1 Introduction to derivative securities

1.1 Some definitions from finance

There are an enormous number of derivative securities being traded in financial markets. Rather than spending time listing them here, we refer to the references and just define those securities that we shall be pricing.

Definition 1.1 *A* forward contract *is an agreement to buy (or sell) an asset on a specified future date for a specified price.*

Forwards are not generally traded on exchanges. It costs nothing to enter into a forward contract. A *future contract* is the same as a forward except that futures *are* normally traded on exchanges and the exchange specifies certain standard features of the contract.

For most of this course, we shall be concerned with the pricing of *options*. Options come in many different types of which the most important are the American and European options. Options were first traded on an organised exchange in 1973, since when there has been a dramatic growth in the options markets. An option is so-called because it gives the holder the right, but not the obligation, to do something.

Definition 1.2 *A* call *option gives the holder the right to buy. A* put *option gives the holder the right to sell.*

A European call *(resp. put)* option *gives the holder the right, but not the obligation, to buy (resp. sell) an asset on a certain date (the* expiration date, exercise date *or* maturity*), for a certain price (*the strike price*).*

An American option is similar except that the holder of an American call, for example, has the right to buy the asset for the specified price *at any time* up until the expiration date. It can be optimal to exercise such an option prior to expiration. American options are harder to analyse than European ones and we shall touch on them at most briefly.

There are two main uses for options, speculation and hedging. In speculation, available funds are invested opportunistically in the hope of making a profit. Hedging is typically engaged in by companies who have to deal habitually in intrinsically risky assets such as foreign exchange, wheat, copper, oil and so on. For hedgers, the basic purpose of an option is to spread risk.

Suppose that I know that in three months time I shall need a million gallons of jet fuel. I am worried by the fluctuations in the price of fuel and so I buy European call options. This way I know the *maximum* amount of money that I shall need to buy the fuel. Let's denote the price of the underlying asset (jet fuel) at time T when the option expires (three months time) by S_T and suppose that the strike price is K. If, at time T, $S_T > K$ then I shall exercise the option. The option is then said to be *in the money*: I am able to buy an asset worth $S_T > K$ dollars for just K dollars. If on the other hand $S_T < K$ then I'll buy my fuel on the open market and throw away the option which is worthless. The option is then said to be *out of the money*. The *payoff* from the option is thus $(\max(S_T, K) - K)$ or $(S_T - K)_+$.

The fundamental problem is to determine how much I should be willing to pay for such an option. You can think of it as an insurance premium. I am insuring myself against the price of jet fuel going up.

In order to answer this question we are going to have to make some assumptions about the way in which markets operate. To formulate these we begin by discussing forward contracts in more detail.

1.2 Pricing a forward

Recall that a forward contract is an agreement to buy (or sell) an asset on a specified future date for a specified price. The pricing problem here is 'What price for the asset should be specified in the contract?'.

Problem 1.3 *Suppose that I enter into a long position on a forward contract, that is I agree to buy the asset for price K at time T. Then the payoff at time T is $(S_T - K)$, where S_T is the asset price at time T, since I (must) buy an asset worth S_T for price K. This payoff could be positive or negative and since the cost of entering into a forward contract is zero this is also my total gain (or loss) from the contract. Our question then is 'what is a fair value of K?'*

At the time when the contract is written, we don't know S_T, we can only guess at it. Typically, we assign a probability distribution to it. A widely accepted model (that we return to in §7) is that stock prices are *log-normally distributed.* That is, there are constants μ and σ so that the *logarithm* of S_T/S_0 (the stock price at time T divided by that at time zero) is normally distributed with mean μ and variance σ^2. In symbols:

$$\begin{aligned}\mathbb{P}\left[\frac{S_T}{S_0} \in [a,b]\right] = \mathbb{P}\left[\log\left(\frac{S_T}{S_0}\right) \in [\log a, \log b]\right] \\ = \int_{\log a}^{\log b} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx\end{aligned}$$

(Notice that stock prices should stay positive, so a, b are positive.)

Our first guess might be that $\mathbb{E}[S_T]$ should represent a fair price to write into our contract. However, it would be a rare coincidence for this to be the market price.

For simplicity, we always assume that market participants can borrow money for the same risk-free rate of interest as they can lend money. Thus, if I borrow \$1 now, my debt at time T will be \$$e^r$, say, and similarly if I keep \$1 in the bank then at time T it will be worth \$$e^r$. (See Hull p.50 for a discussion of this assumption.) We now show that *it is the interest rate and not our log-normal model that forces a choice of K upon us.*

1. Suppose first that $K > S_0 e^r$. Then my counterpart can adopt the following strategy: she borrows \$$S_0$ at time zero and buys with it one unit of stock. At time T, she must repay \$$S_0 e^r$ to the bank, but she has the stock to sell to me for \$$K$, leaving her a *certain* profit of \$$(S_0 e^r - K)$.

2. If $K < S_0 e^r$, then I can reverse this strategy. I sell a unit of stock at time zero for $\$S_0$ and I put the money in the bank. At time T, I have accumulated $\$S_0 e^r$ and I use $\$K$ to buy a unit of stock leaving me with a *certain* profit of $\$(S_0 e^r - K)$.

Unless $K = S_0 e^r$, one of us is guaranteed to make a profit.
Remark. In this argument, I sold stock that I may not own. This is known as short selling. This can, and does, happen: investors can hold negative amounts of stock. For the details of the mechanisms involved see Hull p.48-9.

Definition 1.4 *An opportunity to lock into a risk free profit is called an* arbitrage *opportunity.*

The starting point in establishing a model in modern finance theory is to specify that there are no arbitrage opportunities. (In fact there are people who make their living entirely from exploiting arbitrage opportunities, but such opportunities do not exist for a significant length of time before market prices move to eliminate them.)

Of course forwards are a very special sort of derivative. The argument above won't tell us how to value an option, but the strategy of seeking a price that does not provide either party with an arbitrage opportunity will be fundamental in what follows.

2 Discrete time models I

2.1 The single period binary model

For most of what follows we shall be interested in finding the 'fair' price for a European call option. We begin with a simple example.

Example 2.1 *Suppose that the current price in Swiss Francs (SFR) of \$100 is $S_0 = 150$. Consider a European call option with strike price $K = 150$ at time T. What is a fair price to pay for this option?*

Recall that such an option gives the buyer the right to buy \$100 for 150SFR at time T. Now the true exchange rate at time T is not something we know, but it can be modelled by a random variable. The simplest model is the *single period binary model* where S_T (which here denotes the cost in SFR of \$100 at time T) is assumed to take one of two values with specified probabilities. Let's suppose

$$S_T = \begin{cases} 180 & \text{with probability } p \\ 90 & \text{with probability } 1-p \end{cases}$$

The payoff of the option will be 30=180-150SFR with probability p and 0 with probability $1-p$ and so has expectation $30p$ SFR. Is $30p$ SFR a fair price to pay for the option?
Assume for simplicity that interest rates are zero and that currency is bought and sold at the same exchange rate.

To make things more concrete, suppose that $p = 0.5$. That is, we are modelling S_T as a random variable that takes the values 90SFR and 180SFR with equal probability. We are then asking whether 15SFR is a fair price to pay for the option.

As in §1, by 'fair' we mean that there is no possibility of a risk free profit for either the buyer or the seller of the option. I claim that if the price is 15SFR, then I can make a risk-free profit by the following strategy: I buy the option *and* I borrow \$33.33 and convert it straight into 50 SFR. (I shall have to pay off my loan in dollars at time T.)

My position at time zero is as follows: I have one option (to buy \$100 for 150SFR at time T), I have 35SFR (50 from the conversion of my dollar loan, less 15 paid for the option) and I have a debt of \$33.33.

At time T, one of two things has happened:

1. If $S_T = 180$, then I exercise the option and buy \$100 for 150SFR. I use \$33.33 to pay off my dollar debt, leaving me \$66.67. This I convert back into SFR at the *current* exchange rate. This nets $2/3 \times 180 = 120$SFR. In total then, if $S_T = 180$, at time T I have 35SFR -150SFR (used to exercise the option)+120SFR, which totals 5SFR clear profit.

2. If $S_T = 90$, then I throw away the option (which is worthless) and convert my 35SFR into dollars, netting \$$0.9 \times 35$=\$38.89. I pay off my debt leaving a profit of \$5.56.

Whatever the true exchange rate at time T, I make a profit.

So the price of the option is too low (at least from the point of view of the seller). What is the right price?

Let's think of things from the point of view of the seller. If I am the seller of the option, then I know that at time T, I shall need \$$(S_T - 150)_+$ in order to meet the claim against me. The idea is to calculate how much money I need at time zero (to be held in a combination of dollars and Swiss Francs) to guarantee this.

Suppose then I hold \$$x_1$ and x_2SFR at time zero. I need this holding to be worth at least $(S_T - 150)_+$SFR at time T.

1. If $S_T = 180$, then I shall need at least 30SFR. That is, we must have
$$x_1 + \frac{180}{100} x_2 \geq 30.$$

2. On the other hand, if $S_T = 90$, then the payoff of the option is zero and I just need not to be out of pocket. That is, I want
$$x_1 + \frac{90}{100} x_2 \geq 0.$$

A profit is guaranteed (without risk) for the seller if (x_1, x_2) lies in the interior of the shaded region in Figure 1. On the boundary of the region, there is a positive probability of profit and no probability of loss at all points other than the intersection of the two lines. At that point the seller is guaranteed to have exactly that money required to meet the claim against her at time T.

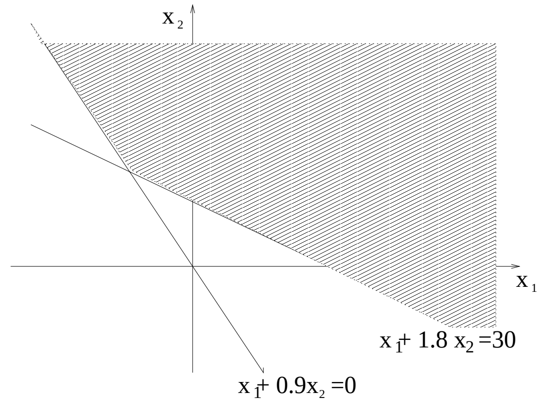

Figure 1.

Solving the simultaneous equations gives that the seller can exactly meet the claim if $x_1 = -30$ and $x_2 = 300/9$. Now to purchase \$300/9 at time zero would require $300/9 \times 150/100 = 50$SFR. the value of the portfolio at time zero is then $50 - 30 = 20$SFR. The seller requires exactly 20SFR at time zero to construct a portfolio that will be worth exactly the payoff of the option at time T.

One can reverse the argument above to show that for any lower price, there is a strategy for which the buyer makes a risk-free profit.

The fair price is 20SFR.

Notice that we did not use the probability, p, of the price S_T going up to 180SFR at any point in the calculation. We just needed the fact that we could *replicate* the claim by this simple portfolio. The seller can *hedge* the *contingent claim* $(S_T - 150)_+$ using the *portfolio* consisting of x_1SFR and $\$x_2$.

One can use the same argument to prove the following result.

Lemma 2.2 *Suppose that the risk free interest rate is such that \$1 deposited now will be worth* $\$e^{r\Delta T}$ *at time* ΔT. *Denote the time zero value of a certain asset by* S_0. *Suppose that the motion of stock prices is such that the value of the asset at time* ΔT *will be either* $S_0 u$ *or* $S_0 d$. *Assume further that*

$$d < e^{r\Delta T} < u.$$

At time zero, the market price of a European call option based on this stock with strike price K *and maturity* ΔT *is*

$$\left(\frac{1 - de^{-r\Delta t}}{u-d}\right)(S_0 u - K)_+ + \left(\frac{ue^{-r\Delta t} - 1}{u-d}\right)(S_0 d - K)_+ .$$

(The proof is exercise 1 on the problem sheet.)

Remark 2.3 *(A ternary model.)*
There were several things about the binary model that were very special. In particular we assumed that we knew that the asset price would be one of just two specified values at time T. *What if we allow* three *values?*

We can try to repeat the analysis above. Again the seller would like to replicate the claim at time T by a portfolio consisting of x_1SFR and $\$x_2$. This time there will be three scenarios to consider, corresponding to the three possible values of S_T. This gives rise to three inequalities:

$$x_1 + \frac{S_T^i}{100} x_2 \geq (S_T^i - 150)_+ \qquad i = 1, 2, 3,$$

where S_T^i are the possible values of S_T (see Figure 2).

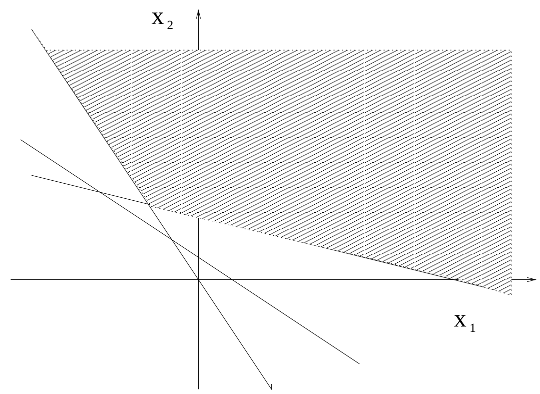

Figure 2.

In order to be *guaranteed* to meet the claim at time T, the seller requires (x_1, x_2) to lie in the shaded region, but at any point in that region, she has a positive probability of making a profit and zero probability of making a loss. There is no portfolio that exactly replicates the claim and there is no unique 'fair' price for the option.

Our market is not *complete*. That is, there are contingent claims that cannot be perfectly hedged.

If we allowed the seller to trade in a third 'independent' asset, then our argument would lead us to three linear equations in three unknowns, corresponding to three non-parallel planes in $\mathbb{R}^3$ which intersect in a single point. That point represents the unique portfolio that exactly replicates the claim at time T.

2.2 A characterisation of 'no arbitrage'

In our binary setting it was easy to find the right price for our option by solving a pair of simultaneous equations in two unknowns. In more complex situations, it is convenient to have a rather different viewpoint. This new viewpoint will also carry over to our continuous models and is possibly the single most important idea in valuation of derivatives. This will take us back into probability theory, but first we need a concise mathematical condition to characterise markets that have no arbitrage opportunities.

Our market will now consist of a finite (but possibly large) number of tradeable assets.

Suppose then that there are N tradeable assets in the market. Their prices at time zero are given by the column vector $S_0 = (S_0^1, S_0^2, \ldots, S_0^N)^T$. Uncertainty about the market is represented by a finite number of possible states in which the market might be at time one that we label $1, 2, \ldots, n$. The security values at time one are given by an $N \times n$ matrix $D = (D_{ij})$, where the coefficient D_{ij} is the value of the ith security at time one if the market is in state j.

In this notation, a portfolio can be thought of as a vector $\theta = (\theta_1, \theta_2, \dots, \theta_n)^T \in \mathbb{R}^N$, whose market value at time zero is the scalar product $S_0.\theta = S_0^1\theta_1 + S_0^2\theta_2 + \dots + S_0^N\theta_N$. The payoff of the portfolio at time one is a vector in $\mathbb{R}^n$ whose ith entry is the value of the portfolio if the market is in state i. We can write the payoff as

$$\begin{pmatrix} D_{11}\theta_1 + D_{21}\theta_2 + \cdots D_{N1}\theta_N \\ D_{12}\theta_1 + D_{22}\theta_2 + \cdots D_{N2}\theta_N \\ \vdots \\ D_{1n}\theta_1 + D_{2n}\theta_2 + \cdots D_{Nn}\theta_N \end{pmatrix} = D^T\theta.$$

Notation

For a vector $x \in \mathbb{R}^n$ we write $x \geq 0$ to mean $x \in \mathbb{R}^n_+$ and $x > 0$ to mean $x \geq 0, x \neq 0$. Notice that $x > 0$ does not require x to be strictly positive in all its coordinates. We write $x \gg 0$, for vectors which are strictly positive in all coordinates, ie for vectors $x \in \mathbb{R}^n_{++}$.

An *arbitrage* is then a portfolio $\theta \in \mathbb{R}^N$ with

$$S_0.\theta \leq 0, D^T\theta > 0 \qquad \text{or } S_0.\theta < 0, D^T\theta \geq 0.$$

Definition 2.4 *A* state price vector *is a vector $\psi \in \mathbb{R}^n_{++}$ such that $S_0 = D\psi$.*

Expanding this gives

$$\begin{pmatrix} S_0^1 \\ S_0^2 \\ \vdots \\ S_0^N \end{pmatrix} = \psi_1 \begin{pmatrix} D_{11} \\ D_{21} \\ \vdots \\ D_{N1} \end{pmatrix} + \psi_2 \begin{pmatrix} D_{12} \\ D_{22} \\ \vdots \\ D_{N2} \end{pmatrix} + \cdots + \psi_n \begin{pmatrix} D_{1n} \\ D_{2n} \\ \vdots \\ D_{Nn} \end{pmatrix}.$$

The vector multiplied by ψ_i is the security price vector if the market is in state i. [We can think of ψ_i as the marginal cost of obtaining an additional unit of wealth at the end of the time period if the system is in state i.]

Theorem 2.5 *There is no arbitrage if and only if there is a state price vector.*

The proof is an application of the following result from convexity theory (for a discussion see Appendix B of Duffie and the references therein). (Recall that $M \subseteq \mathbb{R}^d$ is a cone if $x \in M$ implies $\lambda x \in M$ for all strictly positive scalars λ.)

Theorem 2.6 *Suppose M and K are closed convex cones in $\mathbb{R}^d$ that intersect precisely at the origin. If K is not a linear subspace, then there is a non-zero linear functional F such that $F(x) < F(y)$ for each $x \in M$ and each non-zero $y \in K$.*

Reminder: A *linear functional* on $\mathbb{R}^d$ is a linear mapping $F : \mathbb{R}^d \to \mathbb{R}$. By the *Riesz Representation Theorem*, any bounded linear functional on $\mathbb{R}^d$ can be written as $F(x) = v_0.x$. That is $F(x)$ is the scalar ('dot') product of some fixed vector $v_0 \in \mathbb{R}^d$ with x.

Proof of Theorem 2.5

We take $d = 1 + n$ in Theorem 2.6 and set

$$M = \left\{ \left(-S_0.\theta, D^T\theta \right) : \theta \in \mathbb{R}^N \right\} \subseteq \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{1+n},$$

$$K = \mathbb{R}_+ \times \mathbb{R}^n_+ .$$

Note that K is a cone and M is a linear space.

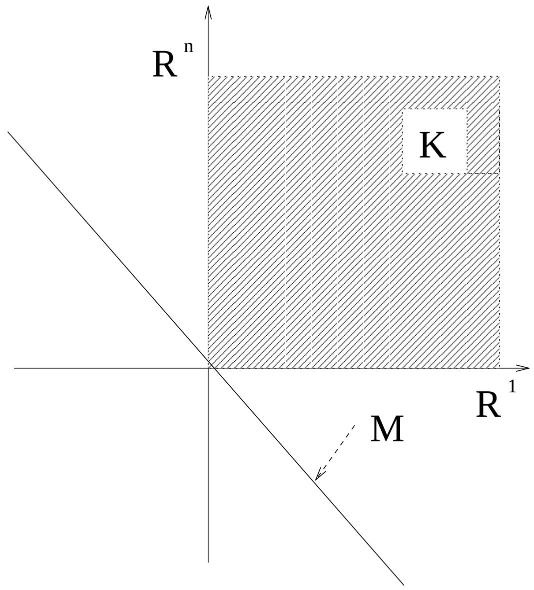

Figure 3.

Evidently, there is no arbitrage if and only if K and M intersect precisely at the origin as shown in Figure 3. We must prove that $K \cap M = \{0\}$ if and only if there is a state price vector.

(i) Suppose first that $K \cap M = \{0\}$. From Theorem 2.6, there is a linear functional $F : \mathbb{R}^d \to \mathbb{R}$ such that $F(z) < F(x)$ for all $z \in M$ and non-zero $x \in K$. The first step is to show that F must vanish on M. We exploit the fact that M is a linear space.

First observe that $F(0) = 0$ (by linearity of F) and so $F(x) \geq 0$ for $x \in K$ and $F(x) > 0$ for $x \in K \backslash \{0\}$. Fix $x_0 \in K$ with $x_0 \neq 0$. Now take an arbitrary $z \in M$. Then $F(z) < F(x_0)$, but also, since M is a linear space, $\lambda F(z) = F(\lambda z) < F(x_0)$ for all $\lambda \in \mathbb{R}$. This can only hold if $F(z) = 0$. $z \in M$ was arbitrary and so F vanishes on M as required.

We now use this to actually explicitly construct the state price vector from F. First we use the Riesz Representation Theorem to write F as $F(x) = v_0.x$ for some $v_0 \in \mathbb{R}^d$. It is convenient to write $v_0 = (\alpha, \phi)$ for some $\alpha \in \mathbb{R}$, $\phi \in \mathbb{R}^n$. Then

$$F(v, c) = \alpha v + \phi.c \qquad \text{for any } (v, c) \in \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^d.$$

Since $F(x) > 0$ for all non-zero $x \in K$, we must have $\alpha > 0$ and $\phi \gg 0$ (consider a vector along each of the co-ordinate axes). Finally, since F vanishes on M,

$$-\alpha S_0.\theta + \phi.D^T\theta = 0 \qquad \forall \theta \in \mathbb{R}^N.$$

Observing that $\phi.D^T\theta = (D\phi).\theta$, this becomes

$$-\alpha S_0.\theta + (D\phi).\theta = 0 \qquad \forall \theta \in \mathbb{R}^N,$$

which implies that $S_0 = D(\phi/\alpha)$. The vector $\psi = \phi/\alpha$ is a state price vector.

(ii) Suppose now that there is a state price vector. We must prove that $K \cap M = \{0\}$.

Writing ψ for the state price vector, $S_0 = D\psi$. Then for any portfolio θ,

$$S_0.\theta = (D\psi).\theta = \psi.(D^T\theta).$$

Now, if $D^T\theta \in \mathbb{R}^n_+$, then, since $\psi \gg 0$, $\psi.(D^T\theta) \geq 0$. If $(-S_0.\theta, D^T\theta) \in K$, $D^T\theta \in \mathbb{R}^n_+$ *and* $-S_0.\theta \geq 0$. That is, we must have $\psi.(D^T\theta) \geq 0$ *and* $-S_0.\theta = -\psi.(D^T\theta) \geq 0$. This happens if and only if $-S_0.\theta = \psi.(D^T\theta) = 0$. That is, $K \cap M = \{0\}$, as required. $\square$

2.3 The risk neutral probability measure.

In order to get back to probability theory, we are going to think of the state price vector rather differently. Recall that all the entries of ψ are strictly positive. Writing $\psi_0 = \sum_{i=1}^n \psi_i$, we can think of

$$\overline{\psi} = \left(\frac{\psi_1}{\psi_0}, \frac{\psi_2}{\psi_0}, \ldots, \frac{\psi_n}{\psi_0}\right)^T \tag{1}$$

as a vector of *probabilities* for being in different states. (Of course they may have nothing to do with our view of how the markets will move.)

First of all, what is ψ_0?

Suppose that the market allows *positive riskless borrowing*, by which we mean that for some portfolio, $\overline{\theta}$,

$$D^T\overline{\theta} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix},$$

i.e. the value of the portfolio at time T is one, no matter what state the market is in. Using the fact that ψ is a state price vector, we calculate that the cost of such a portfolio at time zero is

$$S_0.\overline{\theta} = (D\psi).\overline{\theta} = \psi.(D^T\overline{\theta}) = \sum_{i=1}^n \psi_i = \psi_0.$$

That is ψ_0 *represents the discount on riskless borrowing.*

Now under the probability distribution given by the vector (1), the expected value of the ith security at time T is

$$\mathbb{E}\left[S_T^i\right] = \sum_{j=1}^{n} D_{ij}\frac{\psi_j}{\psi_0} = \frac{1}{\psi_0}\sum_{j=1}^{n} D_{ij}\psi_j = \frac{1}{\psi_0}S_0^i,$$

where in the last equality we have used $S_0 = D\psi$. That is

$$S_0^i = \psi_0\mathbb{E}\left[S_T^i\right], \qquad i = 1, \dots, n. \tag{2}$$

Any security's price is its discounted expected payoff under the probability distribution (1).

This observation gives us a new way to think about the pricing of contingent claims. We shall say that a claim is *attainable* if it can be hedged. That is, if there is a portfolio whose value at time T is exactly C.

Theorem 2.7 *If there is no arbitrage, the unique time zero price of an attainable claim C at time T is $\psi_0\mathbb{E}[C]$ where the expectation is with respect to any probability measure for which $S_0^i = \psi_0\mathbb{E}[S_T^i]$ for all i and ψ_0 is the discount on riskless borrowing.*

Remark. The same value is obtained if the expectation is calculated for any vector of probabilities such that $S_0^i = \psi_0\mathbb{E}[S_T^i]$ since, in the absence of arbitrage, there is only one riskless borrowing rate.

This result says that if I can find a probability vector for which the value of each security now is its discounted expected value at time T then I can find the time zero value of any attainable contingent claim by calculating the expectation. (I will be using the same probability vector, whatever the claim.)

Proof of Theorem 2.7

Since the claim can be hedged, there is a portfolio θ so that $\theta.S_T = C$. Now

$$\theta.S_0 = \theta.(\psi_0\mathbb{E}[S_T]) = \psi_0\sum_{i=1}^{N}\theta_i\mathbb{E}[S_T^i] = \psi_0\mathbb{E}[\theta.S_T],$$

as required. $\square$

Let's go back to our very first option pricing example.

Example 2.1 revisited.

Recall the conditions in that example: the current price in Swiss Francs (SFR) of \$100 is $S_0 = 150$. We suppose that at time T, the price will have moved to either 90SFR or 180SFR. Our problem was to price a European call option with strike price $K = 150$ at time T. (The holder of such an option has the right, but not the obligation, to buy \$100 for 150SFR at time T.) For simplicity, interest rates were assumed to be zero.

We already know the fair price for such an option is 20SFR, but lets see how to obtain this with our new approach.

Under our assumption of zero interest rates, we have $\psi_0 = 1$ and so we are seeking a probability vector such that $\mathbb{E}[S_T] = S_0$. That is, we are looking for a probability vector for which the exchange rate behaves like a fair game. Let p be the probability that $S_T = 180$SFR, then since S_T can only take values 180 and 90 it must be that $\mathbb{P}[S_T = 90] = 1 - p$. For the price to behave like a fair game, we require that

$$180p + 90(1-p) = 150$$

which yields $p = 2/3$. The claim at time T is $C = (S_T - 150)_+$, and so using Theorem 2.7, we see that the fair price (in SFR) is

$$\mathbb{E}[(S_T - 150)_+] = \frac{2}{3}(180 - 150)_+ + \frac{1}{3}(90 - 150)_+ = 20,$$

as before.

Armed with the probability p, it is a trivial matter to value other options, for example a European call with strike price 120SFR instead of 150SFR would be valued at

$$\mathbb{E}[(S_T - 120)_+] = \frac{2}{3}(180 - 120)_+ + \frac{1}{3}(90 - 120)_+ = 40.$$

The probability measure that assigns probability 2/3 to $S_T = 180$SFR and 1/3 to $S_T = 90$SFR is called an *equivalent martingale probability* for the (discounted) price process $\{S_0, \psi_0 S_T\}$. The probabilities are also sometimes called the *risk neutral probabilities*.