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The Accelerated Binomial Option Pricing Model

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Abstract

This paper describes the application of a convergence acceleration technique to the binomial option pricing model. The resulting model, termed the accelerated binomial option pricing model, also can be viewed as an approximation to the Geske-Johnson model for the value of the American put. The new model is accurate and faster than the conventional binomial model. It is applicable to a wide range of option pricing problems.

I. Introduction

The binomial option pricing model, introduced by Cox, Ross, and Rubinstein (1979) and Rendleman and Bartter (1979), is now widely used to value options, particularly where no analytic (closed form) solution exists, as in the benchmark case of the American put option. More recently, Geske and Johnson (1984) introduced a method of valuing American put options based on Geske's (1979) compound option model and utilizing convergence acceleration techniques. As a result, their approach is a more efficient means of valuing such options than the binomial. In this paper, we present a method, called the accelerated binomial option pricing model, which is a hybrid of the binomial and Geske-Johnson models. It can be viewed as a binomial model incorporating the convergence accelerating techniques used by Geske and Johnson: equally it can be seen as a binomial approximation to the continuous time Geske-Johnson model.

Our purpose is to present the accelerated binomial option pricing method, to illustrate its accuracy, and to give some indication of its computational efficiency vis-à-vis other methods. We begin by swiftly reviewing the binomial and Geske-Johnson models, then go on to present the accelerated binomial option pricing model. In Section V of the paper, we discuss extensions of the model to options on dividend paying stocks and to options with multiple state variables. Section VI reports the results of a number of tests of the method's efficiency and Section VII provides a brief conclusion.

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We initially deal with American put options written on nondividend paying stock. We make the usual assumptions—namely that the risk-free interest rate, r, and the annualized standard deviation of the returns of the underlying stock σ, are both constant over the life of the option. We denote time by the index t ($t = 0 \ldots T$, the maturity date of the option), the stock price at t by $S(t)$, and the exercise price by X.

II. The Binomial Option Pricing Model

In the binomial option pricing model, the life of the option is divided into N discrete time periods, during each of which the price of the underlying asset is assumed to make a single move, either up or down. The magnitude of these movements is given by the multiplicative parameters u and d. The risk-neutral probability of an upward movement is given by p, and the one period risk-free rate we denote by q.

The binomial method approximates the continuous change in the option's value through time by valuing the option at a discrete set of nodes, which together make a tree-shaped grid. We identify each node in the tree by $<j, n>$ where j indicates the number of upward stock moves required to generate the option's immediate nonnegative exercise value at that node, given (for a put) by

(1) $$B_{j,n} = \max(0; X - u^j d^{n-j} S),$$

and n counts the number of time steps of the model that have elapsed during the option's life ($n = 0 \ldots N$).

When valuing European options or American call options on nondividend paying stock, it is only necessary to calculate the $N + 1$ terminal exercise values of the option (i.e., the set $B_{jN}, j = 0 \ldots N$ in our notation). Since there is no probability of early exercise in these cases, the intermediate values of the binomial process (for $0 < n < N$) need not be computed. Instead the binomial formula is used to "jump backward" from the terminal values to the initial option value (at node $< 0, 0 >$). In Geske and Shastri's (1985) analysis of approximation methods for valuing options on nondividend paying stock, the use of the binomial formula is chiefly responsible for the binomial model outperforming several finite difference methods in terms of computing demands and expense. However, the finite difference methods become computationally superior relative to the binomial whenever a large number (300) of options are valued.

The application of the binomial method to the problem of valuing American put options on stocks with or without dividend payments will be much less efficient because these puts will always have positive probability of premature exercise. This requires that both the holding value and the exercise value of the option be computed and compared for each node in the process. The more nodes required for penny accuracy, the less efficient will be the binomial method. However, this is where the accelerated binomial method will be most beneficial in reducing the number of nodes that must be computed and compared.

We define the value of the option at the jnth node by

(2) $$V_{jn} = \max(A_{jn}, B_{jn}),$$

where B_{jn} is as before and A_{jn} is the holding value of the option at that node,

(3) $$A_{jn} = (p/q)V_{j+1,n+1} + ((1-p)/q)V_{j,n+1}.$$

The binomial method entails the calculation of the values of all nodes in successively earlier periods, culminating in the value V_{00}, which is the option's value.

III. The Geske and Johnson Compound Option Approach

The Geske-Johnson analytic formula for the value of an American put option, which we denote by ϕ, can be written,

(4) $$\begin{aligned}&\textstyle\sum_{n=1}^{\infty} \text{prob}(S_{ndt} < \bar{S}_{ndt}, S_{mdt} \geq \bar{S}_{mdt} \forall m < n)\\ &\cdot (X - E[S_{ndt}|S_{ndt} < \bar{S}_{ndt}, S_{mdt} \geq \bar{S}_{mdt} \forall m < n])/r^{ndt}.\end{aligned}$$

That is, the value of the option is given by the sum of the discounted conditional exercise values of the option at each instant during its life. The condition in question is that, at instant *ndt*, the stock price, S, should be below its critical value $\bar{S}$, not having fallen below its critical value at any previous instant, *mdt*. To use Expression (4), then, entails the evaluation of an infinite sequence of successively higher order normal integrals, reflecting the fact that, at instant *ndt*, estimating the conditional expectation of the option's exercise value requires the evaluation of an n-variate normal integral.

Geske and Johnson (1984) surmount this difficulty by defining a reduced number of early exercise instants during the life of the option and using Richardson's extrapolation to find an approximation to the infinite sequence of integrals that yields the true option value. Their three-point extrapolation, for example, defines a set of option values, $P(n)$, based on exercise opportunities restricted as follows: $P(1)$ = option value based on exercise opportunities restricted to T; $P(2)$ = option value based on exercise opportunities at T and $T/2$; $P(3)$ = option value based on exercise opportunities at T, $2T/3$, and $T/3$. The limit of this sequence, $P(n)n \rightarrow \infty$, is the option's value, ϕ. The approximation to ϕ is then given by

(5) $$P = P(3) + 3.5 * [P(3) - P(2)] - 0.5 * [P(2) - P(1)],$$

(see Geske and Johnson, (1984), pp. 1518 and 1523).

IV. The Accelerated Binomial Option Pricing Model

The sequence of functions,

(6) $$P_a(S) = \sum_{j}^{N-a} \binom{N-a}{j} (p/q)^j((1-p)/q)^{N-a-j} V_{j,N-a},$$

where V is defined as earlier, converges to $P_N(S)(= V_{00})$ for any binomial model with N periods. Convergence is uniform from below and occurs at $P_m(S)$, where

m is the earliest period in the model for which V_{0m} takes its immediate exercise value rather than its holding value (Breen (1988)). m will always be less than N unless the put should be exercised immediately. In other words, for a binomial option pricing model with fixed N, the value of the option is the limit of the sequence $P_a(S)$. $P_a(S)$ defines a sequence of option values with an increasing number of exercise opportunities. Thus, $P_0(S)$ is a European option, permitting exercise only at period N. $P_1(S)$ is the value of an option permitting exercise at period N and period $N-1$; and so on. That the sequence converges to the option's value is true by definition ($P_N(S) = V_{00}$). That convergence is from below is intuitively clear, insofar as, if this were not so, $P_j(S) > P_N(S)\,(j < N)$ and $P_j(S)$ would be the option's value. But this would imply that exercise opportunities in periods earlier than period $N-j$ would reduce the value of the option—an obvious contradiction.

Consider now the related sequence, $P'_n(S)$, or $P'(n)$ for short, defined as follows: $P'(1) = P_0(S)$; $P'(2)$ = binomial option value permitting exercise at N and $N/2$ only; $P'(3)$ = binomial option value permitting exercise at N, $2N/3$, and $N/3$ only. Again, this sequence converges to the option's value, $P_N(S)$ from below—that is, $P_N(S)$ is the limit of the sequence $P'(n)$ as $n \rightarrow N$. It follows too that $P'(2) > P'(1)$ and that $P'(3) > P'(1)$, though not necessarily that $P'(3) > P'(2)$, although, in practice, this usually seems to be the case.[1]

To apply the Richardson extrapolation technique to the binomial, we proceed by analogy with Geske and Johnson's exposition. The parallels between the sequences P and P' are clear: in both, the number of exercise opportunities increases as we move down the sequence. Thus, we apply Geske and Johnson's Formula (5) to the terms $P'(n)$, $n = 1, 2, 3$. The value of the option is then given as $\max(P', X - S)$.

In practical terms, the resulting accelerated binomial model is very easy to program. To give some idea of its accuracy we refer first to Table 1, where three sets of American option values for the data originally given by Cox and Rubinstein (1985) are shown. These three sets of values are based on, respectively, the unmodified binomial (which we denote by B) with 150 periods; the Geske-Johnson "analytic" (G-J) method using a four-point extrapolation; and the accelerated binomial (AB) method presented here, using a three-point extrapolation over a 150-period model. All three sets of values agree very closely. The largest error in the AB method is of the order of one and a half cents compared with the B values. Clearly a four-point extrapolation would be more accurate.

What is most striking about the AB method, however, is the reduction it brings about in the amount of computation required. Recall that for each nonterminal node in the binomial tree we must calculate both a holding and an

[1]As with the sequence $P(n)$, this means that $P'(n)$ does not converge uniformly to its limit. In discussing the Geske-Johnson model, Omberg ((1987), pp. 463–464) has suggested that uniform convergence of the sequence $P(n)$ would be desirable on the grounds that this would ensure that the convergence acceleration technique performs as intended. For both the sequences $P(n)$ and $P'(n)$, this could be accomplished by ensuring that each term in the sequence permits early exercise at every instant (or period, in the case of $P'(n)$) at which exercise was permitted in forming earlier terms. Thus, the term $P(3)$ in the Geske-Johnson sequence would be amended to permit exercise at T, $3T/4$, $2T/4$, and $T/4$—and analogously for $P'(3)$. In what follows, however, we retain the original specifications of the terms of P and P'.

TABLE 1

Values of American Put Option Using Binomial, Geske-Johnson, and Accelerated Binomial Methods (S = \$40; r = 1.05)

X	σ	T	Binomial	Geske-Johnson	Accelerated Binomial
35.00	0.2000	0.08330	0.01000	0.006200	0.006000
35.00	0.2000	0.33330	0.20000	0.199900	0.198900
35.00	0.2000	0.58330	0.43000	0.432100	0.433800
40.00	0.2000	0.08330	0.85000	0.852800	0.851200
40.00	0.2000	0.33330	1.58000	1.580700	1.574000
40.00	0.2000	0.58330	1.99000	1.990500	1.984000
45.00	0.2000	0.08330	5.00000	4.998500	5.000000
45.00	0.2000	0.33330	5.09000	5.095100	5.102000
45.00	0.2000	0.58330	5.27000	5.271900	5.285000
35.00	0.3000	0.08330	0.08000	0.077400	0.077400
35.00	0.3000	0.33330	0.70000	0.696900	0.698500
35.00	0.3000	0.58330	1.22000	1.219400	1.224000
40.00	0.3000	0.08330	1.31000	1.310000	1.309000
40.00	0.3000	0.33330	2.48000	2.481700	2.476000
40.00	0.3000	0.58330	3.17000	3.173300	3.159000
45.00	0.3000	0.08330	5.06000	5.059900	5.063000
45.00	0.3000	0.33330	5.71000	5.701200	5.698000
45.00	0.3000	0.58330	6.24000	6.236500	6.239000
35.00	0.4000	0.08330	0.25000	0.246600	0.245000
35.00	0.4000	0.33330	1.35000	1.345000	1.350000
35.00	0.4000	0.58330	2.16000	2.156800	2.159000
40.00	0.4000	0.08330	1.77000	1.767900	1.766000
40.00	0.4000	0.33330	3.38000	3.363200	3.383000
40.00	0.4000	0.58330	4.35000	4.355600	4.339000
45.00	0.4000	0.08330	5.29000	5.285500	5.287000
45.00	0.4000	0.33330	6.51000	6.509300	6.505000
45.00	0.4000	0.58330	7.39000	7.383100	7.382000

For the binomial and accelerated binomial, N = 150. Geske-Johnson value is based on four-point extrapolation; Accelerated Binomial value is based on three-point extrapolation. Geske-Johnson values are from Geske and Johnson ((1984), p. 1519). Binomial values are from Cox and Rubinstein ((1985), p. 248).

exercise value, and for each terminal node (at period N), we calculate only an exercise value. The B method, therefore, requires the calculation of $(N + 1)^2$ node values, which, for $N = 150$, is 22801. The AB model, on the other hand, requires only $4N + 10$ node calculations—610 for a 150-period model.

A second source of comparison will be found in Table 2, which shows put option values calculated using the AB method together with values obtained using three other methods—the finite difference method, the G-J method, and Macmillan's (1986) quadratic approximation. Comparing the three other methods against the finite difference values, we can see that there is little to choose between them, although the AB and G-J values are, if anything, marginally more accurate than Macmillan's. A similar conclusion is reached if we compare the AB values in the present Table 1 with those given by Macmillan ((1986), pp. 131–132) for the same data using his own method.

TABLE 2

Values of American Put Option Using Finite Difference, Geske-Johnson, Macmillan's Quadratic Approximation, and Accelerated Binomial Methods (X = 100)

r	σ	t	S	FD	GJ	MQ	AB
1.08	0.2	0.25	80	20.00	20.00	20.00	20.00
1.08	0.2	0.25	90	10.04	10.07	10.01	10.06
1.08	0.2	0.25	100	3.22	3.21	3.22	3.22
1.08	0.2	0.25	110	0.66	0.66	0.68	0.66
1.08	0.2	0.25	120	0.09	0.09	0.10	0.09
1.13	0.2	0.25	80	20.00	20.01	20.00	20.01
1.13	0.2	0.25	90	10.00	9.96	10.00	10.00
1.13	0.2	0.25	100	2.92	2.91	2.93	2.92
1.13	0.2	0.25	110	0.55	0.55	0.58	0.56
1.13	0.2	0.25	120	0.07	0.07	0.08	0.07
1.08	0.4	0.25	80	20.32	20.37	20.25	20.36
1.08	0.4	0.25	90	12.56	12.55	12.51	12.56
1.08	0.4	0.25	100	7.11	7.10	7.10	7.09
1.08	0.4	0.25	110	3.70	3.70	3.71	3.70
1.08	0.4	0.25	120	1.79	1.79	1.81	1.80
1.08	0.2	0.50	80	20.00	19.94	20.00	20.00
1.08	0.2	0.50	90	10.29	10.37	10.23	10.37
1.08	0.2	0.50	100	4.19	4.17	4.19	4.17
1.08	0.2	0.50	110	1.41	1.41	1.45	1.40
1.08	0.2	0.50	120	0.40	0.40	0.42	0.40

FD = finite difference method.
GJ = Geske-Johnson method, based on three-point extrapolation.
MQ = Macmillan's quadratic approximation.
AB = accelerated binomial with N = 150 and three-point extrapolation.

Columns 1 to 7 from Barone-Adesi and Whaley ((1987), Table 4, p. 315).

V. Dividends and Multiple State Variables

The binomial is an efficient means of valuing American calls on dividend paying stock because, as is well known, the early exercise value of the option need only be computed at each ex-dividend date. However, because American put options generally have a positive probability of premature exercise, whether the stock pays dividends or not, the early exercise condition must be checked at all times during the option's life. Unfortunately, the presence of discrete dividend payments excludes, in both the discrete time and continuous time contexts, the application of the three- or four-point Richardson extrapolation technique. This is because the dividend payments' effect on the stock price makes it highly unlikely that sequences such as $P(n)$ and $P'(n)$ will converge to the appropriate limits.

Geske and Johnson ((1984), pp. 1520–1521) surmount this difficulty in a manner that can be extended to the binomial method. They argue as follows. Let $P(D)$ represent the estimated value of a put on a stock that pays a dividend D. Define D' to be the smallest dividend that precludes early exercise at all times prior to the last dividend payment before the option expires. Then, to find the value $P(D)$, they employ the simple linear interpolation,

$$P(D) = P(0) + \frac{D}{D'}[P(D') - P(0)]. \tag{7}$$

Geske and Johnson ((1984), p. 1521) report that this yields very accurate estimates of $P(D)$ when compared with the unmodified binomial method.

To adapt this approach to the accelerated binomial, we must replace the assumption of a fixed cash dividend with a fixed percentage dividend. This is because fixed cash dividends cause considerable computational awkwardness for the binomial method. However, Geske and Shastri ((1985), pp. 62, 64–65) show that if the cash dividend amount is replaced by a percentage dividend calculated as the cash dividend amount expressed as a percentage of the initial stock price, the resulting option values are virtually identical in either case. In Geske and Shastri (1985) this is termed a "fixed dividend yield." In the example reported in Table 3, the fixed 50-cent discrete dividend payments used in the G-J model were replaced, in the B and AB models, by discrete dividend payments of 1.25 percent of the current stock price.

In the case of the binomial, then, we write $P(\delta)$ to be the estimated value of a put on a stock paying a dividend of δ percent. Likewise, δ' is the smallest percentage dividend that ensures no early exercise prior to the last ex-dividend date. Our equation for the value of an American put on dividend paying stock is thus

$$P'(\delta) = P'(0) + \frac{\delta}{\delta'}[P'(\delta') - P'(0)]. \tag{8}$$

We write $P'(0)$ because this is estimated using the method set out in Section IV, while the estimation of $P'(\delta')$ entails the application of the convergence acceleration procedure to the periods of the model that lie between the final ex-dividend date and period N. If the reported dividend percentage exceeds the critical dividend percentage, δ', then we replace δ' with δ in Equation (8). In other words, we estimate the option's value without recourse to the interpolation.[2] The accuracy of this method can be seen in Table 3, which reports option values obtained by this approach, by the Geske-Johnson approach, and by the application of the unmodified binomial.

A second extension of the accelerated binomial is to options whose value depends on two state variables (see Boyle (1988) for a discussion of these). In such cases, the binomial can be replaced by a multinomial distribution of joint asset prices. An immediate consequence, however, is to greatly increase the computational requirements. Consider, for example, a fourth order multinomial modeling the joint distribution of assets $S1$ and $S2$. From any node, the multinomial process can then move to any one of four nodes in the next period, corresponding to each of the possible pairwise combinations of the jump parameters $u1$, $d1$, $u2$, $d2$. Thus, the total number of node values to be calculated in an N period model of this type will be

$$\frac{1}{2}\sum_{n=0}^{N}\sum_{m=0}^{n}(m+1)(m+2).$$

[2]It is also easy to illustrate how the direct application of the accelerated binomial, as set out in Section IV, would lead to inaccuracy in such instances. Assume that the final dividend payment occurred at a period of the model later than the $2N/3$ period. In this case, the values $P'(n)$ for $n = 1, 2, 3$ would all be equal to the binomial value of the corresponding European put, leading to an underestimate of the option's value.

TABLE 3

Values of American Put Option Adjusted for Dividends, Using Binomial, Geske-Johnson, and Accelerated Binomial Methods (S = \$40; r = 1.05)

X	σ	T	Binomial	Geske-Johnson	Accelerated Binomial
35.00	0.2	0.0833	0.01	0.012	0.011
35.00	0.2	0.3333	0.31	0.307	0.307
35.00	0.2	0.5833	0.66	0.658	0.657
40.00	0.2	0.0833	1.11	1.108	1.110
40.00	0.2	0.3333	2.01	2.012	2.010
40.00	0.2	0.5833	2.58	2.572	2.574
45.00	0.2	0.0833	5.41	5.421	5.413
45.00	0.2	0.3333	5.67	5.690	5.688
45.00	0.2	0.5833	6.02	6.030	6.027
35.00	0.3	0.0833	0.11	0.107	0.108
35.00	0.3	0.3333	0.88	0.884	0.884
35.00	0.3	0.5833	1.55	1.545	1.548
40.00	0.3	0.0833	1.56	1.559	1.559
40.00	0.3	0.3333	2.91	2.907	2.908
40.00	0.3	0.5833	3.74	3.744	3.739
45.00	0.3	0.0833	5.50	5.500	5.497
45.00	0.3	0.3333	6.29	6.309	6.293
45.00	0.3	0.5833	6.99	6.998	6.994
35.00	0.4	0.0833	0.31	0.305	0.306
35.00	0.4	0.3333	1.58	1.580	1.579
35.00	0.4	0.5833	2.52	2.528	2.524
40.00	0.4	0.0833	2.01	2.012	2.015
40.00	0.4	0.3333	3.81	3.803	3.811
40.00	0.4	0.5833	4.92	4.912	4.914
45.00	0.4	0.0833	5.70	5.702	5.701
45.00	0.4	0.3333	7.07	7.077	7.076
45.00	0.4	0.5833	8.10	8.091	8.090

Binomial values from Cox and Rubinstein ((1985), p. 249). Geske-Johnson values from ((1984), p. 1521). Dividends are payable in ½, 3½, and 6½ months. Dividend is fixed yield of 1.25 percent for binomial and accelerated binomial, 50-cent dividend for Geske-Johnson.

As a result, even a 50-period model will be very time consuming to estimate. However, the arguments developed in Section IV in respect of the binomial are directly extendable to the more general multinomial case, and thus the convergence acceleration technique can be applied to value options that depend on two (or more) state variables. Table 4 illustrates that the accelerated multinomial is as accurate as the unmodified multinomial approach in valuing American put options on the minimum of two assets.[3]

VI. Computational Efficiency

We have already seen that the accelerated binomial method entails an enormous reduction in the number of nodes of the binomial tree that must be valued. We now turn briefly to the question of how much this reduces the computational requirements of option valuation. Geske and Shastri (1985) have previously compared the efficiency of a number of option pricing techniques: the current

[3]The first three option values in Table 4 can be compared with those given by Boyle ((1988), p. 11).

TABLE 4

Values of American Put Option on the Minimum of Two Assets Using Multinomial and Accelerated Multinomial Methods ($S_1 = S_2 = \$40.00; r = 1.05; \sigma_1 = 0.2; \rho = 0.5$)

X	σ_2	T	Multinomial	Accelerated Multinomial
35.00	0.3	0.5833	1.424	1.425
40.00	0.3	0.5833	3.886	3.883
45.00	0.3	0.5833	7.685	7.678
35.00	0.4	0.3333	1.423	1.422
40.00	0.4	0.3333	3.880	3.875
45.00	0.4	0.3333	7.836	7.833
35.00	0.5	0.0833	0.497	0.496
40.00	0.5	0.0833	2.486	2.485
45.00	0.5	0.0833	6.862	6.862

Multinomial and Accelerated Multinomial use $N = 60$. The latter uses three-point extrapolation.

exercise is much more restricted inasmuch as we confine comparisons to three methods. These are the Black-Scholes (1973) analytic formula for the European put; the binomial; and the accelerated binomial. Both the B and AB methods are applied to value American puts.

We measure efficiency in terms of CPU time required to run the relevant compiled FORTRAN program on a Microvax II. As Geske and Shastri ((1985), p. 46) note, the results will be sensitive to the particular hardware and/or software employed. In addition, tests of efficiency may yield different results depending on the use we want to make of an option valuation program. In order to deal, at least partially, with this latter issue, we make the following sets of comparisons.

(a) For put options on nondividend paying stocks, we report

(i) the average CPU time per single option valuation for the set of options shown in Table 1;

(ii) the average CPU time required to value a series of options on the same asset with a range of ten exercise prices. This average is taken over ten such series.

(b) For put options on dividend paying stocks, we carry out the same set of comparisons.

(c) For options with two state variables, we present the average CPU time per single valuation of the American put options on the minimum of two assets shown in Table 4.

The results of these comparisons are given in panels A, B, and C of Table 5. We can summarize these as follows. For a single put option on nondividend paying stock, the AB method (with $N = 150$) is 16 percent slower than the Black-Scholes (B-S) formula[4] in valuing the corresponding European put. The B method is 46 percent slower than B-S. For a single option, as the number of periods in the model increases from $N = 150$ to 300, the absolute difference

[4]Here, we use a polynomial of degree 5 to approximate the normal integral.

TABLE 5

Average CPU Time (in Minutes, Seconds, Hundredths of Seconds) Required to Value Options

Panel A. Puts on Nondividend Paying Stock

	European	American (*N* = 150)		American (*N* = 300)	
	Black-Scholes	Binomial	Accelerated Binomial	Binomial	Accelerated Binomial
One Put Option[a]	0:05.04	0:07.35	0:05.82	0:15.09	0:10.70
Series of 10 Puts with Varying Exercise Price[b]	0:05.13	0:26.84	0:18.29	0:27.45	0:18.46

Panel B. Puts on Dividend Paying Stock

	European	American (*N* = 150)	
	Black-Scholes	Binomial	Accelerated Binomial
One Put Option[c]	0:05.07	0:07.45	0:06.60
Series of 10 puts with Varying Exercise Price[d]	0:05.19	0:28.13	0:19.84

Panel C. Puts on Minimum of Two Assets

	American (*N* = 60)	
	Multinomial	Accelerated Multinomial
One Put Option[e]	4:14.01	0:58.22

[a] Average time over 27 options shown in Table 1.
[b] Average time over 10 series of 10 puts each.
[c] Average time over 27 options shown in Table 3.
[d] Average time over 10 series of 10 puts each.
[e] Average time over 9 options shown in Table 4.

between B and AB models increases from 1.53 seconds (7.35–5.82) to 4.39 seconds (15.09–10.70); however, the relative difference remains about the same.[5]

The accelerated binomial values a series of options more efficiently than the unmodified binomial. The average CPU time per series of ten options for the AB method is about two-thirds of that required for the B method. The AB model estimates only five vectors of values of the stock price—at N, $2N/3$, $N/2$, $N/3$, and 0. Thus, it is feasible to hold these in memory once they have been estimated and to call them when we wish to value the same option with a different exercise price (or, with a slight modification, when we wish to alter the initial stock price). This is not feasible for the B method which calculates $N + 1$ vectors of stock prices.

Of course, most equity options are written on dividend paying stocks. Nevertheless, the foregoing results are important because they are indicative of the results that are obtained when this method is applied to options on other assets

[5]Using a PC with 80287 co-processor, the differences are greater. Broadly speaking, the average elapsed time taken to value one put option with $N = 150$ is about four times greater for the binomial than for the accelerated binomial.

that do not pay discrete dividends (such as foreign exchange options, options on futures, and so forth).

For put options on dividend paying stock, the AB model outperforms the B model by a small margin in the valuing of a single put option. The gap between the two, however, is more substantial in the valuing of a series of options.

Finally, in Panel C of Table 5, we compare the CPU times for a multinomial model with those for the corresponding accelerated multinomial. The relatively large computing requirements of these models are evident: equally evident is the considerable reduction in CPU time brought about by the use of the acceleration technique.

VII. Conclusion

The only previous attempt to investigate the application of convergence acceleration techniques to the binomial option pricing model is contained in a paper by Omberg (1987). His approach, however, differed from the present one. Whereas we have shown how to accelerate the convergence of a particular binomial model with fixed N, Omberg sought to find a means by which to accelerate the convergence of a sequence of binomial option models with increasing N. However, such a sequence converges in an oscillating, rather than uniform, manner, and Omberg showed that it was impossible to select the parameters of the binomial model in such a way as to ensure uniform convergence.

The accelerated binomial model presented here is easy to program and returns accurate option values when compared with other methods, including the widely used binomial. Tests of the accelerated binomial's efficiency show that it is more efficient than the binomial model. Our results indicate that the more computationally burdensome are the requirements of the unmodified binomial (or multinomial) model, the greater will be the improvement in efficiency arising from the use of the accelerated binomial (or accelerated multinomial). Finally, because our model is a modification to the binomial, it retains all the flexibility of the latter. Thus, the accelerated binomial can be used to value all the variety of options (on foreign exchange, commodities, futures, stocks that pay dividends, and so on) for which the binomial itself is applicable. In addition, it can be extended to price options whose value depends on two (or more) state variables.

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