CHAPTER 2

The Diffusion Equation

In this chapter we study the one-dimensional diffusion equation

$$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2} + p(x,t),$$

which describes such physical situations as the heat conduction in a one-dimensional solid body, spread of a die in a stationary fluid, population dispersion, and other similar processes. In the last section we will also discuss the quasilinear version of the diffusion equation, known as, the Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \gamma \frac{\partial^2 u}{\partial x^2} = p(x,t)$$

which arises in the context of modelling the motion of a viscous fluid as well as traffic flow.

We begin with a derivation of the heat equation from the principle of the energy conservation.

2.1. Heat Conduction

Consider a thin, rigid, heat-conducting body (we shall call it a bar) of length l. Let $\theta(x,t)$ indicate the temperature of this bar at position x and time t, where $0 \leq x \leq l$ and $t \geq 0$. In other words, we postulate that the temperature of the bar does not vary with the thickness. We assume that at each point of the bar the energy density per unit volume ε is proportional to the temperature, that is

$$\varepsilon(x,t) = c(x)\theta(x,t), \tag{2.1.1}$$

where $c(x)$ is called *heat capacity* and where we also assumed that the mass density is constant throughout the body and normalized to equal one. Although the body has been assumed rigid, and with constant mass density, its material properties, including the heat capacity, may vary from one point to another.

To derive the "homogeneous" heat-conduction equation we assume that there are no internal sources of heat along the bar, and that the heat can only enter the bar through its ends. In other words, we assume that the lateral surface of the bar is perfectly insulated so no heat can be gained or lost through it. The fundamental physical law which we employ here is the law of conservation of energy . It says that the rate of change of energy in any finite part of the bar is equal to the total amount of heat flowing into this part of the bar. Let $q(x,t)$ denote the *heat flux* that is the rate at which heat flows through the body at position x and time t, and let us consider the portion of the bar from x to $x+\triangle x$. The rate of change of the total energy of this part of the bar equals the total amount of heat that flows into this part through its ends, namely

$$\frac{\partial}{\partial t}\int_x^{x+\triangle x} c(z)\theta(z,t)dz = -q(x+\triangle x,t)+q(x,t). \tag{2.1.2}$$

We use here commonly acceptable convention that the heat flux $q(x,t)>0$ if the flow is to the right.

In order to obtain the equation describing the heat conduction at an arbitrary point x we shall consider the limit of (2.1.2) as $\triangle x\to 0$. First, assuming that the integrand $c(z)\theta(z,t)$ is sufficiently regular, we are able to differentiate inside the integral. Second, dividing both sides of the equation by $\triangle x$, invoking the Mean-Value Theorem for Integrals, and taking $\triangle x\to 0$ we obtain the equation

$$c(x)\frac{\partial\theta}{\partial t} = -\frac{\partial q}{\partial x} \tag{2.1.3}$$

relating the rate of change of temperature with the gradient of the heat flux. We are ready now to make yet another assumption; a constitutive assumption which relates the heat flux to the temperature. Namely, we postulate what is known as *Fourier's Law of Cooling*, that the heat flows at the rate directly proportional to the (spatial) rate of change of the temperature. If in addition we accept that the heat flows, as commonly observed, from hot to cold we get that

$$q(x,t) = -\kappa(x)\frac{\partial\theta}{\partial x}. \tag{2.1.4}$$

where the proportionality factor $\kappa(x)>0$ is called the *thermal conductivity*. Notice the choice of the sign in the definition of the heat flux guarantees that if the temperature is increasing with x the heat flux is negative and the heat flows from right to left, i.e., from hot to cold.

Combining (2.1.3) and (2.1.4) produces the partial differential equation

$$c(x)\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}(\kappa(x)\frac{\partial\theta}{\partial x}), \qquad 0 < x < l, \tag{2.1.5}$$

governing the heat flow in a inhomogeneous (κ is in general point dependent) one-dimensional body. However, if the bar is made of the same material throughout, whereby the heat capacity $c(x)$ and the thermal conductivity $\kappa(x)$ are point independent, (2.1.5) reduces to

$$\frac{\partial\theta}{\partial t} = \gamma\frac{\partial^2\theta}{\partial x^2}, \qquad 0 < x < l, \tag{2.1.6}$$

where

$$\gamma = \frac{\kappa}{c}. \tag{2.1.7}$$

This equation is known as the *heat equation*, and it describes the evolution of temperature within a finite, one-dimensional, homogeneous continuum, with no internal sources of heat, subject to some initial and boundary conditions. Indeed, in order to determine uniquely the temperature $\theta(x,t)$, we must specify the temperature distribution along the bar at the initial moment, say $\theta(x,0) = g(x)$ for $0 \leq x \leq l$. In addition, we must tell how the heat is to be transmitted through the boundaries. We already know that no heat may be transmitted through the lateral surface but we need to impose boundary conditions at the ends of the bar. There are two particularly relevant physical types of such conditions. We may for example assume that

$$\theta(l,t) = \alpha(t) \tag{2.1.8}$$

which means that the right hand end of the bar is kept at a prescribed temperature $\alpha(t)$. Such a condition is called the *Dirichlet boundary condition*. On the other hand, the *Neumann boundary condition* requires specifying how the heat flows out of the bar. This means prescribing the flux

$$q(l,t) = \kappa(l)\frac{\partial\theta}{\partial x}(l,t) = \beta(t). \tag{2.1.9}$$

at the right hand end. In particular, $\beta(t) \equiv 0$ corresponds to insulating the right hand end of the bar. If both ends are insulated we deal with the *homogeneous Neumann boundary conditions*.

REMARK 2.1. Other boundary conditions like the periodic one are also possible.

2.2. Separation of Variables

The most basic solutions to the heat equation (2.1.6) are obtained by using the *separation of variables* technique, that is, by seeking a solution in which the time variable t is separated from the space variable x. In other words, assume that

$$\theta(x,t) = T(t)u(x), \tag{2.2.1}$$

where $T(t)$ is a x-independent function while $u(x)$ is a time-independent function. Substituting the separable solution into (2.1.6) and gathering the time-dependent terms on one side and the x-dependent terms on the other side we find that the functions $T(t)$ and $u(x)$ must solve an equation

$$\frac{T'}{T} = \gamma\frac{u''}{u}. \tag{2.2.2}$$

The left hand side of equation (2.2.2) is a function of time t only. The right hand side, on the other hand, is time independent while it depends on x only. Thus, both sides of equation (2.2.2) must be equal to the same constant. If we denote the constant as $-\lambda$ and specify the initial condition

$$\theta(x,0) = u(x), \qquad 0 \le x \le l, \tag{2.2.3}$$

we obtain that

$$\theta(x,t) = e^{-\lambda t}u(x) \tag{2.2.4}$$

solves the heat equation (2.1.6) provided we are able to find $u(x)$ and λ such that

$$-\gamma u'' = \lambda u \tag{2.2.5}$$

along the bar. This is an *eigenvalue problem* for the second order differential operator $K \equiv -\gamma \frac{d^2}{dt^2}$ with the *eigenvalue* λ and the *eigenfunction* $u(x)$. The particular eigenvalues and the corresponding eigenfunctions will be determined by the boundary conditions that u inherits from θ. Once we find all eigenvalues and eigenfunctions we will be able to write the general solution as a linear combinations of basic solutions (2.2.4).

Homogeneous Boundary Conditions.

Let us consider a simple Dirichlet boundary value problem for the heat conduction in a (uniform) bar held at zero temperature at both ends, i.e.,

$$\theta(0,t) = \theta(l,t) = 0, \qquad t \geq 0, \tag{2.2.6}$$

where initially

$$\theta(x,0) = g(x), \qquad 0 < x < l. \tag{2.2.7}$$

This amounts, as we have explained earlier, to finding the eigenvalues and the eigenfunctions of (2.2.5) subject to the boundary conditions

$$u(0) = u(l) = 0. \tag{2.2.8}$$

Notice first that as evident from the form of the equation (2.2.5) the eigenvalues λ must be real. Also, it can be easily checked using the theory of second order ordinary linear differential equations with constant coefficients that if $\lambda \leq 0$, then the boundary conditions (2.2.8) yield only the trivial solution $u(x) \equiv 0$. Hence, the general solution of the differential equation (2.2.5) is a combination of trigonometric functions

$$u(x) = a\cos\omega x + b\sin\omega x \tag{2.2.9}$$

where we let $\lambda = \gamma\omega^2$ with $\omega > 0$. The boundary condition $u(0) = 0$ implies that $a = 0$. Because of the second boundary condition

$$u(l) = b\sin\omega l = 0 \tag{2.2.10}$$

ωl must be an integer multiple of π. Thus, the eigenvalues and the eigenfunctions of the eigenvalue problem (2.2.5) with boundary conditions (2.2.8) are

$$\lambda_i = \gamma\left(\frac{i\pi}{l}\right)^2, \qquad u_i(x) = \sin\frac{i\pi}{l}x, \qquad i = 1,2,3,\ldots. \tag{2.2.11}$$

The corresponding basic solutions (2.2.4) to the heat equation are

$$\theta_i(x,t) = \exp\left(-\frac{\gamma i^2\pi^2}{l^2}t\right)\sin\frac{i\pi}{l}x, \qquad i = 1,2,3,\ldots. \tag{2.2.12}$$

By linear superposition of these basic solutions we get a formal series

$$\theta(x,t) = \sum_{i=1}^{\infty} a_i u_i(x,t) = \sum_{i=1}^{\infty} \exp\left(-\frac{\gamma i^2\pi^2}{l^2}t\right)\sin\frac{i\pi}{l}x. \tag{2.2.13}$$

Assuming that the series converges we have a general series solution of the heat equation with the initial temperature distribution

$$\theta(x,0) = g(x) = \sum_{i=1}^{\infty} a_i \sin\frac{i\pi}{l}x. \tag{2.2.14}$$

This is a Fourier sine series on the interval $[0,l]$ of the initial condition $g(x)$[1]. Its coefficients a_i can be evaluated explicitly thanks to the remarkable *orthogonality* property of the eigenfunctions. Indeed, it is a matter of a simple exercise on integration by parts to show that

$$\int_0^l \sin\frac{k\pi}{l}x\sin\frac{n\pi}{l}x dx \neq 0 \tag{2.2.15}$$

only if $n = k$, and that

$$\int_0^l \sin^2\frac{k\pi}{l}x = \frac{l}{2}. \tag{2.2.16}$$

Multiplying the Fourier series of $g(x)$ by the k-th eigenfunction and integrating over the interval $[0,l]$ one gets that

$$a_k = \frac{2}{l}\int_0^l g(x)\sin\frac{k\pi}{l}x dx, \qquad k = 1,2,3,\ldots. \tag{2.2.17}$$

Example 2.2. Consider the initial-boundary value problem

$$\theta(0,t) = \theta(2,t) = 0, \qquad \theta(x,0) = g(x) = \begin{cases} x, & 0 \le x \le 1, \\ -x+2, & 1 \le x \le 2, \end{cases} \tag{2.2.18}$$

for the heat equation for a homogeneous bar of length 2. The Fourier coefficients of $g(x)$ are

$$a_{2k+2} \equiv 0, \quad a_{2k+1} = (-1)^k\frac{8}{(2k+1)^2\pi^2}, \quad k = 0,1,2,\ldots. \tag{2.2.19}$$

[1]Fourier series are introduced and treated extensively in Appendix B

The resulting series solution is

$$\theta(x,t) = 8\sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)^2\pi^2} \exp\left(-\frac{(2i+1)^2\pi^2 t}{4}\right) \sin(i+\frac{\pi}{2})x. \tag{2.2.20}$$

Notice first that although the initial data is piecewise differentiable the solution is smooth for any $t > 0$. Also, as long as the initial profile is integrable (e.g., piecewise continuous) on $[0,2]$ its Fourier coefficients are uniformly bounded, namely:

$$|a_k| \leq \int_0^2 |g(x)\sin k\pi x|\,dx \leq \int_0^2 |g(x)|\,dx \equiv M. \tag{2.2.21}$$

Consequently, the series solution (2.2.20) is bounded by an exponentially decaying time series

$$|\theta(x,t)| \leq M\sum_{i=0}^{\infty} \exp\left(-\frac{(2i+1)^2\pi^2 t}{4}\right). \tag{2.2.22}$$

This means that solution decays to the zero temperature profile, a direct consequence of the fact that both ends are hold at zero temperature.

This simple example shows that in the case of homogeneous boundary conditions any initial heat distributed throughout the bar will eventually dissipate away. Moreover, as the Fourier coefficients in (2.2.20) decay exponentially as $t \to \infty$, the solution gets very smooth despite the fact that the initial data was not. In fact, this is an illustration of the general *smoothing property* of the heat equation.

THEOREM 2.3. *If $u(t,x)$ is a solution to the heat equation with the initial condition such that its Fourier coefficients are uniformly bounded, then for all $t > 0$ the solution is an infinitely differentiable function of x. Also, $u(t,x) \to 0$ as $t \to \infty$, in such a way that there exists $K > 0$ such that $|u(t,x)| < Ke^{-\gamma\pi^2 t/l^2}$ for all $t \geq t_0 > 0$.*

The smoothing effect of the heat equation means that it can be effectively used to de-noise signals by damping the high frequency modes. This, however, means also that it is impossible to reconstruct the initial temperature by measuring the temperature distribution at some later time. The heat equation cannot be run backwards in time. There is no temperature distribution at $t < 0$ which would produce a non-smooth temperature distribution at $t = 0$. Had we tried to run it backwards, we would only get noise due to the fact that the Fourier coefficients grow exponentially as $t < 0$. The backwards heat equation is *ill possed*.

Inhomogeneous Boundary Conditions.

There is a simple *homogenization transformations* that converts a homogeneous heat equation with inhomogeneous Dirichlet boundary conditions

$$\theta(0,t) = \alpha(t), \quad \theta(l,t) = \beta(t), \quad t \geq 0, \tag{2.2.23}$$

into an inhomogeneous heat equation with homogeneous Dirichlet boundary conditions. Suppose

$$\omega(x,t) = \theta(x,t) - \alpha(t) + \frac{\alpha(t) - \beta(t)}{l} x \tag{2.2.24}$$

where $\theta(x,0) = g(x)$. $\theta(x,t)$ is a solution of a homogeneous heat equation if and only if $\omega(x,t)$ satisfies the inhomogeneous equation

$$\frac{\partial \omega}{\partial t} - \frac{\partial^2 \omega}{\partial t^2} = \frac{\alpha' - \beta'}{l} x - \alpha' \tag{2.2.25}$$

subject to the initial condition

$$\omega(x,0) = g(x) - \alpha(0) + \frac{\alpha(0) - \beta(0)}{l} x, \tag{2.2.26}$$

where

$$\omega(0,t) = \omega(0,l) = 0. \tag{2.2.27}$$

Note that $\omega(x,t)$ is a solution to the homogeneous heat equation if and only if the Dirichlet boundary conditions are constant. As the homogeneous boundary conditions are essential in being able to superpose basic solutions (eigensolutions) the Fourier series method can be used now in conjunction with the separation of variables to obtain solutions of (2.2.25).

Example 2.4. Consider

$$\frac{\partial \omega}{\partial t} - \frac{\partial^2 \omega}{\partial t^2} = x \cos t, \quad 0 < x < 1, \quad t > 0, \tag{2.2.28}$$

subject to the initial condition

$$\omega(x,0) = x, \tag{2.2.29}$$

and the following homogeneous boundary conditions:

$$\omega(0,t) = \frac{\partial \omega}{\partial x}(1,t) = 0, \quad t > 0. \tag{2.2.30}$$

First, let us look for a solution of the homogeneous version of (2.2.28) with the given boundary conditions (2.2.30) using the separation of variables method. To this end the reader can easily show that the eigenfunctions are:

$$u_i(x) = \sin \lambda_i x, \quad \lambda_i = \frac{(2i+1)\pi}{2}, \quad i = 0, 1, 2 \ldots . \tag{2.2.31}$$

By the analogy with the form of the solution to the homogeneous heat equation let us suppose a solution of (2.2.28) as a series of eigenfunctions

$$\omega(x,t) = \sum_{i=0}^{\infty} \alpha_i(t) \sin \lambda_i x. \tag{2.2.32}$$

Also, represent the right-hand side of (2.2.28) as a series of eigenfunctions. Namely, write

$$x \cos t = \left(\sum_{i=0}^{\infty} b_i \sin \lambda_i x \right) \cos t, \tag{2.2.33}$$

where

$$b_i = \int_0^1 x \sin \lambda_i x dx = \frac{(-1)^i}{\lambda_i^2}. \tag{2.2.34}$$

Substituting the solution (2.2.32) with (2.2.33) for its right hand side we are able to show that the unknown functions $\alpha_i(t)$ satisfy an inhomogeneous ordinary differential equation

$$\frac{d\alpha_i}{dt} + \lambda_i^2 \alpha_i = \frac{(-1)^i}{\lambda_i^2} \cos t. \tag{2.2.35}$$

Using the method of undetermined coefficients it is easy to obtain its solution

$$\alpha_i(t) = Ae^{-\lambda_i^2 t} + (-1)^i [\frac{\cos t}{1 + \lambda_i^4} + \frac{\sin t}{\lambda_i^2 (1 + \lambda_i^4)}]. \tag{2.2.36}$$

From the initial condition (2.2.29) and using (2.2.32) one can calculate that

$$A = \frac{(-1)^{i+1} (\lambda_i^4 - \lambda_i^2 + 1)}{\lambda_i^2 (\lambda_i^4 + 1)}. \tag{2.2.37}$$

This enables us to construct the solution (2.2.32).

Periodic Boundary Conditions.

Heat flow in a circular ring is governed by the same homogeneous heat equation as is heat conduction in a rod (2.1.6), however, this time subject to *periodic boundary conditions*

$$\theta(\pi,t)=\theta(-\pi,t), \quad \frac{\partial\theta}{\partial x}(\pi,t)=\frac{\partial\theta}{\partial x}(-\pi,t), \quad t\geq 0, \tag{2.2.38}$$

where $-\pi<x<\pi$ is the angular variable, and where we assume that the heat can only flow along the ring as no radiation of heat from one side of the ring to another is permitted[2].

Benefiting from the separation of variables technique we are seeking a solution in the form $\theta(x,t)=e^{-\lambda t}u(x)$. Assuming for simplicity that $\gamma=1$, we arrive, as before, at the associated eigenvalue problem

$$\frac{d^2u}{dx^2}+\lambda u=0, \quad u(\pi)=u(-\pi), \quad u'(\pi)=u'(-\pi). \tag{2.2.41}$$

Its solutions are combinations of trigonometric sine and cosine functions

$$u_i(x)=a_i\cos ix+b_i\sin ix, \qquad i=0,1,2,\ldots, \tag{2.2.42}$$

with the eigenvalues

$$\lambda_i=i^2, \qquad i=0,1,2,\ldots. \tag{2.2.43}$$

The resulting infinite series solution is

$$\theta(x,t)=\frac{1}{2}a_0+\sum_{i=1}^{\infty}e^{-i^2t}\left[a_i\cos ix+b_i\sin ix\right]. \tag{2.2.44}$$

If we postulate the initial condition $\theta(x,0)=g(x)$ the coefficients a_i and b_i must be such that

$$g(x)=\frac{1}{2}a_0+\sum_{i=1}^{\infty}\left[a_i\cos ix+b_i\sin ix\right], \tag{2.2.45}$$

[2]The heat conduction equation for a heated ring can easily be derived from the two-dimensional heat equation

$$\frac{\partial\theta}{\partial t}=\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2} \tag{2.2.39}$$

by rewriting its right hand side in polar coordinates (r,α)

$$\frac{\partial\theta}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\theta}{\partial\alpha^2}, \tag{2.2.40}$$

and assuming that the solution is r independent.

which is precisely the Fourier series of the initial condition $g(x)$ provided

$$a_i = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos ix dx, \quad b_i = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin ix dx, \quad i = 0, 1, 2, \ldots. \tag{2.2.46}$$

2.3. Uniqueness of Solutions

In this section we investigate the uniqueness of solutions to the initial-boundary value problem for the heat equation. To this end let us consider solutions of the homogeneous heat equation

$$\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial x^2}, \qquad 0 < x < l, \quad 0 < t < \infty, \tag{2.3.1}$$

with the initial condition

$$\theta(x, 0) = g(x), \tag{2.3.2}$$

and the boundary conditions

$$\theta(0, t) = \alpha(t), \quad \theta(l, t) = \beta(t). \tag{2.3.3}$$

Suppose that θ_1 and θ_2 are two solutions of (2.3.1) both satisfying the initial condition (2.3.2) and boundary conditions (2.3.3). As the equation (2.3.1) is linear the function $\omega(x,t) \equiv \theta_1 - \theta_2$ is also a solution but with the zero initial profile and the homogeneous boundary conditions.

Let us multiply (2.3.1) by $\omega(x,t)$ and integrate the resulting equation with respect x on the interval $[0, l]$ to obtain

$$\int_0^l \omega\frac{\partial\omega}{\partial t}dx = \int_0^l \frac{\partial^2\omega}{\partial x^2}\omega dx. \tag{2.3.4}$$

Assuming that $\omega(x,t)$ is regular enough, and integrating the right-hand side by parts we reduce the relation (2.3.4) to

$$\frac{1}{2}\frac{d}{dt}\int_0^l \omega^2 dx = \omega\frac{\partial\omega}{\partial x}\Big|_0^l - \int_0^l \left(\frac{\partial\omega}{\partial x}\right)^2 dx = -\int_0^l \left(\frac{\partial\omega}{\partial x}\right)^2 dx \le 0. \tag{2.3.5}$$

Let

$$I(t) \equiv \frac{1}{2}\int_0^l \omega^2 dx \ge 0. \tag{2.3.6}$$

Then,

$$I(t) - I(0) = -\int_0^t \int_0^l \left(\frac{\partial\omega}{\partial x}\right)^2 dxdt \le 0. \tag{2.3.7}$$

However, $I(0) = 0$ implying that $I(t) \leq 0$. On the other hand according to its definition $I(t) \geq 0$. Hence, $I(t) \equiv 0$. This is possibly only if $\omega(x,t) \equiv 0$ proving that $\theta_1(x,t) = \theta_2(x,t)$ everywhere. Note that the same technique can be used to prove uniqueness of solutions to other boundary value problems as long as $\omega \frac{\partial \omega}{\partial x} = 0$ at $x = 0$ and $x = l$.

2.4. Fundamental Solutions

The idea of the *fundamental solution* of a partial differential equation is an extension of the Green's function method for solving boundary value problems of ordinary differential equations. To set the stage for further considerations let us briefly review the main points of the that method[3].

Consider a homogeneous boundary value problem for the linear ordinary differential equation

$$L(u) = \delta(x - \xi), \qquad u(0) = u(l) = 0, \qquad 0 < x < l, \tag{2.4.1}$$

where $L(u)$ denotes a linear second-order differential operator actig on the function $u(x)$ defined on $[0,l]$ interval, while $\delta(x - \xi) \equiv \delta_\xi(x)$ is the (Dirac) delta function at ξ. Note that if the boundary conditions are inhomogeneous we can use the homogenization transformation (2.2.24) to transform the problem into one with the homogeneous boundary conditions and a different inhomogeneous right hand side.

Let $u(x,\xi) = G(x,\xi)$ denote the solution to (2.4.1). This is the Green's function of this particular boundary value problem. Once we found this solution we can use linearity to obtain the general solution of

$$L(u) = f(x), \qquad u(0) = u(l) = 0, \qquad 0 < x < l, \tag{2.4.2}$$

in the form of the *superposition integral*. Indeed, let

$$u(x) \equiv \int_0^l G(x,\xi) f(\xi) d\xi. \tag{2.4.3}$$

[3]Details can be found in Section 1.2

It is easy to see that $u(x)$ solves the boundary value problem (2.4.2) as

$$L(u) = \int_0^l L_x(G)(x,\xi)f(\xi)d\xi = \int_0^l \delta(x-\xi)f(\xi)d\xi = f(x) \tag{2.4.4}$$

and the boundary conditions are satisfied. $L_x(u)$ denotes here the partial differential operator induced by L. We will try to use the same idea in the context of the heat equation.

Consider first the initial value problem for the heat conduction in an infinite homogeneous bar subjected initially to a concentrated unit heat source applied at a point y. We assume for simplicity that the thermal diffusivity $\gamma = 1$. This requires solving the heat equation with the initial condition

$$u(x,0) = \delta(x-y), \qquad -\infty < x < \infty. \tag{2.4.5}$$

To avoid any specific boundary conditions but to guarantee the uniqueness of solutions (see Section 2.3) we require the solution to be square integrable at all times, that is

$$\int_{-\infty}^{\infty} |u(x,t)|^2 \, dx < \infty \qquad \text{for all} \qquad t \geq 0. \tag{2.4.6}$$

This, in fact, implies that the solution vanishes at infinity.

Let us now take the complex separable solution to the heat equation

$$u(x,t) = e^{-k^2 t} e^{\mathrm{i}kx}, \tag{2.4.7}$$

where, as there are no boundary conditions, there are no restrictions on the choice of frequencies k. Mimicking the Fourier series superposition solution when there are infinitely many frequencies allowed we may combine these solutions into a Fourier integral (see Appendix B.5)

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-k^2 t} e^{\mathrm{i}kx} \widehat{\delta}_y(k) dk \tag{2.4.8}$$

to realize, provided we can differentiate under the integral, that it solves the heat equation. Moreover, the initial condition is also satisfied as

$$u(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mathrm{i}kx} \widehat{\delta}_y(k) dk = \delta(x-y), \tag{2.4.9}$$

where $\widehat{\delta}_y(k)$ denotes the Fourier transform of the delta function $\delta(x-y)$, that is

$$\widehat{\delta}_y(k) = \frac{1}{\sqrt{2\pi}} e^{-\mathrm{i}ky}. \tag{2.4.10}$$

Combining (2.4.9) with (2.4.10) we find that the *fundamental solution* of the heat equation is

$$F(x-y,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-k^2 t} e^{\mathrm{i}k(x-y)}dk = \frac{1}{2\sqrt{\pi t}} e^{\frac{-(x-y)^2}{4t}}. \tag{2.4.11}$$

It is worth pointing out here that although the individual component of the Fourier series (2.4.8) are not square integrable the resulting fundamental solution (2.4.11) is. Another interesting derivation of the fundamental solution based on the concept of the similarity transformation can be found in [**Kevorkian**].

REMARK 2.5. It is important to point out here that one of the drawbacks of the heat equation model is - as evident from the form of the fundamental solution - that the heat propagates at infinite speed. Indeed, a very localized heat source at y is felt immediately at the entire infinite bar because the fundamental solution is at all times nonzero everywhere.

With the fundamental solution $F(x-y,t)$ at hand we can now adopt the superposition integral formula (2.4.3) to construct the solution to the heat conduction problem of an infinite homogeneous bar with the an arbitrary initial temperature distribution $u(x,0) = g(x)$ as

$$u(x,t) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4t}} g(y)dy. \tag{2.4.12}$$

That is, the general solution is obtained by a convolution of the initial data with the fundamental solution. In other words, the solution with the initial temperature profile $g(x)$ is an infinite superposition over the entire bar of the point source solutions of the initial strength

$$g(y) = \int_{-\infty}^{\infty} \delta(x-y)g(x)dx. \tag{2.4.13}$$

Inhomogeneous Heat Equation for the Infinite Bar.

The Green's function method can also be used to solve the inhomogeneous heat conduction problem

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = p(x,t), \quad -\infty < x < \infty, \quad t > 0, \tag{2.4.14a}$$

where the bar is subjected to a heat source $p(x,t)$ which may vary in time and along its length. We impose the zero initial condition

$$u(x,0) = 0, \tag{2.4.14b}$$

and some homogeneous boundary conditions. The main idea behind this method is to solve first the heat equation with the concentrated source applied instantaneously at a single moment, and to use the method of superposition to obtain the general solution with an arbitrary source term. We therefore begin by solving the heat equation (2.4.14a) with the source term

$$p(x,t) = \delta(x-y)\delta(t-s). \tag{2.4.15}$$

It represents a unit heat input applied instantaneously at time s and position y. We postulate the same homogeneous initial and boundary conditions as in the general case. Let

$$u(x,t) = G(x-y, t-s) \tag{2.4.16}$$

denote the solution to this problem. We will refer to it as the *general fundamental solution* or a *Green's function*. Thanks to the linearity of the heat equation the solution of the general problem is given by the superposition integral

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} G(x-y,t-s)p(y,s)dyds, \tag{2.4.17}$$

where the forcing term may be also rewritten by the superposition formula as

$$p(x,t) = \int_0^{\infty} \int_{-\infty}^{\infty} p(y,s)\delta(t-s)\delta(x-y)dyds. \tag{2.4.18}$$

If we replace the zero initial condition by $u(x,0) = f(x)$, then once again due to the linearity of the differential equation we may write the solution as a combination of a solution to the homogeneous equation with inhomogeneous initial data and the solution with the homogeneous initial condition but a nonzero forcing term

$$u(x,t) = \int_{-\infty}^{\infty} F(x-y,t)f(y)dy + \int_0^t \int_{-\infty}^{\infty} G(x-y,t-s)p(y,s)dyds. \quad (2.4.19)$$

To find the general fundamental solution in an explicit form let us take the Fourier transform with respect to variable x of both sides of the differential equation (2.4.14a) with the forcing term (2.4.15). Using (2.4.10) we find that

$$\frac{d\widehat{u}}{dt} + k^2\widehat{u} = \frac{1}{\sqrt{2\pi}} e^{-\mathrm{i}ky}\delta(t-s), \quad (2.4.20)$$

where $\widehat{u}(k,t)$ denotes the Fourier transform of $u(x,t)$, and where k is viewed as a parameter. This is an inhomogeneous first order linear ordinary differential equation for the Fourier transform of $u(x,t)$ with the initial condition

$$\widehat{u}(k,0) = 0 \qquad \text{for} \qquad s > 0. \quad (2.4.21)$$

Using the integrating factor method with the integrating factor $e^{k^2 t}$ we obtain that

$$\widehat{u}(k,t) = \frac{1}{\sqrt{2\pi}} e^{k^2(t-s)-\mathrm{i}ky}\sigma(t-s), \quad (2.4.22)$$

where $\sigma(t-s)$ is the usual step function. The Green's function is than obtained by the inverse Fourier transform

$$G(x-y,t-s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mathrm{i}kx}\widehat{u}(k,t)dk. \quad (2.4.23)$$

Using the formula (2.4.11) of the fundamental solution we deduce that

$$\begin{aligned} G(x-y,t-s) &= \frac{\sigma(t-s)}{2\pi} \int_{-\infty}^{\infty} e^{\mathrm{i}k(x-y)+k^2(t-s)}dk \quad (2.4.24) \\ &= \frac{\sigma(t-s)}{2\sqrt{\pi(t-s)}} \exp\left\{-\frac{(x-y)^2}{4(t-s)}\right\}. \end{aligned}$$

The general fundamental solution (Green's function) is just a shift of the fundamental solution for the initial value problem at $t = 0$ to the starting time $t = s$. More importantly, its form shows that the effect of a concentrated heat source applied at the initial moment is the same as that of a concentrated initial temperature.

Finally, the superposition integral (2.4.17) gives us the solution

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} \frac{p(y,s)}{2\sqrt{\pi(t-s)}} \exp\left\{-\frac{(x-y)^2}{4(t-s)}\right\} dsdy \tag{2.4.25}$$

of the heat conduction problem for the infinite homogeneous bar with a heat source.

Heat Equation for the Semi-infinite Bar.

To illustrate how the Green's function method can be applied in the case of the semi-infinite domain we consider the heat equation with the concentrated forcing term

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \delta(x-y)\delta(t), \qquad 0 \le x < \infty, \qquad t > 0 \tag{2.4.26a}$$

and impose the zero initial condition, i.e., $u(x,0) = 0$, and the homogeneous boundary conditions

$$u(0,t) = 0, \quad \lim_{x\to\infty} u(x,t) = 0. \tag{2.4.26b}$$

As we have remarked earlier the effect of such a concentrated instantaneous heat source is the same as that of the concentrated initial distribution. Thus, the only difference between this case and the case of the fundamental solution (2.4.5) is the imposition of the boundary condition at $x = 0$. One possible way to tackle this difficulty is to consider in place of this semi-infinite problem such an infinite domain problem in which the homogeneous "boundary" condition at $x = 0$ is permanently satisfied.

Hence, consider the heat conduction problem for an infinite homogeneous bar with $\delta(t)[\delta(x-y) - \delta(x+y)]$ as the forcing term, homogeneous initial condition, and the homogeneous boundary conditions at infinities. In other words, in the infinite domain we apply a unit strength source at $x = y$, and simultaneously a negative source of unit strength at $x = -y$. This approach is known as the *method of images*. Once again, due to the linearity of the heat equation and that of the forcing term, the temperature profile at $t > 0$ will be the sum of two fundamental solutions $F(x-y,t)$ and $F(x+y,t)$ each corresponding to one of the source terms. In particular, due to the skew-symmetry of these solutions the combined solution will always be vanishing at $x = 0$. Moreover since all the boundary conditions of the original problem (2.4.26a) are satisfied, and since the second source term $\delta(t)\delta(x+y)$ is outside of the original semi-infinite domain, the Green's function

$$G(x-y,t) \equiv F(x-y,t) - F(x+y,t) \tag{2.4.27}$$

is the solution of (2.4.26a), where the fundamental solution F is defined by (2.4.11).

In conclusion, the solution of the inhomogeneous heat conduction problem for a semi-infinite bar

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = p(x,t), \quad 0 \le x < \infty, \quad t > 0, \tag{2.4.28}$$

with the initial condition $u(x,0) = 0$ and the homogeneous boundary conditions (2.4.26b) has the form

$$u(x,t) = \int_0^t \int_0^\infty \frac{p(y,s)}{2\sqrt{\pi(t-s)}} \left\{ \exp\left[-\frac{(x-y)^2}{4(t-s)}\right] - \exp\left[-\frac{(x+y)^2}{4(t-s)}\right] \right\} dyds. \tag{2.4.29}$$

Example 2.6. Suppose that a semi-infinite homogeneous bar is initially heated to a unit temperature along a finite interval $[a,b]$, where $a > 0$. Assume also that at $x = 0$ the temperature is held at zero (by attaching an infinite rod of this temperature) and vanishes at infinity. This corresponds to the following initial value problem for the heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(x,0) = \sigma(x-a) - \sigma(x-b) = \begin{cases} 0, & \text{if } 0 < x < a, \\ 1, & \text{if } a < x < b, \\ 0, & \text{if } x > b, \end{cases} \tag{2.4.30a}$$

with the homogeneous boundary conditions

$$u(0,t) = 0, \quad \lim_{x\to\infty} u(x,t) = 0, \quad t > 0.$$

The method of images and the superposition formula (2.4.12) yield the solution

$$\begin{aligned} u(x,t) &= \frac{1}{2\sqrt{\pi t}} \left(\int_a^b e^{-\frac{(x-y)^2}{4t}} dy + \int_{-a}^{-b} e^{-\frac{(x-y)^2}{4t}} \right) dy \qquad (2.4.31) \\ &= \frac{1}{2}\left\{ \operatorname{erf}\left(\frac{x-a}{2\sqrt{t}}\right) + \operatorname{erf}\left(\frac{x+a}{2\sqrt{t}}\right) \right\} - \frac{1}{2}\left\{ \operatorname{erf}\left(\frac{x-b}{2\sqrt{t}}\right) + \operatorname{erf}\left(\frac{x+b}{2\sqrt{t}}\right) \right\}, \end{aligned}$$

where the *error function*

$$\mathrm{erf} z \equiv \frac{2}{\sqrt{\pi}} \int_0^z e^{-\eta^2} d\eta. \tag{2.4.32}$$

Note that the error function is odd and that its asymptotic value at infinity is 1.

2.5. Burgers' Equation

In this last section we will study the quasilinear version of the diffusion equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \epsilon\frac{\partial^2 u}{\partial x^2} = 0, \qquad \epsilon > 0, \tag{2.5.1}$$

to show how the solution methods developed in previous sections for the heat equation may be used to obtain solutions for other equations. Also, Burgers' equation is a fundamental example of an *evolution equation* modelling situations in which viscous and nonlinear effects are equally important. Moreover, it plays somewhat important role in discussing discontinuous solutions (*shocks*) of the one-dimensional conservation law

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \tag{2.5.2}$$

a topic which will not be discussed here (see for example [**Knobel**], [**Smoller**]).

We start by looking at ways at which the methods for solving the initial-boundary value problems of heat equations can be used to solve (2.5.1).

The Cole-Hopf Transformation.

This is a change of dependent variable $w = W(u)$ which enables us to transform Burgers' equation into the linear diffusion equation studied already in this chapter. Let

$$u \equiv -2\epsilon\frac{w_x}{w}, \tag{2.5.3}$$

where w_x denotes partial differentiation. Calculating all derivatives and substituting them into (2.5.1) yields

$$w_x(\epsilon w_{xx} - w_t) - w(\epsilon w_{xx} - w_t)_x = 0. \tag{2.5.4}$$

In particular, if $w(x,t)$ solves the diffusion equation

$$\epsilon w_{xx} - w_t = 0, \tag{2.5.5}$$

the function $u(x,t)$ given by (2.5.3) satisfies Burgers' equation (2.5.1).

Initial Value Problem on the Infinite Domain.

Let us consider the following initial value problem:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \epsilon\frac{\partial^2 u}{\partial x^2} = 0, \quad u(x,0) = g(x), \quad -\infty < x < \infty, \tag{2.5.6}$$

and suppose that we are looking for the solutions which satisfy the corresponding diffusion equation (2.5.5). According to (2.5.3), the initial condition for the new variable $w(x,0)$ must be such that

$$g(x)w(x,0) = -2\epsilon w_x(x,0). \tag{2.5.7}$$

The general solution of this linear ordinary differential equation for $w(x,0)$ is

$$w(x,0) = A\exp\left[-\frac{1}{2\epsilon}\int_0^x g(s)ds\right], \tag{2.5.8}$$

where A is a constant, and where we assume that the integral exists. Hence, we essentially need to solve the following initial value problem for the homogeneous diffusion equation with the inhomogeneous initial condition:

$$\epsilon w_{xx} - w_t = 0, \quad w(x,0) = h(x), \quad -\infty < x < \infty. \tag{2.5.9}$$

Its solution has the form of (2.4.12) :

$$w(x,t) = \frac{1}{2\sqrt{\pi\epsilon t}}\int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4\epsilon t}} h(y)dy \tag{2.5.10}$$

where we replaced t by ϵt Note that the parameter ϵ may be eliminated from the equation, and so from the solution, by an appropriate scaling of variables. We retain it, however, so we one can later study the asymptotic behavior of solutions when $\epsilon \to 0$. Differentiating with respect to x and using the Cole-Hopf formula (2.5.3) we compute that

$$u(x,t) = \frac{\int_{-\infty}^{\infty} \frac{(x-y)}{t}\exp\left[\frac{-(x-y)^2}{4\epsilon t}\right] h(y)dy}{\int_{-\infty}^{\infty}\exp\left[\frac{-(x-y)^2}{4\epsilon t}\right] h(y)dy}, \tag{2.5.11}$$

where

$$h(y) \equiv \exp\left[-\frac{1}{2\epsilon}\int_0^x g(s)ds\right], \tag{2.5.12}$$

as the constant A cancels out.

Boundary Value Problem on a Finite Interval.

Using separation of variables method, we solve here the following initial-boundary value problem:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \epsilon\frac{\partial^2 u}{\partial x^2} = 0, \quad u(x,0) = g(x), \quad 0 < x < a, \tag{2.5.13a}$$

$$u(0,t) = u(a,t) = 0, \qquad t > 0. \tag{2.5.13b}$$

After Cole-Hope transformation we obtain the corresponding initial-boundary value problem for the diffusion equation:

$$w_t - \epsilon w_{xx} = 0, \quad w(x,0) = Ah(x), \quad 0 < x < a, \tag{2.5.14a}$$

$$w_x(0,t) = w_x(a,t) = 0, \qquad t > 0. \tag{2.5.14b}$$

As the boundary condition are homogeneous the solution $w(x,t)$ can easily be derived (see page 27) as

$$w(x,t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k e^{-(\frac{k\pi}{a})^2 t} \cos\frac{k\pi}{a}x, \tag{2.5.15}$$

where

$$a_k = \frac{2A}{a\pi}\int_0^a h(x)\cos\frac{k\pi}{a}x dx. \tag{2.5.16}$$

From the Cole-Hope transformation formula (2.5.3) one now gets the solution to the initial vale problem (2.5.13)

$$u(x,t) = 2\epsilon\frac{\frac{\pi}{a}\sum_{k=1}^{\infty} k a_k \exp\left[-(\frac{k\pi}{a})^2 t\right]\sin\frac{k\pi}{a}x}{\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \exp\left[-(\frac{k\pi}{a})^2 t\right]\cos\frac{k\pi}{a}x} \tag{2.5.17}$$

EXAMPLE 2.7. Consider Burgers' equation

$$u_t + uu_x - \epsilon u_{xx} = 0, \qquad -\infty < x, \infty, \tag{2.5.18}$$

with the the piecewise initial condition

$$u(x,0) = 2\sigma(x) - 1 = \begin{cases} 1, & \text{if } x < 0 \\ -1, & \text{if } x > 0. \end{cases} \tag{2.5.19}$$

The initial condition of the associated diffusion equation (2.5.5) may now be obtained from (2.5.7):

$$w(x,0) = Ae^{\frac{1}{2\epsilon}|x|}. \tag{2.5.20}$$

The solution $w(x,t)$ takes the form of (2.4.12) with t replaced by ϵt. Namely,

$$w(x,t) = \frac{1}{2\sqrt{\pi\epsilon t}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4\epsilon t}} e^{\frac{|y|}{2\epsilon}} dy. \tag{2.5.21}$$

Therefore, the solution of the original initial value problem is given by (2.5.11):

$$u(x,t) = \frac{\int_{-\infty}^{\infty} \frac{(x-y)}{t} \exp\left[\frac{-(x-y)^2}{4\epsilon t}\right]) e^{\frac{|y|}{2\epsilon}} dy}{\int_{-\infty}^{\infty} \exp\left[\frac{-(x-y)^2}{4\epsilon t}\right] e^{\frac{|y|}{2\epsilon}} dy}. \tag{2.5.22}$$

Integrating independently from $-\infty$ to 0 and from 0 to ∞, and using the substitution

$$\eta = \frac{(x-y\pm t)}{2\sqrt{\epsilon t}},$$

respectively, we finally obtain

$$u(x,t) = \frac{e^{-\frac{x}{\epsilon}} \operatorname{erfc}\left(\frac{x-t}{2\sqrt{\epsilon t}}\right) - \operatorname{erfc}\left(-\frac{x+t}{2\sqrt{\epsilon t}}\right)}{e^{-\frac{x}{\epsilon}} \operatorname{erfc}\left(\frac{x-t}{2\sqrt{\epsilon t}}\right) + \operatorname{erfc}\left(-\frac{x+t}{2\sqrt{\epsilon t}}\right)}, \tag{2.5.23}$$

where the *complimentary error function*

$$\operatorname{erfc}(z) \equiv 1 - \operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-\eta^2} d\eta. \tag{2.5.24}$$