LECTURE NOTES
Mathematical Modeling and its Application in Finance

Lecture 9:
Fixed-Income Portfolio Dedication and Immunization

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OUTLINE

- *Portfolio Dedication Principles*
- *Portfolio Immunization Principles*
- *Mathematical Programming Models*
  - *Reading:*
    - *S. Zenios, Financial Optimization, pp. 15--24*
    - *Fabozzi, Handbook of Fixed Income Securities, ch. 48--49, fifth edition, 1997.*

Portfolio Dedication

Fixed income Asset/Liability management strategy:

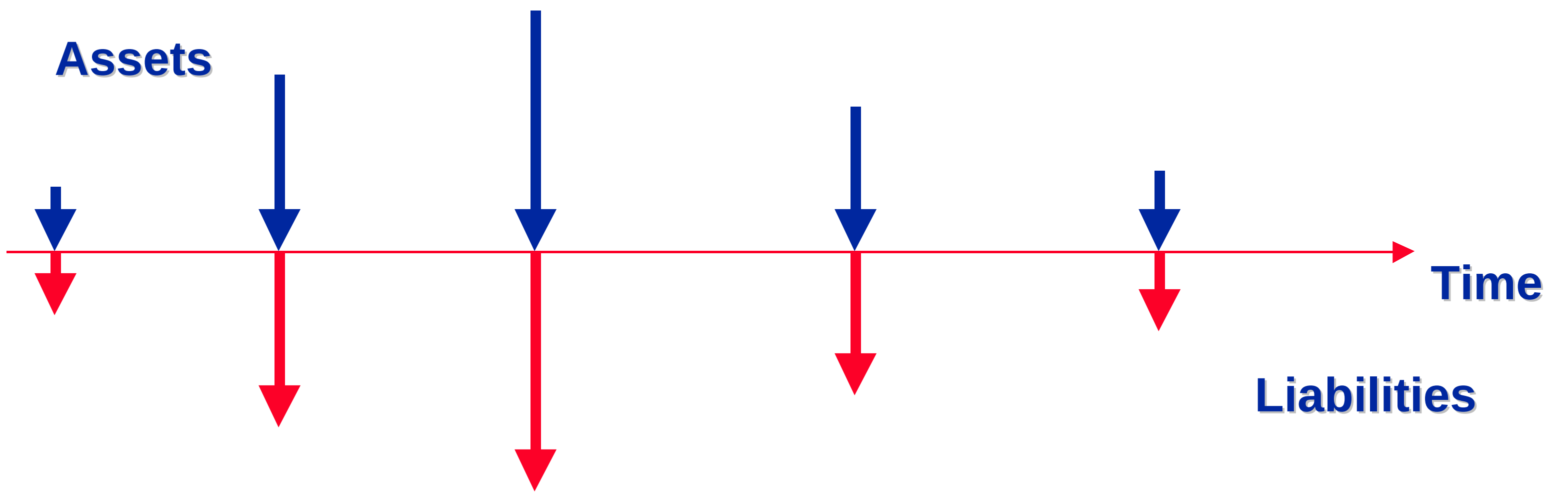
<u>Example 1:</u> Guaranteed Investment Contracts in the 70's:
Upward sloping yield curve
3- to 7-year maturity of GICs with low interest payments
Proceeds reinvested in 10- to 30-year mortgages, public bonds
What happened as interest rates rose in the late 1970s ?

<u>Example 2:</u> Pension fund management may assume very conservative values for reinvestment opportunities. Result?

Example 3: Court settlements.

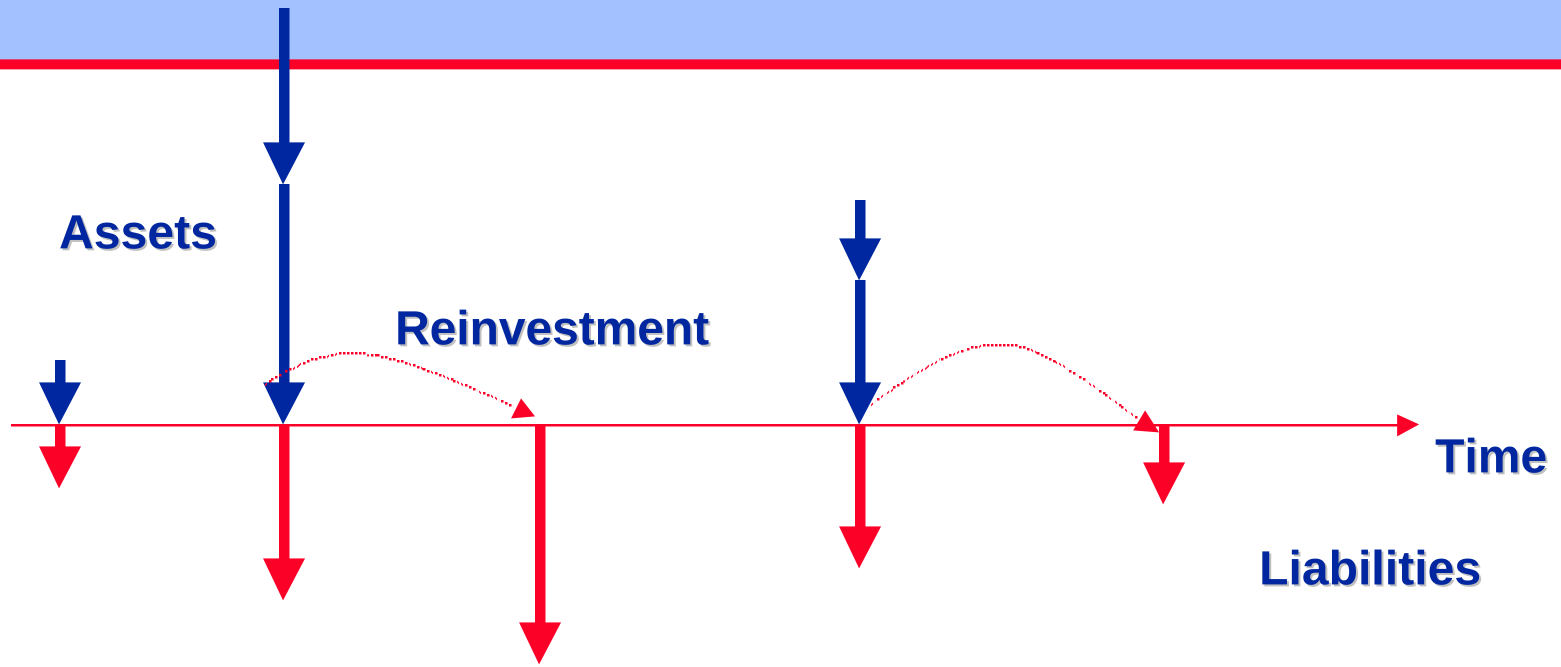
How to manage such risks? CASH-FLOW MATCHING

Cashflow Matching



No interest rate risk.
Can we achieve cashflow matching?

Portfolio Dedication



Maturing assets + Proceeds from reinvestment = Liabilities, at each point in time

Portfolio Dedication

- Determining the Liabilities for a Pension Fund
  - Closed block of retirees:
    - Very accurate estimate---benefits are known (+)
    - Only a small part of the fund's liabilities (-)
    - Declining stream of liabilities due to mortality
  - Retired-lives plus terminated vested participants
    - Broader universe of participants (+)
    - Vested benefits are known (+)
  - Anticipated retiree obligations
    - Even broader universe (+)
    - Benefits are not known with certainty (-)
    - Liabilities increase the first 10-15 years and decline thereafter

Portfolio Dedication

- Setting constraints:
  - Restrict investments to high-quality bonds
  - Restrictions by agency, sector, issuer (diversification of industry specific risk)
- Example:
  - Quality
    - Treasury 20 to 100%
    - Agency 0 to 100%
    - AAA 0 to 100%
    - AA 0 to 100%
    - A 0 to 50%
    - BBB 0
  - Sector
    - Treasury 20 to 100%
    - Agency 0 to 100%
    - Industrial 0 to 30%
    - Utility 0 to 30%
  - Concentration: No more than 10% in one issue of issuer

Portfolio Dedication

- Choosing the reinvestment---rollover---rate:
  - Assuming high rollover rate will prefund liabilities if high-yield securities are available with maturity before the liability dates
  - Assuming low reinvestment rates leads to tighter cashflow matching, at the potential loss of portfolio yield
- Example: Current actuarial practices allow 5 to 8 percent
- Objective function: Minimize cost of portfolio

Portfolio Dedication: mathematical program

p_j : Cost (price) of security j

x_j : Amount invested in security j

L_t : Liability obligation at period t

cf_{jt} : Cashflow generated from security j at period t

r : Reinvestment rate

Constraint: at each period t, the cashflow received from assets + income from reinvestment = the liability obligation

Portfolio Dedication: mathematical program

$$\text{Minimize } \sum_{j=1}^{N} p_j x_j$$

Subject to:

$$(1) \sum_{j=1}^{N} cf_{j1} x_j \geq L_1$$

$$(2) \sum_{j=1}^{N} cf_{j2} x_j + \text{Reinvestments from period 1} \geq L_2$$

$$\text{i.e., } (2) \sum_{j=1}^{N} cf_{j2} x_j + r \left(\sum_{j=1}^{N} cf_{j1} x_j - L_1 \right) \geq L_2$$

$$(2) \sum_{j=1}^{N} (cf_{j2} + r cf_{j1}) x_j \geq L_2 + r L_1$$

$$(3) \sum_{j=1}^{N} (cf_{j3} + r cf_{j2} + r^2 cf_{j1}) x_j \geq L_3 + r L_2 + r^2 L_1$$

Dedication condition: Present value of Assets = Present Value of Liabilities

Example: Portfolio Dedication for a Pension Fund

Total liabilities: $283,758,000 over 35 years (retired lives only)
Reinvestment rate: 5%

Present value of liabilities: $159,818,000

Portfolio cost (market value): $123,160,000

Portfolio yield: 7.83%
Savings (portfolio cost over present value of liabilities: 23%

Portfolio reoptimization.
Bond swaps.
The importance of accurate price databases.

Portfolio Immunization

- Broad fixed-income asset/liability management strategy
  - Actuary F.M.Reddington (1952)
    “the investment of the assets in such a way that the existing business is immune to a general change in the rate of interest”
  - How to measure interest rate sensitivity of assets and liabilities?

P_j : Price (or present value) of bond j

cf_{jt} : Cashflow from bond j at period t

r_j : Yield of bond j

$$P_j = \sum_{t=1}^{T} cf_{jt}(1+r_j)^{-t}$$

Taking derivative with respect to yield we obtain the dollar duration

$$d_j = -\sum_{t=1}^{T} t cf_{jt}(1+r_j)^{-(t+1)}$$

Note: Dollar duration of a portfolio of bonds is additive

$$d_p = \sum_{j=1}^{N} d_j x_j$$

Mathematical Program for Portfolio Immunization

p_j : Cost (price) of security j

x_j : Amount invested in security j

r_j : Yield of security j

D_L : Duration of liability

P_L : Present value of liability

d_j : Duration of security j

Mathematical Program for Portfolio Immunization

$$\text{Maximize Portfolio Yield} \approx \sum_{j=1}^{N} r_j d_j x_j$$

Subject to :

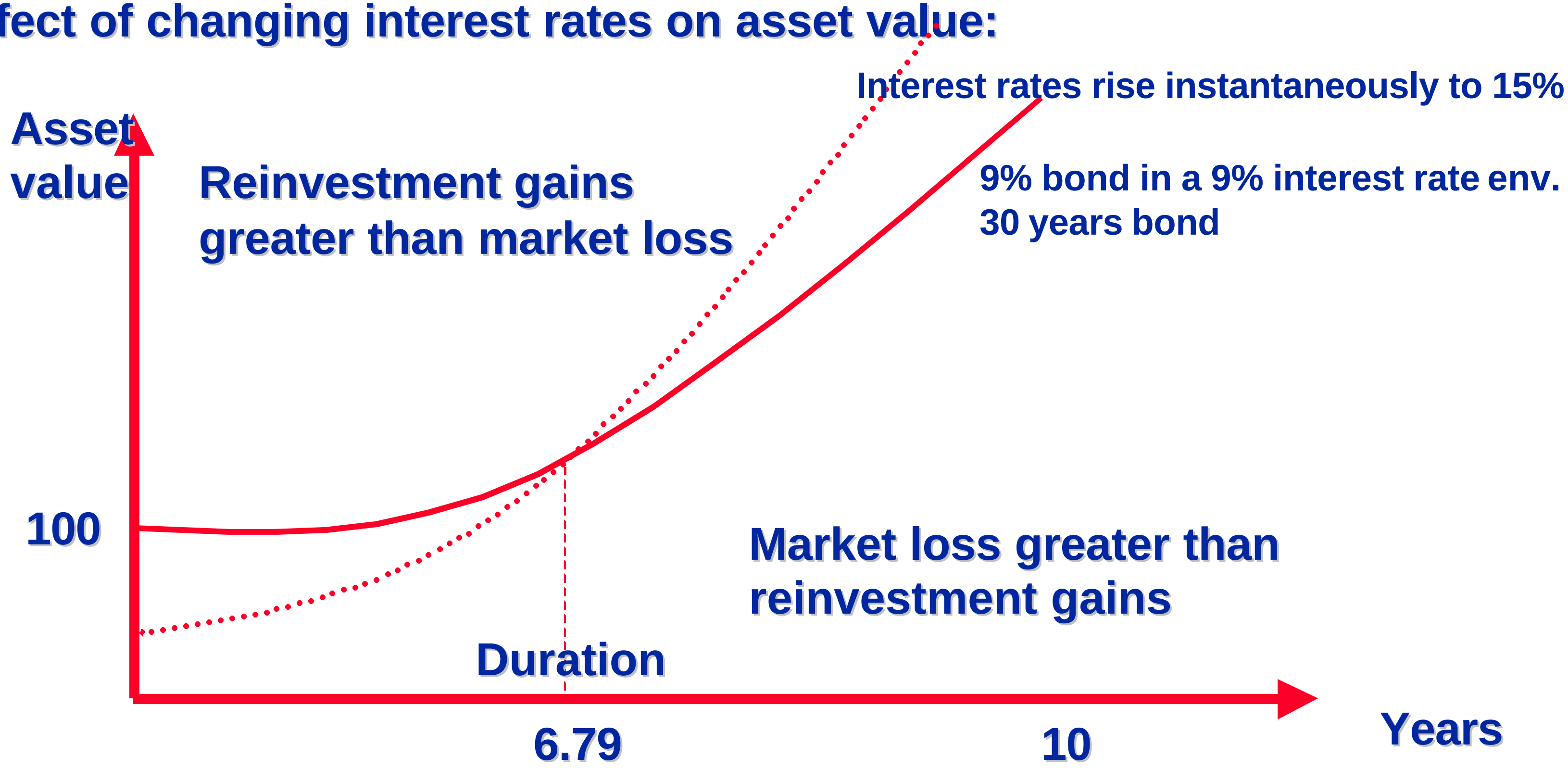
$$\text{(Present value matching)} \sum_{j=1}^{N} P_j x_j = P_L$$

$$\text{(Duration matching)} \sum_{j=1}^{N} d_j x_j = d_L$$

$$x_j \geq 0$$

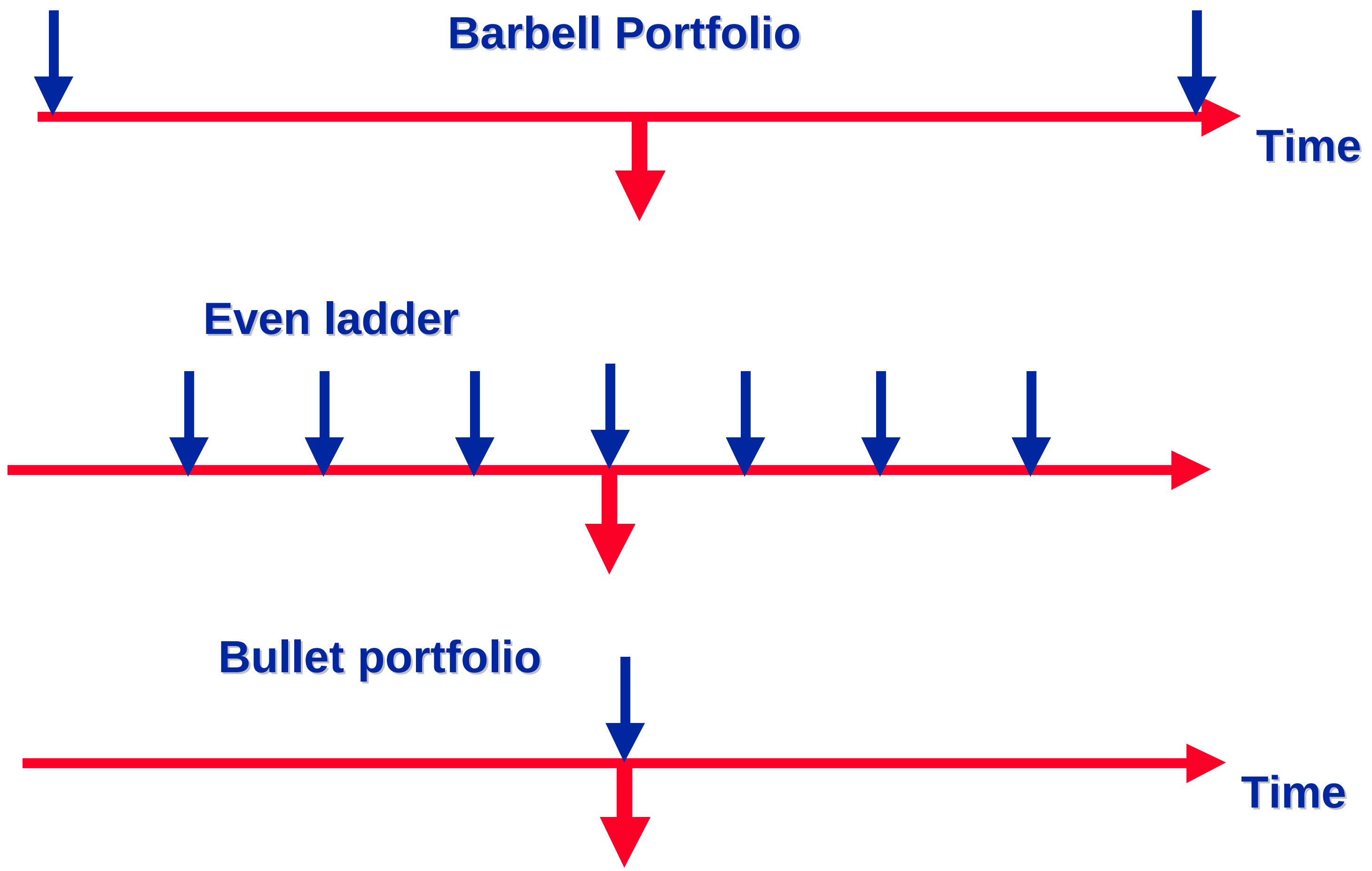
Understanding Portfolio Immunization: Interpretation of duration

Effect of changing interest rates on asset value:

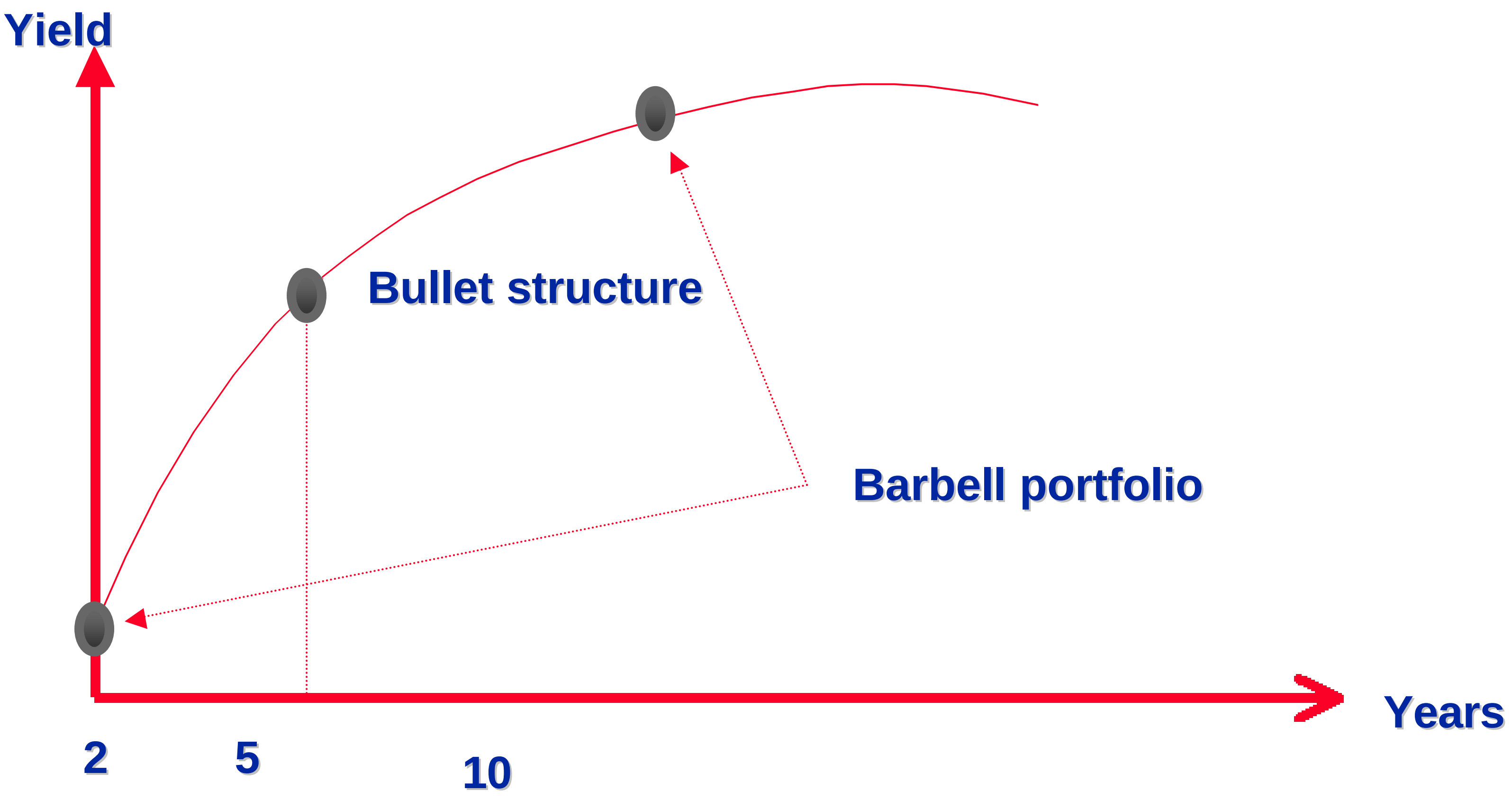


Understanding Portfolio Immunization: Duration matched portfolios

Assume a target (liability) duration of 6.79 years



Yield of structured duration matched portfolios



What is the risk exposure of barbell and even ladder portfolios?

Shape risk.

Add convexity constraints

The price equation is give by: $P_j = \sum_{t=1}^{T} cf_{jt}(1+r_j)^{-t}$

Taking second derivative with respect to yield we obtain the

dollar convexity $Q_j = -\sum_{t=1}^{T} t(t+1)cf_{jt}(1+r_j)^{-(t+2)}$.

The portfolio convexity is give by $Q_P = \sum_{j=1}^{N} Q_j x_j$