time-stationary boundaries  $B$  rather than functions of time  $B(t)$ . The memoryless property of the exponential distribution implies that the passage of time has no effect on either the randomized option value or its optimal exercise boundary. Thus, the time-dependent exercise boundary becomes flat prior to the random maturity. When the Poisson process governing maturity jumps up, the randomized option value jumps down to intrinsic value  $(K - S)^+$ . Thus, one can think of the pent up time decay of the option as being released at the jump time. This release causes the exercise boundary to jump up from  $\underline{S}_1$  to  $K$ , crudely approximating the behavior of the true exercise boundary.

The expectation in (4) can be evaluated in closed form and the result can be maximized over barriers analytically. Since the details are cumbersome, a perhaps simpler approach is to recognize the following relationship between random and fixed maturity<sup>4</sup> put values:

$$P^{(1)}(S) = \sup_B \lambda \int_0^\infty e^{-\lambda t} D(0, S; t; B) dt, \quad (5)$$

where  $D(0, S; T; B)$  is the initial value of a down-and-out put with fixed maturity  $T$ , out barrier  $B$ , and rebate  $K - B$ :

$$D(0, S; t; B) = E_{0,S} \{ e^{-r(\tau_B \wedge T)} [K - S_{\tau_B \wedge T}]^+ \}, \quad S > B.$$

One can immediately observe that the randomized American put value is simply the Laplace-Carson<sup>5</sup> transform of a fixed maturity barrier put, maximized over barriers. Since down-and-out put values satisfy the

Black-Scholes p.d.e. (3), one can take the Laplace-Carson transform of both sides of this p.d.e. to obtain the following simpler *ordinary* differential equation (o.d.e.):

$$\frac{\sigma^2}{2}S^2P_{ss}^{(1)}(S) + rSP_s^{(1)}(S) - rP^{(1)}(S) = \lambda[P^{(1)}(S) - (K - S)^+], \quad S > \underline{S}_1, \quad (6)$$

subject to the following boundary conditions:

$$\lim_{S \uparrow \infty} P^{(1)}(S) = 0, \quad \lim_{S \downarrow \underline{S}_1} P^{(1)}(S) = K - \underline{S}_1, \quad \lim_{S \downarrow \underline{S}_1} P_s^{(1)}(S) = -1. \quad (7)$$

Using standard techniques for solving o.d.e.'s, the randomized value of an American put can be decomposed as:

$$P^{(1)}(S) = \begin{cases} p^{(1)}(S) + b^{(1)}(S) & \text{if } S > \underline{S}_0 \equiv K \\ KR - S + c^{(1)}(S) + b^{(1)}(S) & \text{if } S \in (\underline{S}_1, \underline{S}_0) \\ K - S & \text{if } S \leq \underline{S}_1, \end{cases} \quad (8)$$

where  $p^{(1)}(S)$  is the randomized value of a European put paying  $(K - S)^+$  at the first jump time:

$$p^{(1)}(S) = \left(\frac{S}{K}\right)^{\gamma - \epsilon} (qKR - \hat{q}K), \quad S > K, \quad (9)$$

with  $\gamma \equiv \frac{1}{2} - \frac{r}{\sigma^2}$ ,  $R \equiv \frac{1}{1+rT}$ ,  $\epsilon \equiv \sqrt{\gamma^2 + \frac{2}{R\sigma^2T}}$ , and:

$$p \equiv \frac{\epsilon - \gamma}{2\epsilon}, \quad q \equiv 1 - p, \quad \hat{p} \equiv \frac{\epsilon - \gamma + 1}{2\epsilon}, \quad \text{and } \hat{q} \equiv 1 - \hat{p}, \quad (10)$$

$b^{(1)}(S)$  is the present value of interest received below the critical stock price  $\underline{S}_1$  until the first jump time:

$$b^{(1)}(S) = \left(\frac{S}{\underline{S}_1}\right)^{\gamma - \epsilon} qKRrT, \quad (11)$$

and finally,  $c^{(1)}(S)$  is the randomized value of a European call paying  $(S - K)^+$  at the first jump time:

$$c^{(1)}(S) = \left(\frac{S}{K}\right)^{\gamma + \epsilon} (\hat{p}K - pKR), \quad S < K. \quad (12)$$

The first line of our formula (8) represents the randomized version of a decomposition of the American put value into the European put value and the early exercise premium. This decomposition also holds in the fixed maturity setting as shown previously in Carr, Jarrow, and Myneni (1992), Jacka (1991), and

Kim (1990). Note that the formula (9) for the randomized value of the European put is simpler than the Black Scholes formula in that it does not use any special functions such as the normal distribution function. On the other hand, (9) holds only for out-of-the-money values ( $S > K$ ). In contrast to the Black Scholes put formula which holds for all positive stock prices, formula (9) which values the put when  $S > K$  does not correctly value the put when  $S < K$ . The lack of smoothness in the payoff function implies that Put Call Parity<sup>6</sup> must be used to generate in-the-money values for European puts with random maturity. The second line of our formula (8) reflects this restriction. The third line of (8) sets the randomized put value to exercise value below the critical stock price  $\underline{S}_1$ . Figure 1 graphs the value of an exponential maturity American put against the stock price. The function is twice differentiable at the strike price, but only once differentiable at the exercise boundary, as is the case for a true American put.

Imposing value-matching in (8) at the critical stock price  $\underline{S}_1$  yields the following balance equation:

$$c^{(1)}(\underline{S}_1) = pK RrT. \quad (13)$$

The left hand side is clearly the randomized value of a European call when the stock price is at the critical stock price. The right hand side represents the randomized value of a claim paying interest on the strike price at all stock prices *above* the current stock price level. The critical stock price is chosen so that the call value just matches the present value of the interest flow received above the boundary. Stationarity in the values involved implies that the exercise boundary remains flat at this level until the jump time.

The simple expression (12) for the European call value implies that the balance equation (13) can be explicitly solved for our first approximation to the exercise boundary,  $\underline{S}_1$ :

$$\underline{S}_1 = K \left( \frac{pRrT}{\hat{p} - Rp} \right)^{\frac{1}{\gamma+\epsilon}}. \quad (14)$$

It is worth pointing out that explicit expressions for the critical stock price are rare. Indeed, we will lose this explicitness once constant proportional dividends are introduced.

For future use, note that substituting (14) into (12) implies that the randomized value of a European call is given by a formula similar to that of the randomized early exercise premium in (11):

$$c^{(1)}(S) = \left( \frac{S}{\underline{S}_1} \right)^{\gamma+\epsilon} pKRrT \equiv A^{(1)}(S).$$

Equations (8) and (14) represent the randomized versions of the American put value and critical stock price respectively. While these first approximations are simple and explicit, numerical implementation indicates substantial undervaluation of the put. The reason the randomized value is substantially smaller than the true value is that the owner of a random maturity put must optimize over boundaries without the benefit of knowing when the option will mature.

Clearly, the valuation error can be reduced by lowering the variance of the distribution governing maturity. Unfortunately, if a random variable with an exponential distribution has mean  $T$ , then its variance is  $T^2$ . The next section uses a two parameter distribution for maturity, which permits keeping the mean maturity constant at  $T$ , while reducing the variance as much as desired. As the variance approaches zero, the result is a de facto inversion of the Laplace-Carson transform (8), yielding an accurate approximation of the American put value.

### 3 Erlang Maturity Valuation

Consider an investor who is faced with the problem of allocating his investable wealth among  $n$  different securities. If the security returns are independently and identically distributed (i.i.d.), the variance minimizing allocation is to invest an equal proportion in each security. By the same token, a simple and efficient way to reduce the variance of our option's random maturity is to split it into  $n$  i.i.d sub-periods. If we also assume that each of the  $n$  periods is exponentially distributed with parameter  $\lambda$ , then the maturity date  $\tau$  is Erlang distributed:

$$\text{Prob}\{\tau \in dt\} = \frac{\lambda^n}{(n-1)!} t^{n-1} e^{-\lambda t} dt.$$

In order that the mean maturity be  $T$ , each subperiod must have mean  $\Delta \equiv T/n$ , which implies  $\lambda = 1/\Delta$ . By assuming that the maturity is Erlang distributed instead of exponentially distributed, the variance is reduced by a factor of  $\frac{1}{n}$  to only  $T^2/n$ . Figure 2 shows three Erlang density functions, with each corresponding to a maturity of mean  $T = 1$  year, and with variances of 1, 1/2, and 1/3 respectively. The densities are converging to a Dirac delta function centered at  $T = 1$  year.

Let  $P^{(n)}(S)$  denote the randomized value of an American put option which can be exercised for  $(K - S)^+$  at any time up to and including the  $n$ -th jump time of a standard Poisson process (with intensity  $\lambda = 1/\Delta$ ). To value this put, we use dynamic programming. Accordingly, suppose that  $n - 1$  jumps have occurred and that the investor is holding a put maturing at the next jump time of the Poisson process. This valuation problem was solved in the previous section, with the solution  $P^{(1)}(S)$  given by (5), except that  $T$  must be everywhere replaced by  $\Delta \equiv T/n$ .

We now back up a random time period and think of  $P^{(1)}(S)$  as the random payoff occurring at the end of this random period, provided that no exercise has occurred beforehand. Since exercising yields a payoff of  $(K - S)^+$  as usual, the randomized value of the American put with two jumps to maturity is:

$$P^{(2)}(S) = \sup_{B > 0} E_S \{ e^{-r\tau_B} [K - B]^+ 1(\tau_B < \tau_2) + e^{-r\tau_2} P^{(1)}(S_{\tau_2}) 1(\tau_B \geq \tau_2) \}, \quad S > \underline{S}_2, \quad (15)$$

where  $\tau_2$  denotes the length of the second random period prior to maturity and  $\underline{S}_2$  denotes the unknown optimal exercise boundary over this period. Once again, the stationarity of the barrier  $B$  over the period implies that the expectation in (15) can be evaluated in closed form and the result can be maximized over barriers analytically.

As in the previous section, a perhaps simpler approach is to work with Laplace-Carson transforms. Proceeding by analogy with the previous section, let  $D(S; T - t, B)$  denote the time  $t$  value of a down-and-out put with *fixed* maturity  $T$ , out barrier  $B$ , and which pays a rebate of  $K - B$  at the first passage time to  $B$ , if this occurs before  $T$ , and which pays  $P^{(1)}(S_T)$  at  $T$  otherwise. Then,  $D(S; T - t, B)$  satisfies

the Black Scholes p.d.e.:

$$\begin{aligned} \frac{\sigma^2}{2}S^2D_{ss}(S;T-t,B) + rSD_s(S;T-t,B) - rD(S;T-t,B) &= D_T(S;T-t,B), \\ S \in (B, \infty), t \in (0, T), \end{aligned} \quad (16)$$

subject to the terminal condition  $D(S;0,B) = P^{(1)}(S)$  and the boundary conditions:

$$\lim_{S \uparrow \infty} D(S;T-t,B) = 0, \quad \lim_{S \downarrow B} D(S;T-t,B) = K - B, \quad t \in (0, T).$$

The randomized value of the American put maturing after two more jumps of the Poisson process is related to this fixed maturity claim by:

$$P^{(2)}(S) = \sup_B \lambda \int_0^\infty e^{-\lambda t} D(S;t,B) dt. \quad (17)$$

Taking Laplace-Carson transforms of both sides of the p.d.e. (16) implies that:

$$\frac{\sigma^2}{2}S^2P_{ss}^{(2)}(S) + rSP_s^{(1)}(S) - rP^{(2)}(S) = \lambda[P^{(2)}(S) - P^{(1)}(S)], \quad S > \underline{S}_2, \quad (18)$$

subject to the following boundary conditions:

$$\lim_{S \uparrow \infty} P^{(2)}(S) = 0, \quad \lim_{S \downarrow \underline{S}_2} P^{(2)}(S) = K - \underline{S}_2, \quad \lim_{S \downarrow \underline{S}_2} P_s^{(2)}(S) = -1. \quad (19)$$

This simpler free boundary problem can be solved analytically for both the randomized put value  $P^{(2)}(S)$  and the critical stock price  $S_2$ . The graph of the American put value is similar to Figure 1, but with slightly higher value due to the lower variance in maturity. Figure 3 shows the exercise boundary for a realization in which the first jump happened to occur 0.53 years after issuance, while the put matured with the second jump 0.93 years after issuance. The critical stock price over the earlier of the two periods is below the critical stock price of the later period because the end of period payoff is greater (i.e.,  $P^{(1)}(S) \geq K - S$ ).

More generally, let  $P^{(m)}(S)$  and  $\underline{S}_m$  respectively denote the randomized put value and exercise boundary stair levels with  $m$  random periods to maturity,  $m = 0, 1, \dots, n$ , with  $P^{(0)}(S) \equiv (K - S)^+$  and  $\underline{S}_0 \equiv K$ .

Then  $P^{(m)}(S)$  and  $\underline{S}_m$  jointly solve the following sequence of free boundary problems:

$$\frac{\sigma^2}{2}S^2P_{ss}^{(m)}(S) + rSP_s^{(m)}(S) - rP^{(m)}(S) = \lambda[P^{(m)}(S) - P^{(m-1)}(S)], \text{ for } S \in (\underline{S}_m, \infty), \quad (20)$$

subject to the boundary conditions:

$$\lim_{S \uparrow \infty} P^{(m)}(S) = 0, \lim_{S \downarrow \underline{S}_m} P^{(m)}(S) = K - \underline{S}_m, \lim_{S \downarrow \underline{S}_m} P_s^{(m)}(S) = -1, \text{ for } m = 1, \dots, n. \quad (21)$$

Substituting  $\lambda \equiv \frac{1}{\Delta}$  on the right side of (20) and comparing with the Black Scholes p.d.e. (3) indicates an alternative interpretation of the approximation induced by our randomization procedure. Our randomized put value  $P^{(m)}(S)$  is also the approximation for  $P(T - m\Delta, S; T)$  which arises when time is discretized and the maturity derivative  $P_T(t, S; T) \equiv \frac{\partial P}{\partial T}(t, S; T)$  in (3) is replaced with the finite difference  $\frac{P^{(m)}(S) - P^{(m-1)}(S)}{\Delta} = \frac{\Delta P^{(m)}(S)}{\Delta}$ . Note however that the spatial derivatives are not replaced with their finite differences, in contrast to standard finite difference schemes or the binomial model.<sup>7</sup> As mentioned in the introduction the notion of discretizing one variable while leaving the other continuous is known in the numerical methods literature as semi-discretization or the method of lines.

The accuracy of our approach may be anticipated *a priori* by noting that as the maturity date  $T$  approaches infinity holding the number of periods  $n$  fixed, then  $\lambda \downarrow 0$  and thus the problem (20) describing the randomized put value approaches that of the perpetual put. As a result, the randomized put solution with any number of jumps remaining will converge to the correct perpetual solution. Conversely, as  $n$  gets arbitrarily large with  $T$  held fixed, then the finite difference  $\frac{\Delta P^{(m)}(S)}{\Delta}$  on the right side of (20) converges to the maturity derivative  $P_T(t, S; T)$  in (3). As a result, we conjecture<sup>8</sup> that the solution  $(P^{(n)}(S), \underline{S}_n)$  to our randomized option problem converges to the unknown solution  $(P(0, S; T), \underline{S}(0; T))$  of the American problem (1) or (3).

Recall from Section 2 that our formulas for random maturity option values depended on whether the option was in or out-of-the-money. Similarly, our formula<sup>9</sup> for the randomized put value,  $P^{(n)}(S)$ ,

depends on which interval  $(\underline{S}_i, \underline{S}_{i-1})$  contains the current spot price  $S$ :

$$P^{(n)}(S) = \begin{cases} p_0^{(n)}(S) + b_1^{(n)}(S) & \text{if } S > \underline{S}_0 \equiv K \\ v_i^{(n)}(S) + b_i^{(n)}(S) + A_i^{(n)}(S; 1) & \text{if } S \in (\underline{S}_i, \underline{S}_{i-1}], i = 1, \dots, n \\ K - S & \text{if } S \leq \underline{S}_n. \end{cases} \quad (22)$$

where  $p_0^{(n)}(S)$  is the out-of-the-money<sup>10</sup> value of a European put maturing in  $n$  (random-length) periods:

$$p_0^{(n)}(S) = \left(\frac{S}{K}\right)^{\gamma-\epsilon} \sum_{k=0}^{n-1} \frac{\left(2\epsilon \ln\left(\frac{S}{K}\right)\right)^k}{k!} \sum_{l=0}^{n-k-1} \binom{n-1+l}{n-1} [K R^n q^n p^{l+k} - K \hat{q}^n \hat{p}^{l+k}], \quad S > K, \quad (23)$$

with  $\Delta \equiv T/n$ ,  $\gamma \equiv \frac{1}{2} - \frac{r}{\sigma^2}$ ,

$$R \equiv \frac{1}{1+r\Delta}, \quad \epsilon \equiv \sqrt{\gamma^2 + \frac{2}{R\sigma^2\Delta}}, \quad (24)$$

$p, q, \hat{p}, \hat{q}$  given in (10), and for  $i = 1, \dots, n$ ,  $v_i^{(n)}(S)$  is the randomized value of a short forward position maturing in  $n - i + 1$  periods:

$$v_i^{(n)}(S) = K R^{n-i+1} - S,$$

$b_i^{(n)}(S)$  is the present value of interest received below the boundary for the first  $n - i + 1$  periods:

$$b_i^{(n)}(S) = \sum_{j=1}^{n-i+1} \left(\frac{S}{\underline{S}_{n-j+1}}\right)^{\gamma-\epsilon} \sum_{k=0}^{j-1} \frac{\left(2\epsilon \ln\left(\frac{S}{\underline{S}_{n-j+1}}\right)\right)^k}{k!} \sum_{l=0}^{j-k-1} \binom{j-1+l}{j-1} q^j p^{k+l} R^j K r \Delta, \quad (25)$$

and finally,  $A_i^{(n)}(S; 1)$  is the randomized value<sup>11</sup> of an out-of-the-money European call less interest paid above the boundary over the complementary period:

$$A_i^{(n)}(S; h) \equiv \sum_{j=h}^{n-i+1} \left(\frac{S}{\underline{S}_{n-j+1}}\right)^{\gamma+\epsilon} \sum_{k=0}^{j-1} \frac{\left(2\epsilon \ln\left(\frac{S}{\underline{S}_{n-j+1}}\right)\right)^k}{k!} \sum_{l=0}^{j-k-1} \binom{j-1+l}{j-1} p^j q^{k+l} R^j K r \Delta. \quad (26)$$

The formula in the first line of (22) again reflects the randomized version of the well-known decomposition of the American put value into the value of the corresponding European put and the early exercise premium. The formula in the second line is the randomized version of a new decomposition of the American put value into the value if forced to sell at a given date prior to expiration, and the premia which arise because exercise can occur before or after this date. The final line of (22) indicates that the put

should be exercised immediately if the stock price  $S$  is at or below our approximation for the critical stock price  $\underline{S}_n$ .

The staircase levels comprising the exercise boundary can be determined by recursive solution of an explicit formula. Continuity at the strike price in each period  $m = 1, \dots, n$  implies  $c_1^{(m)}(K) = A_1^{(m)}(K; 1)$ , which in turn implies the following explicit solution for each critical stock price  $\underline{S}_m$ :

$$\underline{S}_m = K \left( \frac{pRKr\Delta}{c_1^{(m)}(K) - A_1^{(m)}(K; 2)} \right)^{\frac{1}{\gamma+\epsilon}}, \quad m = 1, \dots, n, \quad (27)$$

where from (51) in the appendix, the at-the-money call value with  $m$  periods to maturity simplifies to:

$$c_1^{(m)}(K) = \sum_{l=0}^{m-1} \binom{m-1+l}{m-1} [K \hat{p}^m \hat{q}^l - K R^m p^m q^l], \quad m = 1, \dots, n. \quad (28)$$

Since  $A^{(m)}$  in (27) depends on  $\underline{S}_{m-1}$  to  $\underline{S}_1$ , the critical stock prices must be solved recursively, with  $\underline{S}_1 = K \left( \frac{pRr\Delta}{\hat{p}-Rp} \right)^{\frac{1}{\gamma+\epsilon}}$ . For future use, we let  $\underline{S}(\Delta) \equiv \underline{S}_n$  denote the critical stock price at the initial time.

## 4 Implementation

Our solution (22) for the randomized put value  $P^{(n)}(S)$  is a triple sum. Clearly, we need the number of periods  $n$  to be small in order to achieve computational efficiency. This section describes how Richardson extrapolation can be used to provide accurate answers using just a few periods. Richardson extrapolation has been used previously to accelerate valuation schemes for American options. Geske and Johnson (1984) first used Richardson extrapolation in a financial context to speed up and simplify their compound option valuation model. In general, it is not a good idea to extrapolate on the number of time steps in the binomial model (see Rendleman and Bartter (1979) and Cox, Ross, and Rubinstein (1979)) due to oscillatory nature of the convergence. However, Broadie and Detemple (1996) successfully use Richardson extrapolation to accelerate a hybrid of the binomial and Black-Scholes models. Furthermore, Liesen (1997) and Rogers and Stapleton (1997) show that randomizing the length of the time steps in the binomial model permits the successful use of extrapolation. Finally, Huang, Subrahmanyam, and Yu (1996) and Ju (1997) use the approach to accelerate the integral representation of the early exercise premium.

Denote our approximation (22) by a function  $\hat{P}(\Delta)$  of the mean period length  $\Delta$ . Richardson extrapolation can be used when the approximation can be adequately described by the first  $N$  terms in a Taylor series expansion about the origin:

$$\hat{P}(\Delta) = \sum_{n=0}^{N-1} \frac{\partial^n \hat{P}(0)}{\partial \Delta^n} \frac{\Delta^n}{n!} + O(\Delta^N). \quad (29)$$

The explicit nature of our solution (22) can be used to show that our approximation has the requisite smoothness for any  $N$ . If we ignore the terms of  $O(\Delta^N)$  in (29), then the  $N$  coefficients  $\frac{\partial^n \hat{P}(0)}{\partial \Delta^n}$ ,  $n = 0, 1, \dots, N - 1$  can be determined by using any  $N$  values of  $\Delta$  for which  $\hat{P}(\Delta)$  is known. The  $N$  point Richardson extrapolation is then the first coefficient  $\hat{P}(0)$ . From (29), this extrapolation has error of order  $O(\Delta^N)$ .

For example, a 3 point Richardson extrapolation can be obtained by assuming that our approximation is approximately quadratic in the mean period length:

$$\hat{P}(\Delta) \approx \hat{P}(0) + \hat{P}'(0)\Delta + \frac{1}{2}\hat{P}''(0)\Delta^2.$$

Substituting in  $\Delta = T, \Delta = T/2$  and  $\Delta = T/3$  leads to 3 equations in the 3 unknowns  $\hat{P}(0), \hat{P}'(0)$ , and  $\hat{P}''(0)$ . Inverting the system implies that the 3 point extrapolation is given by:

$$\hat{P}^{1:3}(0) \equiv \frac{1}{2}\hat{P}(T) - 4\hat{P}(T/2) + \frac{9}{2}\hat{P}(T/3). \quad (30)$$

Figures 4 and 5 illustrate the idea behind a 3 point extrapolation. From Marchuk and Shaidurov (1983), p. 24, an  $N$  point Richardson extrapolation is the following weighted<sup>12</sup> average of  $N$  randomized put values:

$$\hat{P}^{1:N}(0) \equiv \sum_{n=1}^N \frac{(-1)^{N-n} n^N}{n!(N-n)!} \hat{P}(T/n). \quad (31)$$

The critical stock price can be obtained by imposing either of the smooth pasting conditions in (21) or<sup>13</sup> by Richardson extrapolation:

$$\underline{S}^{1:N}(0) \equiv \sum_{n=1}^N \frac{(-1)^{N-n} n^N}{n!(N-n)!} \underline{S}(T/n). \quad (32)$$

The effectiveness of Richardson extrapolation is illustrated by a typical test case:  $S = 100, K = 100, T = 1, r = 0.1$ , and  $\sigma = 0.3$ . The true value based on the binomial method with 2000 time steps appears to be 8.3378. Table 1 shows that for this test case, the unextrapolated values approach the true value very slowly from below. In contrast, the extrapolated put values converge rapidly to this true value, with penny accuracy obtained in only 5 points. Table 2 elaborates on the calculation of the first two unextrapolated values in Table 1. Besides indicating typical values of some of the variables, it should aid in the reproduction of the results of Table 1.

Broadie and Detemple (1996) and Ju (1997) conduct extensive numerical simulations of a wide array of methods for valuing American options. Both papers conclude that three approaches dominate other methods in terms of speed and accuracy. These three methods are the lower and upper bound approximation (LUBA) in Broadie and Detemple (1996), the piecewise exponential boundary approximation in Ju (1997), and the randomization approach discussed in this paper. Of these three methods, LUBA has the singular advantage of providing bounds as well as an accurate approximation. The randomization approach is unique in that the exercise boundary is given by a recursion rather than root finding, when dividends are constant or zero. Finally, Ju's exponential boundary approach appears to deliver the best combination of speed and accuracy, although speed comparisons at each accuracy level were not conducted.

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## 5 Extension to Positive Dividends and American Calls

It is reasonable to assume that the dividend stream from the underlying asset is continuous over time if the asset underlying the option is an index or a basket with a large number of stocks. Merton (1973) generalized the Black-Scholes analysis to continuously-paid dividends which are either constant or proportional to the price of the underlying. He did not permit a dividend rate which is linear in the spot price, presumably due to the difficulty in generating analytic solutions under this assumption. While we are

also unable to deal with a linear dividend rate, this section develops formulas for randomized American option values when the dividend payout rate has both a fixed and a proportional component. We also show that our approximation to the put's critical stock price is still given by an explicit formula when dividends are constant, but must be determined numerically when there is a proportional component to the dividend flow. Finally, we show how to find the randomized values of American calls on dividend paying stocks.

We assume that the underlying stock pays dividends continuously until the fixed maturity  $T$ . To obtain a truly fixed component  $\phi$  of this dividend flow, we follow Roll (1979) in assuming that this component has been escrowed out of the stock price. In other words, the time  $t$  stock price  $S_t$  decomposes into:

$$S_t = \frac{\phi}{r}[1 - e^{-r(T-t)}] + s_t, \quad t \in [0, T], \quad (33)$$

where the first term is the present value at  $t$  of the constant flow  $\phi$  until  $T$ , and the residual  $s_t$  is the *stripped price*, reflecting the stripping off of the fixed component of the dividend flow from the stock price. We assume that the risk-neutralized process for the stripped price  $\{s_t, t \in [0, T]\}$  is the following geometric Brownian motion:

$$s_t = s \exp \left[ \left( r - \delta - \frac{\sigma^2}{2} \right) t + \sigma W_t \right], \quad t \in [0, T], \quad (34)$$

where  $\{W_t, t \in [0, T]\}$  is a standard Brownian motion, and from (33), the initial value is:

$$s = S - \frac{\phi}{r}[1 - e^{-rT}]. \quad (35)$$

Thus, the dollar dividend rate  $d_t$  has both a fixed and a proportional component:

$$d_t = \phi + \delta s_t, \quad t \in [0, T]. \quad (36)$$

The parameter  $\phi$  captures the stickiness of dividends in the short run, while  $\delta$  captures the tendency for dividends to increase with stock prices in the long run. If  $\delta = 0$ , then  $\phi$  is the constant dividend rate, while if  $\phi = 0$ , then  $\delta$  is the constant dividend *yield*, since  $s_t = S_t$  from (33).

## 5.1 Positive Dividends and American Puts

We generalize the previous analysis by letting  $P(t, s; T)$  denote the value of an American put as a function of the current time  $t$ , the current *stripped* price  $s$ , and the maturity date  $T$ . We also define the *critical stripped price*  $\underline{s}(t)$  as the largest stripped price  $s$  at which the American put value  $P(t, s; T)$  equals its exercise value  $K - s - \frac{\phi}{r}[1 - e^{-r(T-t)}]$ , for  $t \in [0, T]$ . From (33), the critical stock price  $\underline{S}(t)$  is now defined by:

$$\underline{S}(t) \equiv \frac{\phi}{r}[1 - e^{-r(T-t)}] + \underline{s}(t), \quad t \in [0, T]. \quad (37)$$

In the random maturity setting, the underlying stock pays dividends continuously until the option matures. Recalling that  $R \equiv \frac{1}{1+r\Delta}$  is the discount factor over a single period of random length, the random maturity analog of (35) is:

$$s = S - \phi\Delta(R + R^2 + \dots + R^n) = S - \frac{\phi}{r}R(1 - R^n). \quad (38)$$

We define  $P^{(m)}(s)$  as our approximation for the American put value when  $m$  random periods remain,  $m = 1, \dots, n$ . Our approximation for the critical stripped price,  $\underline{s}_m$ , is the largest  $s$  satisfying  $P^{(m)}(s) = K - s - \frac{\phi}{r}R(1 - R^m)$ ,  $m = 1, \dots, n$ .

The values of European options maturing in  $n$  random-length periods are:

$$p^{(n)}(s) = \begin{cases} \left(\frac{s}{K}\right)^{\gamma-\epsilon} \sum_{k=0}^{n-1} \frac{(2\epsilon \ln(\frac{s}{K}))^k}{k!} \sum_{l=0}^{n-k-1} \binom{n-1+l}{n-1} [K R^n q^n p^{k+l} - K D^n \hat{q}^n \hat{p}^{k+l}] & \text{if } s > K \\ K R^n - s D^n + c^{(n)}(s) & \text{if } s \leq K \end{cases} \quad (39)$$

$$c^{(n)}(s) = \begin{cases} s D^n - K R^n + p^{(n)}(s) & \text{if } s > K \\ \left(\frac{s}{K}\right)^{\gamma+\epsilon} \sum_{k=0}^{n-1} \frac{(2\epsilon \ln(\frac{K}{s}))^k}{k!} \sum_{l=0}^{n-k-1} \binom{n-1+l}{n-1} [K D^n \hat{p}^n \hat{q}^{k+l} - K R^n p^n q^{k+l}] & \text{if } s \leq K, \end{cases} \quad (40)$$

where now  $\gamma \equiv \frac{1}{2} - \frac{r-\delta}{\sigma^2}$ ,  $R$ ,  $\epsilon$ ,  $p$ ,  $q$ ,  $\hat{p}$ ,  $\hat{q}$ , are again given by (24) and (10), while:

$$D \equiv \frac{1}{1 + \delta\Delta}. \quad (41)$$

For  $\delta = 0$  and  $\phi \geq rK$ , American puts are not rationally exercised early. Consequently, the randomized put value  $P^{(n)}(s)$  is given by (39) in this case. For  $\delta > 0$  or  $\phi < rK$ , the randomized put value decomposes

as:

$$P^{(n)}(s) = \begin{cases} p_0^{(n)}(s) + b_1^{(n)}(s) & \text{if } s > \underline{s}_0 \equiv K \\ v_i^{(n)}(s) + b_i^{(n)}(s) + A_i^{(n)}(s; 1) & \text{if } s \in (\underline{s}_i, \underline{s}_{i-1}], i = 1, \dots, n \\ K - S & \text{if } s \leq \underline{s}_n, \end{cases} \quad (42)$$

where for  $i = 1, \dots, n$ ,  $v_i^{(n)}(s)$  is the randomized value of a short forward position maturing in  $n - i + 1$  periods:

$$v_i^{(n)}(s) = KR^{n-i+1} - sD^{n-i+1} - \phi R \frac{R^{n-i+1} - R^n}{1 - R}, \quad (43)$$

$b_i^{(n)}(s)$  is the present value of the interest less dividends (net interest) received when below the boundary for the first  $n - i + 1$  periods:

$$b_i^{(n)}(s) = \sum_{j=1}^{n-i+1} \left( \frac{s}{\underline{s}_{n-j+1}} \right)^{\gamma-\epsilon} \sum_{k=0}^{j-1} \frac{\left( 2\epsilon \ln \left( \frac{s}{\underline{s}_{n-j+1}} \right) \right)^k}{k!} \sum_{l=0}^{j-k-1} \binom{j-1+l}{j-1} \cdot [q^j p^{k+l} R^j (Kr - \phi) - \hat{q}^j \hat{p}^{k+l} D^j \underline{s}_{n-j+1} \delta] \Delta, \quad (44)$$

while  $A_i^{(n)}(s; 1)$  represents the randomized value of a European call less the net interest paid above the boundary over the complementary period, after accounting for the smoothness at the exercise boundary in every period:

$$A_i^{(n)}(s; h) = \sum_{j=h}^{n-i+1} \left( \frac{s}{\underline{s}_{n-j+1}} \right)^{\gamma+\epsilon} \sum_{k=0}^{j-1} \frac{\left( 2\epsilon \ln \left( \frac{\underline{s}_{n-j+1}}{s} \right) \right)^k}{k!} \sum_{l=0}^{j-k-1} \binom{j-1+l}{j-1} \cdot [p^j q^{k+l} R^j (Kr - \phi) - \hat{p}^j \hat{q}^{k+l} D^j \underline{s}_{n-j+1} \delta] \Delta. \quad (45)$$

Continuity in  $s$  at the strike price in each period  $m = 1, \dots, n$  again implies  $c_1^{(m)}(K) = A_1^{(m)}(K; 1)$ , which in turn implies that each critical stripped price  $\underline{s}_m$  implicitly solves:

$$c_1^{(m)}(K) - A_1^{(m)}(K; 2) = \left( \frac{K}{\underline{s}_m} \right)^{\gamma+\epsilon} [pR(Kr - \phi) - \hat{p}D\underline{s}_m\delta] \Delta, \quad m = 1, \dots, n, \quad (46)$$

where from (40), the at-the-money call value on the left hand side (LHS) of (46) simplifies to:

$$c_1^{(m)}(K) = \sum_{l=0}^{m-1} \binom{m-1+l}{m-1} [K D^m \hat{p}^m \hat{q}^l - K R^m p^m q^l] \quad m = 1, \dots, n. \quad (47)$$

It is straightforward to recursively solve (46) numerically for each critical stripped price  $\underline{s}_m$ , since  $\underline{s}_m$  does not appear on the LHS. Setting  $\delta = 0$  in (46) implies the following explicit solution for the critical stripped prices when the dividend rate is constant at  $\phi$ :

$$\underline{s}_m = K \left( \frac{pR(Kr - \phi)\Delta}{c_1^{(m)}(K) - A_1^{(m)}(K; 2)} \right)^{\frac{1}{\gamma + \epsilon}}, \quad m = 1, \dots, n, \quad (48)$$

where the call value  $c_1^{(m)}(K)$  is now given by (28). This solution is a good initial guess when numerically solving (46). From (38), each critical stock price  $\underline{S}_m$  is determined by:

$$\underline{S}_m = \frac{\phi}{r} R(1 - R^m) + \underline{s}_m, \quad m = 1, \dots, n, \quad (49)$$

where  $\underline{s}_m$  is given by (48) when  $\delta = 0$  and solves (46) otherwise. Letting  $\underline{S}(\Delta) \equiv \underline{S}_n$  denote the initial critical stock price as a function of the mean period length  $\Delta$ , one can use Richardson extrapolation (32) to approximate the initial critical stock price for an American put on a dividend paying stock.

## 5.2 Positive Dividends and American Calls

When there is no fixed component to the dividend (i.e.  $\phi = 0$ ), an American put call symmetry result can be used to easily value American calls on stocks with a constant dividend yield  $\delta$ . Let  $P(S, K; \delta, r)$  and  $C(S, K; \delta, r)$  denote the respective values of American puts and calls with fixed maturity  $T$ . Working in the binomial model, McDonald and Schroder (1990) show that:

$$C(S, K; \delta, r) = P(K, S; r, \delta).$$

In words, the call value can be obtained from the put valuation formula by switching the stock price and strike price, and also by switching the riskfree rate and dividend yield. This result is proved in the Black Scholes model by Schroeder (1997) and Carr and Chesney (1997), who also prove the corresponding result for critical stock prices:

$$\bar{S}(\delta, r) = \frac{K^2}{\underline{S}(r, \delta)}.$$

In words, the critical stock price for an American call can be obtained from that of an American put by switching the riskfree rate and dividend yield, and then obtaining the geometric reflection in the strike.

It can be shown that these symmetry results also hold for randomized option values and critical stock prices. Furthermore, randomized American calls can be valued directly when there is also a fixed component to the dividend flow. The appendix presents the formulas for the call value and critical stock price in this case.

## 6 Summary

We implemented a new approach to valuing American options, which is fast, accurate, and flexible. The approach is to value options which mature by definition at the  $n$ -th jump time of a standard Poisson process. Between jump times, the memoryless property of the exponential distribution implies that the option value and exercise boundary are time-stationary. In contrast, at jump times, the option value jumps down and the exercise boundary jumps nearer to the strike price. The local time-stationarity yields semi-explicit solutions for the option value and critical stock price, while the jump behavior roughly captures the global behavior of these values. As we let the number of jump times approach infinity, keeping the mean maturity fixed, our numerical results indicate that the randomized option value appears to converge smoothly from below to the true American option value. This convergence is dramatically enhanced through the use of Richardson extrapolation.

## Appendix

This appendix collects all the formulas needed to calculate random maturity values of European and American puts and calls when the underlying has a continuous payout with a fixed component  $\phi$  and a proportional component  $\delta$ . Letting  $s = S - \frac{\phi}{r}[1 - e^{-rT}]$ , the  $N$ -point Richardson extrapolation of the randomized European put formula is:

$$p^{1:N}(s) \equiv \sum_{n=1}^N \frac{(-1)^{N-n} n^N}{n!(N-n)!} p^{(n)}(s),$$

where:

$$p^{(n)}(s) = \begin{cases} \left(\frac{s}{K}\right)^{\gamma-\epsilon} \sum_{k=0}^{n-1} \frac{(2\epsilon \ln(\frac{s}{K}))^k}{k!} \sum_{l=0}^{n-k-1} \binom{n-1+l}{n-1} [K R^n q^n p^{k+l} - K D^n \hat{q}^n \hat{p}^{k+l}] & \text{if } s > K \\ K R^n - s D^n + c^{(n)}(s) & \text{if } s \leq K, \end{cases} \quad (50)$$

and where:

$$\begin{aligned} \gamma &\equiv \frac{1}{2} - \frac{r - \delta}{\sigma^2}, \quad \Delta \equiv \frac{T}{n}, \quad R \equiv \frac{1}{1 + r\Delta}, \quad D \equiv \frac{1}{1 + \delta\Delta}, \quad \epsilon \equiv \sqrt{\gamma^2 + \frac{2}{R\sigma^2\Delta}}, \\ p &\equiv \frac{\epsilon - \gamma}{2\epsilon}, \quad q \equiv 1 - p, \quad \hat{p} \equiv \frac{\epsilon - \gamma + 1}{2\epsilon}, \quad \text{and } \hat{q} \equiv 1 - \hat{p}. \end{aligned}$$

The  $N$ -point Richardson extrapolation of the randomized put formula is  $P^{1:N}(s) \equiv \sum_{n=1}^N \frac{(-1)^{N-n} n^N}{n!(N-n)!} P^{(n)}(s)$ ,

where:

$$P^{(n)}(s) = \begin{cases} p_0^{(n)}(s) + b_1^{(n)}(s) & \text{if } s > \underline{s}_0 \equiv K \\ v_i^{(n)}(s) + b_i^{(n)}(s) + A_i^{(n)}(s; 1) & \text{if } s \in (\underline{s}_i, \underline{s}_{i-1}], i = 1, \dots, n \\ K - S & \text{if } s \leq \underline{s}_n, \end{cases}$$

where for  $i = 1, \dots, n$ ,  $v_i^{(n)}(s) = K R^{n-i+1} - s D^{n-i+1} - \frac{\phi}{r} R (R^{n-i+1} - R^n)$ ,

$$\begin{aligned} b_i^{(n)}(s) &= \sum_{j=1}^{n-i+1} \left(\frac{s}{\underline{s}_{n-j+1}}\right)^{\gamma-\epsilon} \sum_{k=0}^{j-1} \frac{\left(2\epsilon \ln\left(\frac{s}{\underline{s}_{n-j+1}}\right)\right)^k}{k!} \sum_{l=0}^{j-k-1} \binom{j-1+l}{j-1} \\ &\quad \left[ q^j p^{k+l} R^j (K r - \phi) - \hat{q}^j \hat{p}^{k+l} D^j \underline{s}_{n-j+1} \delta \right] \Delta, \end{aligned}$$

$$\begin{aligned} A_i^{(n)}(s; h) &= \sum_{j=h}^{n-i+1} \left(\frac{s}{\underline{s}_{n-j+1}}\right)^{\gamma+\epsilon} \sum_{k=0}^{j-1} \frac{\left(2\epsilon \ln\left(\frac{\underline{s}_{n-j+1}}{s}\right)\right)^k}{k!} \sum_{l=0}^{j-k-1} \binom{j-1+l}{j-1} \\ &\quad \left[ p^j q^{k+l} R^j (K r - \phi) - \hat{p}^j \hat{q}^{k+l} D^j \underline{s}_{n-j+1} \delta \right] \Delta. \end{aligned}$$

If  $\delta = 0$ , the critical stripped prices are given by  $\underline{s}_m = K \left( \frac{pR(Kr - \phi)\Delta}{c_1^{(m)}(K) - A_1^{(m)}(K; 2)} \right)^{\frac{1}{\gamma + \epsilon}}$ ,  $m = 1, \dots, n$ .

If  $\delta > 0$ , the critical stripped prices solve:

$$\sum_{l=0}^{m-1} \binom{m-1+l}{m-1} [K D^m \hat{p}^m \hat{q}^l - K R^m p^m q^l] - A_1^{(m)}(K; 2) = \left( \frac{K}{\underline{s}_m} \right)^{\gamma + \epsilon} [pR(Kr - \phi) - \hat{p}D\underline{s}_m\delta]\Delta, \quad m = 1, \dots, n.$$

Letting  $\underline{s}(T/n) \equiv \underline{s}_n$  denote the solution obtained by recursing on  $\bar{s}_m$ , the  $N$ -point Richardson extrapolation of the put's initial critical stock price is  $\underline{S}^{1:N} \equiv \frac{\phi}{r}[1 - e^{-rT}] + \sum_{n=1}^N \frac{(-1)^{N-n} n^N}{n!(N-n)!} \underline{s}(T/n)$ .

Similarly, letting  $s = S - \frac{\phi}{r}[1 - e^{-rT}]$ , the  $N$ -point Richardson extrapolation of the randomized European call formula is  $c^{1:N}(s) \equiv \sum_{n=1}^N \frac{(-1)^{N-n} n^N}{n!(N-n)!} c^{(n)}(s)$ , where:

$$c^{(n)}(s) = \begin{cases} sD^n - KR^n + p^{(n)}(s) & \text{if } s > K \\ \left(\frac{s}{K}\right)^{\gamma + \epsilon} \sum_{k=0}^{n-1} \frac{(2\epsilon \ln(\frac{K}{s}))^k}{k!} \sum_{l=0}^{n-k-1} \binom{n-1+l}{n-1} [KD^n \hat{p}^n \hat{q}^{k+l} - KR^n p^n q^{k+l}] & \text{if } s \leq K, \end{cases} \quad (51)$$

and where again:

$$\begin{aligned} \gamma &\equiv \frac{1}{2} - \frac{r - \delta}{\sigma^2}, \quad \Delta \equiv \frac{T}{n}, \quad R \equiv \frac{1}{1 + r\Delta}, \quad D \equiv \frac{1}{1 + \delta\Delta}, \quad \epsilon \equiv \sqrt{\gamma^2 + \frac{2}{R\sigma^2\Delta}}, \\ p &\equiv \frac{\epsilon - \gamma}{2\epsilon}, \quad q \equiv 1 - p, \quad \hat{p} \equiv \frac{\epsilon - \gamma + 1}{2\epsilon}, \quad \text{and } \hat{q} \equiv 1 - \hat{p}. \end{aligned}$$

For  $\delta = 0$  and  $\phi \leq rK$ , early exercise is not optimal so the randomized call value is given by (51).

For  $\delta > 0$  or  $\phi > rK$ , the  $N$  point Richardson extrapolation of the randomized call value is  $C^{1:N}(s) \equiv \sum_{n=1}^N \frac{(-1)^{N-n} n^N}{n!(N-n)!} C^{(n)}(s)$ , where:

$$C^{(n)}(s) = \begin{cases} S - K & \text{if } s \geq \bar{s}_n \\ -v_i^{(n)}(s) + \alpha_i^{(n)}(s) + B_i^{(n)}(s; 1) & \text{if } s \in [\bar{s}_{i-1}, \bar{s}_i], i = 1, \dots, n \\ c_0^{(n)}(s) + \alpha_1^{(n)}(s) & \text{if } s < \bar{s}_0 \equiv K \end{cases}$$

where for  $i = 1, \dots, n$ ,  $-v_i^{(n)}(s) = sD^{n-i+1} + \frac{\phi}{r}R(R^{n-i+1} - R^n) - KR^{n-i+1}$  is the initial value of a long forward position maturing in  $n - i + 1$  periods,

$$\begin{aligned} \alpha_i^{(n)}(s) &= \sum_{j=1}^{n-i+1} \left( \frac{s}{\bar{s}_{n-j+1}} \right)^{\gamma + \epsilon} \sum_{k=0}^{j-1} \frac{(2\epsilon \ln(\frac{\bar{s}_{n-j+1}}{s}))^k}{k!} \sum_{l=0}^{j-k-1} \binom{j-1+l}{j-1} \\ &\quad \left[ \hat{p}^j \hat{q}^{k+l} D^j \bar{s}_{n-j+1} \delta - p^j q^{k+l} R^j (Kr - \phi) \right] \Delta, \end{aligned}$$

is the initial value of dividends less interest received above the boundary for the first  $n - i + 1$  periods, while:

$$B_i^{(n)}(s; h) = \sum_{j=h}^{n-i+1} \left( \frac{s}{\bar{s}_{n-j+1}} \right)^{\gamma-\epsilon} \sum_{k=0}^{j-1} \frac{\left( 2\epsilon \ln \left( \frac{s}{\bar{s}_{n-j+1}} \right) \right)^k}{k!} \sum_{l=0}^{j-k-1} \binom{j-1+l}{j-1} \left[ \hat{q}^j \hat{p}^{k+l} D^j \bar{s}_{n-j+1} \delta - q^j p^{k+l} R^j (Kr - \phi) \right] \Delta.$$

$B_i^{(n)}(s; h)$  is the initial value of a European put less the excess of dividends over interest received below the boundary over the complementary period, after accounting for the smoothness of the exercise boundary in every period. Continuity in  $s$  at  $K$  in each period implies that  $\bar{s}_m$  solves:

$$p_0^{(m)}(K) - B_1^{(m)}(K; 2) = \left( \frac{K}{\bar{s}_m} \right)^{\gamma-\epsilon} [\hat{q} D \bar{s}_m \delta - q R (Kr - \phi)] \Delta, \quad m = 1, \dots, n, \quad (52)$$

where from (50),  $p_0^{(m)}(K) = \sum_{l=0}^{m-1} \binom{m-1+l}{m-1} [K R^m q^m p^l - K D^m \hat{q}^m \hat{p}^l]$ . If  $\phi = rK$ , (52) can be solved, and  $\bar{s}_m = K \left( \frac{K \hat{q} D \delta \Delta}{p_0^{(m)}(K) - B_1^{(m)}(K; 2)} \right)^{\frac{1}{\gamma-\epsilon-1}}$ ,  $m = 1, \dots, n$ . This solution is a good initial guess when solving (52) numerically. Recursively solving for each  $\bar{s}_m$  results in  $\bar{s}(T/n) \equiv \bar{s}_n$ . The  $N$ -point Richardson extrapolation of the call's critical stock price is  $\bar{S}^{1:N}(T) \equiv \frac{\phi}{r} [1 - e^{-rT}] + \sum_{n=1}^N \frac{(-1)^{N-n} n^N}{n!(N-n)!} \bar{s}(T/n)$ .

## Footnotes

1. I thank the referee for this insight.
2. However, given the speed of modern computers, they argue that its inherent accuracy makes it the method of choice among those tested.
3. As usual, the first passage time is considered to be infinite if the boundary is never touched.
4. Note that the randomized value obtained in this paper is strictly smaller than the value of an exponentially weighted portfolio of true American puts, i.e.  $P^{(1)}(S) < \lambda \int_0^\infty e^{-\lambda t} P(0, S; t) dt$ . The reason is that the optimization over boundaries for our contract must be done with a random maturity. In contrast, the given integral simply averages American values over maturities, where each American value  $P(0, S; t)$  is calculated by optimizing over a fixed maturity  $t$ . I thank the editor, Kerry Back, for correcting a mistake on this point in an earlier draft.
5. The Laplace-Carson transform differs from the standard Laplace transform only by the introduction of a constant  $\lambda$  in the kernel. See Rubinstein and Rubinstein (1993), pgs. 512–517 for the properties of this transform.
6. Put Call Parity holds so long as the options and a forward contract mature at the same jump time.
7. The binomial model uses a forward finite difference for the maturity derivative leading to an explicit scheme. The appearance of a backward difference for the maturity derivative indicates that our randomization procedure may be considered as the limiting case of a fully implicit scheme, where the size of each space step is infinitesimally small. Surprisingly, this implicit scheme has a semi-explicit solution for an American option and a fully explicit solution for a European or barrier option.
8. While numerical implementation of our solution will prove to be consistent with this conjectured convergence, a formal proof of convergence remains an open question.

9. Note that (22) is closely related to the value of a fixed maturity American option when the variance rate is gamma distributed. See Madan and Chang (1997) for a closed form solution for European options.
10. See (51) and (50) in the appendix for the randomized values of European calls and in-the-money European puts respectively.
11. This value also accounts for the smoothness at the exercise boundary in every period.
12. The weights always sum to unity and alternate in sign. In general, higher order approximations involve weights with greater absolute value. As a result, implementing higher order extrapolations on a computer requires double precision to control roundoff error.
13. We prefer the former method when accuracy is important and the latter method when speed matters.

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Table 1: Convergence of Randomized Put Value to American without and with Richardson

Extrapolation

$S = 100, K = 100, T = 1, r = 0.1, \delta = 0, \sigma = 0.3$

Number of Steps $n$ or Points $N$	Unextrapolated Put Value $P^{(n)}$	Extrapolated Put Value $P^{1:N}$
1	7.0405	7.0405
2	7.6175	8.1946
3	7.8353	8.3089
4	7.9505	8.3257
5	8.0220	8.3311
6	8.0709	8.3333
7	8.1065	8.3345
8	8.1335	8.3353
9	8.1548	8.3358
10	8.1720	8.3362
11	8.1862	8.3365
12	8.1981	8.3367
13	8.2082	8.3369
14	8.2169	8.3370
15	8.2246	8.3371

Table 2: Intermediate Put Values

Variable	n=1	n=2
$\Delta$	1.0000	0.5000
$\gamma$	-0.6111	-0.6111
$R$	0.9091	0.9524
$\epsilon$	4.9818	6.8586
$p$	0.5613	0.5446
$q$	0.4387	0.4554
$\hat{p}$	0.6617	0.6175
$\hat{q}$	0.3383	0.3825
$\underline{S}_1$	77.9724	80.7216
$\underline{S}_2$	N/A	77.2941
$v_1^{(n)}(S)$	-9.0909	-9.2971
$b_1^{(n)}(S)$	0.9917	1.0176
$A_1^{(n)}(S; 1)$	15.1397	15.8970
$P^{(n)}(S)$	7.0405	7.6175

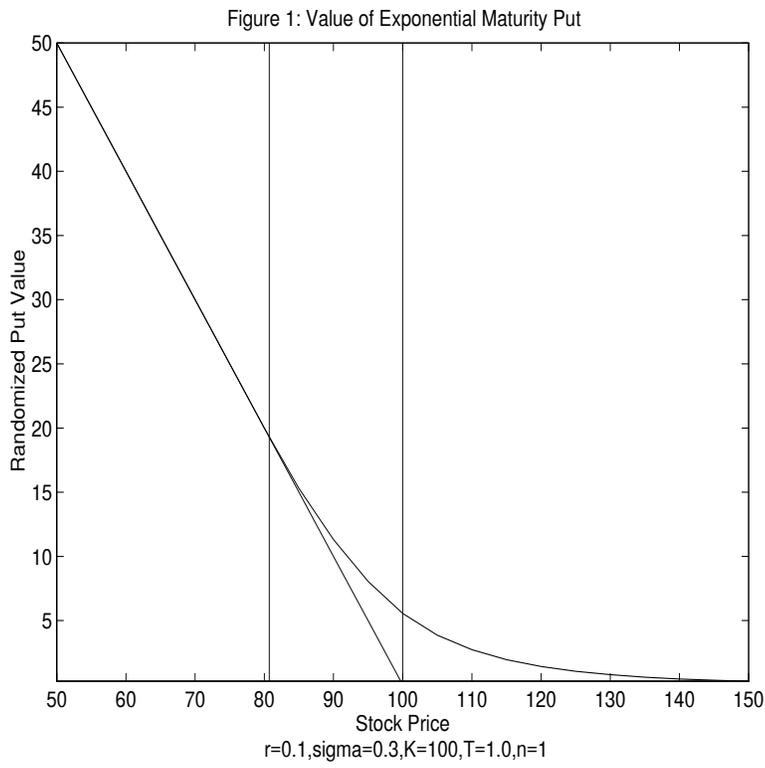


Figure 1: Value of Exponential Maturity Put

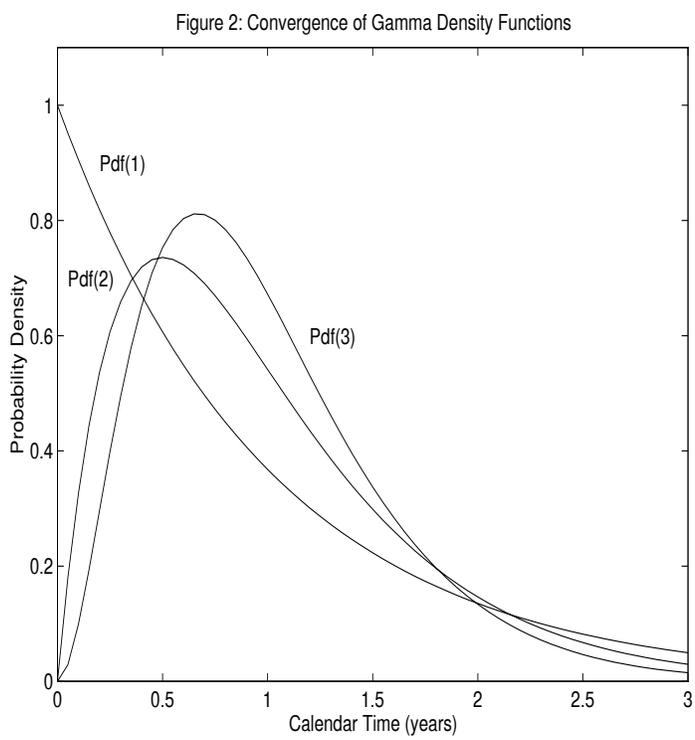


Figure 2: Convergence of Gamma Density Functions

Figure 3 Exercise Boundary of Erlang Maturity Put

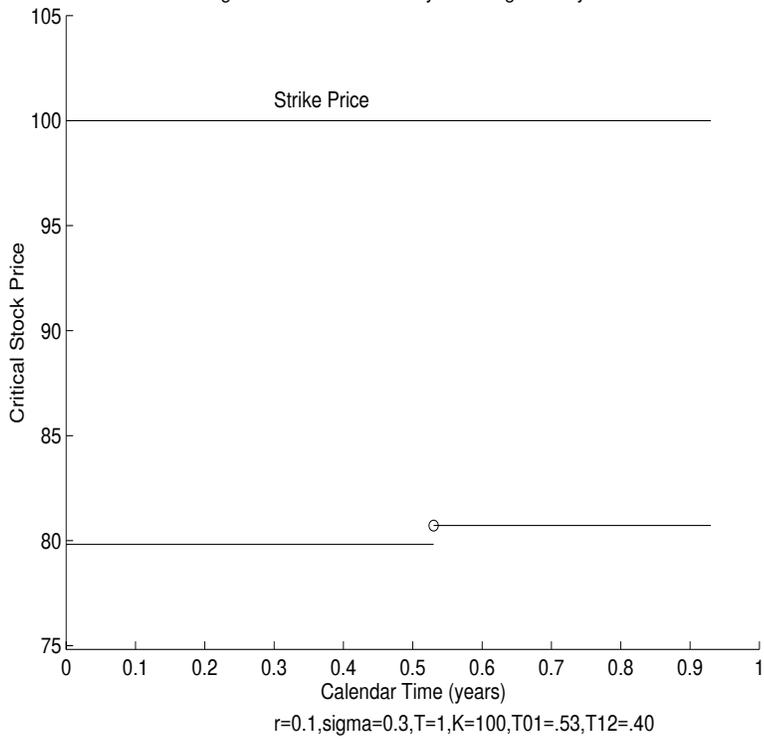


Figure 3: Exercise Boundary of Erlang Maturity Put

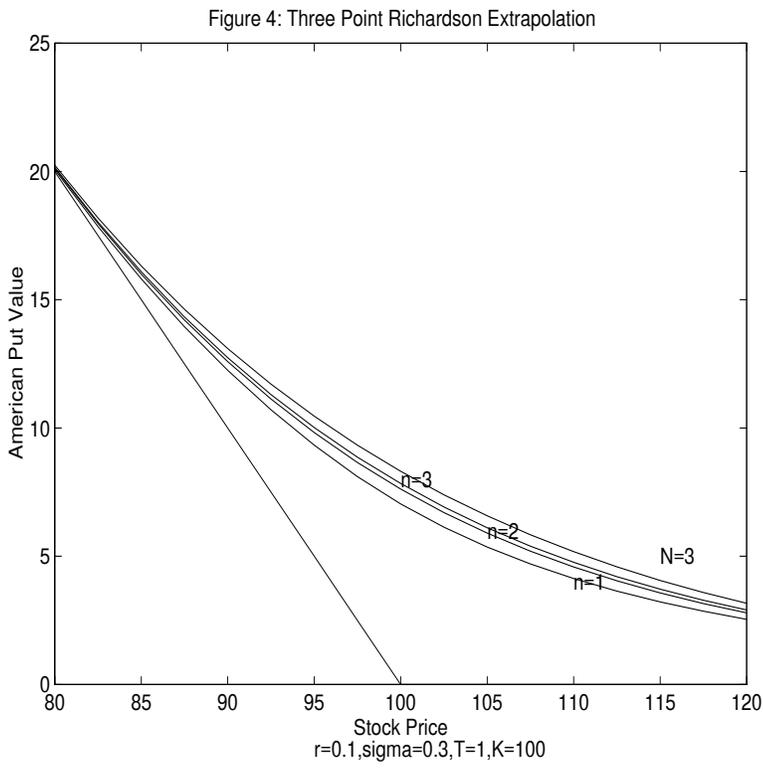


Figure 4: Three Point Richardson Extrapolation

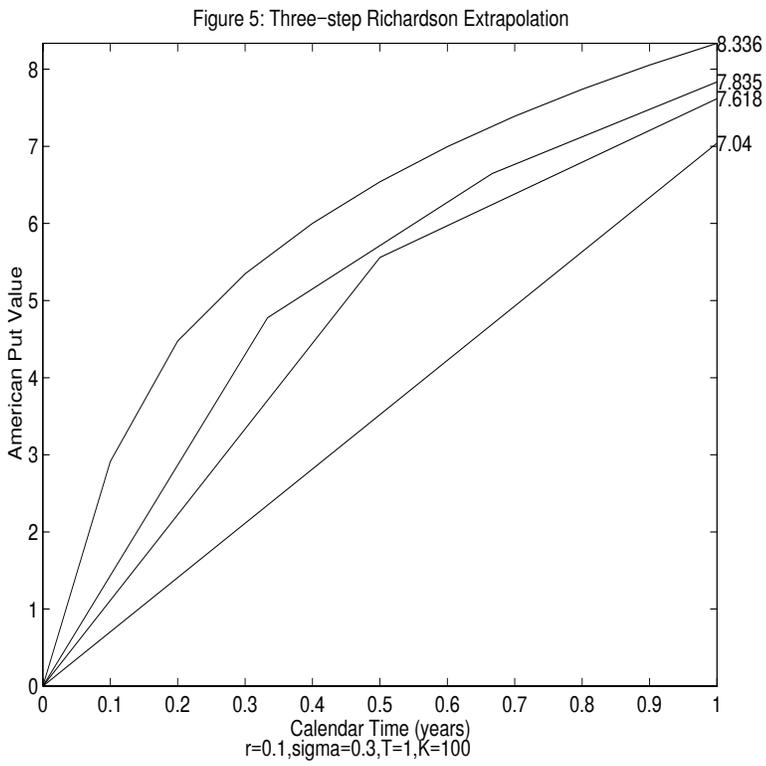


Figure 5: Three-step Richardson Extrapolation