A Note on the No Arbitrage Condition for International Financial Markets

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Abstract: We consider an international financial market model that consists of N currencies. The purpose is to derive a no arbitrage condition which is not affected by the choice of numéraire between the N currencies. As a result, we show that a finiteness condition for an arbitrary chosen currency and the no arbitrage condition for the basket currency are necessary and sufficient for the no arbitrage property of all the N currencies.
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1 Introduction

After the pioneering work by Harrison–Kreps [12], many researchers have studied the relationship between the existence of an equivalent martingale measure and the no arbitrage property. In this setting most authors use a fixed asset as numéraire. But in the case of an international economy model, an a priori choice of numéraire poses some problems to characterize the no arbitrage property.

Suppose that an international financial market consists of N currencies. Then from the viewpoint of economic efficiency, we should require that if we take any of the N currencies as numéraire and express the others in function of this choice, the so obtained price process should not allow arbitrage profits. This problem is not trivial. Delbaen–Schachermayer [5] and [7] have given an example of a two currency model with the property that when the first is chosen as numéraire, there is no arbitrage but on the contrary there is an arbitrage profit when the second is chosen as numéraire.

The purpose of this note is to derive a compact condition which guarantees the no arbitrage property, regardless of the currency chosen as numéraire. In particular we show that if a finiteness condition with respect to an arbitrary chosen currency holds, then the no arbitrage property with respect to a basket currency is necessary and sufficient for the no arbitrage property to hold, regardless which of the N currencies is chosen as numéraire.

The paper is organized as follows. In section 2, we summarize the fundamental results from Delbaen–Schachermayer [5]–[11] used in this note. In section 3, we explain the multi-

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currency international financial market model. In section 4, we investigate the finiteness condition, which turns out to be invariant for the choice of numéraire and for the choice of equivalent probability measure. Finally in section 5, we derive the compact no arbitrage condition.

Without loss of generality, we use the interval $[0, 1]$ as the time interval for the finite time horizon model. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le 1}, \boldsymbol{P})$ be a filtered probability space satisfying the usual conditions. The bold face characters denote column vectors, and the $'$ denotes the transpose. We introduce the notation $\boldsymbol{H}\cdot\boldsymbol{S}$ for the vector stochastic integral and we refer to Jacod [15] for details.

2 Summary of Results on Arbitrage Theory

We consider a financial market consisting of N assets numbered from 1 to N. Asset number 1 is chosen as numéraire. The price of asset k, $2 \le k \le N$, at time t is denoted by S_t^k, of course we have $S_t^1 = 1$. The process $\boldsymbol{S}=(S^1, \cdots, S^N)$ is supposed to be a continuous, vector-valued semi-martingale such that each coordinate is strictly positive.

Definition 2.1 *Let a be a positive real number. An $\boldsymbol{S}$-integrable predictable vector-valued process $\boldsymbol{H}=(H^1, \cdots, H^N)$ is called a-admissible if $\boldsymbol{H}_0 = \mathbf{o}$ and $(\boldsymbol{H}\cdot\boldsymbol{S})_t \ge -a$, P-a.s. for all $0 \le t \le 1$. The predictable process $\boldsymbol{H}$ is called admissible if it is a-admissible for some $a \in \boldsymbol{R}$.*

Definition 2.2 *We say that the vector-valued semi-martingale $\boldsymbol{S}$ satisfies the no arbitrage condition, (NA), for general admissible integrands, if for all $\boldsymbol{H}$ admissible we have that*

$$(\boldsymbol{H}\cdot\boldsymbol{S})_1 \ge 0, \ \textit{a.s. implies } (\boldsymbol{H}\cdot\boldsymbol{S})_1 = 0, \ \textit{a.s.}. \tag{2.1}$$

For the history and the use of this condition we refer to Delbaen–Schachermayer [6] and Harrison–Pliska [13].

If $\boldsymbol{M}$ is a continuous d-dimensional vector-valued local martingale, then the bracket process $\langle \boldsymbol{M}, \boldsymbol{M}\rangle$ is defined as a continuous process taking values in the space of $d \times d$ matrices. The elements are described by the usual brackets $\langle M_i, M_j\rangle$ where M_i denotes the i-th coordinate of $\boldsymbol{M}$. The Kunita–Watanabe inequality states that the process $\langle \boldsymbol{M}, \boldsymbol{M}\rangle$ takes values in the cone of positive definite symmetric matrices and the process is increasing in the sense that $\langle \boldsymbol{M}, \boldsymbol{M}\rangle_t - \langle \boldsymbol{M}, \boldsymbol{M}\rangle_s$ is a positive definite symmetric matrix for $s < t$.

Using a control measure λ, we can describe $\langle \boldsymbol{M}, \boldsymbol{M}\rangle$ as a process having a Radon–Nykodim derivative with respect to λ. For λ we can take the predictable increasing process $\lambda_t = \mathrm{trace}\langle \boldsymbol{M}, \boldsymbol{M}\rangle_t = \sum_{i=1}^d \langle M_i, M_i\rangle_t$. The process $\langle \boldsymbol{M}, \boldsymbol{M}\rangle$ can then be written as $\langle \boldsymbol{M}, \boldsymbol{M}\rangle_t = \int_0^t \boldsymbol{L}_u d\lambda_u$ where $\boldsymbol{L}$ is a predictable process having values in the cone of positive definite symmetric matrices. There is an easy way to see this, by using the following construction of the Radon–Nykodim derivative. For each $n \ge 1$, we define the process $\boldsymbol{L}^n$ as follows.

$$\begin{aligned}
\boldsymbol{L}_u^n &= \frac{\langle \boldsymbol{M}, \boldsymbol{M}\rangle_{\frac{k}{2^n}} - \langle \boldsymbol{M}, \boldsymbol{M}\rangle_{\frac{k-1}{2^n}}}{\lambda_{\frac{k}{2^n}} - \lambda_{\frac{k-1}{2^n}}}, \qquad \frac{k}{2^n} < u \le \frac{k+1}{2^n}, \ 1 \le k \le 2^n - 1 \\
\boldsymbol{L}_u^n &= 0, \qquad 0 \le u \le \frac{1}{2^n}
\end{aligned}$$

One shows that $d\lambda$ a.e., $\boldsymbol{L}^n \to \boldsymbol{L}$ on $[0,1]\times\Omega$, and $\int_0^1 \|\boldsymbol{L}_u^n - \boldsymbol{L}_u\| d\lambda_u \to 0$, in probability, for any matrix norm. The Kunita–Watanabe inequality shows that each $\boldsymbol{L}_u^n$ is a positive definite symmetric matrix and hence the same remains true for $\boldsymbol{L}_u$.

Using power series, we define the processes $\boldsymbol{I}_d - \exp(-n\boldsymbol{L})$. These processes are still predictable and when $n\to\infty$, the limit of $\boldsymbol{I}_d - \exp(-n\boldsymbol{L})$ tends to the projection $\mathcal{P}$ on the range of $\boldsymbol{L}$, $\mathcal{P}$ is therefore predictable, see also lemma 4.2 below. Remark that the kernel of $\mathcal{P}$ is also the kernel of $\boldsymbol{L}$. For a continuous semi-martingale $\boldsymbol{S}$, the following was proved in Delbaen–Schachermayer [10].

Theorem 2.3 *If $\boldsymbol{S}$ is a continuous vector-valued semi-martingale decomposed as*

$$\boldsymbol{dS}_t = \boldsymbol{dM}_t + \boldsymbol{dA}_t$$

where $\boldsymbol{M}$ is a continuous local martingale and $\boldsymbol{A}$ is a continuous process of bounded variation, (the Doob–Meyer decomposition of $\boldsymbol{S}$), then

(a) if $\boldsymbol{S}$ satisfies NA for general admissible integrands, there is a predictable vector process $\boldsymbol{h}$ such that

$$\boldsymbol{dA}_t = \boldsymbol{d}\langle \boldsymbol{M}, \boldsymbol{M}\rangle_t \boldsymbol{h}_t, \tag{2.2}$$

where $\boldsymbol{d}\langle \boldsymbol{M}, \boldsymbol{M}\rangle$ is the matrix measure and $\boldsymbol{h}$ is a vector-valued predictable process.

(b) Under the same hypothesis,

$$\tau = \inf\left\{ t \,\middle|\, \int_0^t \boldsymbol{h}_u' \boldsymbol{d}\langle \boldsymbol{M}, \boldsymbol{M}\rangle_u \boldsymbol{h}_u = \infty \right\} > 0, \ \textit{a.s.} \tag{2.3}$$

(c) The continuous semi-martingale $\boldsymbol{S}$ admits an equivalent local martingale measure, i.e. satisfies EMM, if and only if the following two conditions hold

(i) $\boldsymbol{S}$ has the NA property for general admissible integrands.

(ii) $\boldsymbol{S}$ satisfies the finiteness property

$$\int_0^1 \boldsymbol{h}_u' \boldsymbol{d}\langle \boldsymbol{M}, \boldsymbol{M}\rangle_u \boldsymbol{h}_u < \infty, \ \textit{a.s.}, \tag{2.4}$$

where $\boldsymbol{h}$ is defined by (2.2)

Remark 2.4 *The local martingale process*

$$L_t = \exp\left(- \int_0^t \boldsymbol{h}_u' \boldsymbol{dM}_u - \frac{1}{2}\int_0^t \boldsymbol{h}_u' \boldsymbol{d}\langle \boldsymbol{M}, \boldsymbol{M}\rangle \boldsymbol{h}_u \right) \tag{2.5}$$

is not necessarily a martingale and hence the obvious Girsanov–Maruyama transformation does not give an equivalent local martingale measure for $\boldsymbol{S}$ (see Delbaen–Schachermayer [9] and Schachermayer [16]).

Remark 2.5 *Under the hypothesis of the theorem, $\boldsymbol{H}$ is $\boldsymbol{S}$-integrable if and only if*

$$(1) \quad \int_0^1 \boldsymbol{H}_u' d\langle \boldsymbol{M}, \boldsymbol{M}\rangle_u \boldsymbol{H}_u = \int_0^1 \boldsymbol{H}_u' \boldsymbol{L}_u \boldsymbol{H}_u d\lambda_u < \infty, \quad \textit{a.s.}$$

and

$$(2) \quad \int_0^1 |\boldsymbol{H}_u' d\boldsymbol{A}_u| = \int_0^1 |\boldsymbol{H}_u' \boldsymbol{L}_u \boldsymbol{h}_u| d\lambda_u < \infty, \quad \textit{a.s.}$$

It is clear that in such expressions, we may replace $\boldsymbol{h}$ by its projection $\mathcal{P}\boldsymbol{h}$ and we see that $\boldsymbol{H}$ is $\boldsymbol{S}$-integrable if and only if $\mathcal{P}\boldsymbol{H}$ is $\boldsymbol{S}$-integrable. This follows easily from the fact that ker($\mathcal{P}$) = ker($\boldsymbol{L}$) and Range($\mathcal{P}$) = Range($\boldsymbol{L}$).

If $\boldsymbol{S}$ is a continuous vector-valued semi-martingale, then we denote by $\mathcal{M}^e(\boldsymbol{P})$ the set of all equivalent probability measures $\boldsymbol{Q}$ under which $\boldsymbol{S}$ becomes a (vector-valued) $\boldsymbol{Q}$-*local* martingale. We will also make us of the following sets:

$$\mathcal{K}_1 = \{(\boldsymbol{H}\cdot\boldsymbol{S})_1 | \boldsymbol{H} \text{ is 1-admissible}\}, \tag{2.6}$$
$$\mathcal{K} = \{(\boldsymbol{H}\cdot\boldsymbol{S})_1 | \boldsymbol{H} \text{ is admissible}\}. \tag{2.7}$$

We can easily see that $\mathcal{K}_1 \subset \mathcal{K}$ and $\mathcal{K} = \cup_{a>0}\mathcal{K}_a = \cup_{\lambda>0}\lambda\mathcal{K}_1$. The following theorems are proved in Delbaen–Schachermayer [8] and [11] (see also Jacka [14], Ansel–Stricker [3]). It makes use of the set of maximal elements defined as follows

Definition 2.6 *We say that $f \in \mathcal{K}_1$ is maximal in $\mathcal{K}_1$ if $g \in \mathcal{K}_1$ and $g \geq f$ imply $g = f$. The element $f \in \mathcal{K}$ is maximal in $\mathcal{K}$ if $g \in \mathcal{K}$ and $g \geq f$ imply $g = f$.*

Remark 2.7 *The (NA) property is equivalent to the statement that the zero function is maximal in $\mathcal{K}$ (or in $\mathcal{K}_1$). If $\boldsymbol{S}$ satisfies the (NA) property with respect to general admissible integrands, then $f \in \mathcal{K}_1$ is maximal in $\mathcal{K}_1$ if and only if f is maximal in $\mathcal{K}$.*

Theorem 2.8 *If $\boldsymbol{S}$ satisfies EMM, i.e. admits an equivalent local martingale measure, then for an element $f \in \mathcal{K}_1$ the following are equivalent :*

(a) f is a maximal element in $\mathcal{K}_1$.

(b) f is a maximal element in $\mathcal{K}$.

(c) There is an equivalent local martingale measure $\boldsymbol{Q} \in \mathcal{M}^e(\boldsymbol{P})$ such that $\boldsymbol{E}_Q[f] = 0$.

(d) There is a 1-admissible integrand $\boldsymbol{H}$ and there is an equivalent local martingale measure $\boldsymbol{Q} \in \mathcal{M}^e(\boldsymbol{P})$ such that $f = (\boldsymbol{H}\cdot\boldsymbol{S})_1$ and the process $\boldsymbol{H}\cdot\boldsymbol{S}$ is a $\boldsymbol{Q}$-uniformly integrable martingale.

Theorem 2.9 *If $\boldsymbol{S}$ satisfies EMM, i.e. admits an equivalent local martingale measure, then for an admissible integrand $\boldsymbol{H}$ such that $V = c + \boldsymbol{H} \cdot \boldsymbol{S}$ satisfies $V_t > 0$, a.s., for $0 \leq t \leq 1$, the following are equivalent :*

(a') $f = (\boldsymbol{H}\cdot\boldsymbol{S})_1$ is a maximal element in $\mathcal{K}$.

(b') There is an equivalent local martingale measure $\boldsymbol{Q} \in \mathcal{M}^e(\boldsymbol{P})$ such that $\sup\{\boldsymbol{E}_R[V_1] | \boldsymbol{R} \in \mathcal{M}^e(\boldsymbol{P})\} = \boldsymbol{E}_Q[V_1] < \infty$.

(c') There is an equivalent local martingale measure $\boldsymbol{Q} \in \mathcal{M}^e(\boldsymbol{P})$ such that V is a $\boldsymbol{Q}$-uniformly integrable martingale.

(d') The process $\frac{\boldsymbol{S}}{V}$ has an equivalent local martingale measure.

Theorem 2.10 *Suppose that $\boldsymbol{S}$ satisfies EMM, i.e. admits an equivalent local martingale measure. If $f_1, \cdots, f_n$ are maximal in $\mathcal{K}$, then $f_1 + \cdots + f_n$ is maximal in $\mathcal{K}$.*

Corollary 2.11 *Suppose that $\boldsymbol{S}$ satisfies EMM, i.e. admits an equivalent local martingale measure. If $f_1, \cdots, f_n$ are elements in $\mathcal{K}$ such that for each $j \leq n$, there is $\boldsymbol{Q}^j \in \mathcal{M}^e(\boldsymbol{P})$ with $\boldsymbol{E}_{Q^j}[f_j] = 0$, then there is $\boldsymbol{Q} \in \mathcal{M}^e(\boldsymbol{P})$ such that $\boldsymbol{E}_Q[f_j] = 0$ for each $j \leq n$.*

Proof. By the theorem, each f_j, $j \le n$ is maximal and hence $f_1 + \cdots + f_n$ is maximal. This implies the existence of $\boldsymbol{Q} \in \mathcal{M}^e(\boldsymbol{P})$ such that $\boldsymbol{E}_Q[f_1 + \cdots + f_n] = 0$. Since the elements f_j are in $\mathcal{K}$ and since $\boldsymbol{Q} \in \mathcal{M}^e(\boldsymbol{P})$, we necessarily have $\boldsymbol{E}_Q[f_j] \le 0$. But this implies that $\boldsymbol{E}_Q[f_j] = 0$ for each j. $\square$

From Theorem 2.9–2.10 and Corollary 2.12, it follows that

Corollary 2.12 *If $V^j = c^j + \boldsymbol{H}^j \cdot \boldsymbol{S} > 0$, $j = 1, \cdots, J < \infty$ are stochastic integrals such that for each j, there is an equivalent probability measure $\boldsymbol{Q}^j \in \mathcal{M}^e(\boldsymbol{P})$ for which V^j is a $\boldsymbol{Q}^j$-uniformly integrable martingale, then there is an equivalent probability measure $\boldsymbol{Q} \in \mathcal{M}^e(\boldsymbol{P})$ such that for all $j \le J$, V^j is a $\boldsymbol{Q}$-uniformly integrable martingale.*

3 An International Financial Market Model

We consider an international financial market model consisting of N currencies numbered from 1 to N. For each currency k there is a positive (mostly stochastic) interest rate r^k such that

$$\int_0^1 r_u^k du < \infty, \ a.s., \ 1 \le k \le N. \tag{3.1}$$

Without loss of generality, we assume that the currency 1 is the domestic currency which is used as numéraire to express the other values. The exchange rate of currency k for the *domestic* currency 1 is described by E_t^k. From the definition, we have $E_t^1 = 1$. E^k and r^k are supposed to be adapted processes and each E^k is a continuous, strictly positive semi-martingale. Following e.g. Harrison–Kreps [12] and Artzner–Delbaen [1], we define the following *discounted* exchange rates

$$S_t^1 = 1, \tag{3.2}$$

$$S_t^k = \exp\left(-\int_0^t r_u^1 du\right) \exp\left(\int_0^t r_u^k du\right) E_t^k, \ 2 \le k \le N. \tag{3.3}$$

The process S^k describes, in terms of currency 1, the relative value of one unit of currency k, deposited at time 0 and continuously compounded at interest rate r^k. From the definition it follows that $S^k > 0, \ a.s., \ 1 \le k \le N$. If we choose currency k as the numéraire, the discounted vector-valued process becomes $(\frac{S^1}{S^k}, \cdots, \frac{S^N}{S^k})$. More generally, for a positive constant weight vector $\boldsymbol{\alpha}=(\alpha^1, \cdots, \alpha^N)$, $\alpha^j > 0$, we may define the basket currency B by

$$B_t = \sum_{k=1}^N \alpha^k S_t^k. \tag{3.4}$$

When the basket currency is used as numéraire, the discounted vector-valued process is expressed by the process $(\frac{S^1}{B}, \cdots, \frac{S^N}{B})$. Notice that an admissible strategy $\boldsymbol{H}$ for $\boldsymbol{S}$, i.e. with respect to the domestic currency, is not necessarily admissible when currency k is used as numéraire. That is, $\boldsymbol{H}$ is not necessarily admissible for the vector-valued process $\frac{1}{S^k}\ \boldsymbol{S}$. It follows that the NA-property depends on the currency used as numéraire.

4 Finiteness Property

As stated in Section 2 in general the NA property is not sufficient to guarantee the existence of an equivalent local martingale measure. Stronger conditions are needed. For general locally bounded semi-martingales, such a condition is the so-called No Free Lunch with Vanishing Risk (or $NFLVR$) property. However in the case of a continuous price process, we can relax the assumption and split the $NFLVR$ condition in two separate conditions. The first is the already mentioned (NA) property, the second is the finiteness condition, which can be seen as the integrability of the risk premium process $\boldsymbol{h}$.

As will be shown below, the finiteness condition does not depend on the choice of the probability measure. In other words, if the probability measure $\boldsymbol{P}$ is replaced by an equivalent probability measure $\boldsymbol{Q}$, then the Doob–Meyer decomposition under $\boldsymbol{Q}$ again satisfies the finiteness condition. We start with some obvious results from linear algebra.

Lemma 4.1 *If $\boldsymbol{A} : \boldsymbol{R}^d \to \boldsymbol{R}^d$ is a symmetric linear operator, then the Moore–Penrose inverse is given by*

$$\boldsymbol{A}^{-1} = \lim_{n\to\infty} n\boldsymbol{A} \int_0^\infty \exp\Big(-(\boldsymbol{I}_d + \boldsymbol{A}^2 n)x\Big)\,dx \tag{4.1}$$

Proof. Since $\boldsymbol{A}$ is a symmetric linear operator, there exists an orthogonal basis of $\boldsymbol{R}^d$ in which $\boldsymbol{A}$ is represented by a diagonal matrix. In this basis, we have

$$\boldsymbol{A} = \begin{pmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & \lambda_r & & \\ & & & 0 & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{A}^{-1} = \begin{pmatrix} \lambda_1^{-1} & & & & \\ & \ddots & & & \\ & & \lambda_r^{-1} & & \\ & & & 0 & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}. \tag{4.2}$$

Now observe that

$$n\lambda \int_0^\infty \exp\Big(-x(1+n\lambda^2)\Big)\,dx = \frac{n\lambda}{1+n\lambda^2} \to \begin{cases} \frac{1}{\lambda}, & \text{if } \lambda \neq 0, \\ 0, & \text{if } \lambda = 0. \end{cases} \tag{4.3}$$

□

Lemma 4.2 *If $\phi : (E, \mathcal{E}) \to \mathcal{S}(\boldsymbol{R}^d)$ is a measurable mapping from a measurable space $(E, \mathcal{E})$ into the vector space of symmetric operators on $\boldsymbol{R}^d$. Then*

(a) $\phi^{-1} : E \to \mathcal{S}(\boldsymbol{R}^d)$ is still measurable

(b) $\mathcal{P} : E \to \mathcal{S}(\boldsymbol{R}^d)$, where $\mathcal{P}$ is projection on Range(ϕ), is measurable.

Proof. (a)

$$\phi^{-1} = \lim_{n\to\infty} n\phi \int_0^\infty \exp\Big(-x(\boldsymbol{I}_d + n\phi^2)\Big)\,dx \tag{4.4}$$

expresses ϕ^{-1} as a limit of measurable expressions of ϕ, hence ϕ^{-1} is measurable.
(b) $\mathcal{P} = \phi^{-1}\phi$ is the product (matrix product) of two measurable mappings and hence is measurable. □

In the same way, we can show the following corollary.

Corollary 4.3 *If $\boldsymbol{Ax} = \boldsymbol{b}$ is a linear system then the couple $(\boldsymbol{x}, \boldsymbol{y})$ such that $\boldsymbol{Ax} + \boldsymbol{y} = \boldsymbol{b}$ and $\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2$ is minimal, depends in a measurable way on "$\boldsymbol{A}$" and "$\boldsymbol{b}$".*

Proof. Take normal equations and apply Lemma 4.2. □

Lemma 4.4 *If $\boldsymbol{S}$ is a vector-valued continuous semi-martingale and if $\boldsymbol{S}$ satisfies the finiteness condition under $\boldsymbol{P}$, then for each probability measure $\boldsymbol{Q}$, equivalent to $\boldsymbol{P}$, the process $\boldsymbol{S}$ still satisfies the finiteness condition under $\boldsymbol{Q}$.*

Proof. Let Z be the martingale defined by $Z_t = \boldsymbol{E}_P\left[\frac{dQ}{dP}\,|\mathcal{F}_t\right]$. We, of course, can take a cadlag version for Z (we remark that we did not make a continuity assumption on the filtration $(\mathcal{F}_t)_{0\le t\le 1}$ and we therefore cannot state that Z is continuous). Suppose that under $\boldsymbol{P}$, the semi-martingale $\boldsymbol{S}$ is decomposed as $\boldsymbol{dS}_t = \boldsymbol{dM}_t + \boldsymbol{d}\langle\boldsymbol{M},\boldsymbol{M}\rangle_t\boldsymbol{h}_t$. Because $\boldsymbol{S}$ satisfies the finiteness property under $\boldsymbol{P}$, we have that

$$\int_0^1 \boldsymbol{h}'\boldsymbol{d}\langle\boldsymbol{M},\boldsymbol{M}\rangle\boldsymbol{h} < \infty.$$

The Girsanov–Maruyama formula says that under $\boldsymbol{Q}$, the martingale part becomes

$$\boldsymbol{M}_t - \int_0^t \frac{1}{Z_{u-}} d\langle\boldsymbol{M}, Z\rangle_u$$

and the predictable part is given by

$$\boldsymbol{dA}_t = \boldsymbol{d}\langle\boldsymbol{M},\boldsymbol{M}\rangle_t\boldsymbol{h}_t + \frac{1}{Z_{t-}}d\langle\boldsymbol{M}, Z\rangle_t \tag{4.5}$$

Because $\boldsymbol{M}$ is continuous, we may decompose the martingale Z as $dZ = \boldsymbol{\phi}\cdot\boldsymbol{dM} + dN$, where N is a local martingale, strongly orthogonal to the continuous martingale $\boldsymbol{M}$ and where $\boldsymbol{\phi}$ is $\boldsymbol{M}$-integrable, i.e. $\int_0^1 \boldsymbol{\phi}'\boldsymbol{d}\langle\boldsymbol{M},\boldsymbol{M}\rangle\boldsymbol{\phi} < \infty$. We consequently obtain $d\langle\boldsymbol{M}, Z\rangle_t = \boldsymbol{d}\langle\boldsymbol{M},\boldsymbol{M}\rangle_t\boldsymbol{\phi}_t$. The process Z is bounded away from zero (see Dellacherie–Meyer [4]) and therefore the finiteness property under $\boldsymbol{Q}$ follows from the finiteness property under $\boldsymbol{P}$ and from the $\boldsymbol{M}$-integrability of $\boldsymbol{\phi}$.

Lemma 4.5 *The finiteness condition does not depend on the choice of numéraire, i.e. if $\boldsymbol{S}$ satisfies the finiteness condition and if $\rho = c + \boldsymbol{H}\cdot\boldsymbol{S} > 0$ is a stochastic integral, then $\frac{\boldsymbol{S}}{\rho}$ also satisfies the finiteness condition.*

Proof. By definition we have that $d\rho = \boldsymbol{H}\cdot\boldsymbol{dS}$. Then from the generalized Itô's lemma and from $\boldsymbol{dS} = \boldsymbol{dM} + \boldsymbol{dA}$, we deduce:

$$\begin{aligned} d\left(\frac{1}{\rho}\right) &= -\frac{1}{\rho^2}d\rho + \frac{1}{\rho^3}(d\rho)^2 \\ &= -\frac{1}{\rho^2}\boldsymbol{H}'\boldsymbol{dS} + \frac{1}{\rho^3}\boldsymbol{H}'\boldsymbol{d}\langle\boldsymbol{M},\boldsymbol{M}\rangle\boldsymbol{H}. \end{aligned} \tag{4.6}$$

Hence

$$\begin{aligned} d\left(\frac{\boldsymbol{S}}{\rho}\right) &= \frac{1}{\rho}\boldsymbol{dS} + \boldsymbol{S}d\left(\frac{1}{\rho}\right) + d\langle\boldsymbol{S},\frac{1}{\rho}\rangle \\ &= \frac{1}{\rho}\boldsymbol{dM} + \frac{1}{\rho}\boldsymbol{d}\langle\boldsymbol{M},\boldsymbol{M}\rangle\boldsymbol{h} - \frac{1}{\rho^2}\boldsymbol{S}\boldsymbol{H}'\boldsymbol{dS} \\ &\quad + \frac{1}{\rho^3}\boldsymbol{S}\boldsymbol{H}'\boldsymbol{d}\langle\boldsymbol{M},\boldsymbol{M}\rangle\boldsymbol{H} - \frac{1}{\rho^2}\boldsymbol{d}\langle\boldsymbol{M},\boldsymbol{M}\rangle\boldsymbol{H} \\ &= \frac{1}{\rho}\boldsymbol{dM} - \frac{1}{\rho^2}\boldsymbol{S}\boldsymbol{H}'\boldsymbol{dM} - \frac{1}{\rho^2}\boldsymbol{S}\boldsymbol{H}'\boldsymbol{d}\langle\boldsymbol{M},\boldsymbol{M}\rangle\boldsymbol{h} + \frac{1}{\rho}\boldsymbol{d}\langle\boldsymbol{M},\boldsymbol{M}\rangle\boldsymbol{h} \\ &\quad + \frac{1}{\rho^3}\boldsymbol{S}\boldsymbol{H}'\boldsymbol{d}\langle\boldsymbol{M},\boldsymbol{M}\rangle\boldsymbol{H} - \frac{1}{\rho^2}\boldsymbol{d}\langle\boldsymbol{M},\boldsymbol{M}\rangle\boldsymbol{H}. \end{aligned} \tag{4.7}$$

The martingale part $\boldsymbol{N}_t$ of (4.7) is given by

$$\boldsymbol{dN} = \frac{1}{\rho}\boldsymbol{dM} - \frac{1}{\rho^2}\boldsymbol{SH}'\boldsymbol{dM} = \frac{1}{\rho}(\boldsymbol{I} - \frac{1}{\rho}\boldsymbol{SH}')d\boldsymbol{M}. \tag{4.8}$$

From this we can calculate $\boldsymbol{d}\langle\boldsymbol{N},\boldsymbol{N}\rangle$:

$$\boldsymbol{d}\langle\boldsymbol{N},\boldsymbol{N}\rangle = \frac{1}{\rho^2}\left(\boldsymbol{I} - \frac{1}{\rho}\boldsymbol{SH}'\right)\boldsymbol{d}\langle\boldsymbol{M},\boldsymbol{M}\rangle\left(\boldsymbol{I} - \frac{1}{\rho}\boldsymbol{HS}'\right).$$

The bounded variation part $\boldsymbol{B}_t$ of (4.7) is given by

$$\begin{aligned}\boldsymbol{dB} \;&=\; -\frac{1}{\rho^2}\boldsymbol{SH}'\boldsymbol{d}\langle\boldsymbol{M},\boldsymbol{M}\rangle\boldsymbol{h} + \frac{1}{\rho}\boldsymbol{d}\langle\boldsymbol{M},\boldsymbol{M}\rangle\boldsymbol{h}\\ &\quad+\frac{1}{\rho^3}\boldsymbol{SH}'\boldsymbol{d}\langle\boldsymbol{M},\boldsymbol{M}\rangle\boldsymbol{H} - \frac{1}{\rho^2}\boldsymbol{d}\langle\boldsymbol{M},\boldsymbol{M}\rangle\boldsymbol{H}.\end{aligned} \tag{4.9}$$

Since $\rho > 0$, $\boldsymbol{U} = \frac{1}{\rho}\boldsymbol{H}$ is well defined. By the exponential formula we know that ρ_t is given by

$$\rho_t = \exp\left(\int_0^t \boldsymbol{U}'\boldsymbol{dM} - \frac{1}{2}\int_0^t \boldsymbol{U}'\boldsymbol{d}\langle\boldsymbol{M},\boldsymbol{M}\rangle\boldsymbol{U}\right). \tag{4.10}$$

The equation (4.9) can be rewritten as

$$\begin{aligned}\boldsymbol{dB} \;&=\; -\frac{1}{\rho}\boldsymbol{SU}'\boldsymbol{d}\langle\boldsymbol{M},\boldsymbol{M}\rangle\boldsymbol{h} + \frac{1}{\rho}\boldsymbol{d}\langle\boldsymbol{M},\boldsymbol{M}\rangle\boldsymbol{h}\\ &\quad+\frac{1}{\rho}\boldsymbol{SU}'\boldsymbol{d}\langle\boldsymbol{M},\boldsymbol{M}\rangle\boldsymbol{U} - \frac{1}{\rho}\boldsymbol{d}\langle\boldsymbol{M},\boldsymbol{M}\rangle\boldsymbol{U}\\ &=\; (\boldsymbol{I}-\boldsymbol{SU}')\,\boldsymbol{d}\langle\boldsymbol{M},\boldsymbol{M}\rangle\frac{\boldsymbol{h}}{\rho} - (\boldsymbol{I}-\boldsymbol{SU}')\,\boldsymbol{d}\langle\boldsymbol{M},\boldsymbol{M}\rangle\frac{\boldsymbol{U}}{\rho}\\ &=\; (\boldsymbol{I}-\boldsymbol{SU}')\,\boldsymbol{d}\langle\boldsymbol{M},\boldsymbol{M}\rangle\left(\frac{\boldsymbol{h}-\boldsymbol{U}}{\rho}\right).\end{aligned} \tag{4.11}$$

Next we shall show that $\boldsymbol{dB}$ can be written as $(\boldsymbol{I}-\boldsymbol{SU}')\,\boldsymbol{d}\langle\boldsymbol{M},\boldsymbol{M}\rangle\,(\boldsymbol{I}-\boldsymbol{SU}')\,\boldsymbol{g}$, for some predictable process $\boldsymbol{g}$. As shown in Section 2, we can write $\boldsymbol{d}\langle\boldsymbol{M},\boldsymbol{M}\rangle = \boldsymbol{L}d\lambda$ for some control measure λ. Substitute this for (4.11), we have

$$\boldsymbol{dB} = (\boldsymbol{I}-\boldsymbol{SU}')\,\boldsymbol{L}\left(\frac{\boldsymbol{h}-\boldsymbol{U}}{\rho}\right)d\lambda. \tag{4.12}$$

To simplify notation let $\boldsymbol{C}{=}\boldsymbol{I}{-}\boldsymbol{SU}'$. We then have the following trivial inclusion on the ranges of different operators, $Range(\boldsymbol{CLC}') \subset Range(\boldsymbol{CL})$. However we also have $Range(\boldsymbol{CL}) \subset Range(\boldsymbol{CLC}')$. Indeed if $\boldsymbol{y} \perp Range(\boldsymbol{CLC}')$, we have $\boldsymbol{y}'\boldsymbol{CLC}' = \mathbf{o}'$. Then $\boldsymbol{y}'\boldsymbol{CLC}'\boldsymbol{y} = 0$ and $\boldsymbol{C}'\boldsymbol{y} \in \ker(\boldsymbol{L}) = Range(\boldsymbol{L})^{\perp}$. This means that $\boldsymbol{y}'\boldsymbol{CL} = \mathbf{o}$ and hence $\boldsymbol{y} \perp Range(\boldsymbol{CL})$. Therefore $Range(\boldsymbol{CL}) = Range(\boldsymbol{CLC}')$. Let $\boldsymbol{D}{=}(\boldsymbol{I}{-}\boldsymbol{SU}')\boldsymbol{L}$ $(\boldsymbol{I}{-}\boldsymbol{U}'\boldsymbol{S})$. The projection $\mathcal{P}'$ on $Range(\boldsymbol{D})$ is predictable and $\boldsymbol{D}$ is bijective on the $Range(\boldsymbol{D}) = Range(\mathcal{P}')$.

The Moore–Penrose inverse $\boldsymbol{D}^{-1}$ is predictable and $\boldsymbol{D}^{-1}\boldsymbol{D} = \boldsymbol{D}\boldsymbol{D}^{-1} = \mathcal{P}$. By taking $\boldsymbol{g} = \boldsymbol{D}^{-1}(\boldsymbol{I}-\boldsymbol{SU}')\boldsymbol{L}\frac{\boldsymbol{h}-\boldsymbol{U}}{\rho}$, we have a predictable process $\boldsymbol{g}$ such that $\boldsymbol{dB} =$

$\boldsymbol{d}\langle\boldsymbol{N},\boldsymbol{N}\rangle\boldsymbol{g}$. We should check the finiteness property for $\boldsymbol{g}$.

$$\begin{aligned}\boldsymbol{g}'\boldsymbol{d}\langle\boldsymbol{N},\boldsymbol{N}\rangle\boldsymbol{g} &= \boldsymbol{g}'\boldsymbol{C}\boldsymbol{L}\boldsymbol{C}'\boldsymbol{g}d\lambda \\ &= \boldsymbol{g}'\boldsymbol{C}\boldsymbol{L}\left(\frac{\boldsymbol{h}-\boldsymbol{U}}{\rho}\right)d\lambda \\ &\leq (\boldsymbol{g}'\boldsymbol{C}\boldsymbol{L}\boldsymbol{C}'\boldsymbol{g})^{\frac{1}{2}}\left[\left(\frac{\boldsymbol{h}-\boldsymbol{U}}{\rho}\right)'\boldsymbol{L}\left(\frac{\boldsymbol{h}-\boldsymbol{U}}{\rho}\right)\right]^{\frac{1}{2}}d\lambda.\end{aligned}$$

The last inequality follows from the Cauchy–Schwarz inequality for positive definite bilinear forms. Hence, again by the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\int_0^1 \boldsymbol{g}_s'\boldsymbol{d}\langle\boldsymbol{N},\boldsymbol{N}\rangle_s\boldsymbol{g}_s &\leq \int_0^1\left(\frac{\boldsymbol{h}_s-\boldsymbol{U}_s}{\rho_s}\right)'\boldsymbol{L}_s\left(\frac{\boldsymbol{h}_s-\boldsymbol{U}_s}{\rho_s}\right)d\lambda \\ &= \int_0^1\left(\frac{\boldsymbol{h}_s-\boldsymbol{U}_s}{\rho_s}\right)'\boldsymbol{d}\langle\boldsymbol{M},\boldsymbol{M}\rangle_s\left(\frac{\boldsymbol{h}_s-\boldsymbol{U}_s}{\rho_s}\right) \\ &\leq \int_0^1\frac{1}{\rho_s^2}(\boldsymbol{h}_s'\boldsymbol{d}\langle\boldsymbol{M},\boldsymbol{M}\rangle_s\boldsymbol{h}_s+\boldsymbol{U}_s'\boldsymbol{d}\langle\boldsymbol{M},\boldsymbol{M}\rangle_s\boldsymbol{U}_s). \qquad (4.13)\end{aligned}$$

Since $\inf_{0\leq t\leq 1}\rho_t>0$, we have $\int_0^1\frac{1}{\rho_s^2}\boldsymbol{U}_s'\boldsymbol{d}\langle\boldsymbol{M},\boldsymbol{M}\rangle_s\boldsymbol{U}_s<\infty$ from (4.10) and from $\rho_1>0$. This together with the finiteness property (2.4) yields the desired result. □

Remark 4.6 *From Lemma 4.4 and 4.5, the finiteness condition is invariant for the choice of numéraire and for the choice of equivalent probability measure. Hence from Theorem 2.3 (c), if the finiteness condition is satisfied under $\boldsymbol{P}$, the NA property becomes equivalent to the existence of an equivalent local martingale measure for the process $\boldsymbol{S}$.*

5 The Main Theorem

We show that under the finiteness condition for an arbitrary chosen currency, the no arbitrage condition for a basket currency is necessary and sufficient for the no arbitrage property to hold with respect to all the N currencies.

Theorem 5.1 *If $\boldsymbol{S}$ is a continuous vector-valued semi-martingale that satisfies the finiteness condition, then the following are equivalent:*

(a) For all j, $1\leq j\leq N$, there is an equivalent probability measure $\boldsymbol{Q}^j\in\mathcal{M}^e(\boldsymbol{P})$ such that S^j is a $\boldsymbol{Q}^j$-uniformly integrable martingale.

(b) There is an equivalent probability measure $\boldsymbol{Q}\in\mathcal{M}^e(\boldsymbol{P})$ such that $\boldsymbol{S}$ is a $\boldsymbol{Q}$-uniformly integrable vector martingale.

(c) For all j, $1\leq j\leq N$, $\frac{\boldsymbol{S}}{S^j}$ satisfies the NA property with respect to general admissible integrands.

(d) If B is a basket currency, $B=\alpha_1\,\boldsymbol{S}_1+\cdots+\alpha_N\,\boldsymbol{S}_N$, where the α_i are strictly positive constants, then $\frac{\boldsymbol{S}}{B}$ satisfies the NA property with respect to general admissible integrands.

Proof. $(a) \Rightarrow (b)$: Follows from Corollary 2.12. $(a) \Leftrightarrow (c)$: Follows from $(c') \Leftrightarrow (d')$ in Theorem 2.9 and Remark 4.6. $(b) \Rightarrow (d)$: Suppose now that there is an equivalent measure $\boldsymbol{Q} \in \mathcal{M}^e(\boldsymbol{P})$ such that B is a $\boldsymbol{Q}$-uniformly integrable martingale. Hence from Theorem 2.3 (c) and $(c') \Leftrightarrow (d')$ in Theorem 2.9, $\frac{\boldsymbol{S}}{B}$ satisfies NA. $(d) \Rightarrow (a)$: If $\frac{\boldsymbol{S}}{B}$ satisfies NA, from Theorem 2.3 (c), Remark 4.6 and $(c') \Leftrightarrow (d')$ in Theorem 2.9, there exists $\boldsymbol{Q} \in \mathcal{M}^e(\boldsymbol{P})$ such that B is a $\boldsymbol{Q}$-uniformly integrable martingale. Since each S^k is a $\boldsymbol{Q}$-local martingale and $S^k \leq \frac{B}{\min_{1\leq j\leq N}\alpha^j}$, S^k is a $\boldsymbol{Q}$-uniformly integrable martingale. □

Remark 5.2 *Theorem 5.1 shows how important the existence of a martingale measure (instead of a local martingale measure) is when dealing with different currencies and numéraires. Another application of the theorem is that of different stocks and the use of an index as numéraire. We also want to point out that the N financial assets were interpreted as currencies. They can also represent arbitrary financial assets. The model is therefore much more general than the title indicates.*

The NA properties for general admissible integrands follow, in the continuous case, from the no free lunch with vanishing risk $NFLVR$ property for simple admissible integrands (see Delbaen–Schachermayer [6]). So the theorem can also be stated using $NFLVR$ for simple admissible integrands. In this case, the finiteness property follows from $NFLVR$.

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