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An Empirical Examination of the Pricing of American Put Options

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Abstract

This study is an *ex post* performance test comparing the accuracy of an American model to a European model for valuing listed options. Specifically, the Geske and Johnson American put valuation model is compared with the Black and Scholes European put model. On average, both models undervalue, relative to market prices, put options. However, the Geske and Johnson model values are significantly closer to market prices than are the Black and Scholes values.

I. Introduction

The development of models capable of valuing American put options has occurred only relatively recently. The valuation model for the European put option was derived by Black and Scholes (1973) for the case in which the stock price follows geometric Brownian motion. Numerical solutions to the American put value were produced by Parkinson (1977), Brennan and Schwartz (1977), and Cox, Ross, and Rubinstein (1979). Geske and Shastri (1985b) compared the efficiency of these numerical approaches, with and without dividends, and explained why an analytic solution may be more efficient than those numerical approaches in terms of the number of critical stock price calculations required for penny accuracy. Johnson (1983) developed an analytic approximation, which was extended to handle dividends by Blomeyer (1986), but it cannot be made arbitrarily accurate. In a recent paper, Geske and Johnson (1984) derived an analytic solution to the American put value and demonstrated that their values are virtually identical to the Cox, Ross, and Rubinstein values. Although Geske and Johnson (GJ) required numerical procedures to evaluate their analytic formula, the computational expense for this method is lower than that for numerical approaches because far fewer critical stock prices need be computed for penny accuracy. As pointed out in GJ (1984) and Geske and Shastri (1985b), the numerical approaches would presumably yield equally accurate prices. Very recently,

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Omberg (1986) developed a method, based on a suboptimal exercise policy, for valuing American puts. Omberg maintained that the GJ interpolation scheme for handling dividends can produce large errors. However, Omberg considered a range of parameter values so wide that a large number of the cases he considered would never be encountered in any real data samples. As we shall show, for reasonable parameter values, the GJ method seems to be sufficiently accurate (see also GJ (1984), p. 1521).

Very little testing of put pricing models has been reported in the literature. Parkinson (1977) compared his numerical solutions with over-the-counter put quotations from the *New York Times* and found that the American put model undervalued puts by a few percentage points. Brennan and Schwartz (1977) compared their numerical solutions with prices for 55 puts traded in the New York dealer market from May 1966 to May 1969. They reported the American put model undervalued puts by 25 to 40 percent. Severe data restrictions limited such early studies. With the advent of trading in listed options on the Chicago Board Options Exchange (CBOE), the development of the Berkeley Options Data Base, and the derivation of the faster GJ technique for valuing American puts, it recently has become possible to use a huge amount of reliable market data, free of nonsynchronous trading problems, to test market efficiency and the accuracy of the put valuation formula. Shastri and Titman (1984) have recently reported preliminary evidence on the efficiency issue, while this study examines the accuracy issue.

This study is a comparative *ex post* performance evaluation of the GJ approach and the Black and Scholes (BS) European model using CBOE listed put option transactions. Of course, we would expect the GJ approach, which explicitly allows for the possibility of early exercise, to be superior to the BS model, which does not. However, sometimes traders believe that models determine market prices simply by being popular and frequently used. Since the BS model may currently be the most widely used, this paper provides a useful test of this argument. More importantly, this paper shows how well the GJ approach (or any equivalent numerical approach) values puts. A total of 10,295 put option transactions, occurring during 13 trading days chosen from the months of June through August 1978, are examined in this study. Not only does this study use an enormous amount of data, free of the nonsynchronous trading problem, but, since the GJ technique can allow explicitly for dividends, this study includes long-lived puts on dividend-paying stocks. The degree to which the model values differ from the market prices is compared to the degree to which the options are in or out of the money and the time remaining to option maturity. This pricing bias is reported in graphical and tabular form.

In Section II, we briefly discuss the GJ model and the effects of cash dividends on American put option pricing. The empirical methodology is discussed in Section III, and the results are presented in Section IV. Section V is a summary.

II. Geske-Johnson Model

This study uses the theoretical model developed by GJ (1984). In evaluating their analytic formula, we use their sequential extrapolation from four exercise points to the infinite limit, which yields the following approximation to their solution

$$(1)\quad P = P_4 + 29/3\left[P_4 - P_3\right] - 23/6\left[P_3 - P_2\right] + 1/6\left[P_2 - P_1\right],$$

where P is the American put price, P_1 is the price of a put that can only be exercised at maturity (i.e., a European put), P_2 is the price of a put that can be exercised halfway to maturity or at maturity, and P_3 and P_4 are the prices of puts that can be exercised at three and four equally spaced points, respectively. Equation (1) is given in footnote 6, page 1520 of GJ (1984). If there are dividends during the life of the put, we use the GJ interpolation procedure, which modifies both their formula and the evaluation technique for the dividends. For dividends above some critical value, the put can be valued directly. For dividends below this critical value, linear interpolation is performed between the value for a put with no dividend and the value for a put with the critical dividend. See GJ (1984) for more details.

The BS European put values are adjusted for dividends, when they are paid during the life of the option, by subtracting the discounted value of the dividends from the stock price and assuming that the put can only be exercised at maturity. The existence of cash dividends should reduce the magnitude of the difference between the BS European put values and the GJ American put values because cash dividends reduce the probability of early exercise of the American put option. This point was demonstrated first by Geske and Shastri (1985a) and is illustrated in Figure 1 where GJ and BS values are plotted for options written on stocks with and without cash dividends. These options have maturity ranges of one to nine months and are in the money with a stock price of \$40 and an exercise price of \$45. The cash dividends are \$1 and the stock goes *ex dividend* in ½, 3½, and 6½ months. A large dividend was chosen to accentuate the effects of dividends.

The top two plots in Figure 1 are for puts written on stocks with dividends, and the bottom two plots are for puts on stocks without dividends. As expected, the relative (but not necessarily absolute) difference in GJ and BS values is greater for the no dividend case because dividends tend to delay exercise and therefore make the early exercise feature less valuable. The BS plot for the dividend case has negatively sloped segments because the particular puts considered are deep in the money, and the loss from having to wait an extra month to exercise (the interest effect) dominates the gain from having more time to maturity (the volatility effect). As the time to maturity increases past an *ex dividend* date, the put value rises because the stock price is expected to fall on the *ex dividend* day. The European puts for the non-dividend case are not as deep in the money because there are no *ex dividend* dates on which the stock price is expected to fall. For the parameters used in Figure 1, these European put values happen to increase monotonically with time to maturity.

III. Methodology

All put option transactions occurring during the months of June through Au-

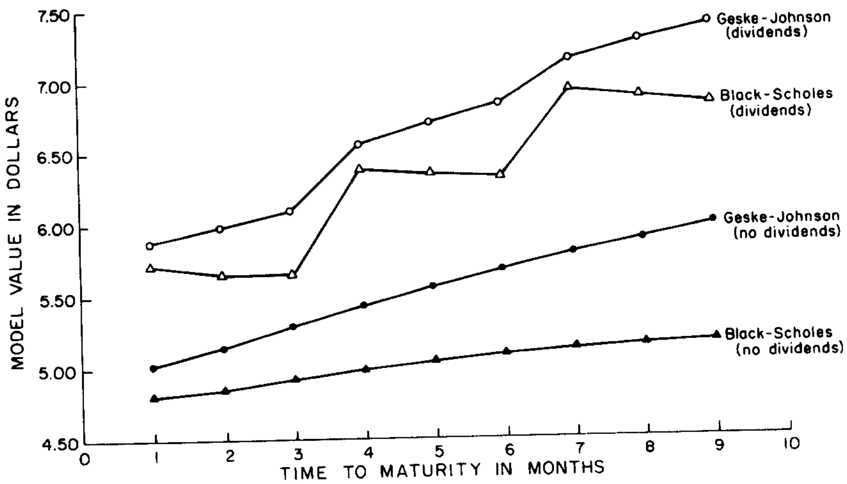

FIGURE 1

Model Values for Put Options Written on Stocks with and without Dividends
$S = \$40, X = \$45, r = 10\%, \sigma = 0.3,$
$D = \$1, T_D = ½, 3½, 6½$ months

gust 1978 on the CBOE for Avon Products (AVP), Eastman Kodak (EK), General Motors (GM), and Honeywell (HON) were obtained from the Berkeley Options Data Base.[1] From this data set, one trading day per week was randomly chosen for evaluation, which resulted in a total of 13 trading days. Options that were out of the money by more than \$5.00 with premia of less than fifty cents were in a restricted trading status and were excluded from the sample. After filtering the data, 10,295 option transactions remained for evaluation. These options had a median time to maturity of 80 days and a maturity range of 5 to 270 days. Using the GJ interpolation scheme to handle multiple cash dividends, we are able to examine long-lived options.

The risk-free bond equivalent yield was calculated from the *Wall Street Journal* quotation for a Treasury bill maturing at approximately the same time as the option expiration date. Dividend information was obtained from *Moody's Dividend Record*. The stock price used in the BS and GJ models was the market price of the stock, net of any dividends discounted at the risk-free rate from the transaction observation date to the *ex dividend* date. The stock return standard deviation estimates were calculated using the Parkinson (1980) extreme value method with weekly stock price ranges obtained from *Barron's*. Each stock return volatility was calculated using stock price range data from the 20 weeks immediately preceding the week of the option transaction.[2]

[1] During this time period, put options were also traded on IBM stock. These options were not examined in this study for two reasons. First, the trading volume of IBM options was about equal to the combined volume of options written on the other four companies, and inclusion of the IBM options would place undue emphasis on the options of one company. Secondly, this study examines the absolute difference between model and market values. During this study period, IBM stock traded in a range of \$255 to \$301, and the equities of the other four companies traded in the range of \$51 to \$73. Consequently, due to the very large premia, the IBM options would completely dominate the graphs and tables.

[2] Our choice of a 20-week estimation period is somewhat arbitrary. We have used data that

The *ex post* performance of each model is evaluated by comparing Y, the degree to which the market and model prices differ, to Z, the degree to which the option is in or out of the money, where

$$Y = P_{\text{market}} - P_{\text{model}} \tag{2}$$

$$\text{and } Z = \left[\frac{X - S}{X}\right] \times 100\% . \tag{3}$$

We chose to define Y in absolute rather than relative terms because the relative bias can be enormous for very low-priced puts. The absolute difference gives a better indication of whether or not price deviations are economically significant, i.e., greater than transactions costs. This is because transactions costs are not simply a percentage of the amount bought or sold, but contain a fixed element as well. Specifically, the bid-ask spread does not become negligible as the price of the put gets small.

IV. Results

Using transaction-by-transaction data, we compare the BS European put values and the GJ American put values to market prices. In Figure 2, median values of the pricing bias Y are plotted for two percent ranges of Z, the degree to which the option is in ($Z \geq 0$) or out of the money ($Z < 0$). For in-the-money options, the median pricing bias for the GJ model is substantially lower than the median BS model pricing bias. Due to early exercise, none of the options is in the money by more than 17.5 percent. For out-of-the-money options, the differences between the two model pricing biases are small because of the decreased probability of early exercise of these options.

In the upper part of Table 1, median and mean values of Y are reported for each model for all options, for out-of-the-money options, and for in-the-money options. Within each cell in Table 1, median values of Y are reported with mean values reported directly below these median values. The differences between mean values is reported in the column headed by "ΔY" with t statistics in parentheses. All mean values reported in Table 1 are significant at the 1 percent level. Since the mean values can be influenced strongly by a few outliers, the median value may be a better measure of the model pricing bias.

Examining the upper part of the table, the cell in the upper left-hand corner at the intersection of the row and column labeled "all options" reports the pricing bias for the GJ model and the BS model for all 10,295 option transactions. The median pricing bias of 38.6 cents for the GJ model and 46.7 cents for the BS model indicates that both models tend to undervalue put options, with the GJ values being closer, on average, to market values than the BS values. The values

would have been available to the option trader. For comparison, we also calculated stock return standard deviations using stock price ranges from the 10 weeks preceding and the 10 weeks following the observation week. Although there were small differences between these two methods for each company's volatility estimates for each of the 13 trading weeks, the average volatility for each company was virtually identical for the two methods. Additionally, the average of each company's volatility estimates was very close to the stock return standard deviations calculated by Fischer Black for July 1978 and reported in Cox and Rubinstein (1985).

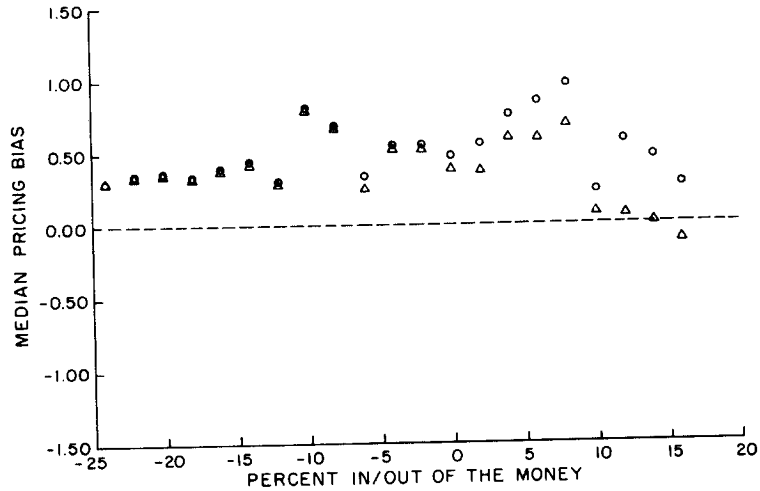

FIGURE 2

Median pricing bias for the Black-Scholes model (*) and the Geske-Johnson model (Δ). Each median pricing bias value is calculated for a 2-percent range of the degree to which the option is "in" or "out of the money." The symbols plotted in this graph are centered within this range.

TABLE 1

Summary of Differences between Market Prices and Model Prices[a]
Median and Mean Values of $Y = P_{market} - P_{model}$

	All Options				Out of the Money				In the Money			
	Geske-Johnson	Black-Scholes	ΔY	N	Geske-Johnson	Black-Scholes	ΔY	N	Geske-Johnson	Black-Scholes	ΔY	N
All Options	0.386	0.467		10295	0.412	0.457		7223	0.313	0.493		3072
	0.408	0.503	0.095		0.439	0.484	0.045		0.336	0.547	0.211	
	(23.8%)	(26.5%)	(18.79)		(32.6%)	(35.3%)	(9.16)		(7.5%)	(11.4%)	(17.41)	
Options Written on												
AVP	0.477	0.560		1595	0.441	0.481		1197	0.590	0.747		398
	0.486	0.567	0.081		0.469	0.502	0.033		0.536	0.764	0.228	
			(7.94)				(3.70)				(7.88)	
EK	0.487	0.576		4107	0.487	0.546		3051	0.491	0.703		1056
	0.499	0.584	0.085		0.507	0.546	0.039		0.479	0.694	0.215	
			(11.77)				(5.37)				(12.10)	
GM	0.448	0.535		2010	0.513	0.556		1224	0.141	0.404		786
	0.400	0.550	0.150		0.518	0.589	0.081		0.216	0.474	0.258	
			(13.18)				(6.69)				(12.67)	
HON	0.228	0.268		2583	0.225	0.248		1751	0.249	0.378		832
	0.223	0.297	0.074		0.246	0.284	0.038		0.174	0.324	0.150	
			(7.11)				(4.28)				(5.69)	

[a] Median values of Y are reported with mean values of Y below the median values. All mean values of Y are significant at the 1-percent level. The values in brackets are median values of Y expressed as a percent of the market price. The column headed by "ΔY" contains differences between mean values with t values in parentheses. The column labeled "N" is the number of observations.

reported in parentheses are median values of the pricing bias, Y, expressed as a percent of the market price. For example, the median degree of undervaluation for the GJ model is 23.9 percent. Note, however, that nearly 70 percent of our sample consists of out-of-the-money puts, where the error is relatively large. The in-the-money puts must have values close to $X - S$, and so are easier to value than the out-of-the-money puts, which are not similarly constrained. For all out-of-the-money options, the median pricing bias of 41.2 cents for the GJ model is 4.5 cents lower than the BS model median pricing bias. For all in-the-money options, there is a substantial difference between the pricing biases of the two models. The median values of Y are 31.3 cents and 49.3 cents for the GJ and BS models, respectively.[3] The absolute difference between the GJ and BS models is largest for the in-the-money options, as expected. In the lower part of Table 1, the model pricing biases are reported separately for options on each of the four companies. In every case, the mean and median biases are lower for the GJ values than for the BS values.

The probability of early exercise of an American put is positively related to the degree to which the option is in the money and to the time to option maturity, and negatively related to the magnitude of cash dividends with *ex dividend* dates prior to the expiration of the puts. Consequently, the magnitude of the difference in pricing bias between the European put and American put models should be large for in-the-money, long term to maturity options, and should decrease as cash dividends are introduced. We examine these conjectures in Section I of Table 2 by stratifying the options into four maturity ranges of 45 days each and a fifth range for options with maturities of greater than 180 days. Examination of the reported pricing biases in this section indicates that median and mean values of Y are lower for the GJ model over all maturity ranges for both in- and out-of-the-money options. For in-the-money options, there is a substantial difference between the pricing biases of the two models. For options with maturities of less than 45 days, the size of the pricing bias is substantially lower than the size of the bias for options with longer maturities. Also, for options with maturities greater than 45 days, there does not appear to be a systematic relationship between the time to option maturity and the magnitude of the pricing bias. However, the size of the GJ pricing bias does appear to become a smaller proportion of the BS pricing bias as the maturity increases. Dividends play a part in explaining these biases.

All options with maturities of less than 60 days were written on stocks that did not go *ex dividend* prior to the option expiration date, and all options with maturities greater than 88 days were on stocks with dividends. Within the maturity range of 60 to 88 days, the options were written on stocks with and without dividends, and we examine these options in the last two sections of Table 2. In Section II of Table 2, for options on stocks without dividends, the mean value of Y for the GJ model of 36.9 cents is 82 percent of the BS value of 45 cents. For options on stocks with dividends, in Section III, the mean GJ model bias of 63.4 is 89 percent of the BS bias. When broken down by in and out of the money, the GJ bias is also proportionally smaller than the BS bias for options on stocks without dividends than for options on stocks with dividends. These findings provide

[3] Our reported pricing biases may appear relatively large when compared with previous studies of call option pricing model performance that have used an implied variance estimation technique (e.g., MacBeth and Merville (1979)). By construction, an implied variance technique will provide a small model pricing bias since the model is used to estimate one of its own input parameters. Additionally, the magnitude and direction of the reported pricing bias will be influenced by the choice of the implied variance weighting technique.

TABLE 2

Summary of Differences between Market Prices and Model Prices for Options with Different Times to Maturity[a]
Median and Mean Values of $Y = P_{market} - P_{model}$

	All Options				Out of the Money				In the Money			
	Geske-Johnson	Black-Scholes	ΔY	N	Geske-Johnson	Black-Scholes	ΔY	N	Geske-Johnson	Black-Scholes	ΔY	N
I. *Time to Maturity*[b]												
Less than 45 Days	0.197	0.227		2581	0.219	0.221		1486	0.125	0.259		1095
	0.177	0.220	0.043		0.217	0.222	0.005		0.122	0.217	0.095	
			(8.15)				(0.88)				(9.73)	
45 to 89 Days	0.541	0.608		3639	0.547	0.583		2575	0.521	0.702		1064
	0.524	0.599	0.075		0.539	0.565	0.026		0.487	0.682	0.195	
			(9.05)				(3.49)				(8.93)	
90 to 134 Days	0.371	0.485		1900	0.380	0.434		1453	0.325	0.728		447
	0.382	0.492	0.110		0.382	0.437	0.055		0.384	0.673	0.289	
			(13.08)				(7.11)				(12.28)	
135 to 179 Days	0.515	0.645		1200	0.515	0.587		963	0.536	1.043		237
	0.524	0.665	0.141		0.541	0.618	0.077		0.457	0.856	0.399	
			(8.37)				(5.13)				(6.94)	
More than 179 Days	0.477	0.635		975	0.466	0.580		746	0.535	0.977		229
	0.499	0.714	0.215		0.517	0.647	0.130		0.441	0.932	0.491	
			(10.71)				(6.56)				(9.24)	
II. *Options on Stocks without Dividends*												
60 to 88 Days	0.360	0.494		881	0.441	0.504		541	0.293	0.486		340
	0.369	0.450	0.081		0.418	0.453	0.035		0.291	0.445	0.154	
			(3.32)				(1.97)				(2.73)	
III. *Options on Stocks with Dividends*												
60 to 88 Days	0.656	0.747		1801	0.663	0.706		1405	0.655	0.832		396
	0.634	0.714	0.080		0.628	0.658	0.030		0.674	0.913	0.239	
			(9.09)				(3.69)				(10.91)	

[a] Median values of Y are reported with mean values of Y below median values. All mean values of Y are significant at the 1-percent level. The column headed "ΔY" contains differences between mean values with t values in parentheses. The column labeled "N" is the number of observations.

[b] All options with less than 60 days to maturity were written on stocks that did not go *ex dividend* prior to the option expiration date. All options with more than 88 days to maturity were written on stocks that went *ex dividend* prior to the option expiration date.

empirical support for the Geske and Shastri observation that the existence of dividends should decrease the relative magnitude of the difference between European and American put option model values.

In order to obtain a consistent sample in which all options exhibit dividend effects, we examine the effects of increasing time to option maturity on the proportional biases of the two models by comparing the biases reported in Section III to the biases in Section I for options with maturities greater than 89 days. For example, for all options with maturities of 60 to 88 days written on stocks with dividends, the mean value of Y for the GJ model was 89 percent of the BS value. For options with maturities greater than 179 days, the mean GJ bias is reduced to 70 percent of the BS bias. Comparing the proportional model bias for out-of-the-money options within the same maturity ranges, we find that the mean GJ model bias decreases from 95 percent to 80 percent of the BS value. For in-the-money options, there is a substantial decrease in the proportional bias from 74 percent to

47 percent. These results empirically support the conjecture that the superiority of the performance of the American put model over the European put model should increase with the time to option maturity and should be greatest for in-the-money options

Although we computed thousands of GJ values, many of them requiring linear interpolation to handle the dividends, we saw no evidence of the failure of the GJ interpolation scheme mentioned in Omberg (1986). In addition, we checked the GJ values against the binomial values. We found that the discrepancies between the two methods were negligible for out-of-the-money puts, a penny or two at-the-money, and two to four cents deep in-the-money. We therefore conclude that the large errors found by Omberg occur only for fairly extreme parameter values. Nonetheless, we would caution the reader that any treatment of dividends is ultimately an approximation since we do not ordinarily know in advance precisely what the dividends will be.

Although there has been little empirical investigation of American put option pricing models, several previous studies have provided direct comparisons between model values and market prices for American call options. These studies have reported that systematic call option model pricing biases do exist. Black (1975) (see also Merton (1976)) reported that the BS call option pricing model undervalued out-of-the-money options and overvalued in-the-money options. Merton also noted that the discrepancies were largest for short term to maturity options. MacBeth and Merville (1979) observed an exercise price bias that was exactly opposite to that reported by Black and Merton. They also noted that the discrepancies were a decreasing function of the time to option maturity. Blomeyer and Klemkosky (1983) examined both the BS European model and Roll (1977) American call option model and found that both models undervalued out-of-the-money options while pricing other options fairly well. Rubinstein (1985) has noted that the direction of the out-of-the-money pricing bias shifted in October 1977. The magnitude and direction of the exercise price and time to maturity bias has not been consistent from study to study. However, the existence of systematic pricing biases for American call options has been well documented. This study has provided evidence that systematic model pricing biases exist also for American put options.

V. Summary and Conclusions

This study has provided an *ex post* performance evaluation comparing the accuracy of an American model and a European model for valuing listed options. The Geske and Johnson model was used to generate the American values, and the Black and Scholes model provided the European values. Overall, both models tended to undervalue put options, although the GJ values were substantially closer to market prices than were the BS values. One possible explanation for model undervaluation is that there is some small probability of a large price drop superimposed on the Brownian motion.

The performance of the GJ model was superior to the BS model performance, with the exception of out-of-the-money options with less than 45 days to maturity where the differences were slight. The superiority of the GJ model increases with the time to option maturity and is greatest for in-the-money options. We also found that the existence of cash dividends decreases the difference between American and European option model values. The performance difference between the GJ and BS models is substantial, and a researcher or trader generally will be handicapped severely by using the Black and Scholes model in lieu of the more sophisticated Geske and Johnson approach.

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