A Simple Unified Model for Pricing Derivative Securities with Equity, Interest-Rate, Default & Liquidity Risk

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“A complex system that works is invariably found to have evolved from a simple system that worked.” - John Gall

Outline

1. Motivation & features
2. Technical structure of the pricing lattice
3. Default modeling
4. Liquidity
5. Correlated default
6. Numerical Examples

Economic Objectives

- A pricing model with multiple risks, which enables security pricing for *hybrid* derivatives with default risk.

- Extraction of *stable* default probability functions for state-dependent default.

- A hybrid defaultable model combining the features of both, *structural* and *reduced-form* approaches.

- Using information from the equity *and* bond markets in a no-arbitrage framework.

- Using *observable* market inputs, so as to value complex securities via *relative pricing*. e.g: debt-equity swaps, distressed convertibles.

- Managing *credit portfolios* and baskets, e.g. collateralized debt obligations (CDOs).

Essentially, the viewpoint taken by the model is that all credit risk can be determined from market risk state variables.

Technical Objectives

- A risk-neutral setting in which the joint process of interest rates and equity are modeled together with the boundary conditions for security payoffs, *after* accounting for default.

- The model is embedded on a recombining lattice for fast computation time of *polynomial complexity*.

- Cross-sectional spread data permits calibration of an *implied default probability* function which dynamically changes on the state space defined by the pricing lattice.

Default Modeling Issues

- *Anticipated vs Unanticipated* Default - Our model contains both default types
- The short-term spread puzzle.
- Structural model with jumps vs hazard-rate models
- *Statistical vs Risk-Neutral* default probabilities.
- *Asymmetric Information*
  - About firm value (Duffie & Lando, 2001)
  - About default barrier (CreditMetrics)

Motivating Example - Jump to Default Equity

- Based on Merton (1976), solution by Samuelson (1972), the call option price on defaultable equity:

$$\begin{aligned}\text{Call on defaultable equity} &= \exp(-\xi T)\ BS[S_0 e^{\xi T}, K, T, \sigma, r] \\ &= BS[S_0, K, T, \sigma, r+\xi]\end{aligned}$$

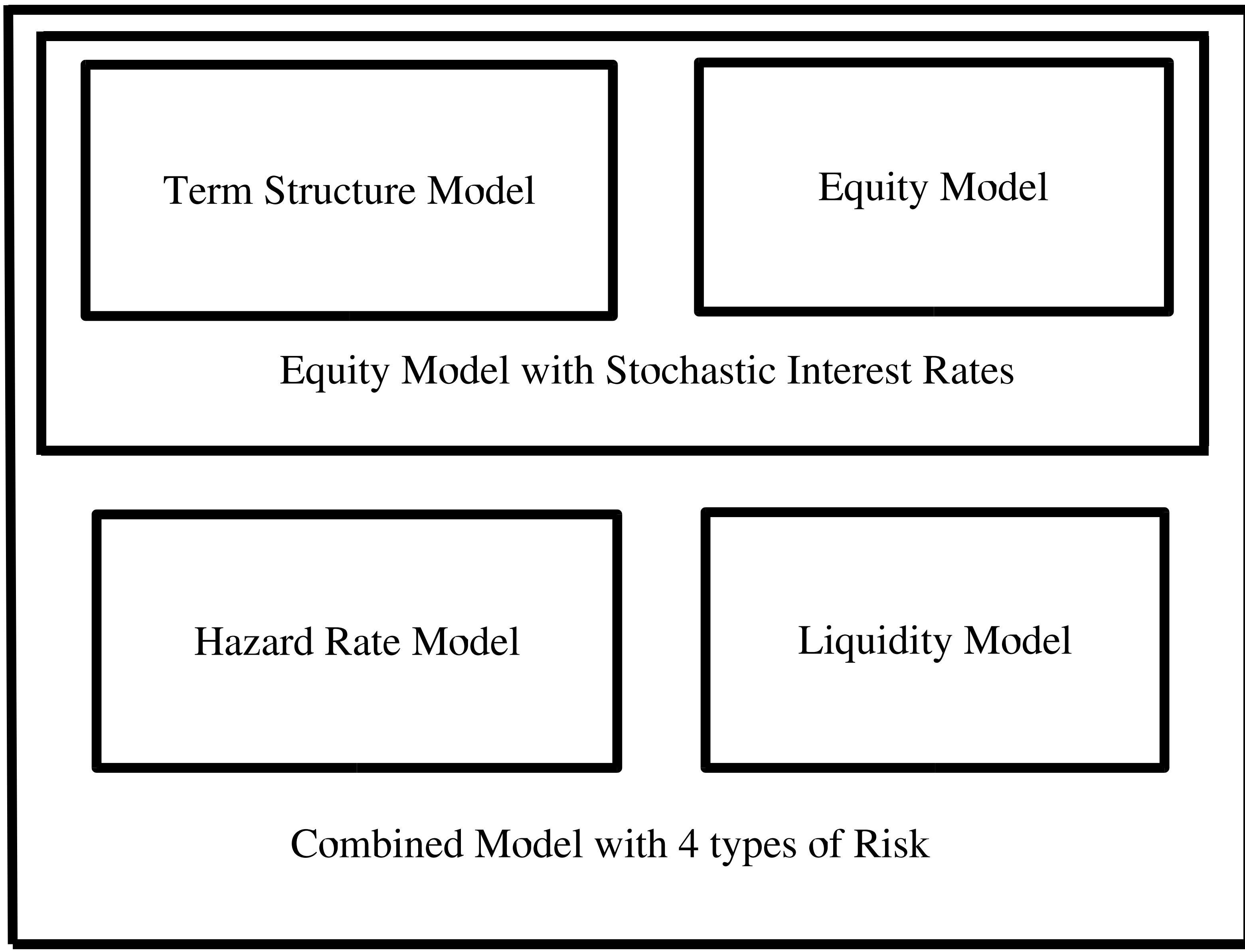
- Discrete-time setting:

$$s(t+h) = \begin{cases} uS(t) & \text{w/prob} \quad q\ exp(-\xi h) \\ dS(t) & \text{w/prob} \quad (1-q)\ exp(-\xi h) \\ 0 & \text{w/prob} \quad 1 - exp(-\xi h) \end{cases}$$

- Risk-neutral martingale measure, after default adjustment:

$$q = \frac{\exp(rh) - d\exp(-\xi h)}{u\exp(-\xi h) - d\exp(-\xi h)} = \frac{\exp[(r+\xi)h] - d}{u-d}$$

- Numerical values: $r = 0.10$, $\sigma = 0.20$, $\xi = 0.01$, and $h = 0.25$. Then, the risk-neutral probability $q = 0.766203$. If there were no defaults, i.e. $\xi = 0$, then $q = 0.740548$. [See `binbsdef.xls`]


Term Structure Model
Equity Model
Equity Model with Stochastic Interest Rates
Hazard Rate Model
Liquidity Model
Combined Model with 4 types of Risk

Related Papers for Hybrid Securities

- Schonbucher (2002): Stochastic r, with exogenous λ (discrete time).
- Das and Sundaram (2000): Stochastic r, with endogenous λ (discrete time).
- Carayannopoulos and Kalimipalli (2000): Stochastic S, with endogenous λ (discrete time).
- Davis and Lischka (1999): Stochastic r and S, with exogenous hazard rate λ (continuous time).
- Jarrow (2001): Stochastic r, S (dividend process), and liquidity π, with exogenous λ (continuous time).

Subsumed Models - 1

- If r is switched off and the hazard rate model is also switched off, we get the Cox-Ross-Rubinstein (CRR, 1979) model.

- If the hazard rate model is switched off, we get the Amin and Bodurtha (1995) framework.

- If the equity process and default processes are turned off, we get the model of Heath, Jarrow and Morton (1990).

- If the interest rate model is turned off, then we are left with the convertible pricing model of Carayannopoulos and Kalimipalli (2001).

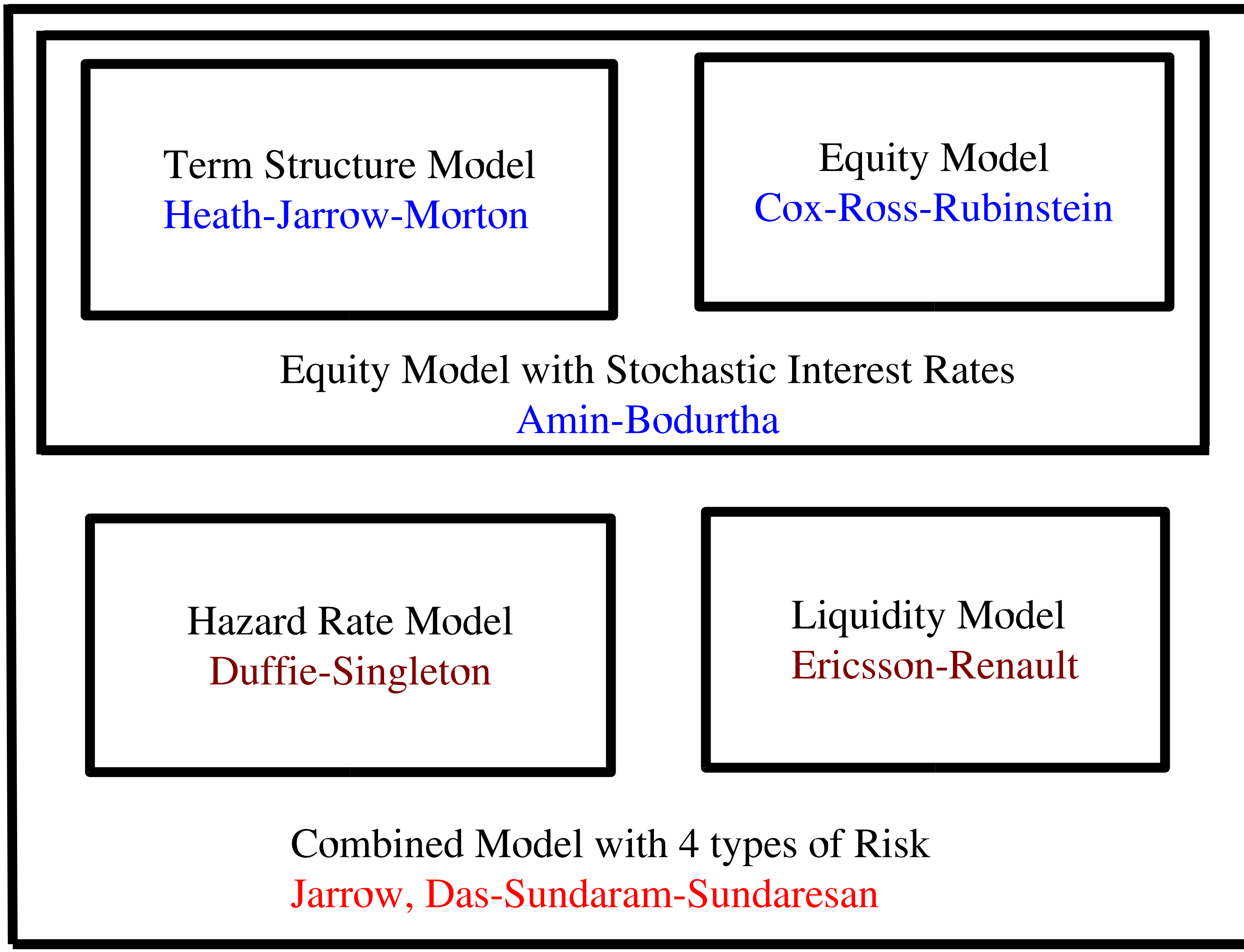
Subsumed Models - 2

- Switching off only the equity component of the model leaves us with the defaultable debt models of Madan and Unal (1995), Duffie and Singleton (1999) and Schonbucher (1998). And in discrete-time implementation form it corresponds to the model of Das and Sundaram (2000), and Schonbucher (2002), which is similar to the earlier paper by Davis and Lischka (1999).

- Turning off only the interest rate process results in the discrete time version of Merton (1974), but more-so, Cox and Ross (1976), but with non-linear default barriers.

- The model is closest to the work of Jarrow (2001) and Jarrow-Lando-Yu (1999) but with endogenous PDs, and allowance for unanticipated default.

Many of these papers do not explicitly involve the equity process and its link to default modeling within a hazard rate model. This approach, therefore, is able to create a link between structural and reduced form models.


Term Structure Model
Heath-Jarrow-Morton
Equity Model
Cox-Ross-Rubinstein
Equity Model with Stochastic Interest Rates
Amin-Bodurtha
Hazard Rate Model
Duffie-Singleton
Liquidity Model
Ericsson-Renault
Combined Model with 4 types of Risk
Jarrow, Das-Sundaram-Sundaresan

Term Structure Model - Recombinant HJM

- Time interval $[0, T^*]$. Periods of length h.
- Assuming no arbitrage, there exists an equivalent martingale measure Q for all assets, *including* defaultable ones. .
- Forward rates follow the stochastic process:

$$f(t+h,T) \; = \; f(t,T) + \alpha(t,T)h + \sigma(t,T)X_f\sqrt{h}, \tag{1}$$

- Time–t price of a default-free zero-coupon bond of maturity $T \geq t$.

$$P(t,T) \; = \; \exp\left\{-\sum_{k=t/h}^{T/h-1} f(t,kh)\cdot h\right\} \tag{2}$$

Martingale Condition

- Let $B(t)$ be the time–t value of a "money-market account" that uses an initial investment of \$1, and rolls the proceeds over at the default-free short rate:

$$B(t) = \exp\left\{\sum_{k=0}^{t/h-1} r(kh) \cdot h\right\}. \tag{3}$$

- Let $Z(t,T)$ denote the price of the default-free bond discounted using $B(t)$:

$$Z(t,T) = \frac{P(t,T)}{B(t)}. \tag{4}$$

- $Z(t,T)$ is a martingale under Q, for any $t < T$, i.e. $Z(t,T) = E^t[Z(t+h,T)]$:

$$E^t\left[\frac{Z(t+h,T)}{Z(t,T)}\right] = 1. \tag{5}$$

Risk-Neutral Drift & Equity Process

- Risk-neutral drift:

$$\sum_{k=t/h+1}^{T/h-1} \alpha(t,kh) \;=\; \frac{1}{h^2}\ln\left(E^t\left[\exp\left\{-\sum_{k=t/h+1}^{T/h-1}\sigma(t,kh)X_f h^{3/2}\right\}\right]\right). \qquad (6)$$

- Define the risk-neutral discrete-time equity process as follows:

$$\ln\left[\frac{S(t+h)}{S(t)}\right] = r(t)h + \sigma_s X_s(t)\sqrt{h} \qquad (7)$$

Joint Defaultable Stochastic Process - Hexanomial Tree

Joint process requires a probability measure over the random shocks $[X_f(t), X_s(t)]$ *AND* the default shock.

This probability measure is chosen to (i) obtain the correct correlations, (ii) ensure that normalized equity prices and bond prices are martingales *after default*, and (iii) still ensure the lattice is recombining.

Our lattice model is hexanomial, i.e. from each node, there are 6 emanating branches or 6 states. The following table depicts the states:

X_f	X_s	Probability
1	1	$\frac{1}{4}(1+m_1)[1-\lambda(t)]$
1	-1	$\frac{1}{4}(1-m_1)[1-\lambda(t)]$
-1	1	$\frac{1}{4}(1+m_2)[1-\lambda(t)]$
-1	-1	$\frac{1}{4}(1-m_2)[1-\lambda(t)]$
1	$-\infty$	$\frac{\lambda(t)}{2}$
-1	$-\infty$	$\frac{\lambda(t)}{2}$

where $\lambda(t)$ is the probability of default at each node of the tree.

We solve for m_1 and m_2 to satisfy the conditions above.

Lattice Recombination

For recombination, it is essential that the drift of the process for equity prices be zero. Hence, we write the risk-neutral stochastic process for equity prices as follows:

$$\ln\left[\frac{S(t+h)}{S(t)}\right] = \sigma_s X_s(t)\sqrt{h} \tag{8}$$

where the probability measure over $X_s(t)$ is such that

$$E[\exp(\sigma_s X_s(t)\sqrt{h})] = \exp[r(t)h].$$

Verification Calculations

To begin with, we first compute the following intermediate results:

$$
\begin{aligned}
E(X_f) &= \frac{1}{4}[1+m_1+1-m_1-1-m_2-1+m_2](1-\lambda(t)) \\
&\quad +\frac{\lambda(t)}{2}[1-1] \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
Var(X_f) &= \frac{1}{4}[1+m_2+1-m_1+1+m_2+1-m_2](1-\lambda(t)) \\
&\quad +\frac{\lambda(t)}{2}[1+1] \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
E(X_s|\text{no def}) &= \frac{1}{4}(1-\lambda(t)) \times [1+m_1-1+m_1+1+m_2-1+m_2] \\
&= \frac{m_1+m_2}{2}(1-\lambda(t))
\end{aligned}
$$

Solving for m_1, m_2

Now, we compute the two conditions required to determine m_1 and m_2. Use the expectation of the equity process to determine one equation. We exploit the fact that under risk-neutrality the equity return must equal the risk free rate of interest. This leads to the following:

$$
\begin{aligned}
E\left[\frac{S(t+h)}{S(t)}\right] &= E[\exp(\sigma_s X_s(t)\sqrt{h})] \\
&= \frac{1}{4}(1-\lambda(t))[e^{\sigma_s\sqrt{h}}(1+m_1)+e^{-\sigma_s\sqrt{h}}(1-m_1) \\
&\quad +e^{\sigma_s\sqrt{h}}(1+m_2)+e^{\sigma_s\sqrt{h}}(1-m_2)]+\frac{\lambda(t)}{2}[0] \\
&= \exp(rh)
\end{aligned}
$$

Hence the stock return is set equal to the riskfree return.

$$
\begin{aligned}
m_1+m_2 &= \frac{\frac{4e^{r(t)h}}{1-\lambda(t)}-2(a+b)}{a-b}=A \\
a &= \exp(\sigma_s\sqrt{h}) \\
b &= \exp(-\sigma_s\sqrt{h})
\end{aligned}
$$

Correlation Specification

The correlation between $[X_f(t), X_s(t)]$ is defined as ρ, where $-1 \leq \rho \leq 1$.

$$\begin{aligned} Cov[X_f(t), X_s(t)] &= \frac{1}{4}(1-\lambda(t))[1+m_1-1+m_1-1-m_2+1-m_2] \\ &\quad + \frac{\lambda(t)}{2}[-\infty+\infty] \\ &= \frac{m_1-m_2}{2}(1-\lambda(t)). \end{aligned}$$

Setting this equal to ρ, we get the equation

$$m_1 - m_2 = \frac{2\rho}{1-\lambda(t)} = B.$$

Solving the two equations for A and B gives:

$$\begin{aligned} m_1 &= \frac{A+B}{2} \\ m_2 &= \frac{A-B}{2} \end{aligned}$$

These values may now be substituted into the probability measure in the table above.

Embedding Default Risk

As noted before, $\lambda(t) = 1 - e^{-\xi(t)h}$, and we express the hazard rate $\xi(t)$ as:

$$\xi[\mathbf{f}(t), S(t), t, y; \theta] \quad \in \quad [0, \infty)$$

This is a function of the term structure of forward rates, and the stock price at each node and time. y may be (for example) the firm's target D/E ratio.

Note that this function may be as general as possible.

We impose the condition $\xi(t) \geq 0$.

θ is a parameter set that defines the function.

A similar *endogenous* hazard rate extraction has been implemented in Das and Sundaram (2000), Carayannopoulos and Kalimipalli (2001), and Acharya, Das and Sundaram (2002).

Cross sectional Calibration of Risk-Neutral Hazard Rates

For pricing purposes, we need the expected risk-neutral hazard rate of default, i.e. $E[\xi(t)]$, not the physical one, $E[\xi_P(t)]$.

An example of a fitting function is:

$$\begin{aligned} \xi(t) &= \exp[a_0 + a_1 r(t) - a_2 \ln S(t) + a_3(t - t_0)] \\ &= \frac{\exp[a_0 + a_1 r(t) + a_3(t - t_0)]}{S(t)^{a_2}} \end{aligned}$$

In this version of the hazard rate function, with $S(t)$ replaced with its natural logarithm, we get that as $S(t) \to 0$, $\xi(t) \to \infty$, and as $S(t) \to \infty$, $\xi(t) \to 0$. For this parameterization, it is required that $a_2 \geq 0$.

Calibrating multiple debt issues

If the issuer has several issues, with varying subordination levels, each of these may be calibrated separately, using the exact same procedure as in the previous subsection.

Given that we know the recovery rates for each issued security, based on subordination and other conditions, we can search over default probabilities that best fit all the different bonds issued by one issuer.

Of course, the real problem is that we often wish to price the issue rather than use it for calibration. In such a case, a comparable issuer's debt may be used to determine default probabilities, which can then be used for pricing.

Correlated default analysis

The model may be used to price a credit basket security. There are many flavors of these securities, and some popular examples are n^{th} to default options, and collateralized debt obligations(CDOs).

Use Monte Carlo simulation, under the risk-neutral measure, based on the parameters fitted on the lattice.

Given a basket of N bonds of distinct issuers, we may simulate default times (τ_i) for each issuer $(i = 1...N)$ based on their stock price correlations.

Thus, we first compute a stock price or stock return covariance matrix, denoted $\Sigma_S \in R^{N \times N}$. Under the risk-neutral measure, the return for all stocks is $r(t)$.

Incorporating Liquidity

- Adapt the approach of Ericsson and Renault (1998).
- At every node on the tree, there is a small probability, denoted $\pi(.)$, that the security may require liquidation at a non-zero transaction cost κ.
- The function π may be set to be correlated with the probability of default. The regression framework of Altman, Brooks, Resti and Sironi (2002) suggests one such linear relationship.
- Or, a non-linear form:

$$\pi = \frac{1}{1+e^{-w}}, \quad w = \gamma_0 + \gamma_1 \xi$$

  In this setting, $\pi \in [0,1]$, and if $\gamma_1 = 0$, then liquidity risk is independent of default risk. If $\gamma_1 > 0$, liquidity risk is positively correlated with default risk, and vice versa.

Credit Default Swaps (CDS)

Default swaps are one of the useful instruments we may employ to calibrate the model.

The price of a default swap is quoted as a spread rate per annum.

Therefore, if the default swap rate is 100 bps, paid quarterly, then the buyer of the insurance in the default swap would pay 25 bps of the notional each quarter to the seller of insurance in the default swap.

The present value of all these payments must equal the expected loss on default anticipated over the life of the default swap.

In the event of default, the buyer of protection in the default swap receives the par value of the bond less the recovery on the bond.

In many case, this is implemented by selling the bond back to the insurance seller at par value.

Numerical and Empirical Implementations

- Term structure of default swap spreads.
- Calibration of default function to CDS data: industrial & financial companies.
- Distressed callable, convertible debt.
- Historical default function for high and low quality credit companies.

Numerical Example - Term Structure of Default Swap Spreads

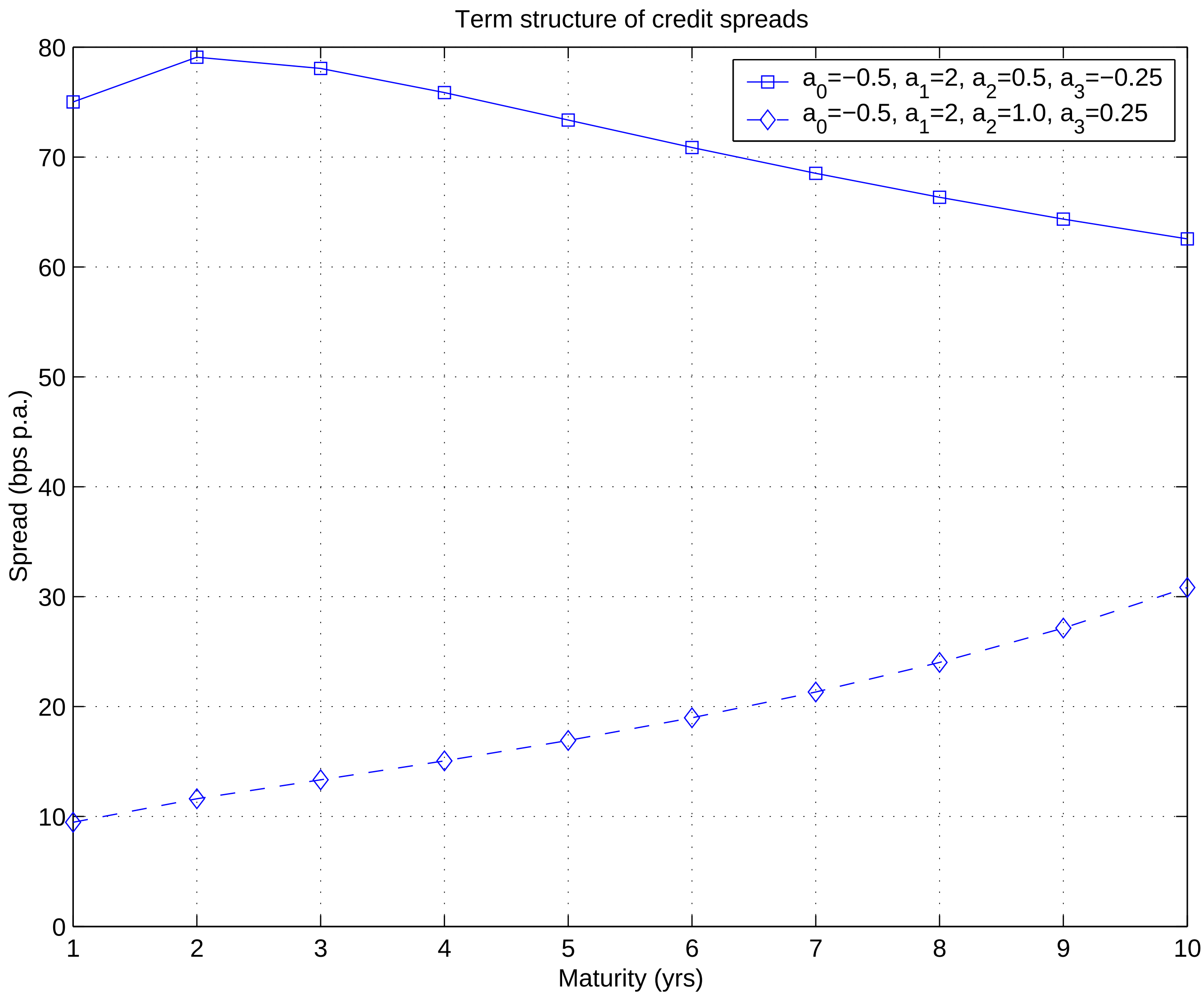
Hazard Rate:

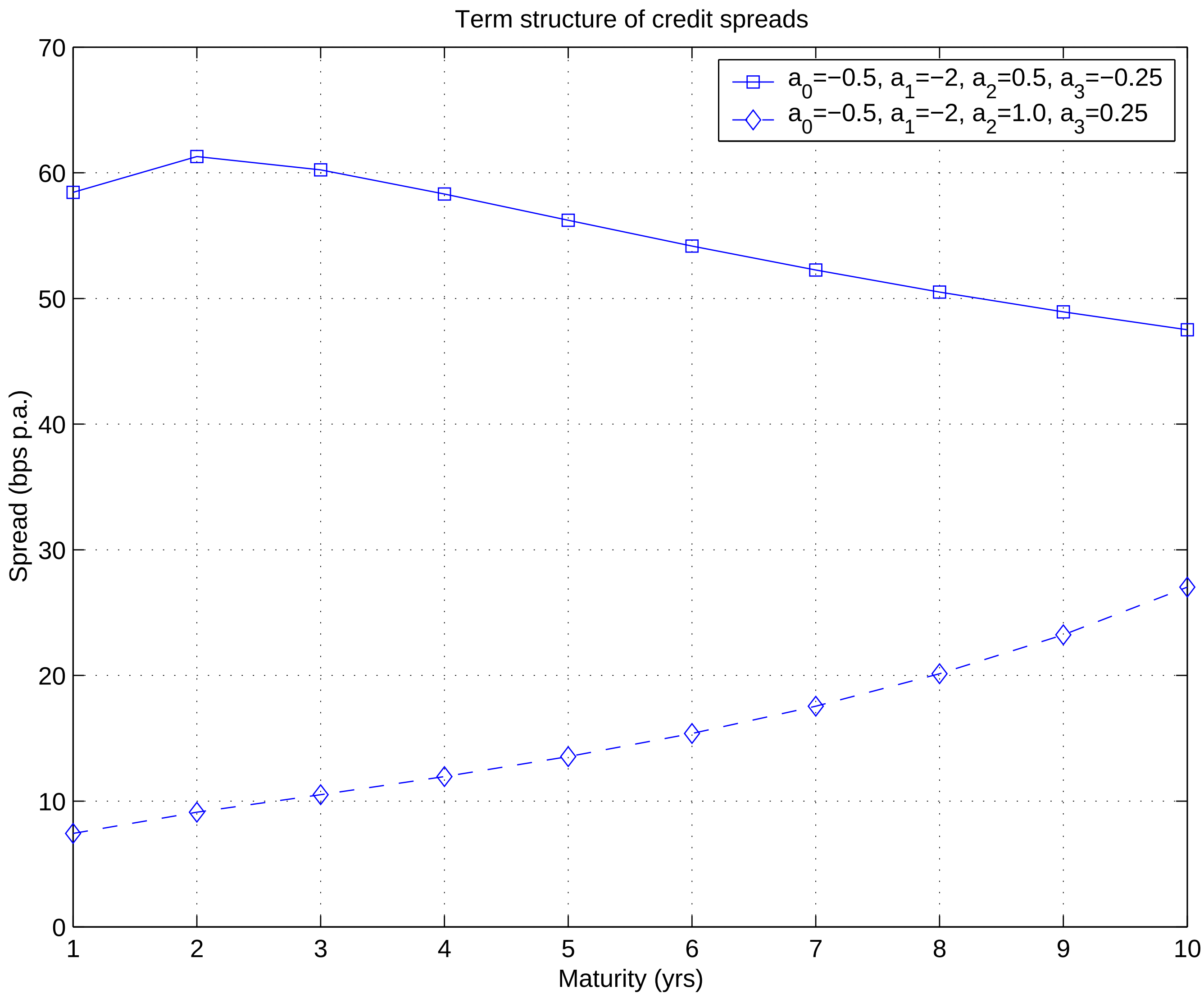
$$\xi(t) = \exp[a_0 + a_1 r(t) + a3(t - t_0)]/S(t)^{a_2}$$

Periods in the model are quarterly, indexed by i.

The forward rate curve is very simple and is just $f(i) = 0.06 + 0.001i$. The forward rate volatility curve is $\sigma_f(i) = 0.01 + 0.0005i$.

The initial stock price is 100, and the stock return volatility is 0.30. Correlation between stock returns and forward rates is 0.30, and recovery rates are a constant 40%.


Term structure of credit spreads
Spread (bps p.a.)
Maturity (yrs)
a0=−0.5, a1=2, a2=0.5, a3=−0.25
a0=−0.5, a1=2, a2=1.0, a3=0.25
0
10
20
30
40
50
60
70
80
1
2
3
4
5
6
7
8
9
10


Term structure of credit spreads
a_0=−0.5, a_1=−2, a_2=0.5, a_3=−0.25
a_0=−0.5, a_1=−2, a_2=1.0, a_3=0.25
Spread (bps p.a.)
Maturity (yrs)
0
10
20
30
40
50
60
70
1
2
3
4
5
6
7
8
9
10

Calibrating the default function to CDS data

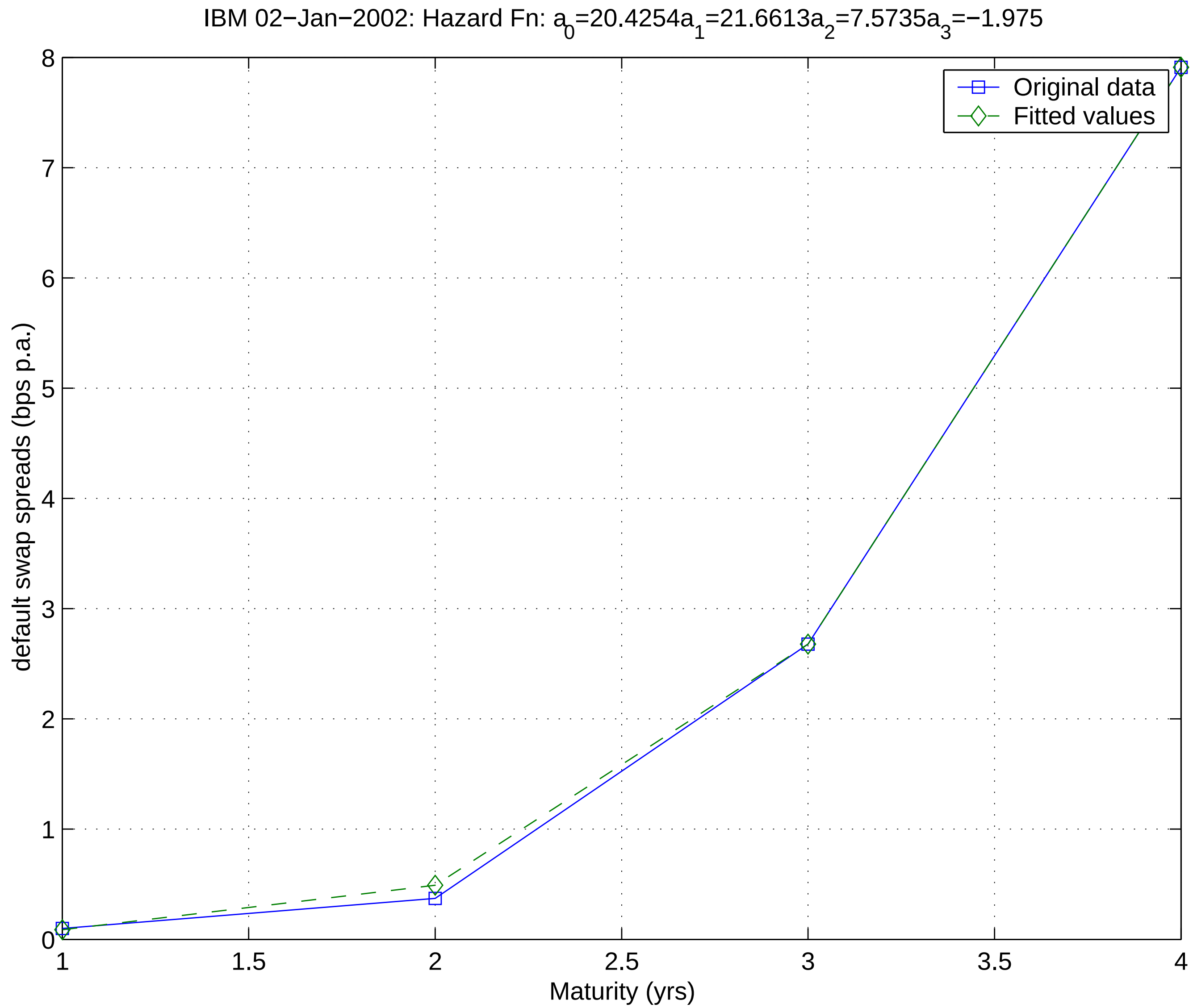
Calibration to the term structure of default swap spreads of IBM.

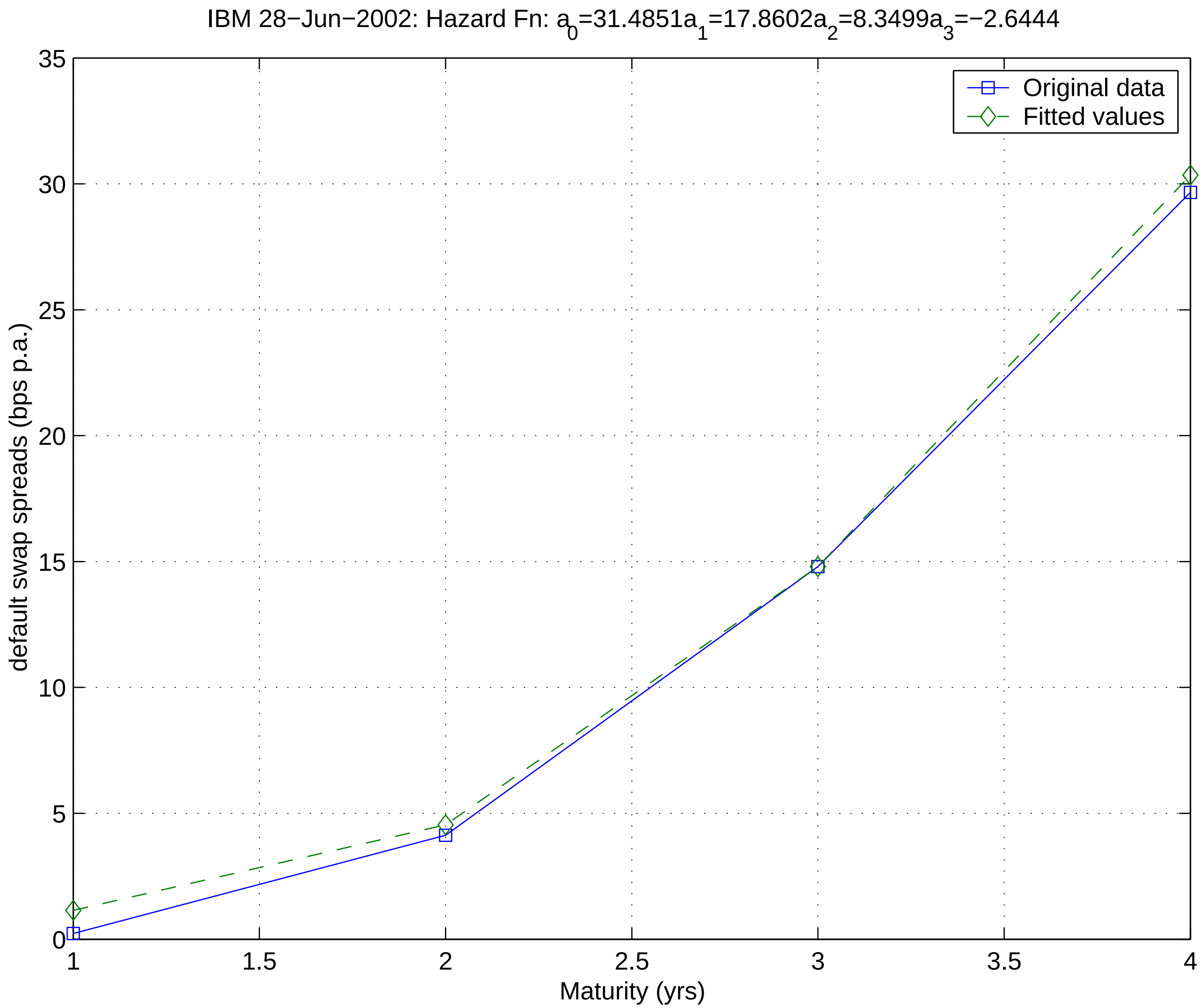
Two dates: 02-Jan-2002 and 28-Jun-2002.

The stock price on the 2 dates was $72.00 and $121.10 respectively. Stock return volatility was roughly 40% on both dates.

Recovery rates on default were assumed to be 40% and the correlation between short rates (i.e. 3 month tbills) and the stock return of IBM was computed over the period January 2000 to June 2002; it was found to be almost zero, i.e. 0.01528.

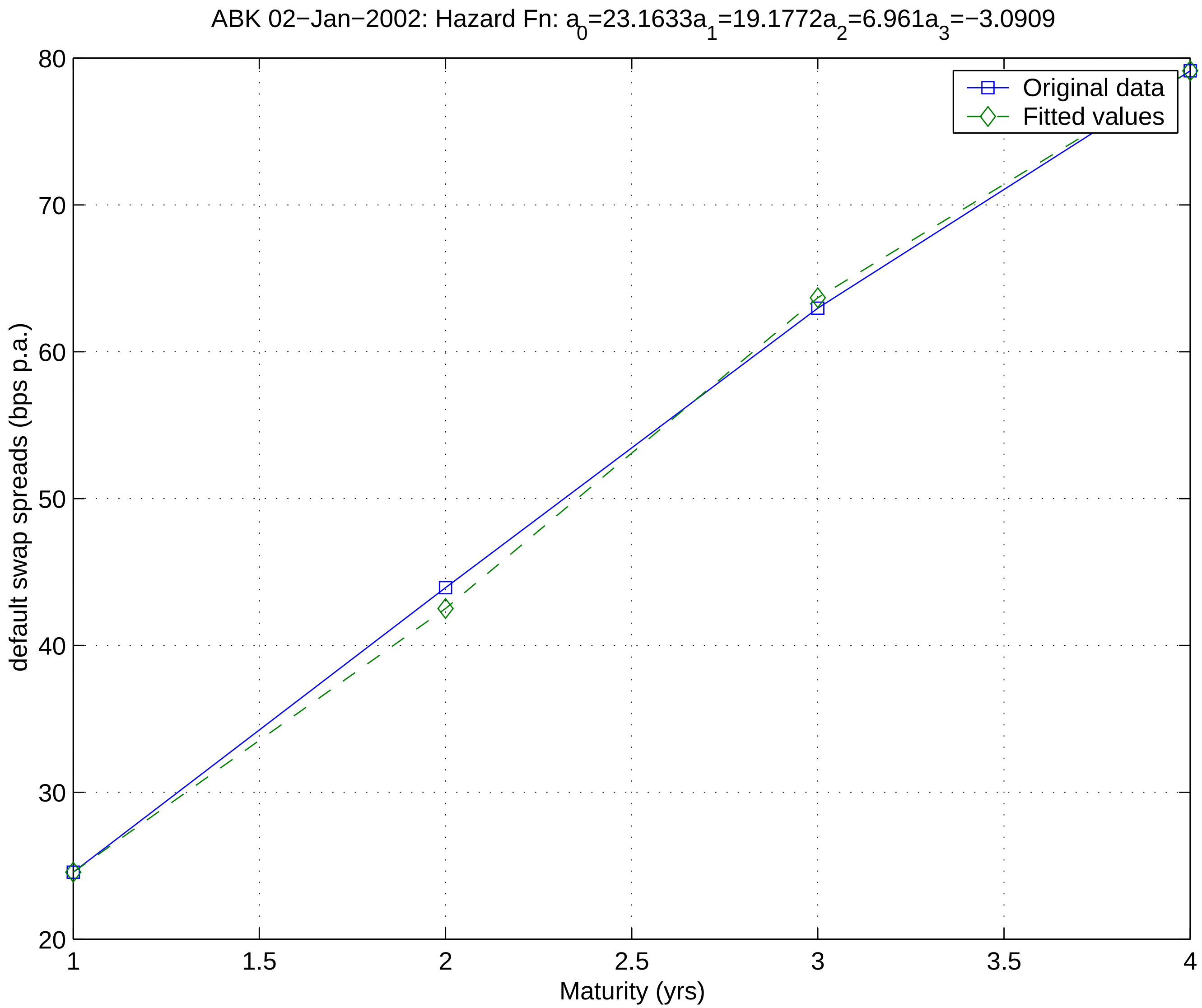
The yield curves for the chosen dates were extracted from the historical data pages provided by the Federal Reserve Board. We converted these into forward rates required by our model. Forward rate volatilities were set to the average historical volatility over the periods January 2000 to June 2002.


IBM 02−Jan−2002: Hazard Fn: a0=20.4254a1=21.6613a2=7.5735a3=−1.975
Original data
Fitted values
default swap spreads (bps p.a.)
Maturity (yrs)
0
1
2
3
4
5
6
7
8
1
1.5
2
2.5
3
3.5
4

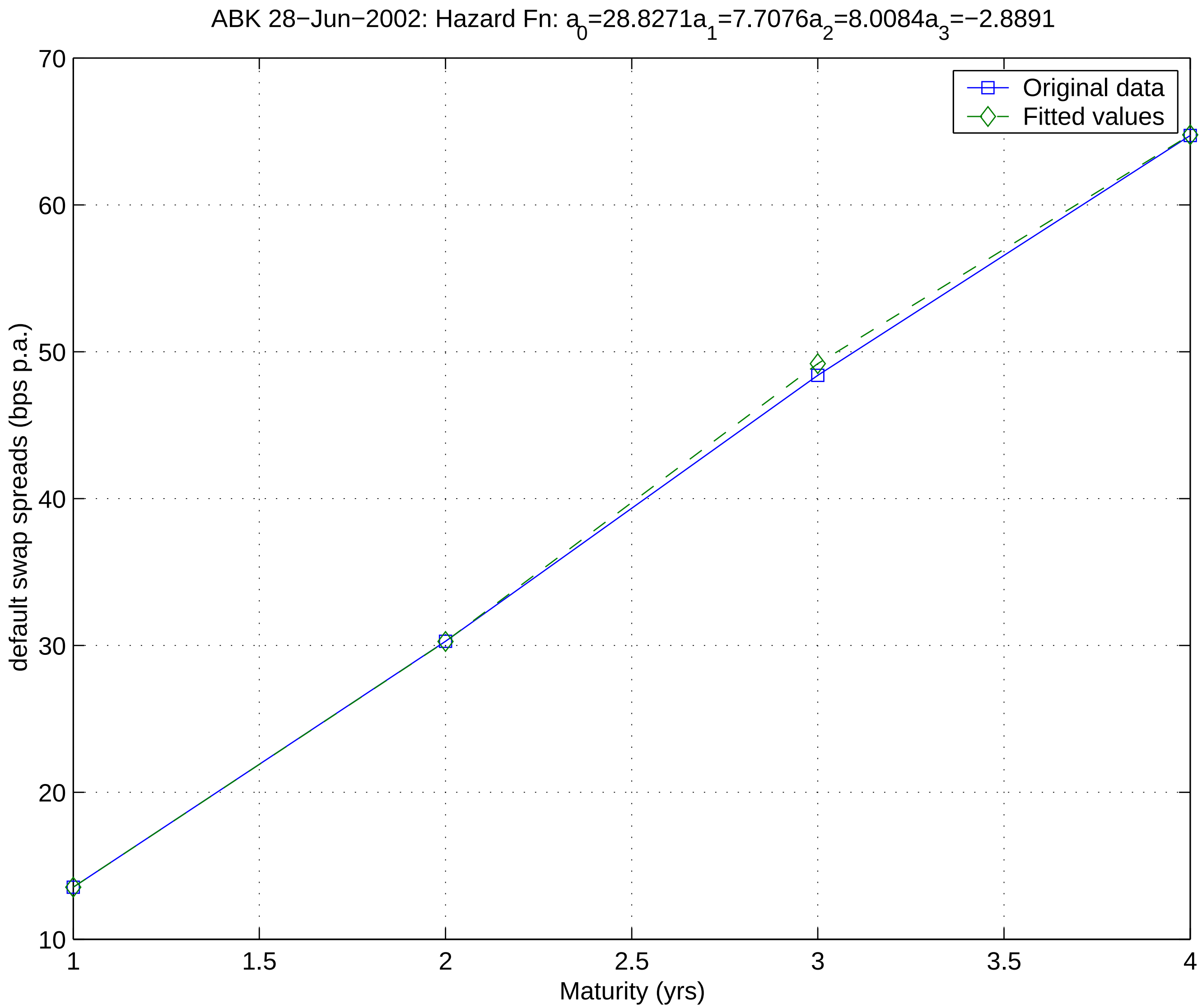

IBM 28−Jun−2002: Hazard Fn: a0=31.4851a1=17.8602a2=8.3499a3=−2.6444
Original data
Fitted values
default swap spreads (bps p.a.)
Maturity (yrs)
0
5
10
15
20
25
30
35
1
1.5
2
2.5
3
3.5
4

Calibrating Financial Companies

- Analysis of a financial company, namely AMBAC Inc (ticker symbol: ABK).
- It is postulated that default processes in the finance sector are different because firms have extreme leverage.
- However, the model calibrates just as easily to the default swap rates for AMBAC.
- For comparison, we calibrated the model on the same dates as we did for IBM.



ABK 02−Jan−2002: Hazard Fn: a0=23.1633a1=19.1772a2=6.961a3=−3.0909
default swap spreads (bps p.a.)
Maturity (yrs)
Original data
Fitted values
80
70
60
50
40
30
20
1
1.5
2
2.5
3
3.5
4


ABK 28−Jun−2002: Hazard Fn: a0=28.8271a1=7.7076a2=8.0084a3=−2.8891
Original data
Fitted values
default swap spreads (bps p.a.)
Maturity (yrs)
70
60
50
40
30
20
10
1
1.5
2
2.5
3
3.5
4

Distressed Callable-Convertibles

- A combination of equity and interest rate risk impact the callability and convertibility of the bond.

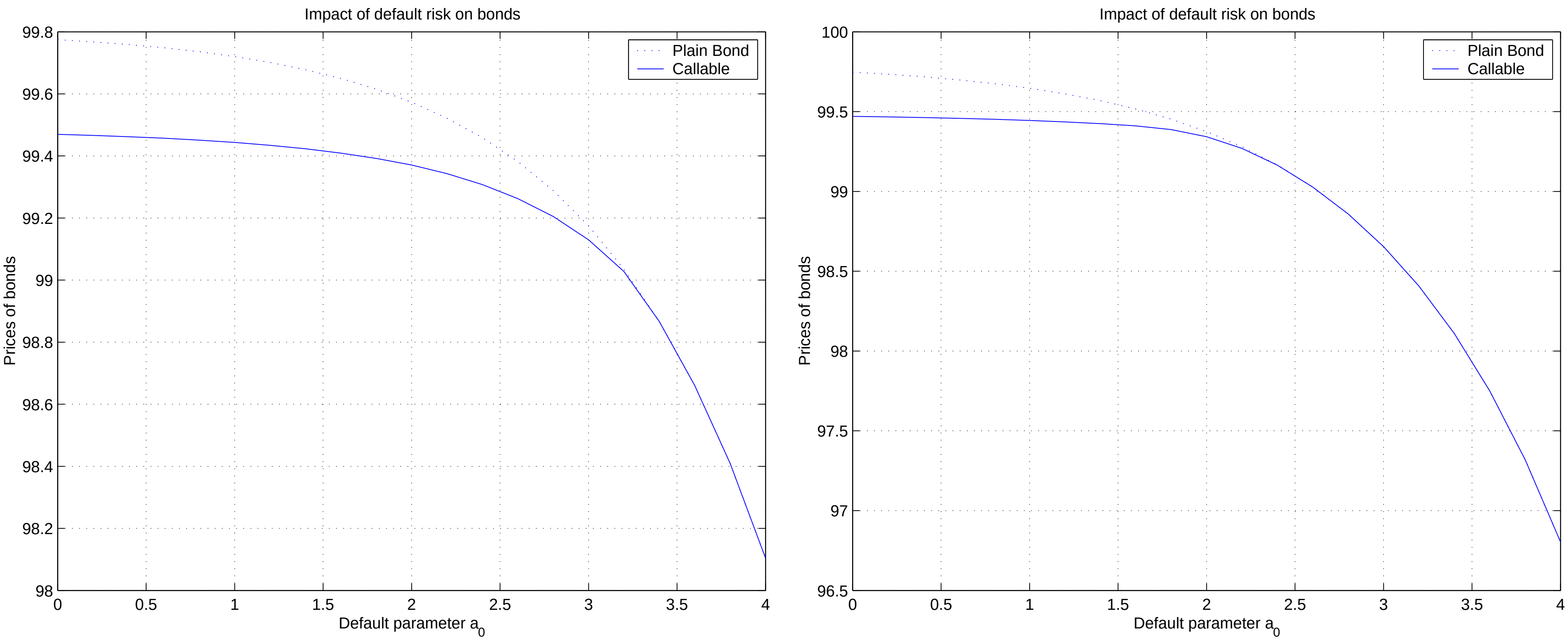
- Default risk affects both call and convertible option values.

- Duration of the bond may increase or decrease with default risk.
  - Default risk shortens effective maturity.
  - It also reduces the impact of the call and convert options, thereby extending maturity.

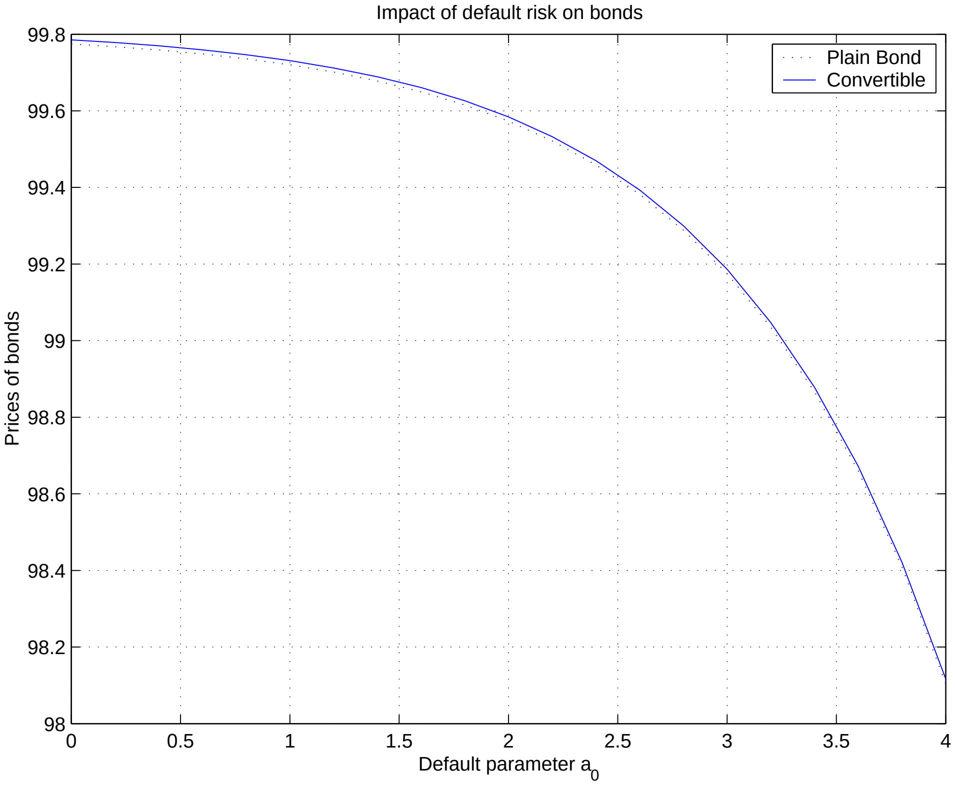
- We examine varying levels of stock volatility (20% and 40%).

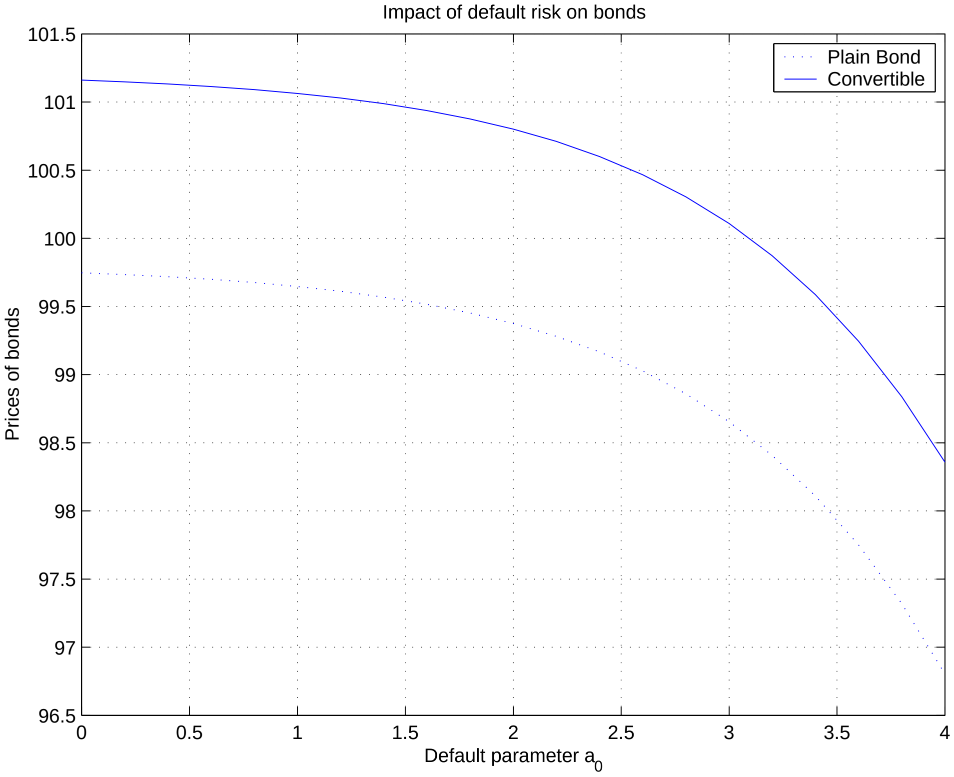
- 3 cases: (a) callable only, (b) convertible only, (c) callable-convertible.

[`dss.xls`]

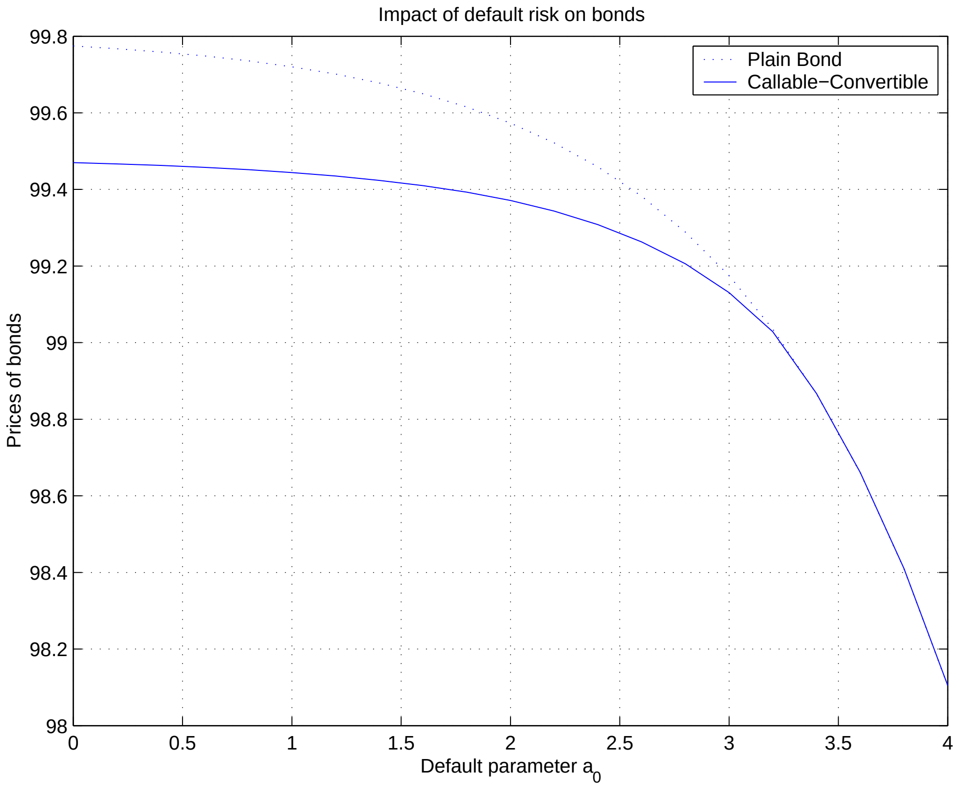


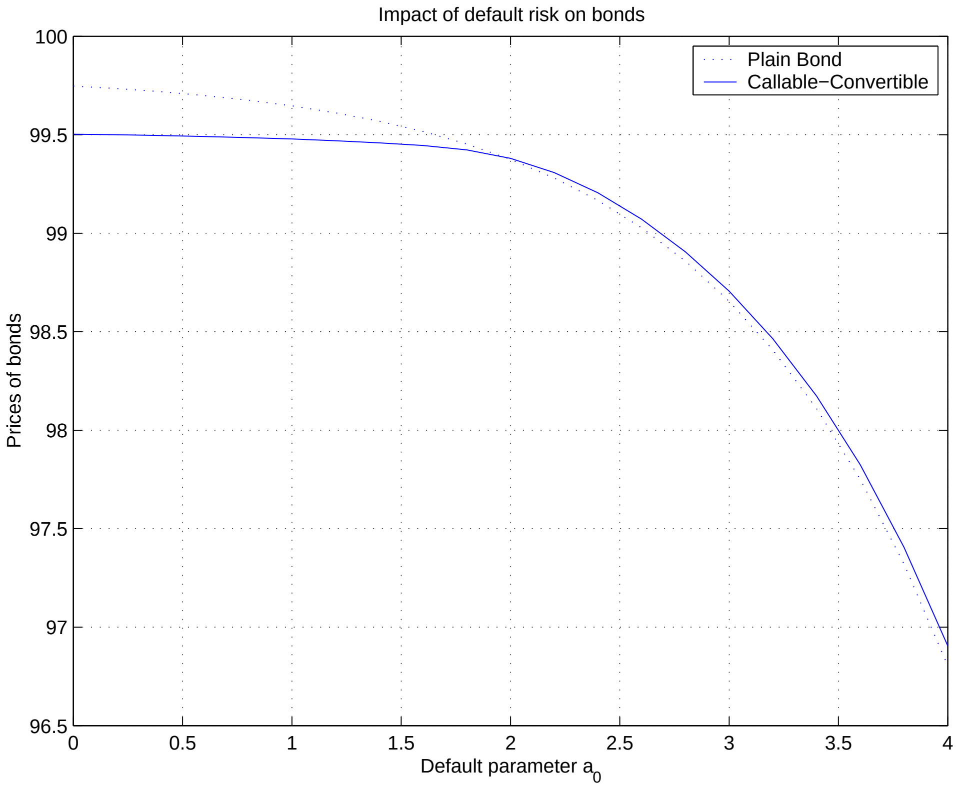
Callable bonds





Convertible bonds





Callable-Convertible bonds

Historical Time-series & Cross-sectional Calibration

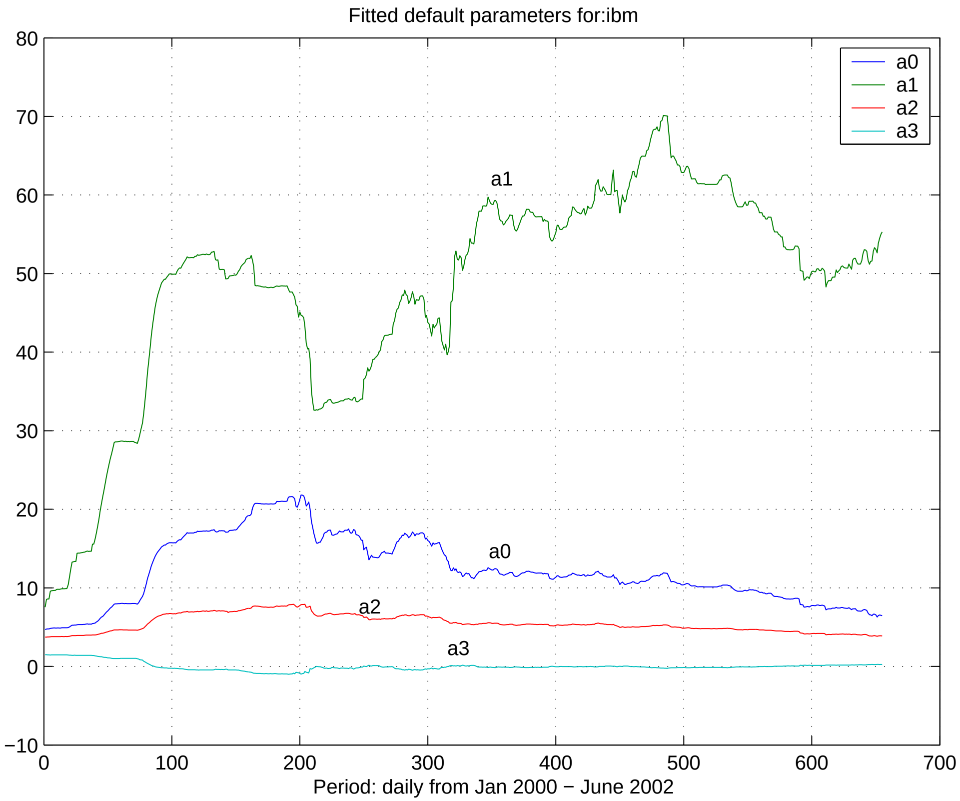
- Calibrate 2 firms: IBM and UAL (high & low credit quality)

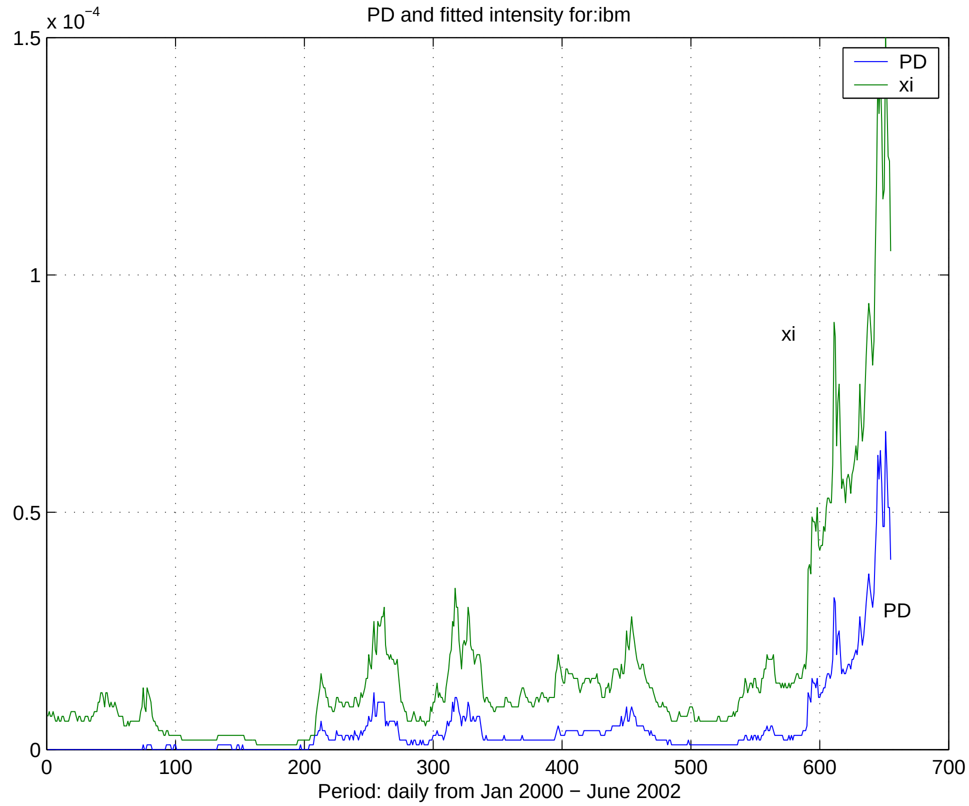
- Period: January 2000 - June 2002 (655 trading days)

- For each date, we fit the four parameters $\{a_0, a_1, a_2, a_3\}$ of the default function to the cross-sectional data on default swap spreads, the stock price and volatility, as well as the current term structure of interest rates.

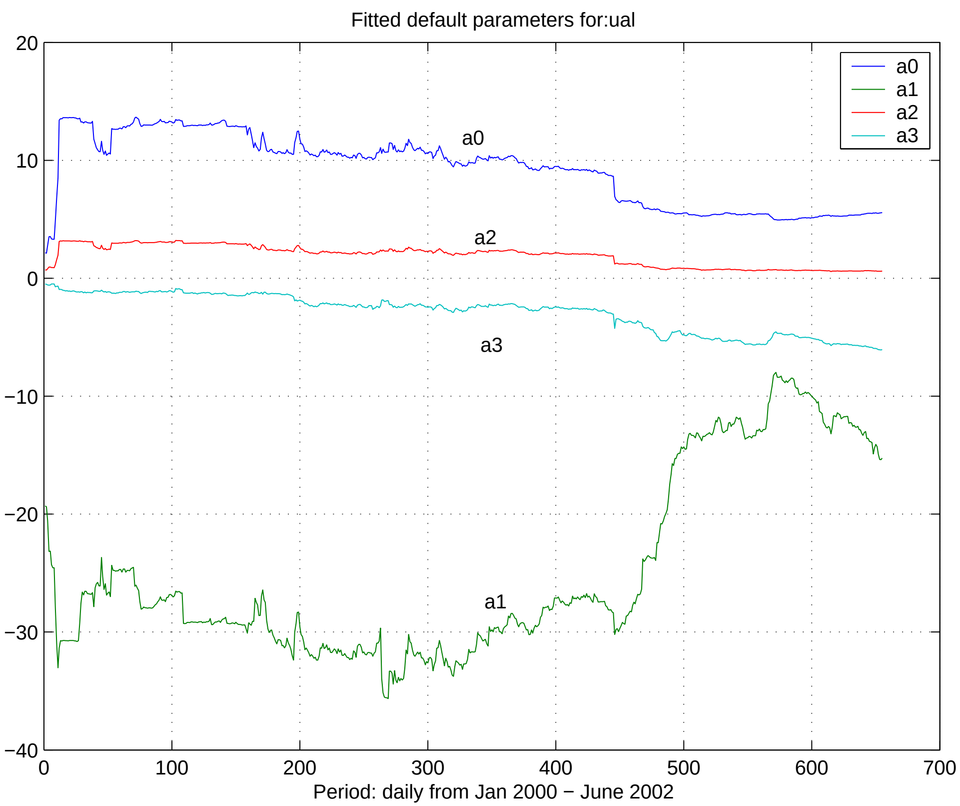
- Compute ξ for initial parameters of the lattice each day.

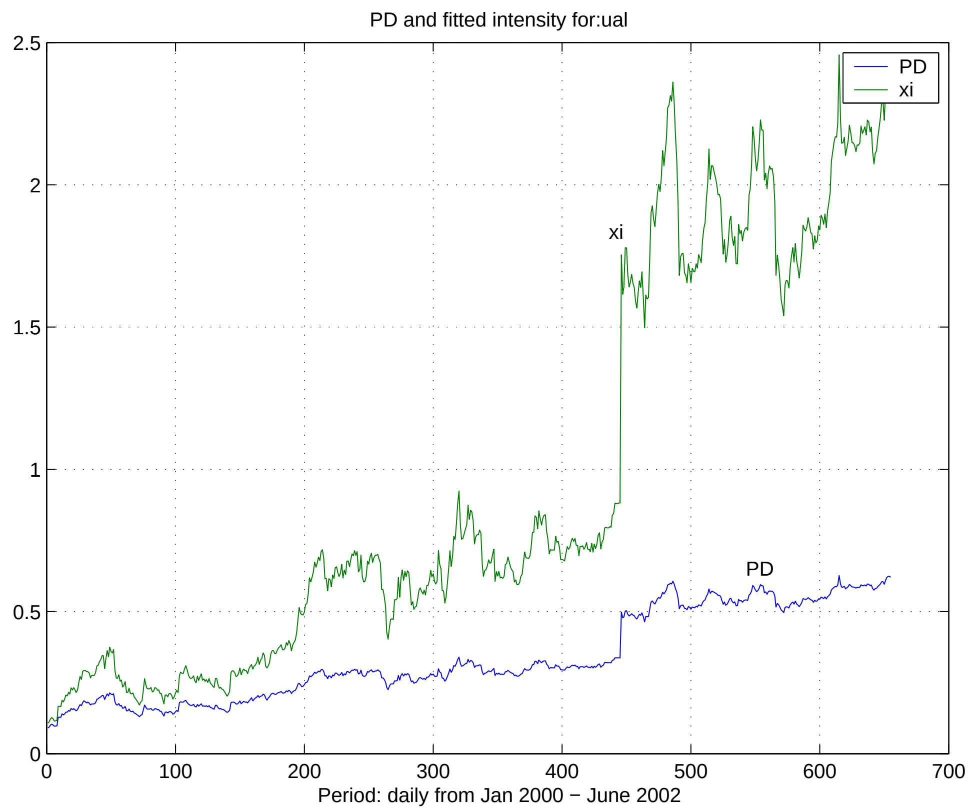
- Compare parameters and ξ to state variables.





Calibration for IBM





Calibration for UAL

Correlation of ξ and state variables

Panel A: IBM

	PD	ξ	$S(0)$	σ	$f(0,0)$
PD	1.0000	0.9873	-0.7641	0.3816	-0.4191
ξ	0.9873	1.0000	-0.7917	0.3991	-0.4744
$S(0)$	-0.7641	-0.7917	1.0000	-0.3991	0.3192
σ	0.3816	0.3991	-0.3991	1.0000	-0.7229
$f(0,0)$	-0.4191	-0.4744	0.3192	-0.7229	1.0000

Panel B: UAL

	PD	ξ	$S(0)$	σ	$f(0,0)$
PD	1.0000	0.9942	-0.9761	0.9510	-0.9242
ξ	0.9942	1.0000	-0.9495	0.9619	-0.9172
$S(0)$	-0.9761	-0.9495	1.0000	-0.8775	0.8928
σ	0.9510	0.9619	-0.8775	1.0000	-0.8938
$f(0,0)$	-0.9242	-0.9172	0.8928	-0.8938	1.0000

Summary

ECONOMIC OBJECTIVES	*TECHNICAL GOALS*
Hybrid risks model for securities with default risk	Combine structural and reduced-form approaches
Extraction of stable PD functions	Recombining lattice for fast computation
Using observable market inputs from the equity *and* bond markets, so as to value complex securities via *relative pricing* in a no-arbitrage framework, e.g: debt-equity swaps, distressed convertibles	A risk-neutral setting in which the joint process of interest rates and equity are modeled together with the boundary conditions for security payoffs, *after* accounting for default.
	Simple cross-sectional calibration