The Second Fundamental Theorem of Asset Pricing: A New Approach

Robert J. Battig
Robert A. Jarrow
Cornell University

This article presents a new definition of market completeness that is independent of the notions of no arbitrage and equivalent martingale measures. Our definition has many advantages, all shown herein. First, it preserves the Second Fundamental Theorem of Asset Pricing, even in complex economies. Second, under our definition, the market can be complete yet arbitrage opportunities exist. This is important in practice, and stands in contrast to the traditional definitions. Third, under the assumption of no arbitrage and when used in the standard models, our definition is equivalent to the traditional one.

Most of modern finance theory is based on the first and second fundamental theorems of asset pricing. The first fundamental theorem relates the notion of no arbitrage to the existence of an equivalent martingale measure, while the second fundamental theorem relates the notion of market completeness to uniqueness of the equivalent martingale measure [see Harrison and Kreps (1979) and Harrison and Pliska (1981)].

For economies that involve only a finite number of assets, these economic notions of no arbitrage and market completeness are equivalent to their probabilistic counterparts [see Dalang, Morton, and Willinger (1990), Delbaen (1992), Lakner (1993), Delbaen and Schachermayer (1994), and Schachermayer (1994) on the first fundamental theorem and Harrison and Pliska (1981) and Battig (1997) on the second]. For economies involving an infinite number of assets with discontinuous sample paths, the first fundamental theorem has not yet been extended, and the second fundamental theorem fails. Indeed, Artzner and Heath (1995) provide an example of a complete economy where there are an infinite number of assets and an infinite number of equivalent martingale measures (market completeness but nonuniqueness of an equivalent martingale measure). Although the mathematics generating their counterexample is well understood, the economic reasoning underlying its failure is not.

The purpose of this article is to propose a new approach to market completeness that maintains the second fundamental theorem, even in complex

---

This research was partially supported by NSF grant DMS-93-01070. Alfred P. Sloan Foundation doctoral dissertation Fellow, 1997–1998. Address correspondence to Robert A. Jarrow, Johnson Graduate School of Mangagement, Cornell University, Ithaca, NY 14853, or email: raj15@cornell.edu.

The Review of Financial Studies Winter 1999 Vol. 12, No. 5, pp. 1219–1235

economies. This equivalence is maintained by redefining the meaning of a complete market to one that is more basic, and to one that is independent of either the notions of no arbitrage or equivalent martingale measures.[1] As shown below, this new definition is important for understanding the economic reasoning behind the Artzner and Heath (1995) counterexample. However, this new definition is also important for practice, as arbitrage opportunities are often sought in complete markets. This consideration is impossible under the existing definitions.[2]

Indeed, under the existing definitions, the notion of a complete market has been studied by first fixing an equivalent martingale measure [see, Harrison and Pliska (1981), Ansel and Stricker (1994), Artzner and Heath (1995)]. By the first fundamental theorem, the existence of an equivalent martingale measure implies no arbitrage opportunities. Thus, in the existing literature, a complete market must necessarily be arbitrage free.

In contrast, our definition of market completeness is independent of any particular probability measure. This is an important property.[3] Under our definition, a market can be complete and yet arbitrage opportunities exist. In fact, this measure independence is the key insight of our article, and the essence of our article's contribution to the literature.

The formulation of our definition starts with a specification of a collection of events that all traders agree cannot occur, the set of traded assets, and a set of trading strategies. The trading strategies are kept simple. They consist of holding only a finite number of assets at any point in time and only a finite number of trades are allowed over the trading horizon. The trading dates can be stopping times. The space of potentially attainable contingent claims is the space of bounded random variables. There can be an arbitrary number of traded assets, one of which is a money market account. Traded assets have prices, and trading strategies have known costs of construction.

The economy consists of a collection of traders. Each trader assesses their own personal value to the set of attainable contingent claims.[4] These personal values satisfy a minimal consistency condition — any claim that makes zero payments except on null events has a zero price. We allow both risk-averse and risk-neutral traders.

---

[1] For articles on related topics see Jarrow and Madan (1997b) and Jarrow, Jin, and Madan (1997). Jarrow, Jin, and Madan (1997) employ a similar definition, but only in a static economy. This article can be viewed as the continuous trading extension of Jarrow, Jin, and Madan (1997).

[2] Except, of course, for the trivial finite state, finite time economies (e.g., binomial model) where much of the profession's intuition for the separation of no arbitrage and market completeness originates. One way to think about this article is that it provides the appropriate generalization of the finite state, finite time economy to more complex economies, while still maintaining the original intuition.

[3] In fact, Battig (1997) has an example of an economy where the existence of an equivalent martingale measure precludes the possibility of market completeness.

[4] The introduction of traders and their personal valuations is purely for pedagogical purposes. The entire setup can be done abstractly without the introduction of these concepts. This comment will become self-evident in the subsequent sections.

In our setup, a trader views two random variables (contingent claims) as approximately equal if the value he assigns to the differences in their payoffs across states is close to zero. In the market at large, two random variables are deemed approximately equal if all traders view them as such. This definition for two random variables being approximately equal, therefore, is seen to depend only on the collection of null events. This is a key insight.

A market is said to be complete if all the potentially attainable contingent claims can be approximated (in the above sense of closeness) via a trading strategy. Due to the meaning of approximately equal, this definition of market completeness is independent of the notion of no arbitrage and is independent of any particular probability measure.

Using this new definition of market completeness, two topological dual pairs exist: (random variables − personal values) and (trading strategies − personal values of trading strategies). There is a linear mapping linking these two dual pairs, that mapping taking trading strategies to random variables. This mapping has an adjoint. The linear mapping and its adjoint are the infinite dimensional analogue of a matrix and its transpose. A straightforward application of mathematics to this mapping and its adjoint yields the second fundamental theorem. That is, the equivalence between market completeness and uniqueness of a valuation operator that prices attainable claims by their cost of construction.

Under appropriate additional hypotheses, this is equivalent to the uniqueness of the equivalent (local) martingale measure, thus yielding an elegant proof of the standard form of the second fundamental theorem. In this context, an equivalent (local) martingale measure is a measure that transforms the traded assets into (local) martingales and whose null sets are precisely those events considered impossible by all traders. Under this definition of market completeness, the economy considered by Artzner and Heath (1995) is *incomplete*, so it is no longer a counterexample to the second fundamental theorem. It is seen that Artzner and Heath use the wrong definition for the closeness of two random variables.

In addition, this article studies the relation between our definition of market completeness and that used in the existing literature. It is shown that the two definitions give equivalent characterizations of market completeness when there is only a finite set of assets trading or asset prices have continuous sample paths. These are the standard structures used in the financial economics literature [see Black and Scholes (1973), Heath, Jarrow, and Morton (1992), and Jarrow and Madan (1995)].

Hence, the standard technique for proving market completeness — showing the non-singularity of the appropriately defined volatility matrix[5] —

[5] See, for example, Jarrow and Madan (1995) and Battig (1997). In Battig (1997), the volatility matrix characterization of completeness is obtained under the new definition of completeness, without reference to the standard definition.

works for our definition as well. The two definitions differ, however, in more complex economies. For more complex economies, our definition is shown to be stronger (i.e., our definition of completeness implies the existing literature's definition, but the converse is not true).[6] This distinction will prove useful as more complex economies involving an infinite number of assets are explored in the finance literature.[7]

An outline for this article is as follows. Section 1 presents the model. Section 2 presents our new definition of market completeness. Section 3 presents the revised second fundamental theorem of asset pricing. Section 4 reviews the old definition of market completeness. Section 5 relates it to the new definition. Section 6 clarifies the Artzner and Heath (1995) counterexample. Section 7 concludes. Proofs are contained in the appendix.

1. The Model

This section presents the details of the model. We start with a filtration $F = (\mathcal{F}_t)_{t\in[0,1]}$ on a measurable space $(\Omega, \mathcal{F})$ and a collection $\mathcal{N}$ of events in F. The filtration $F = (\mathcal{F}_t)_{t\in[0,1]}$ models the evolution of information over time and the elements of $\mathcal{N}$ are events that all traders agree cannot occur. The events in $\mathcal{N}$ are referred to as the null sets.[8] $\mathcal{N}$ could be the null sets of a statistical measure P, but it is not necessary to refer to any measure in the subsequent theory. The trading horizon is continuous, finite, and represented by the time interval [0,1].

Let $A \cup \{\Delta\}$ be an index set representing the traded primary assets. These assets trade in frictionless and competitive markets. We separate out one asset, the money market account $\{\Delta\}$, for easy reference. Otherwise A represents an arbitrary (possibly infinite) set of risky traded securities.

The family of price processes for $A \cup \{\Delta\}$ is denoted by $V = \{(Z_t^\alpha)_{t\in[0,1]}\}_{\alpha\in A\cup\{\Delta\}}$. $(Z_t^\alpha)_{t\in[0,1]}$ are adapted cadlag processes with $Z_t^\Delta \equiv 1$. Without loss of generality, we set the money market account's value constant and equal to one for all time t. This is equivalent to the price processes $(Z_t^\alpha)_{t\in[0,1]}$ already being normalized by the (random) value of the money market account. The analysis proceeds for these normalized price processes.

Traders are allowed to invest in the money market account plus a finite number of the risky assets in A using self-financing, (stopping time) simple trading strategies. More precisely, the set of allowable trading strategies is

[6] An example is given in Battig (1997).

[7] This is not a vacuous or unimportant set of topics. The APT model of Ross (1976), which contains an infinite number of assets, is one such economy. Markets with price processes having discontinuous sample paths and continuous densities for jump amplitudes are also an important class of examples [see Merton (1976)].

[8] If $B \in \mathcal{N}$ *and* $A \subseteq B$, *then* $A \in \mathcal{N}$ and $\mathcal{N}$ is closed under the taking of countable unions. We also assume that the filtration is right-continuous and that $\mathcal{F}_0$ contains $\mathcal{N}$.

denoted by

$$\widetilde{Y} = \left\{ (x, (H^\alpha)_{\alpha \in A}) \mid x \in \Re, \quad H_t^\alpha = \sum_{i=1}^{n_\alpha} h_{i-1}^\alpha 1_{(\tau_{i-1}^\alpha, \tau_i^\alpha]}(t) \right\}, \qquad (1)$$

where $0 \leq \tau_0^\alpha \leq \cdots \leq \tau_{n_\alpha}^\alpha \leq 1$ are stopping times, $1_{(\tau_{i-1}^\alpha, \tau_i^\alpha]}(t) = 1$ if $t \in (\tau_{i-1}^\alpha, \tau_i^\alpha]$ and 0 otherwise, $h_i^\alpha \in L^\infty(\mathcal{F}_{\tau_i^\alpha}, \mathcal{N})$ and $H^\alpha \equiv 0$ except for finitely many $\alpha \in A$.

$L^\infty(\mathcal{F}_{\tau_i^\alpha}, \mathcal{N})$ denotes the bounded $\mathcal{F}_{\tau_i^\alpha}$-measurable random variables. In $L^\infty(\mathcal{F}_{\tau_i^\alpha}, \mathcal{N})$, random variables differing only on null sets are considered equal.

In the definition of $\widetilde{Y}$, x represents the time 0 value of the entire portfolio. H_t^α represents the units of asset α held at time t for $t \in (0, 1]$ During the time interval $(0, \tau_1^\alpha]$, h_0^α units are held. At time τ_1^α a rebalancing occurs and the holdings change to h_1^α. The next rebalancing occurs at time τ_2^α when the holdings are changed to h_2^α, and so forth until final liquidation at time 1. Requiring that $h_i^\alpha \in L^\infty(\mathcal{F}_{\tau_i^\alpha}, \mathcal{N})$ means that the holdings over the time interval $(\tau_i^\alpha, \tau_{i+1}^\alpha]$ are bounded and can only be based on information available at the beginning τ_i^α of the interval.

Note that a self-financing condition is implicit in our trading strategies because after time 0 we do not get to choose the holdings in the money market account. Once we decide on the trading strategy $(x, (H^\alpha)_{\alpha \in A})$, the self-financing condition requires us to hold $x + \sum_{\alpha \in A} \int_0^t H_u^\alpha dZ_u^\alpha - \sum_{\alpha \in A} H_t^\alpha Z_t^\alpha$ dollars in the money market account at time $t \in (0, 1]$.[9]

The payoff to a trading strategy $(x, (H^\alpha)_{\alpha \in A})$ at time 1 is denoted by

$$\widetilde{T}(x, (H^\alpha)_{\alpha \in A}) = x + \sum_{\alpha \in A} \int_0^1 H_u^\alpha dZ_u^\alpha. \qquad (2)$$

This represents the initial cost of constructing the portfolio, x, plus the gains/losses on the risky assets over [0,1]. The sum on the right side of Equation (2) is finite, as only a finite number of the assets can be held by the trader at any time. The integral is well-defined since the holdings are constant for almost all times.[10]

We are interested in studying market completeness. Consequently we need to define the space of potentially attainable contingent claims. We

[9] The value of the portfolio at time t is $V_t = H_t^\Delta + \sum_{\alpha \in A} H_t^\alpha Z_t^\alpha$. The self-financing condition is $dV_t = \sum_{\alpha \in A} H_t^\alpha dZ_t^\alpha$. This implies that $V_t = x + \sum_{\alpha \in A} \int_0^t H_u^\alpha dZ_u^\alpha$. Combined, these yield the expression in the text.

[10] This integral is well defined as $\int_0^1 H_u^\alpha dZ_u^\alpha = \sum_{i=1}^{n_\alpha} h_{i-1}^\alpha (z_{\tau_i}^\alpha - z_{\tau_{i-1}}^\alpha)$, even if $(z^\alpha)_{\alpha \in A}$ are not semimartingales.

restrict ourselves to the set of bounded random variables,[11] denoted by $L^\infty(\mathcal{F}_1, \mathcal{N})$, where, as before, two random variables are considered equal if they only differ on a null set; that is, on a set in $\mathcal{N}$. The set $C = L^\infty(\mathcal{F}_1, \mathcal{N})$ represents the space of potentially attainable contingent claims.

It is possible that the trading strategy operator $\widetilde{T}$ defined in Equation (2) generates random variables which are not bounded and therefore are not in C.[12] For this reason we restrict the domain of $\widetilde{T}$ to be

$$Y = \widetilde{Y} \cap \widetilde{T}^{-1}(C), \tag{3}$$

and we denote the restriction of $\widetilde{T}$ to Y by T.

2. The New Definition of Market Completeness

Given the above structure, this section introduces our new definition of market completeness. The notion we would like to achieve (based on an analogy to the finite state, finite time model) is that the trading strategies generate all the contingent claims, that is, the image of T equals all of C. Note that this definition is independent of the notion of no arbitrage or the concept of an equivalent martingale measure. It only depends on the null sets $\mathcal{N}$ that determine when two claims are considered identical.

However, we cannot reasonably expect that all contingent claims can be obtained as outcomes of the class Y of (stopping time) simple trading strategies.[13] But it is possible that sequences (nets) of simple trading strategies could approximate arbitrary elements in C. This generalization to market completeness is the one we pursue.

To maintain the independence of the definition of market completeness from the notion of no arbitrage and the concept of an equivalent martingale measure, the meaning of "approximate" needs to be formulated carefully.

In this regard, denote by M the space of signed measures on $(\Omega, \mathcal{F}_1)$ which assign zero mass to events in $\mathcal{N}$. The set M represents the possible contingent claims valuation measures held by traders, called "valuation measures" for short. A trader using the valuation measure $\mu \in M$ assigns

[11] This assumption is less restrictive than it first appears. Unbounded random variables could be considered in this context by first normalizing prices by the aggregate value of all traded assets in the economy. Then the normalized prices Z_t^α are bounded by construction. This "trick" has been previously used by Jarrow and Madan (1997a) and Sin (1996).

[12] For instance, suppose $z_1^{\alpha^*}$ is unbounded and take $x = 0$, $H^\alpha \equiv 1$ if $\alpha = \alpha^*$ and identically zero otherwise.

[13] For instance, in the traditional Black–Scholes model, any bounded claim X can be represented as $x + \int_0^1 H_u^X \, dZ_u$, where Z_u denotes the deflated stock price (following geometric Brownian motion), $x \in \Re$, and H_u^X is an appropriate predictable process making $\int_0^1 H_u^X \, dZ_u$ into a martingale (under the unique equivalent martingale measure for Z_u). Furthermore, x and H_u^X are unique and thus choosing H_u^X, which is not stopping time simple, gives a claim X which cannot be attained with strategies from Y.

to a contingent claim $X \in C$ the value

$$\langle X, \mu \rangle = \int X d\mu. \tag{4}$$

This valuation measure $\mu \in M$ is identified as belonging to a particular trader. Under this interpretation, the valuation measure implicitly incorporates the traders' beliefs and preferences (risk aversion).

The set of valuations determined by M has two implicit assumptions. First, the fact that any $\mu \in M$ assign zero mass to events in $\mathcal{N}$ means that all traders agree on the null events. Second, since the measures are signed, this means that there can be strictly positive random variables with negative personal value. For example, the contingent claim 1_E with $E \notin \mathcal{N}$ could have a negative personal valuation, that is, $\mu(E) \leq 0$. The set of nonnegative measures in M whose null set are precisely $\mathcal{N}$ is denoted by M_{++}.

For a given trader, represented by a $\mu \in M$, two contingent claims X and Y can be viewed as approximately equal if

$$\left| \int (X - Y) d\mu \right| < \varepsilon \text{ for small } \varepsilon > 0.$$

This criteria states that two random variables X and Y are approximately equal to trader $\mu \in M$ if he values a claim paying the differences in their payoffs across states as approximately zero.

This measure of closeness can be used to define a topology on C, denoted by τ^μ.[14] This topology is trader (measure) dependent. To eliminate the dependence on a single trader, we endow C with the coarsest topology that is finer than τ^μ for all $\mu \in M$. We denote this new topology by τ.[15] By finer, we mean that τ has all the open sets that are contained in τ^μ for all $\mu \in M$. So if X is approximately close to Y in the new topology τ, then X is approximately close to Y for every trader $\mu \in M$. The converse of this statement for an individual trader is not true. That is, if trader $\mu \in M$ views X as close to Y, other traders may not view it as such and, therefore, X may not be close to Y in the τ topology.

In this topology, two contingent claims are approximately equal if all traders believe the values of these two claims are close. Hence, this measure of closeness depends on the entire set of traders (measures) in M and is therefore independent of any particular trader (measure).

---

[14] The topology is defined by basic open sets of the form $B(X; \varepsilon) = \{Y \in C \mid |\int (X - Y) d\mu| < \varepsilon\}$ for $X \in C$ and $\varepsilon > 0$.

[15] This new topology has another interpretation. τ is the weakest topology on C, making all the linear functionals $\langle \cdot, \mu \rangle$, $\mu \in M$, continuous. Since C is the topological dual of M (endowed with the total variation norm), this is the weak* topology on C.

Finally, prior to our definition, let

$$\mathcal{A}_1 = \left\{ x + \sum_{\alpha \in A} \int_0^1 H_u^\alpha dZ_u^\alpha \mid (x, (H^\alpha)_{\alpha \in A}) \in Y \right\} = Im\, T$$

$$\mathcal{A}_1^0 = \left\{ \sum_{\alpha \in A} \int_0^1 H_u^\alpha dZ_u^\alpha \mid (0, (H^\alpha)_{\alpha \in A}) \in Y \right\},$$

where $\mathcal{A}_1$ is the space of claims attainable by trading in the fundamental assets via stopping time simple strategies, while $\mathcal{A}_1^0$ denotes the contingent claims attainable at zero initial cost.

Definition 1. *The market is complete if $\mathcal{A}_1 = Im\, T$ is dense in C with respect to the topology τ.*

This definition is independent of the notions of no arbitrage and an equivalent martingale measure. Roughly, it says that the market is complete if, given an $X \in C$, there is a trading strategy whose time 1 value all traders (i.e., all $\mu \in M$) consider close to X.

3. The Second Fundamental Theorem of Asset Pricing

This section presents the generalization of the second fundamental theorem of asset pricing. Prior to this, however, we need to introduce some additional notation for the valuation of a trading strategy $(x, (H^\alpha)_{\alpha \in A}) \in Y$. For a trader $\mu \in M$, this value is given by

$$(T^*\mu)(x, (H^\alpha)_{\alpha \in A}) = \int T(x, (H^\alpha)_{\alpha \in A}) d\mu. \tag{5}$$

Equation (5) simply says that a trader $\mu \in M$ values a trading strategy $(x, (H^\alpha)_{\alpha \in A})$ by valuing the time 1 payoff $T(x, (H^\alpha)_{\alpha \in A})$ generated by this strategy.

On the other hand, it costs x dollars to form the trading strategy $(x, (H^\alpha)_{\alpha \in A})$. These two values could be different, representing a situation where the trader's personal value for a strategy differs from its cost. This is a type of arbitrage opportunity for the trader. We want to exclude this type of arbitrage opportunity by considering only those traders $\mu \in M$ whose values in Equation (5) equal x.

More abstractly, T^* can be viewed as an operator mapping an element $\mu \in M$ into the space χ of potential valuations of trading strategies in Y. Let $\pi_0(x, (H^\alpha)_{\alpha \in A}) = x$ represent the linear functional mapping trading strategies into their cost of construction and denote by $\mathcal{P}_{+/-}$ the set of signed measures $\mu \in M$ such that $T^*\mu = \pi_0$. The signed measures in the set $\mathcal{P}_{+/-}$ preclude these simple arbitrage opportunities.

$$
\begin{tikzcd}
X \in C \arrow[r] & \mu \in M \begin{cases} \supset M^{++} \\ \supset \mathcal{P}_{+/-} \end{cases} \arrow[d, "T^{*}\mu = \int T d\mu"] \\
y = (x, (H^{\alpha})_{\alpha \in A}) \in Y \arrow[u, "T(y) = x + \sum_{\alpha \in A} \int_0^1 H_u^{\alpha} dZ_u^{\alpha}"] \arrow[r] & \chi
\end{tikzcd}
$$

Figure 1
Duality mappings between the various sets of values, valuations, and trading strategies
In this figure, $X \in C$ is the set of contingent claim time 1 values, $\mu \in M$ is the set of possible time 0 valuations (signed measures), M^{++} is the set of nonnegative measures, $(x, (H^{\alpha})_{\alpha \in A}) \in Y$ is the set of trading strategies, $\mathcal{P}_{+/-}$ is the set of signed measures such that $T^{*}\mu = \pi_0$ where $\pi_0(x, (H^{\alpha})_{\alpha \in A}) = x$, Z_t^{α} is the price process for asset α, and $\mathcal{X}$ is the set of possible time 0 valuations of trading strategies.

The key theorem in our article uses the special topological duality that exists between the various constructs formulated. The linear mapping $T\colon Y \to C$ takes a trading strategy $(x, (H^{\alpha})_{\alpha \in A})$ and maps it into a random variable. The space of random variables (contingent claims) C is in duality with the space of possible values M via Equation (4). Equation (4) gives the price of a random variable. Continuing, the linear mapping $T^{*}\colon M \to \chi$ takes a particular valuation measure on the random variables and maps it into a valuation operator on trading strategies. The space of valuation operators on trading strategies χ is in duality with the space of trading strategies Y via Equation (5).[16] Equation (5) gives the cost of a trading strategy. In fact, it can be shown that T^{*} is the adjoint of T. This duality pairing, with the associated topologies, is a well-studied construct in mathematics. This duality mapping is illustrated in Figure 1.

From the construct, one can easily obtain the following theorem.

Theorem 1 (generalized second fundamental theorem of asset pricing). *Let there exist $Q \in \mathcal{P}_{+/-}$. The market is complete if and only if Q is unique in $\mathcal{P}_{+/-}$.*

[16] Formally, $\mathcal{X}$ is the vector space of (linear) functions mapping $Y \to R$ generated by $\int T d\mu$ for all $\mu \in M$ and π_0.

Proof. See the appendix.

This generalization of the second fundamental theorem states that if there exists a measure in the set $\mathcal{P}_{+/-}$, then the market is complete if and only if this signed measure is unique.

It generalizes the earlier theorems in two ways. First, the condition that $Q \in \mathcal{P}_{+/-}$ is a very weak no arbitrage condition, much weaker than that which appears in the literature [see Dalang, Morton, and Willinger (1990), Delbaen (1992), Lakner (1993), Delbaen and Schachermayer (1994), and Schachermayer (1994)]. Second, since the measure Q need not be positive, other types of arbitrage opportunities can exist under the hypothesis of this theorem and yet the market may be complete.

4. The Definition of Market Completeness with Respect to an Equivalent (Local) Martingale Measure

This section presents the definition of market completeness with respect to an equivalent (local) martingale measure. Prior to this definition, we need some additional structure.

We let V denote the set of price processes and assume that they are locally bounded. Also, let $\mathcal{M}_{++}/\mathcal{M}_{++}^{loc}$ denote the probability measures in M turning all the price processes into martingales/local martingales and whose null sets are precisely $\mathcal{N}$. The elements of $\mathcal{M}_{++}/\mathcal{M}_{++}^{loc}$ are referred to as equivalent martingale measures/equivalent local martingale measures.

Definition 2. *For $Q \in \mathcal{M}_{++}^{loc}$, the market is Q-complete if $\mathcal{A}_1 = Im\,T$ is dense in C with respect to the $L^1(\mathcal{F}_1, Q)$ topology.*

The market is said to be Q-complete with respect to an equivalent local martingale measure Q if for any contingent claim $X \in C$, there exists a sequence of trading strategies such that their values converge to X in the $L^1(\mathcal{F}_1, Q)$ sense.

This notion of closeness is distinct from the τ topology used in our definition of market completeness. These different notions of closeness are equivalent if and only if M is finite dimensional.[17] This simple observation shows why in the case where M is finite dimensional — the finite state, finite time economy — choosing a definition of closeness is unnecessary. More importantly, it also shows why, in the infinite dimensional case (e.g., the Black–Scholes economy), the specification of a definition for closeness is essential.

[17] The backward implication is obvious. On the other hand, if the two topologies coincide one easily concludes that $C = L^1(\mathcal{F}_1, Q)$. This is only possible if $\dim_{(C)<\infty}$ which is equivalent to $\dim(M) < \infty$. Indeed, if $\dim_{(C)<\infty}$ one can inductively construct a sequence of disjoint sets A_i $i \geq 1$, whose Q mass is positive but decreases to zero fast enough so that $\sum_i i 1_{A_i} \in \{L^1(\mathcal{F}_1, Q) - C\}$.

Finally, we need to make precise the meaning of no arbitrage for our setup:

Definition 3. *The market has no arbitrage (NA) if* $\mathcal{A}_1^0 \cap C_+ = \{0\}$*, where* C_+ *denotes the random variables in* C *which are nonnegative except possibly on a set in* $\mathcal{N}$*.*

The market is said to satisfy *NA* if it is not possible to find a trading strategy with zero initial cost that generates a time 1 value, which is always nonnegative and strictly positive on a set which is not null.

This definition is weaker than that needed to obtain versions of the first fundamental theorem of asset pricing when the time set is infinite [see Dalang, Morton, and Willinger (1990), Lakner (1993), Delbaen and Schachermayer (1994), and Schachermayer (1994)]. It is weaker because it does not involve the approximation of arbitrage opportunities of this sort via (nets) sequences.

For our purposes, *NA* is a sufficient restriction. It guarantees that if $X \in \mathcal{A}_1$, then any two trading strategies attaining X must have the same cost of construction, that is, $X = T(x, (H^\alpha)_{\alpha \in A}) = T(\widetilde{x}(\widetilde{H}^\alpha)_{\alpha \in A})$ implies $x = \widetilde{x}$. Contingent claims are then unambiguously priced by their initial cost of construction.

5. The Relationship Between Market Completeness and Q-Completeness

This section clarifies the relation between the two definitions of market completeness. This is done through a sequence of theorems and lemmas. The first theorem gives a sufficient condition for Q-completeness.

Theorem 2. *Let NA hold and let there exist a* $Q \in \mathcal{M}_{++}^{loc}$*. If* Q *is unique in* $\mathcal{M}_{++}^{loc}$ *then the market is* Q*-complete. When* $Q \in \mathcal{M}_{++}$ *then NA automatically holds. Also, it then suffices only that* Q *be unique in* $\mathcal{M}_{++}$*.*

Proof. See the appendix.

Theorem 2 shows that under the no arbitrage hypothesis, existence and uniqueness of a local martingale measure implies that the market is Q-complete.

However, the Artzner and Heath (1995) counterexample shows that, in general, the converse of Theorem 2 does not hold. They give an example where the market is Q-complete, yet there exists an infinite set of martingale measures.

To understand the relation between market completeness and Q-completeness the following lemma is useful.

Lemma 3. *If NA holds then* $\mathcal{M}_{++}^{loc} = \mathcal{P}_{+/-} \cap M_{++}$*.*

Proof. See the appendix.

This lemma shows the relation between local martingale measures, positive measures, and measures in the set $\mathcal{P}_{+/-}$.

Using this lemma, we see that given NA, if a local martingale measure Q exists (the hypothesis of Theorem 1) and Q is unique in $\mathcal{P}_{+/-}$ (market completeness by Theorem 1), then Q is unique in the class of local martingale measures as well. Using Theorem 2, this insight proves the following theorem.

Theorem 4. *Let NA hold and let there exist a $Q \in \mathcal{M}_{++}^{loc}$. If the market is complete, then the market is Q-complete.*

As the Artzner and Heath example shows, the converse of Theorem 4 does not hold in general. In fact, even uniqueness of Q in $\mathcal{M}_{++}^{loc}$ (stronger than Q-completeness) is not generally sufficient for completeness. The reason is that although uniqueness of Q in $\mathcal{M}_{++}^{loc}$ implies uniqueness in the subset of measures in $\mathcal{P}_{+/-}$ which are positive, there could exist another measure in $\mathcal{P}_{+/-}$ which is not positive. If so, the market is not complete by Theorem 1. Hence, we see that the new definition of market completeness is stronger than Q-completeness. See Battig (1997) for technical examples illustrating these points.

Under additional hypotheses, uniqueness of Q in $\mathcal{M}_{++}^{loc}$ does imply uniqueness of Q in $\mathcal{P}_{+/-}$. These additional hypotheses are given in our last theorem, which is a result of Battig (1997):

Theorem 5. *Let NA hold. Let V be finite or let all the elements of V be processes with continuous sample paths. The market is complete if and only if the market is Q-complete.*

Theorem 5 states that if there are a finite number of price processes [e.g., Jarrow and Madan (1995)] or if the price processes have continuous sample paths [e.g., Black and Scholes (1973), Heath, Jarrow, and Morton (1992)], then the two notions of market completeness are equivalent, given that the NA hypothesis holds.

Since, in practice, we always work under the NA hypothesis, the two notions of completeness coincide for the typical models seen in the literature. This is an important observation. It implies that the standard procedures for testing for Q-completeness involving the volatility matrix of the price processes also gives market completeness under our new definition. The standard procedures guaranteeing Q-completeness involve proving invertibility of the price processes volatility matrix [see Jarrow and Madan (1995) or Battig (1997)].

6. The Artzner and Heath Counterexample Revisited

This section revisits the Artzner and Heath counterexample of the second fundamental theorem under Q-completeness, and shows that it is not a counterexample to our new definition of market completeness. This section is slightly more abstract than the preceding sections, due to the specification of the details in the Artzner and Heath example.

Recall that Artzner and Heath's example (reproduced below) gives an economy that is Q-complete, but the martingale measure is not unique. We show below that the example provided is not complete according to our definition, hence by Theorem 1, we know that there should be more than one martingale measure. The resolution is in recognizing that Artzner and Heath use a different notion of closeness of random variables than our topology τ.

Prior to studying their example, it is instructive to first provide an equivalent characterization of our new definition of market completeness. This characterization uses the fact that T^* is the adjoint of T.

Theorem 6. *The market is complete if and only if T^* mapping M into χ is injective.*

Proof. See the appendix.

This theorem gives us a procedure for checking to see if the Artzner and Heath counterexample is complete with respect to our new definition. We need only investigate the operator T^* and show that it is not injective, that is, its kernel is nontrivial. This verification is done below.

Let Z be the set of nonnegative integers and N the set of all integers. There is discrete trading with only two trading dates times 0 and 1.

We work with the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_0, \mathcal{F}_1\}, P)$, where the state space $\Omega = Z - \{0\}$ is the set of positive integers. The information sets are $\mathcal{F}_0 = \{0, Z - \{0\}\}$, $\mathcal{F}_1 = \mathcal{F} =$ *all subsets of* $(Z - \{0\})$. The probability measure satisfies $P(i) > 0$ *for all* $i \in Z - \{0\}$. Otherwise the probability measure is left unspecified.

We consider a countably infinite number of traded assets indexed by $i \in N$. Their bounded price processes (hence $\mathcal{M}_{++}^{loc} = \mathcal{M}_{++}$) are given by the following expression. The notation $(Z_t^i(j)$ indicates the price of asset $i \in N$ at time t (for $t = 0, 1$) given state $j \in Z$.

$$
\begin{aligned}
Z_0^i &= i \text{ } for \text{ } i \in N \\
Z_1^0(j) &= \frac{c}{(p+q)}(1_{-1}(j) + 1_1(j)) \\
Z_1^i(j) &= \frac{c(q^{i+1} - p^{i+1})}{(pq)^i(q-p)} 1_i(j) \\
&\quad + \frac{c(p^i - q^i)}{(pq)^i(q-p)} 1_{i+1}(j) \text{ for } i \in N,\ j \in Z - \{0\} \\
Z_1^i(j) &= Z_1^{-i}(-j) \text{ for } -i \in N,\ j \in Z - \{0\},
\end{aligned}
$$

where $c = p/(1-p) + q/(1-q)$ for $p \in (0, 1)$ and $q \in (p, 1)$.

In words, this expression states the following. At time 0, all traded assets are worth a dollar. The zeroth asset pays $(c/[p+q])$ if either state 1 or -1 occurs. The ith asset's payoff at time 1 under state j for $i > 0$ is given by $c(q^{i+1} - p^{i+1})/(pq)^i(q-p)$ if state i occurs and by $c(q^i - p^i)/(pq)^i(q-p)$ if state $i+1$ occurs. The ith asset's payoffs are moving to the right along the positive integer indexed assets and to the left along the negative integer indexed assets.

The vector spaces χ, Y, C, and M are infinite dimensional here:

$$\begin{aligned} Y &= \left\{(x, (H^\alpha)_{\alpha\in A}) \in \Re \times \Re^A \mid (H^\alpha)_{\alpha\in A} \text{ has finite support}\right\} \\ C &= \ell^\infty(\Omega) = \{f\colon \Omega \to \Re \mid f \text{ is bounded}\} \\ M &= \ell^1(\Omega) = \left\{ f\colon \Omega \to \Re \mid \sum_{\omega\in\Omega} |f(\omega)| < \infty \right\} \end{aligned}$$

and χ is the topological dual of Y when Y is endowed with the coarsest topology making $\{T^*\mu\}_{\mu\in M} \cup \{\pi_0\}$ continuous linear functionals on Y.

The spaces for the random variables C and the potential valuation measures M are well understood. M is the topological dual of C.

Also, for $(x, (H^\alpha)_{\alpha\in A}) \in Y$ and $\mu \in M$ we have

$$\begin{aligned} T(x, (H^\alpha)_{\alpha\in A}) &= x + \sum_{\alpha\in A} H^\alpha (Z_1^\alpha - Z_0^\alpha) \\ (T^*(\mu))(x, (H^\alpha)_{\alpha\in A}) &= \mu(\Omega) + \sum_{\alpha\in A} H^\alpha \int (Z_1^\alpha - Z_0^\alpha) d\mu. \end{aligned}$$

The linear functional T maps trading strategies into time 1 payoffs, and the linear functional T^* gives the time 0 value of the trading strategy with initial cost x.

For this example we can explicitly find ker T^*. Indeed, a signed measure μ on $Z - \{0\}$ is in ker T^* if and only if $\mu(\Omega) = 0$ and $\int Z_1^i - Z_0^i d\mu = 0$ for $i \in Z$.

First, this implies that $\int Z_1^i d\mu = 0$ for $i \in Z$. Second, using the fact that $Z_1^i(j) = Z_1^{-i}(-j)$ we see that μ must solve the following equations:

$$\begin{aligned} Z_1^0(-1)\mu(-1) + Z_1^0(1)\mu(1) &= 0 \\ Z_1^i(i)\mu(i) + Z_1^i(i+1)\mu(i+1) &= 0 \text{ for } i \geq 1 \\ Z_1^i(i)\mu(-i) + Z_1^i(i+1)\mu(-i-1) &= 0 \text{ for } i \geq 1. \end{aligned}$$

On the other hand, any μ satisfying these equations automatically has total mass zero and so $\mu \in \ker T^*$.

Note that $\mu^*(i) = -\mu^*(-i) = q^i - p^i$ for $i \geq 1$ defines a signed measure solving the above equations and that any other such measure is a scalar multiple of μ^*. Hence

$$\ker T^* = \{\gamma \mu^* \mid \gamma \in \Re\}.$$

The kernel for T^* is nontrivial, hence T^* is not injective and by Theorem 6 we do not have completeness.

7. Conclusion

This article presents a new definition of market completeness. This new definition is independent of the notions of no arbitrage and equivalent martingale measures. Even in complex economies, like that contained in the Artzner and Heath (1995) counterexample, this definition preserves the second fundamental theorem of asset pricing — the market is complete if and only if a (suitably defined) valuation operator is unique.

Our new definition of market completeness is consistent with practice (and the finite state, finite time economy) since it allows the existence of arbitrage opportunities in complete markets. For the standard models used in the literature [e.g., Black and Scholes (1973), Heath, Jarrow, and Morton (1992), Jarrow and Madan (1995)] the new definition of market completeness is shown to be equivalent to the traditional definition. This is an important observation as it leaves intact all of the existing theorems and techniques for proving market completeness in the standard models.

Appendix

Rather than presenting proofs in their full technicality, we sketch the important ideas and provide the interested reader with detailed references.

Theorems 1 and 6

Recall from linear algebra that a linear operator (matrix) between finite-dimensional vector spaces is onto if and only if the adjoint operator (transpose matrix) is one to one.

This result can be generalized in the following way. Suppose Y, C, M, and $\mathcal{X}$ are arbitrary vector spaces and $T\colon Y \to C$ is a linear operator. If Y and $\mathcal{X}$, as well as C and M, can be placed in duality and if $T\colon Y \to C$ is continuous when Y and C are endowed with the topologies arising from their respective dualities, then there is a well-defined adjoint operator $T^*\colon M \to X$ and *Im* T is dense (w.r.t. the topology arising from the duality) if and only if T^* is injective. See Grothendieck (1973: 82), particularly Proposition 26.

One easily sees that these hypotheses are met in our case by the vector spaces Y, C, M, and $\mathcal{X}$ and our linear operator $T\colon Y \to C$ defined in Section 1. In fact, the duality for C and M is given by Equation (4). Furthermore, the topology on C arising from the duality is the topology τ that was used in the definition of market completeness and hence Theorem 6 follows.

The definition of $\mathcal{P}_{+/-}$ implies that if $Q \in \mathcal{P}_{+/-}$, then $\mathcal{P}_{+/-} = \{Q + \mu \mid \mu \in \ker T^*\}$. Theorem 1 now follows from Theorem 6.

Theorem 2

The proof is by contrapositive. By arguments similar to the ones we used above for Theorem 6, one can show that Q-completeness is equivalent to one-to-oneness of T^*: $M^{(Q,\infty)} \to X$, where $M^{(Q,\infty)}$ is the subspace of M consisting of the signed measures whose Radon-Nikodym derivative with respect to Q is bounded.

If the market is not Q-complete one can find a nonzero measure $\mu \in M^{(Q,\infty)} \cap \ker T^*$ such that $Q + \mu$ is a positive measure, that is, $Q + \mu \in M_{++}$, which differs from Q. Furthermore, $Q + \mu \in \mathcal{M}^{loc}_{++}$ and if $Q \in \mathcal{M}_{++}$, then $Q + \mu \in \mathcal{M}_{++}$. Finally, the assertion that NA holds automatically if $Q \in \mathcal{M}_{++}$ is well known; if $X \in \mathcal{A}^0_1 \cap C_+$ then $X \geq 0$ and $E_Q X = 0$, hence $X = 0$.

Lemma 3

Let $\mathcal{A}$ denote the collection of value processes associated with the attainable claims as defined in the text. If NA holds, one can show that these processes are bounded and hence are (bounded) martingales under any $Q + \mu \in \mathcal{M}^{loc}_{++}$. But then

$$\begin{aligned}(T^*Q)(x, (H^\alpha)_{\alpha\in A}) &= E_Q T(x, (H^\alpha)_{\alpha\in A}) = E_Q\left(x + \sum_{\alpha\in A}\int_0^1 H^\alpha_u \, dZ^\alpha_u\right) \\ &= x = \pi_0(x, (H^\alpha)_{\alpha\in A})\end{aligned}$$

for $(x, (H^\alpha)_{\alpha\in A}) \in Y$, which shows that $\mathcal{M}^{loc}_{++} \subseteq \mathcal{P}_{+/-} \cap M_{++}$.

For the reverse containment, one notes that $Q \in \mathcal{P}_{+/-} \cap M_{++}$ is a probability measure, and since the elements of V are locally bounded, the argument is completed by showing that for fixed $\alpha^* \in A$ and stopping time τ such that $Z^\alpha_{\tau\wedge t}$ is bounded, $Z^\alpha_{\tau\wedge t}$ is a Q-martingale.

References

Ansel, J. P., and C. Stricker, 1984, “Couverture des Actifs Contingents et Prix Maximum,” *Ann. Inst. Henri Poincare–Probabilite et Statistique*, 30, 303–315.

Artzner, P., and D. Heath, 1995, “Approximate Completeness with Multiple Martingale Measures,” *Mathematical Finance*, 5, 1–11.

Battig, R., 1997, “Completeness of Securities Market Models — An Operator Point of View,” forthcoming in *Annals of Applied Probability*.

Black, F., and M. Scholes, 1973, “The Pricing of Options and Corporate Liabilities,” *Journal of Political Economy*, 81, 637–654.

Dalang, R. C., A. Morton, and W. Willinger, 1990, “Equivalent Martingale Measures and No-Arbitrage in Stochastic Securities Market Models,” *Stochastics and Stochastic Reports*, 29, 185–201.

Delbaen, F., 1992, “Representing Martingale Measures When Asset Prices are Continuous and Bounded,” *Mathematical Finance*, 2, 107–130.

Delbaen, F., and W. Schachermayer, 1994, “A General Version of the Fundamental Theorem of Asset Pricing,” *Mathematische Annalen*, 300, 463–520.

Grothendieck, A., 1973, “Notes on Mathematics and Its Applications,” in *Topological Vector Spaces*, Gordon and Breach Science Publishers, Montreux, Switzerland.

Harrison, J. M., and D. M. Kreps, 1979, "Martingales and Arbitrage in Multiperiod Securities Markets," *Journal of Economic Theory*, 20, 381–408.

Harrison, J. M., and S. R. Pliska, 1981, "Martingales and Stochastic Integrals in the Theory of Continuous Trading," *Stochastic Processes and Their Applications*, 11, 215–260.

Heath, D., R. Jarrow, and A. Morton, 1992, "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation," *Econometrica*, 60, 77–105.

Jarrow, R. A., S. Jin, and D. B. Madan, 1997, "The Second Fundamental Theorem of Asset Pricing Theory," working paper, Cornell University, forthcoming in *Mathematical Finance*.

Jarrow, R. A., and D. B. Madan, 1995, "Option Pricing Using the Term Structure of Interest Rates to Hedge Systematic Discontinuities in Asset Returns," *Mathematical Finance*, 5, 311–336.

Jarrow, R. A., and D. B. Madan, 1997a, "Arbitrage, Rational Bubbles and Martingale Measures," working paper, Cornell University.

Jarrow, R. A., and D. B. Madan, 1997b, "Hedging Contingent Claims on Semimartingales," working paper, Cornell University, forthcoming in *Finance and Stochastics*.

Lakner, P., 1993, "Martingale Measure for a Class of Right-Continuous Processes," *Mathematical Finance*, 3, 43–53.

Merton, R. C., 1976, "Option Pricing When Underlying Stock Returns Are Discontinuous," *Journal of Financial Economics*, 3, 125–144.

Ross, S., 1976, "The Arbitrage Theory of Capital Asset Pricing," *Journal of Economic Theory*, 13, 341–360.

Schachermayer, W., 1994, "Martingale Measures for Discrete-Time Processes with Infinite Horizon," *Mathematical Finance*, 4, 25–55.

Sin, C. A., 1996, "Strictly Local Martingales and Hedge Ratios on Stochastic Volatility Models," Ph.D. dissertation, Cornell University.