# Improving Lattice Schemes Through Bias Reduction

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**Abstract**

We propose a simple modification of lattice schemes which reduces the bias of lattice option prices with respect to continuous time and state option prices. The modification is generic and is applied here to binomial and trinomial trees used to price American options. Unlike the typical lattice approaches which minimize the distance between the approximating and target distributions by matching the first moments of the distributions, we propose a lattice design minimizing the distance between the computed and theoretical European price. This lattice is then used to price American options. We present a numerical study showing the benefits of the proposed modification in terms of speed and accuracy.

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# 1 Introduction

The binomial tree introduced by Cox, Ross and Rubinstein in 1979 (hereafter CRR) is one of the most important innovation to have appeared in the option pricing literature. Beyond its original use as a tool to approximate the prices of European and American options in the Black Scholes (1973) framework, it is also widely used as a pedagogical device to introduce various key concepts in option pricing.

In the literature, many solutions have been proposed to improve CRR binomial tree pricing. In Hull and White (1988), a control variate approach based on the European Black-Scholes price is suggested to improve the quality of the American price evaluation, while Tian (1993) proposed to force higher moments of the discrete distribution to match the moments of the underlying continuous distribution. Others, such as Broadie and Detemple (1995) and Tian (1999), proposed modifications smoothing out the jagged "price vs. number of time steps" curve of the CRR approach, enabling the use of Richardson extrapolation. Tian's (1999) method is essentially a CRR tree modified by a tilt parameter, while Broadie and Detemple (1995) use the Black-Scholes price instead of the usual continuation value at the penultimate time step before the option's maturity. Other suggestions from Boyle (1988) and Kamrad and Ritchken (1991) are to replace the binomial tree by a trinomial tree. Figlewski and Gao (1999) for their part proposed a trinomial tree with sections of finer meshing, allowing a greater accuracy at a negligible additional computational cost.

In all of the approaches above, the lattice is designed so as to minimize the discrepancy between the approximate (discrete) and target (continuous) distributions by matching, exactly or approximately, their first few moments. The rationale for this is that for any fixed number of time steps, a moment-matching lattice is believed to produce better option price estimates. We suggest a different avenue to lattice design in this paper, which relies on a change of probability measure. Specifically, we show how to modify lattice schemes in general, so that a target different from the usual moment matching objective is achieved. For example, one objective examined in this study is to build a lattice minimizing *the difference between the lattice-based price and the Black-Scholes, analytic price* for a European option. The new lattice is then used to price the corresponding *American* option. For a fixed number of time steps, such a modified lattice induces a discrete distribution whose moments show some departure with the moments of the continuous, target distribution; in the limit however, the distribution associated to the modified lattice converges to the continuous distribution, as is the case for the unmodified lattice schemes. Figure 1 displays the improvement obtained with the modified CRR binomial tree for European and American put options. For the European option, the modified CRR approach yields much smaller price biases with respect to the analytic solution, than the unmodified CRR approach. In the American case, the biases are also improved by the modification, though the improvement is more moderate. Note that all lattice approaches in this paper, whether with or without our modification, are implemented with the Broadie and Detemple (1996) idea of using Black-Scholes prices at the penultimate time step.

Technically, the modification is implemented by replacing the expectations under the risk neutral probability measure, by an equivalent expression under an alternative probability measure. This equivalent expression is specified by the Radon-Nikodym derivative. The family of alternative probability measures considered here is characterized by a unique parameter that corresponds to the drift term of the stock price process. A one-dimensional numerical search yields the parameter that minimizes the price discrepancy. Since this minimization can be performed with trees of very small dimension (say, 10 or 20 time steps), the additional work is typically very small in respect of the precision gain.

The approach is generic in the sense that it can be adapted to the most lattice schemes available in the literature. Examples treated here are the CRR and Jarrow-Rudd binomial trees, Kamrad and Ritchken's trinomial tree, and the trinomial equivalent of the explicit difference, p.d.e. approach. The paper is divided as follows. After this introduction, Section 2 presents background concepts on the changes of measures considered in this study. We next show in Section 3 how to apply these measure changes in order to modify existing lattice schemes. Section 4 contains the details regarding the numerical implementation and the results of our numerical study.

## 2 Background

Under the assumption of complete and arbitrage free markets, Harisson and Kreps (1979) have shown the existence of a unique risk neutral probability measure $Q$ which allows the computation of European style option prices as expected values discounted at the risk free rate. More formally, the price of a European option, under a constant risk free rate assumption, can be written as :

$$V_0 = \mathrm{E}^Q \left[ e^{-rT} f(S_T, K) \right] \tag{1}$$

where $S_T$ is the stock price at time $T$, the maturity date of the option, $K$ is the strike price, $r$ is the constant risk free rate and $f(S_T, K) = \max\left[\phi(S_T - K), 0\right]$ with $\phi = 1$ for call options and -1 for put options is the payoff function of the option. For an American style option, the situation is more complex because of the early exercise possibility. It can however still be written as an expected value:

$$V_0 = \sup_{\tau} \mathrm{E}^Q \left[ e^{-r\tau} f(S_T, K) \right] \tag{2}$$

where the supremum is over all stopping times $\tau \leq T$.

This way of expressing the expected values in the option price formula is not unique. Indeed, it is well known that, through Radon-Nikodym theorem (see Baxter and Rennie 1996), the expectation under one probability measure $Q$ can be expressed as an expectation under another equivalent probability measure $Q^*$. For example, for the European option case, the price could be written as

$$V_0 = \mathrm{E}^{Q^*} \left[ e^{-rT} f(S_T, K) \frac{dQ}{dQ^*} \right] \tag{3}$$

where $\frac{dQ}{dQ^*}$ is the Radon-Nikodym derivative. Theoretically, both expressions lead to identical prices. However, in practice, since these expectations must often be assessed numerically, equation (1) and (3) may obtain different estimated values. For example, in the Monte Carlo simulation context, the estimated price obtained under the alternative measure may have a different variance that the one obtained with the risk-neutral measure. This is the rational for the importance sampling approach which is commonly encountered in the Monte Carlo simulation literature.

In order to compute option prices using an alternative measure such as the one given by equation (3), an expression for the Radon-Nikodym derivative must be available. One way to obtain such an expression is through the stochastic process specified for the underlying security. In the Black-Scholes context, the dynamics of the stock price under the risk neutral measure $Q$ is specified as

$$dS_t = rS_t \; dt + \sigma S_t \; dW_t \tag{4a}$$

with $S_0 = s_0$ and $W$ is a standard Brownian motion under the risk neutral measure $Q$. To keep the problem tractable, we restrict the change of measure considered here to the class of measures $\left\{Q^\lambda : \lambda \in \mathbb{R}\right\}$ preserving the geometric Brownian motion structure of the stock price. Indeed, as shown in Appendix A, the new dynamics for the stock price under $Q^\lambda$ is

$$dS_t^\lambda = \lambda S_t^\lambda \; dt + \sigma S_t^\lambda \; dW_t^\lambda, \tag{5a}$$

with $S_0^\lambda = s_0$ and the Radon-Nikodym derivative, in this particular case, expressed as a likelihood ratio $\frac{dQ}{dQ^\lambda} = L\left(S_T^\lambda, \lambda\right)$ where

$$L\left(S_T^\lambda, \lambda\right) = \exp\left[\frac{r-\lambda}{\sigma^2}\ln\left(\frac{S_T^\lambda}{s_0}\right) + \frac{1}{2}\left(\lambda - r\right)\frac{r+\lambda-\sigma^2}{\sigma^2}T\right]. \tag{6}$$

Equipped with these expressions for the likelihood ratio and the dynamics of the stock price, it is now possible to compute theoretically and/or numerically expectations under alternative probability measures. In the next section, we see how these formula can be applied to modify existing lattice approaches such as binomial or trinomial trees for example.

# 3 Implementing change of measures to lattice approaches

Consider a lattice with $n$ time steps of length $T/n$. The stock price at time $i\frac{T}{n}$ $(i = 0, 1, ..., n)$ at the $j$th node of the lattice is denoted by $s_{i,j}$. The probability, under the risk neutral measure $Q$, of reaching node $s_{i+1,k}$ from node $s_{i,j}$ is represented by $q_{i,j\to k}$. A lattice scheme simply specifies how the prices and probabilities can be computed given the length of the time step, the initial stock price, the interest rate and the volatility parameter. These specifications are usually obtained by imposing the equality between the first two moments of the approximate and target distribution.

To adapt a lattice to the alternative probability measure $Q^\lambda$, it suffices to note that the change of measure from $Q$ to $Q^\lambda$ preserves the geometric Brownian motion structure of the stock price. The difference is the drift coefficient which is no longer $r$ but $\lambda$. It is therefore straightforward to determine the stock prices and the transition probabilities of the $Q^\lambda-$lattice by replacing $r$ by $\lambda$ in the design for $s_{i,j}$ and $q_{i,j\to k}$. We will therefore define $s_{i,j}^\lambda$ as the stock price at the $i$th time step and the $j$th node of the $Q^\lambda-$lattice and $q_{i,j\to k}^\lambda$as the $Q^\lambda-$probability of reaching node $s_{i+1,k}^\lambda$ from node $s_{i,j}^\lambda$.

In the original $Q-$lattice, European and American option prices can be obtained by working backward from the end of grid. Indeed, using the familiar dynamic programming principle, the option value at the $i$th time step and the $j$th node is

$$v_{i,j} = \max\left\{\phi\left(s_{i,j} - K\right), e^{-r\frac{T}{n}}\sum_k v_{i+1,k}q_{i,j\to k}\right\},\ i < n \tag{7a}$$

with $v_{n,j} = \max\left\{\phi\left(s_{n,j} - K\right), 0\right\}$.Unfortunately, a well known draw back of lattices such as the binomial or trinomial tree is the jagged convergence pattern exhibited by the computed price. We will therefore adopt here a simple modification proposed in Broadie and Detemple (1995) which considerably smooths out the convergence pattern. The modified algorithm directly starts the computations at time step $n-1$ and computes the Black-Scholes price instead of the continuation value $v_{n-1,j}$ i.e.

$$v_{i,j} = \max\left\{\phi\left(s_{i,j} - K\right), e^{-r\frac{T}{n}}\sum_k v_{i+1,k}q_{i,j\to k}\right\},\ i < n-1 \tag{8a}$$

with $v_{n-1,j} = BS\left(s_{n,j}; r, \sigma, K, \frac{T}{n}\right)$ denoting the Black-Scholes price for a European style option with initial stock price $s_{n,j}$, strike price $K$, time to maturity $\frac{T}{n}$, risk free rate $r$ and volatility coefficient $\sigma$.

Using this general framework, it is straightforward to adapt the algorithm for the $Q^\lambda-$lattice. Indeed, the discretized equivalent for the likelihood ratio (6) simply becomes

$$l_{i,j}\left(\lambda\right) = \exp\left[\frac{r-\lambda}{\sigma^2}\ln\left(\frac{s_{i,j}^\lambda}{s_0}\right) + \frac{1}{2}\left(\lambda - r\right)\frac{r+\lambda-\sigma^2}{\sigma^2}i\frac{T}{n}\right] \tag{9}$$

and the option values will be computed with

$$v_{i,j}^\lambda = \max\left\{\phi\left(s_{i,j}^\lambda - K\right) l_{n,j}\left(\lambda\right), e^{-r\frac{T}{n}}\sum_k v_{i+1,k}^\lambda q_{i,j\to k}^\lambda\right\},\ i < n-1. \tag{10a}$$

with $v_{n-1,j}^\lambda = BS\left(s_{n-1,j}^\lambda; r, \sigma, K, \frac{T}{n}\right) l_{n-1,j}\left(\lambda\right)$. Note that for the American case, since the prices in the tree are those with respect to $Q^\lambda$, it is important to multiply the early exercise value with the likelihood ratio in the dynamic programming equation. Working backward to the tree will lead to an estimate of the option price. Appendix B shows how specific lattice schemes such as CRR (1979) or Kamrad and Ritchken (1991) can be implemented using the above equations.

In practice, the value of $\lambda$ must be assessed numerically. The idea is to build a lattice with a small number of time steps and use a numerical procedure to find a value $\lambda^*$ which achieves a target objective. For example, the value of $\lambda$ could be set by minimizing the discrepancy between the estimated European option price $v_0^{\lambda^*}$ and the Black and Scholes formula $BS\left(s_0; r, \sigma, K, T\right)$. As shown in the next section, this value of $\lambda^*$ might not be unique and, interestingly, a given value of $\lambda^*$ is not sensible the number of time steps. In other words, a value of $\lambda$ working well for 10 time steps will also work well for 20 or more time steps. This suggests the use of an alternative objective function. One could, for example, compute prices of American options with two lattices using $n_1$ and $n_2$ time steps ($n_1 = 10$ and $n_2 = 20$ for example) and choose a $\lambda$ minimizing the distance between the two computed prices. These ideas will be examined in the section presenting the result of numerical experiments.

# 4 Numerical results

## 4.1 A detailed example

This section provides a simple numerical example using the CRR (1979) lattice with a Black-Scholes price at the penultimate time step (CRR-BS hereafter). This example will provide some intuition on how the expected values and probability distributions are altered under the alternative measure. The following parameter values are used: $s_0 = 100, r = 0.05, \sigma = 0.4, T = 1$ and $n = 3$. With these and the formulas for the CRR (1979) lattice given in Appendix B we have $u = 1.2597$, $d = 0.7937$, $q^{\lambda=r} = 0.4785$ and $q^{\lambda=\lambda^*} = 0.6042$ for $\lambda^* = 0.218$. This last value was found using a numerical search algorithm minimizing the distance between the price obtained with the $Q^{\lambda}-$ CRR-BS lattice with 10 time steps and a Black-Scholes price.

Since in this framework the stock prices remain identical under all measures, it is instructive to compute the discrete distribution at time step $n-1$ for the stock price using $\lambda = r$ and $\lambda = 0.218$ that is the value of $\lambda$ which as been identified numerically to provide accurate prices. Using these distributions, it is then possible to make helpful comparisons showing how the proposed modification changes the computed values. Casual calculations show that the probability to reach state $j$ from the initial stock price in $n-1$ steps can be written as:

$$\chi_j^{\lambda} = \frac{(n-1)!}{j!\,(n-1-j)!}\left(q^{\lambda}\right)^j\left(1-q^{\lambda}\right)^{n-1-j} \times l_{n-1,j}(\lambda).$$

Table 1 reports the computed probabilities $\chi_j^{\lambda=r}$ and $\chi_j^{\lambda=\lambda^*}$ with the associated option prices at step $n-1$. From the numbers in this table it is easy to see that the expected value and standard deviation (discounted) are 100 and 32.2560 when computed with $\chi_j^{\lambda=r}$ while they are 99.98 and 32.2696 when computed with the distribution $\chi_j^{\lambda=\lambda^*}$. The European put option price computed with both measures are, respectively, 13.3989 and 13.1151 while the Black-Scholes price is 13.1459. This shows that the discrete distribution that was originally built to exactly match the first and second moments of the target distribution is modestly altered in order to remove the large bias in terms of price from the original CRR-BS approach. This alteration of the probability distribution has a very small effect on the discounted expected stock price. Indeed, the changes in probability for large stock prices are compensated by the changes in probabilities for low stock prices. However, this is not the case when computing the expected value of the put option payoff which has a low sensitivity to the probability changes associated to large stock prices and a high sensitivity to the changes associated with low stock prices.

Table 2 examines the computed American put option prices as the number of time step is increased using both the CRR-BS binomial tree and $Q^{\lambda^*}$–CRR-BS binomial tree. These numbers can be compared to benchmark values obtained from the Black-Scholes formula for the European put and a 15000 step CRR-BS binomial tree for the American put. As it can be seen, the option prices computed with 20 or 30 steps $Q^{\lambda^*}$–CRR-BS lattice are more accurate than those obtained with 100 time steps with the CRR-BS lattice.

The results presented in this table are of course specific to the chosen example. Furthermore, the additional work required by the algorithm is not taken into account. We will therefore present in the next subsections a numerical study that will assess the performance of the algorithm in terms of computer time and precision for a large test pool of option contracts.

## 4.2 The optimization algorithm

To be continued ...

### 4.3 Numerical study

In order to obtain a more general assessment on the quality of the method, we perform an analysis similar to Broadie and Detemple (1996). The analysis begins by choosing a large test pool of options using randomly selected parameter values based on pre-determined distributions. For each option, the prices using the $Q-$lattice and the $Q^{\lambda^*}-$lattice are computed and compared to a benchmark value. We keep track of the computation times and the pricing errors.

The following distributions are used for the parameter values, with each parameter value drawn independently of the others: $\phi$ equals 1 or $-1$ with probability $1/2$ for each value, $T$ is distributed uniformly between 0.1 and 1.0 year with a probability of 0.75, and uniformly between 1.0 and 3.0 years with a probability of 0.25; $S_0 = 100$ and $K$ is uniform between 70 and 130; $r$ is, with probability 0.8, uniform between 0 and 0.1, and with probability 0.2, equal to 0 and $\sigma$ is uniformly distributed between $(0, 1; 0.6)$.

Using the simulated parameter values, the lattice prices and the benchmarks are computed and compared. We use the root mean square error as our measure of aggregate pricing error. Specifically,

$$RMS(m) = \sqrt{\frac{1}{m}\sum_{i=1}^{m} e_i^2}, \tag{11}$$

where $e_i = |C_i(lattice) - C_i|/C_i$; $C_i$ is the benchmark obtained for the $i^{th}$ option; and $C_i(lattice)$ is the $i^{th}$ price obtained with the $Q-$lattice or the $Q^{\lambda}-$lattice. The variable $m$ stands for the size of the test pool. In our comparison study, we use $m = 1\ 000$ and eliminate the cases where $C_i < 0.50$ to avoid large relative errors caused by a small divider.

To be continued ....

## A Radon-Nikodym derivative

Define

$$W_t^{\lambda} = W_t - \frac{\lambda - r}{\sigma} t \text{ for any } t \geq 0 \tag{12}$$

and note that the replacement of $W_t$ by $W_t^{\lambda} + \frac{\lambda - r}{\sigma} t$ in Equation (4) leads to Equation (5). We would like that the distribution of $W_t^{\lambda}$ to be $N(0, t)$ under the measure $Q^{\lambda}$ which is equivalent to ask that

$W_t$ is $N(\frac{\lambda-r}{\sigma}t,t)$ under the measure $Q^\lambda$. Since $W_t$ is $N(0,t)$ under measure $Q$ and $N(\frac{\lambda-r}{\sigma}t,t)$ under the measure $Q^\lambda$, the change of measure is expressed as a ratio of density functions :

$$L\left(W_t,\lambda\right)=\frac{\frac{1}{\sqrt{2\pi t}}\exp\left[-\frac{1}{2}\frac{W_t^2}{t}\right]}{\frac{1}{\sqrt{2\pi t}}\exp\left[-\frac{1}{2}\frac{\left(W_t-\frac{\lambda-r}{\sigma}t\right)^2}{t}\right]}=\exp\left(-\frac{\lambda-r}{\sigma}W_t+\frac{1}{2}\left(\frac{\lambda-r}{\sigma}\right)^2t\right). \tag{13}$$

By a simple change of variable, the likelihood ratio is expressed as a function of the $Q^\lambda-$Brownian motion :

$$L\left(W_T^\lambda,\lambda\right)=\exp\left[-\frac{\lambda-r}{\sigma}W_T^\lambda-\frac{1}{2}\left(\frac{\lambda-r}{\sigma}\right)^2T\right]. \tag{14}$$

Finally, since

$$S_T^\lambda=s_0\exp\left[\left(\lambda-\frac{\sigma^2}{2}\right)T+\sigma W_T^\lambda\right] \tag{15}$$

implies that

$$W_T^\lambda=\frac{\ln\left(\frac{S_T^\lambda}{s_0}\right)-\left(\lambda-\frac{\sigma^2}{2}\right)T}{\sigma}, \tag{16}$$

the likelihood ratio can also expressed as a function of the stock price under $Q^\lambda$:

$$\begin{aligned}L\left(S_T^\lambda,\lambda\right)&=\exp\left[-\frac{\lambda-r}{\sigma}\frac{\ln\left(\frac{S_T^\lambda}{s_0}\right)-\left(\lambda-\frac{\sigma^2}{2}\right)T}{\sigma}-\frac{1}{2}\left(\frac{\lambda-r}{\sigma}\right)^2T\right]\\&=\exp\left[\frac{r-\lambda}{\sigma^2}\ln\left(\frac{S_T^\lambda}{s_0}\right)+\frac{1}{2}\left(\lambda-r\right)\frac{r+\lambda-\sigma^2}{\sigma^2}T\right].\end{aligned}$$

# B Lattice scheme under alternative measures

## B.1 Binomial trees

Many binomial trees are designed on similar basis. The stock price at $i$th time step and the $j$th node is

$$s_{i,j}=s_0u^jd^{i-j}, i\in\{0,1,...,n\}, j\in\{0,1,...,i\} \tag{17}$$

where $u$ and $d$ are the multiplicative constants for up and down movements in the tree. The probability, under the risk neutral measure $Q$, of an upward move is $q$, i.e.

$$q_{i,j\to k}=\begin{cases}q & \text{if } k=j+1\\1-q & \text{if } k=j\\0 & \text{otherwise.}\end{cases} \tag{18}$$

To choose the different constants $u$, $d$ and $q$, many authors proposed to match the two first moments of the binomial stock price with those of the target continuous time stochastic process which lead to the two following equations related respectively to the expectation and the variance:

$$qu + (1-q) = e^{r\frac{T}{n}} \tag{19a}$$

$$qu^2 + (1-q)\,d^2 - e^{2r\frac{T}{n}} = e^{2r\frac{T}{n}}\left(e^{\sigma^2\frac{T}{n}} - 1\right). \tag{19b}$$

Because there is three variables and two equations, there is some freedom to assess a value to one of the variable. This lead to the different versions of the binomial tree.

### B.1.1 Cox, Ross and Rubinstein

In the binomial tree proposed by Cox, Ross and Rubinstein (1979), we have

$$u = \exp\left[\sigma\sqrt{\frac{T}{n}}\right] \text{ and } d = \exp\left[-\sigma\sqrt{\frac{T}{n}}\right]. \tag{20}$$

and

$$q = \frac{\exp\left[r\frac{T}{n}\right] - d}{u-d} = \frac{\exp\left[r\frac{T}{n}\right] - \exp\left[-\sigma\sqrt{\frac{T}{n}}\right]}{\exp\left[\sigma\sqrt{\frac{T}{n}}\right] - \exp\left[-\sigma\sqrt{\frac{T}{n}}\right]} \tag{21}$$

To adapt this tree to the measure $Q^\lambda$, note that the stock price remains the same

$$s_{i,j}^\lambda = s_0 u^j d^{i-j} = s_0 \exp\left[(2j-i)\,\sigma\sqrt{\frac{T}{n}}\right] \tag{22}$$

since it does not depends on $r$. The probability $q^\lambda$ of an upward move becomes

$$q^\lambda = \frac{\exp\left[\lambda\frac{T}{n}\right] - d}{u-d} = \frac{\exp\left[\lambda\frac{T}{n}\right] - \exp\left[-\sigma\sqrt{\frac{T}{n}}\right]}{\exp\left[\sigma\sqrt{\frac{T}{n}}\right] - \exp\left[-\sigma\sqrt{\frac{T}{n}}\right]}. \tag{23}$$

The likelihood ratio for the $i$th time step and the $j$th node is

$$l_{i,j}(\lambda) = \exp\left[\frac{r-\lambda}{\sigma}(2j-i)\sqrt{\frac{T}{n}} + \frac{1}{2}(\lambda - r)\frac{\lambda + r - \sigma^2}{\sigma^2}i\frac{T}{n}\right]. \tag{24}$$

### B.1.2 Jarrow and Rudd

In the binomial tree proposed by Jarrow and Rudd (1983), the constants are

$$u = \exp\left[\left(r - \frac{\sigma^2}{2}\right)\frac{T}{n} + \sigma\sqrt{\frac{T}{n}}\right],\ d = \exp\left[\left(r - \frac{\sigma^2}{2}\right)\frac{T}{n} - \sigma\sqrt{\frac{T}{n}}\right] \text{ and } q = \frac{1}{2}. \tag{25}$$

To adapt this binomial tree under the measure $Q^\lambda$, it suffices to modify the multiplicative constants which are now given by :

$$u_\lambda = \exp\left[\left(\lambda - \frac{\sigma^2}{2}\right)\frac{T}{n} + \sigma\sqrt{\frac{T}{n}}\right] \text{ and } d_\lambda = \exp\left[\left(\lambda - \frac{\sigma^2}{2}\right)\frac{T}{n} - \sigma\sqrt{\frac{T}{n}}\right]. \tag{26}$$

The transition probabilities remains the same because there are not function of $r$. The likelihood ratio becomes

$$l_{i,j}(\lambda) = \exp\left[-\frac{\lambda - r}{\sigma}\sqrt{\frac{T}{n}}(2j - i) - \frac{1}{2}\left(\frac{\lambda - r}{\sigma}\right)^2 i\frac{T}{n}\right]. \tag{27}$$

## B.2 Trinomial trees

### B.2.1 Kamrad and Ritchken

The trinomial tree proposed by Kamrad and Ritchken (1991) match the two first moments of the stock return to their theoritical value. In this particular set up, the stock price at $i$th time step and the $j$th node is

$$s_{i,j} = s_0 \exp\left[(i - j)\,\sigma\sqrt{\frac{3}{2}}\sqrt{\frac{T}{n}}\right], i \in \{0, 1, ..., n\}, j \in \{0, 1, ..., 2i\} \tag{28}$$

The transition probabilities, under the risk neutral measure $Q$, are

$$q_{i,j\to k} = \begin{cases} \frac{1}{3} + \frac{1}{3\sigma}\left(r - \frac{\sigma^2}{2}\right)\sqrt{\frac{T}{n}} & \text{if } k = j+1 \\ \frac{1}{3} & \text{if } k = j \\ \frac{1}{3} - \frac{1}{3\sigma}\left(r - \frac{\sigma^2}{2}\right)\sqrt{\frac{T}{n}} & \text{if } k = j-1 \\ 0 & \text{otherwise.} \end{cases} \tag{29}$$

To adapt this tree to the change of measure, note that the stock price remains the same $s_{i,j}^{\lambda} = s_{i,j}$ since it does not depends on $r$. The transition probabilities becomes

$$q_{i,j\to k}^{\lambda} = \begin{cases} \frac{1}{3} + \frac{1}{3\sigma}\left(\lambda - \frac{\sigma^2}{2}\right)\sqrt{\frac{T}{n}} & \text{if } k = j+1 \\ \frac{1}{3} & \text{if } k = j \\ \frac{1}{3} - \frac{1}{3\sigma}\left(\lambda - \frac{\sigma^2}{2}\right)\sqrt{\frac{T}{n}} & \text{if } k = j-1 \\ 0 & \text{otherwise.} \end{cases} \tag{30}$$

The likelihood ratio for the $i$th time step and the $j$th node is

$$l_{i,j}(\lambda) = \exp\left[\frac{r-\lambda}{\sigma}\alpha(i-j)\sqrt{\frac{T}{n}} + \frac{1}{2}(\lambda - r)\frac{r+\lambda-\sigma^2}{\sigma^2}i\frac{T}{n}\right]. \tag{31}$$

### B.2.2 The explicit finite difference approach

It can be shown that he explicit finite difference method is equivalent to the trinomial tree approach. The stock price at $i$th time step and the $j$th node is

$$s_{i,j} = s_0 + (j-i)\Delta_s, i \in \{0,1,...,n\}, j \in \{0,1,...,2i\} \tag{32}$$

where $\Delta_s$ is the distance between two consecutive stock price. The transition probabilities, under the risk neutral measure $Q$, are

$$q_{i,j\to k} = \begin{cases} \frac{1}{2}\left(\sigma^2 j^2 - rj\right)\frac{T}{n} & \text{if } k = j+1 \\ 1 - \sigma^2 j^2 \frac{T}{n} & \text{if } k = j \\ \frac{1}{2}\left(\sigma^2 j^2 + rj\right)\frac{T}{n} & \text{if } k = j-1 \\ 0 & \text{otherwise.} \end{cases} \tag{33}$$

To adapt this tree to the change of measure, note that the stock price remains the same $s_{i,j}^{\lambda} = s_{i,j}$ since it does not depends on $r$. The transition probabilities becomes

$$q_{i,j\to k} = \begin{cases} \frac{1}{2}\left(\sigma^2 j^2 - \lambda j\right)\frac{T}{n} & \text{if } k = j+1 \\ 1 - \sigma^2 j^2 \frac{T}{n} & \text{if } k = j \\ \frac{1}{2}\left(\sigma^2 j^2 + \lambda j\right)\frac{T}{n} & \text{if } k = j-1 \\ 0 & \text{otherwise.} \end{cases} \tag{34}$$

The likelihood ratio for the $i$th time step and the $j$th node is

$$l_{i,j}(\lambda) = \exp\left[\frac{r-\lambda}{\sigma^2}\ln\left(\frac{s_0 + (j-i)\Delta_s}{s_0}\right) + \frac{1}{2}(\lambda - r)\frac{r+\lambda-\sigma^2}{\sigma^2}i\frac{T}{n}\right]. \tag{35}$$

### B.3 The Markov chain

The Markov chain method has one distinct feature. Unlike the traditional lattice and finite difference methods, the Markov chain approach allows one to de-couple the partitioning of time and state. In other words, one can use time steps suitable for a particular contingent claim without being unduly constrained to have a particular set of state values. This feature proves to be extremely useful when early exercise or path dependency in present and/or when the postulated underlying price dynamic is a discrete-time stochastic process. This design feature provides three useful operational properties. First, the approach yields a simple recursive matrix formula for valuing options (European, American and exotic). Second, the transition probability matrix associated with the Markov chain is highly sparse, which makes it possible to significantly reduce storage requirement and computation time with the use of sparse matrix techniques. Third, the method can be easily adapted to different theoretical models.

We refer to Duan and Simonato (2001) for a full description of the method. The adaptation of the state space and the transition matrix to the model under the alternative probability measure $Q^{\lambda}$ is straight forward. This change of measure allows to reduce the dimension of the state space without affecting the number of exercise dates, a feature that is not possible using the trees. This is particularly important in the pricing of American option since too few exercise dates result in an underpricing of the option.

To be continued .........

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**Table 1: Probability distributions under alternative measures**

| $j$ | $s_{n-1,j}$ | $l_{n-1,j}(\lambda=r)$ | $l_{n-1,j}(\lambda=\lambda^*)$ | $\chi_j^{\lambda=r}$ | $\chi_j^{\lambda=\lambda^*}$ | $BS\left(s_{n-1,j}^{\lambda};r,\sigma,K,\frac{T}{n}\right)$ |
|---|---|---|---|---|---|---|
| 2 | 158.7055 | 1.0000 | 0.6394 | 0.2290 | 0.2335 | 0.2004 |
| 1 | 100.0000 | 1.0000 | 1.0385 | 0.4991 | 0.4967 | 8.3140 |
| 0 | 63.0098 | 1.0000 | 1.6867 | 0.2719 | 0.2641 | 35.5233 |

$s_{n-1,j}$ is the stock price at time step $n-1$ in state $j$; $l_{n-1,j}(\lambda)$ is the likelihood ratio at time step $n-1$ in state $j$; $\chi_j^{\lambda}$ is the probability of reaching state $j$ in $n-1$ steps from the initial stock price; $BS\left(s_{n-1,j}^{\lambda};r,\sigma,K,\frac{T}{n}\right)$ is the Black-Scholes value computed at step $n-1$. Parameter values: $s_0=100, K=100, r=0.05, \sigma=0.4, T=1$ and $\lambda^*=0.218$.

**Table 2:**
**$Q^{\lambda^*}-$ CRR-BS prices for put options**

| | **European put** | | **American put** | |
|---|---|---|---|---|
| $n$ | $v_{0,0}^{\lambda=r}$ | $v_{0,0}^{\lambda=\lambda^*}$ | $v_{0,0}^{\lambda=r}$ | $v_{0,0}^{\lambda=\lambda^*}$ |
| 10 | 13.2563 | 13.1507 | 13.7544 | 13.6698 |
| 20 | 13.2027 | 13.1475 | 13.7158 | 13.6728 |
| 30 | 13.1842 | 13.1468 | 13.7015 | 13.6724 |
| 40 | 13.1748 | 13.1465 | 13.6936 | 13.6717 |
| 50 | 13.1691 | 13.1464 | 13.6893 | 13.6719 |
| 60 | 13.1652 | 13.1463 | 13.6864 | 13.6719 |
| 70 | 13.1625 | 13.1462 | 13.6843 | 13.6718 |
| 80 | 13.1604 | 13.1462 | 13.6826 | 13.6717 |
| 90 | 13.1588 | 13.1461 | 13.6811 | 13.6714 |
| 100 | 13.1576 | 13.1461 | 13.6799 | 13.6712 |
| 200 | 13.1517 | 13.1460 | 13.6742 | 13.6699 |
| 300 | 13.1498 | 13.1460 | 13.6723 | 13.6694 |
| 400 | 13.1488 | 13.1459 | 13.6712 | 13.6690 |
| 500 | 13.1482 | 13.1459 | 13.6705 | 13.6688 |
| 1000 | 13.1471 | 13.1459 | 13.6691 | 13.6683 |
| 2000 | 13.1465 | 13.1459 | 13.6684 | 13.6680 |
| 3000 | 13.1463 | 13.1459 | 13.6682 | 13.6679 |
| 4000 | 13.1462 | 13.1459 | 13.6680 | 13.6678 |
| 5000 | 13.1461 | 13.1459 | 13.6679 | 13.6678 |

$v_{0,0}^{\lambda=r}$ is the option price computed with the original CRR-BS lattice while $v_{0,0}^{\lambda=\lambda^*}$ is the option price computed with the $Q^{\lambda^*}-$ CRR-BS lattice. The benchmark values for the European and American put options are 13.1459 and 13.6677 and are obtained with the Black Scholes formula and a 15,000 step CRR-BS lattice. Parameter values: $s_0 = 100, K = 100, r = 0.05, \sigma = 0.40, T = 1$ and $\lambda^* = 0.2152$.

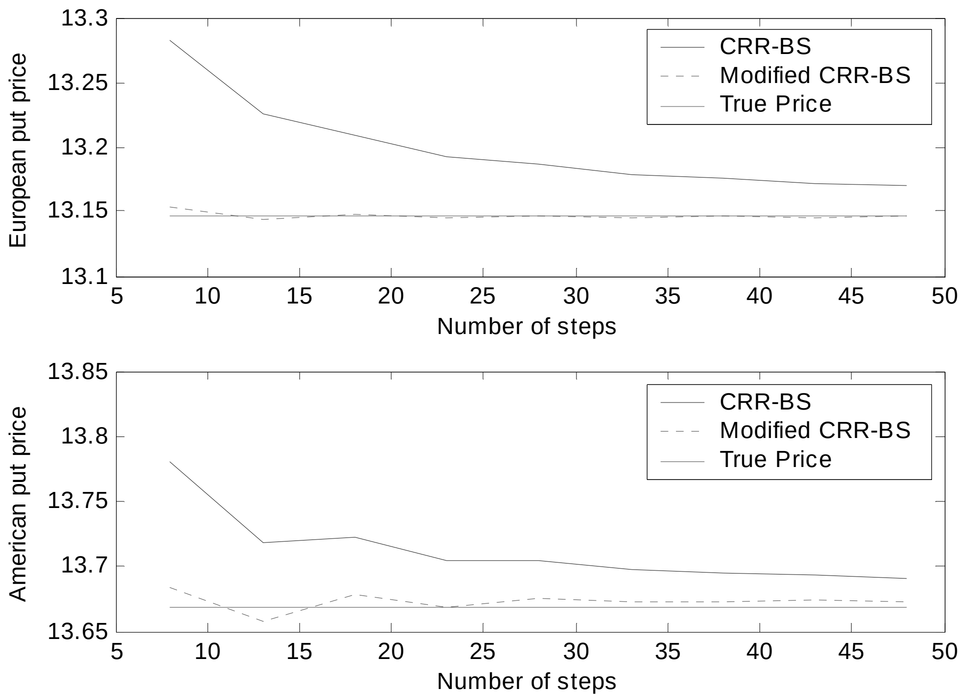


Figure 1: CRR-BS are prices computed using the Cox, Ross and Rubinstein (1979) binomial tree with the Black-Scholes price at the penultimate time step; Modified CRR-BS are prices computed using the Cox, Ross and Rubinstein (1979) binomial tree with a Black-Scholes price at the penultimate time step and the modification proposed in this study. True price is the Black-Scholes price for the European case and a 15 000 step CRR-BS tree for the American case. Parameters: $s_0 = 100, K = 100, T = 1, \sigma = 0.4, r = 0.05$.