Bermudan Swaption Hedging in Black-Karasinski Model

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Abstract

We discuss the hedging of Bermudan swaptions in Black-Karasinski short rate model. We are most interested in the hedging portfolio value distribution when the volatility term in BK model follows a diffusion process. We will propose several slightly different hedging strategies and look at the hedging performance under different scenarios.

Contents

1 Introduction

Interest rate derivatives market is the largest derivatives market in the world. According to ISDA 2007 mid-year market survey[4], the notional amount outstanding of interest rate derivatives grew by 21 percent to $347.09 trillion. Corporations and financial institutions use interest rate derivatives to hedge their interest rate risk, such as mortgage. Hedge funds use these tools to hedge interest rate risks and/or do speculation. Swaption (to be define in next section) is among the most widely used interest rate derivatives.

However, in classical textbooks on interest rate derivatives, there is little discussion on the hedging of swaptions. Some of the existing textbooks[1] only discuss the delta hedging with bonds. However, there is also a stochastic volatility effect in the swaption market. Therefore, we would like to explore the delta- and vega-hedging issues of the commonly used instrument, Bermudan swaption, under a stochastic volatility setting. Bermudan swaption is similar to a Bermudan stock option, which means it can be exercise on a series of dates. We would like to try several hedging strategies and obtain the hedging portfolio value distributions. And by looking at the PnL fluctuation along each hedging path, we can have an idea of the robustness of these hedging strategies.

Because of time limit, we will implement our pricing model in Black-Karasinski short rate model. Although the current market standard model is market model, short rate model can still give a more intuitive understanding of the derivatives' behaviors due to its simplicity.

This report is organized as follows. Section 2 gives backgrounds of the instruments involved. Section 3 outlines Black-Karasinski model and the pricing and greeks computation of swaptions in this model. Section 4 describes the hedging strategies to be used in experiments. Section 5 presents the numerical results. And Section 6 concludes this report.

2 Instruments

The building brick of all of the following interest rate derivatives is the *spot rate curve*. This curve is simply a series of spot rates at different tenors. We can calculate the discount factors $P(0, T_i)$ at these tenors by continuous compounding. In order to obtain the discount factors on tenors outside these know ones, we assume quadratic instantaneous forward rate. By doing so, we can obtain discount factors at any time, i.e. the *discount curve*.

In interest rate modeling, we use the concept of *short rate*. The short rate, usually written r_t is the interest rate at which an entity can borrow money for an infinitesimally short period of time from time t, i.e.

$$P(t, T) = E_t[e^{-\int_t^T r_s ds}]$$

The most basic interest rate derivative is an *interest rate swap*. A swap is a OTC agreement between two counterparties to exchange interest rate payment on a prespecified notional amount. One counterparty (the fixed *payer*) agrees to pay periodically the other counterparty (the fixed *receiver*) a fixed coupon, usually semi-annually, in exchange for receiving periodic LIBOR, usually quarterly. The payment from the fixed payer is call the *fixed leg* of the swap while the payment from the fixed receiver is called the *floating leg*. The value of a swap is the difference between the two legs.

Suppose we have a discount curve. The start date of the swap is T_{start} and the maturity date is T_{mat}. The coupon dates of this swap are $T_1 < \ldots < T_n$.

The DV01 is defined as

$$L = \sum_{j=1}^{n} \alpha_j P(0, T_j) \tag{1}$$

where α_j is the day count fraction between the coupon payment dates.

If the coupon rate is , then the value of the fixed leg is

$$PV_{fixed} = CL \tag{2}$$

And the value of floating leg is[2]

$$PV_{floating} = P(0, T_{start}) - P(0, T_{mat}) \tag{3}$$

The *break-even coupon rate* is the one that makes the values of fixed and floating legs equal

$$C_{break-even} = \frac{PV_{floating}}{L} \tag{4}$$

European swaptions are OTC-traded European calls (*payers*) and puts (*receivers*) on forward swap rates. For example, a 5.50% 1Y→5Y ("1 into 5") receiver swaptions gives the holder the right to receive 5.50% on a 5 year swap starting in 1 year.

Bermudan swaptions are similar to the European ones. But as in the equity option case, "Bermudan" means the holder has the right to exercise the option on a set of dates instead of one.

We will discuss the pricing of swaptions after we introduce the Black-Karasinski short rate model in the next section.

As we are concerned about hedging, we will use European swaptions to hedge the vega risk of a Bermudan swaption and use swaps to hedge its residual delta risk. Details will be discussed in Section 4.

3 Pricing Under Black-Karasinski Model

Black-Karasinski short rate model (BK model) was proposed by Fischer Black and Piotr Karasinski in 1991. In this model, the logarithm of instantaneous short rate $\ln(r(t))$ evolves under the risk-neutral measure according to[1]

$$d\ln(r(t)) = [\theta(t) - a(t)\ln(r(t))]\,dt + \sigma(t)dW(t) \tag{5}$$

In our implementation, we assume θ is a time-dependent parameter in order to fit the discount curve but a and σ are time-independent.

Since in traditional Black's model, which is the basis of market quotes of volatilities, the pricing formulas for caps and swaptions are based on the assumption of lognormal rates, it seemed reasonable to choose the same distribution for the instantaneous short rate process. Moreover, the short rate will not go negative as in Hull-White model. Though this model is relatively simple, there is no closed-form formulas for bonds. We will use a trinomial tree[3] to price swaptions. Further, since there is only finite states in the trinomial tree, this procedure also partially overcomes a major drawback of BK model, which is that the expectation of a money market account value is infinite.

In the trinomial tree implementation, we calculate the branching probabilities to match the first and second moments of the underlying distribution of the short rate process. Then we use the parameter $\theta(t)$ to match the discount factors at different tenors. In market practice, the parameters a (*mean-reversion speed*) and σ (*volatility term*) are calibrated to the caps and/or swaptions prices. Here we just take them as given. Because the tree needs to be generated until the maturity of the underlying swap, the tree construction will be the most time-consuming part in our computation.

We will illustrate in detail how to price a payer swaption in a tree[1].

Consider an interest rate swap first resetting on T_{start} and paying at $T_1 < \ldots < T_n := T$. Assume the swaption holder has the right to enter this swap at any of the reset times $T_{start} = T'_h < T'_{h+1} < \ldots < T'_k := T_{mat}$ with $T_k < T$, all of which are on payment dates. If there is only one reset time, i.e. $T_{start} = T_{mat}$, this will be an European swaption. Therefore we can price the two swaptions in the same framework.

In order to value a Bermudan swaption, we have to discretize time. We build a equally-spaced time grid $\tilde{T}_1, \ldots, \tilde{T}_m$ containing both the payment dates and the reset dates. For $i \in \{1, \ldots, m\}$ denote the short rate value on a generic spatial node j at time $\tilde{T}_i$ by $r_{i,j}$. Denote by $P_{i,j}(T_s)$ the bond price $P(\tilde{T}_i, T_s)$ at time $\tilde{T}_i$ and spatial node j. At each node, we use p_u, p_m, p_d to stand for upward, middle and downward branching risk-neutral probabilities respectively.

The pricing algorithm is as follows

1. Set $i = m$.

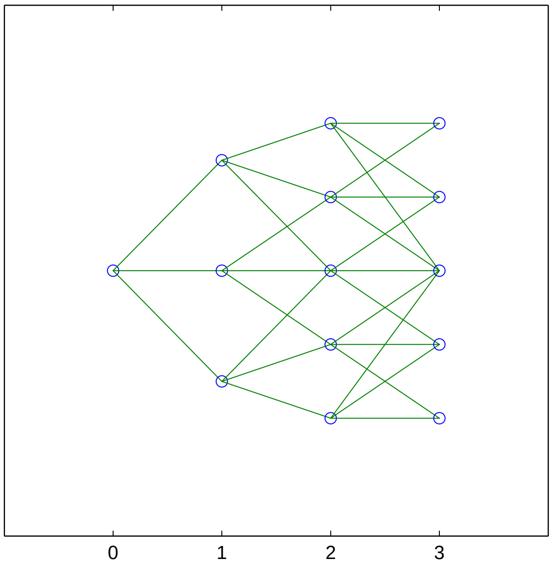

Figure 1: a typical BK trinomial tree

2. (Adding a new zero-coupon bond) Set $P_{i,j}(\tilde{T}_i) = 1$ for all j.
3. (Backward induction)

$$\begin{aligned} & P_{i-1,j}(T_s) \\ =& e^{-r_{i-1,j}(\tilde{T}_i - \tilde{T}_{i-1})}[p_u P_{i,j+1}(T_s) + p_m P_{i,j}(T_s) + p_d P_{i,j-1}(T_s)] \end{aligned} \tag{6}$$

   for all $T_s \geq \tilde{T}_i$.
4. If $\tilde{T}_{i-1}$ is not a payment date then decrease i by one and go back to the preceding step. Otherwise go on to the next step.
5. (A payment date reached) If $\tilde{T}_{i-1} > T_{mat}$, decrease i by one and go to step 2. Other wise move on to step 6.
6. (Exercise time reached) Based on the bonds' values, we can calculate the value of the swap with first reset date $\tilde{T}_{i-1}$ and last payment date T at each spatial node j. If $\tilde{T}_{i-1} = T_{mat}$ then define the backwardly-Cumulated value from Continuation (CC) of the swaption as the swap value at each node j at current time level in the tree

$$CC_{i-1,j} := IRS_{i-1,j}$$

   Else, when $\tilde{T}_{i-1} < T_{mat}$, if the underlying swap value is larger than CC, then set CC equal to the swap value. If $\tilde{T}_{i-1} = T_{start}$, decrease i

by one and go to step 10. Decrease i by one and move on to the next step.

7. (Same as step 2) Set $P_{i,j}(\tilde{T}_i) = 1$ for all j.

8. (Backward induction) We use the same equation as in step 3 for the backward induction of bond prices and replace the bond price in that equation with CC for continuation value backward induction. More precisely,

$$CC_{i-1,j} = e^{-r_{i-1,j}(\tilde{T}_i - \tilde{T}_{i-1})}[p_u CC_{i,j+1} + p_m CC_{i,j} + p_d CC_{i,j-1}] \quad (7)$$

9. If $\tilde{T}_{i-1}$ is not a payment date or exercise date then decrease i by one and go back to the preceding point. Otherwise, if $\tilde{T}_{i-1}$ is a exercise date go to step 6. If $\tilde{T}_{i-1}$ is a payment date decrease i by one and go to step 7.

10. We have reached the first execution date. We will continue the backward induction of CC until time 0. The terminal CC is the price for the swaption.

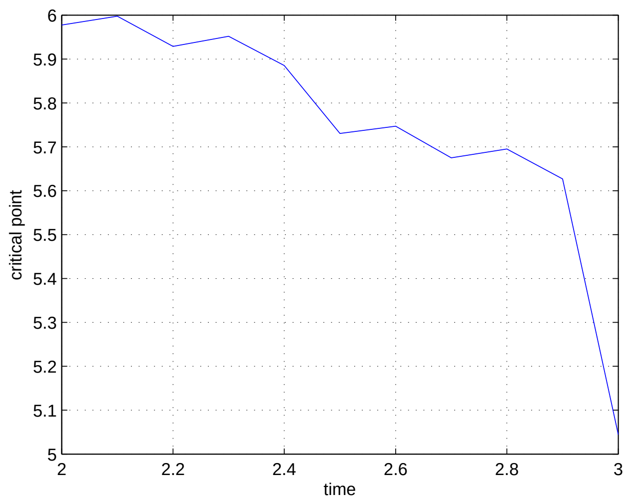

Figure 2: execution boundary for a Bermudan swaption

Figure 2 shows a typical execution boundary for a Bermudan swaptions with 10 execution opportunities per year. Note there is a sudden jump in the critical point before maturity.

One way to check the validity of the pricing results is put-call parity for swaptions

$$PV_{receiver} - PV payer = PV_{swap\ paying\ K} \quad (8)$$

Once we have the pricing function, we can compute the greeks (in a relatively naive way). We bump the yield curve by 1bp to calculate the delta. And we bump the volatility term in BK model to calculate the vega. Swaps' delta can be calculated in a similar way.

4 Hedging Strategies

In the following numerical experiments, we will hedge a long position in a Bermudan swaption in two ways. We first try to delta-hedge our long position in Bermudan swaption only with the next forward starting underlying swap. Then we try first vega-hedging with a portfolio of European swaptions on each execution dates (because Bermudan swaptions have exposure to a series of forward volatilities) and use the next forward starting underlying swap to hedge the residual delta risk. In constructing the vega hedging portfolio, we will experiment *execution-probability*-weighted portfolio. At each execution date in the trinomial tree, we count the execution nodes and divide it by the total number of nodes on that date to obtain the conditional execution probabilities. The probabilities are conditional in the sense that we assume that the Bermudan swaption will be executed according to these conditional probabilities on each execution date and once it is executed, it will not exist any more.

In summary, the hedging strategies are

1. delta-hedge only with the next forward starting underlying swap.
   (a) Construct a position in a swap starting from next execution date to eliminate delta risk.
   (b) Proceed to next hedging date. If it happens to be an execution date, unwind the previous swap position and enter a new forward starting swap position. Otherwise rebalance the swap position. Calculate financing cost. Repeat this step until the maturity of the Bermudan swaption and go to the next step.
   (c) Unwind all the position in hedging portfolio and calculate the final PnL of this hedging path.
2. vega-hedge with a portfolio of execution-probability-weighted European swaption on each execution date and delta-hedge residual risk with the next forward starting underlying swap.
   (a) Construct a position in a portfolio of European swaptions to eliminate vega risk and enter a swap starting from next execution date to hedge residual delta risk.
   (b) Proceed to next hedging date. If it happens to be an execution date, unwind the previous swaptions and swap positions, and enter a new portfolio of swaptions and a new forward starting swap

position. Otherwise rebalance the swaptions and swap positions. Calculate financing cost. Repeat this step until the maturity of the Bermudan swaption and go to the next step.

(c) Unwind all the positions and calculate the final PnL of this hedging path.

In our hedging simulation we generate the shocks to the curve and the volatility term in the following way. The volatility follows a zero-drift diffusion process

$$d\sigma = \eta \sigma dW \tag{9}$$

η is the vol of vol. This assumption is similar to the famous SABR model in interest rate derivatives modeling.

And on the yield curve, we generate a parallel shift

$$dr = \sigma r(1) dW \tag{10}$$

$r(1)$ is the first known value on the yield curve.

Because the BK model is based on constant vol assumption and we are generating random vol shocks, we anticipate the replicating cost would be higher than the price return from this model. And since we are long a position in Bermudan swaptions, we expect the average PnL of the hedging portfolio to be positive. What's more, because of discrete hedging on an equally-spaced time grid, there might be big fluctuations along each path.

In practice, swaptions traders use underlying swaps to hedge and they hedge the vega risk on a portfolio level. Traders rebalance their hedging when needed, e.g. residual risks growing bigger than a certain level. The credit risk involved with swaption is handled by a separate counterparty risk group.

5 Numerical Results

We use the yield curve in Table 1 and the parameters listed in Table 2. The notional is fixed at one. And we hedge a long position until the swaption maturity.

Tenor	Spot Rate
1	5%
2	5.75%
3	6.25%
4	6.75%

Table 1: initial yield curve

Parameter	Value
σ_0, initial value of vol term in BK model	0.1
η, vol of vol	0.1
α, mean-reversion term in BK model	0.15
dt, time step length in trinomial tree	0.05
T_{start}, Bermudan swaption starting date	2
T_{mat}, Bermudan swaption maturity	3
T, underlying swap maturity	4
K, Bermudan swaption strike	5%
M, Bermudan execution times per year	4
N, underlying swap payment per year	2

Table 2: parameters in numerical experiments

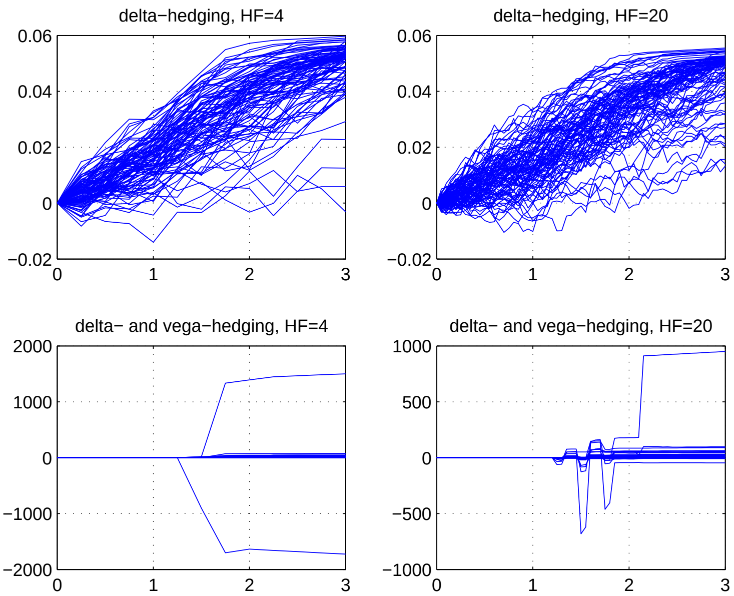

Figure 3: hedging portfolio PnL paths

We test the two hedging strategies mentioned above under two settings, hedging frequency per year (parameter HF) being 4 and 20. Thus there are four combinations. For each scenario, we generate 200 paths.

Figure 3 shows the PnL paths under the four scenarios. It is interesting to see that with vega-hedging, the PnL's variance becomes extremely large, and increasing hedging frequency does not solve this problem. We think this is due to either the dubious nature of the weight for the portfolio, i.e. the execution probability, or the discrete hedging routine. With delta-hedging only, the PnL seems much better. There is no big fluctuation.

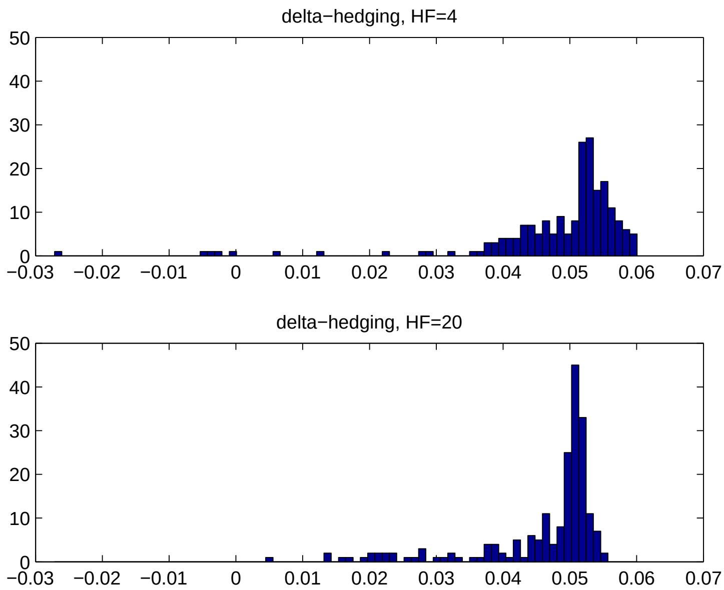

Figure 4: hedging portfolio PnL histogram for delta-hedging strategy

Figure 4 plots histograms of the hedging portfolio PnL at maturity for delta-hedging cases. The price of the Bermudan swaption returned from the BK trinomial tree, which is discounted to time 0, is 0.0467. The mean PnL at maturity with $HF = 4$ is 0.0484, and standard deviation is 1.18×10^{-2}. The mean PnL at maturity with $HF = 20$ is 0.0460, and standard deviation is 9.57×10^{-3}. From the histograms, it is also clear that increasing hedging frequency makes the distribution more concentrated.

The hedging costs are not higher than the premium from our limited experiments. We can see a lot of path PnL's ending below the premium of the Bermudan swaption. Under stochastic vol setting, a delta-neutral portfolio is exposed to gamma and vega risks. But vega risk is much much smaller than delta risk (several scales' difference). That's why a delta-hedging strategy is more robust than a bad delta- and vega-hedging strategy. The lower mean hedging cost is due to the mechanism of the vol process in Equation 9. The volatility is distributed as lognormal with mean at σ_0. In this distribution, the probability of under the mean is bigger than the probability of above the mean. With limited sampling (because of computation burden), this results in a average volatility smaller than σ_0 over a path. This may explain why the mean PnL of the delta-hedging strategy is smaller than the premium of the Bermudan swaption. If we increase the path number, we may see big volatilities appear and raising the average hedging cost.

We also did some preliminary tests with equally-weighted vega-hedging portfolios and obtained similar results to the execution-probability-weighted portfolio case. Hedging vega risk with the European swaption on next execution date only also yields bad results, which confirms that Bermudan swaptions are subject to the risks of a series of forward volatilities, not only at a single point.

6 Conclusion

This report investigates the hedging issues related to a Bermudan swaption. We construct our pricing and greeks computation on Black-Karasinski short rate model. We generate parallel shocks to the whole yield curve and shocks to volatility term in BK model. And in the hedging simulation, we test both delta-hedging only strategy and delta- and vega-hedging strategy.

In the delta-hedging only case, we use the next forward starting swap to hedging our delta risk and leave the vega risk unhedged. This strategy seems to be quite robust based on our limited observations. And by increasing hedging frequency, we succeed to reduce the PnL variance of the hedging portfolio. The discounted mean PnL's at maturity are lower than the price of the Bermudan swaption. This result can be explained by the model and simulation assumptions.

In the delta- and vega-hedging case, we use the next forward starting swap and the European swaptions matured on the execution dates of the Bermudan swaption. Either with equally-weighting or execution-probability-weighting, the hedging portfolio PnL's seems not quite satisfied. In the bigger sample of execution-probability-weighting case, the mean PnL is too big and there are a lot of blowups on the paths. We think this is due to the portfolio construction of the European swaptions.

Most of interest rate modeling textbooks shed little light on the hedging issues of swaptions. We produce some simulation results under a toy model. The further work could be done by using more sophisticated pricing models, e.g. market models, and more realistic simulation assumptions/market data. Finding the proper vega hedging portfolios is still a very important problem to be solved.

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