A SIMPLE COUNTER-EXAMPLE TO SEVERAL PROBLEMS IN THE THEORY OF ASSET PRICING.

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Abstract. We give an easy example of two strictly positive local martingales which fail to be uniformly integrable, but such that their product is a uniformly integrable martingale. The example simplifies an earlier example given by the second author. We give applications in Mathematical Finance and we show that the phenomenon is present in many incomplete markets.

1. Introduction and Known Results.

Let $S = M + A$ be a continuous semimartingale, which we interpret as the discounted price process of some traded asset; the process M is a continuous local martingale and the continuous process A is of finite variation. It is obviously necessary that $dA \ll d\langle M, M\rangle$, for otherwise we would invest in the asset when A moves but M doesn't and make money risklessly. Thus we have for some predictable process α:

$$dS_t = dM_t + \alpha_t \, d\langle M, M\rangle_t. \tag{1}$$

It has long been recognised (see Harrison Kreps 1979) that the absence of "arbitrage" (suitably defined) in this market is equivalent to the existence of some probability $\mathbb{Q}$, equivalent to the reference probability $\mathbb{P}$, under which S becomes a local martingale; see Delbaen and Schachermayer (1994) for the definition of "arbitrage" and a precise statement and proof of this fundamental result. Such a measure $\mathbb{Q}$ is then called

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an equivalent local martingale measure or ELMM. The set of all such ELMM is then denoted by $\mathbf{M}^e(S)$, or $\mathbf{M}^e$ for short. The result of that paper applies to the more general situation of a locally bounded semimartingale S, but in the situation of continuous S, perhaps the result can be proved more simply (see e.g. Delbaen Schachermayer (1995b) for the case of continuous processes and its relation to no "arbitrage"). In particular, it is tempting to define $\mathbb{Q}$ by looking at the decomposition (1) of S and by setting

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} = \mathcal{E}(-\alpha \cdot M)_t \tag{2}$$

provided the exponential local martingale $\mathcal{E}(-\alpha\cdot M)$ is a true martingale. Is it possible for there to exist an equivalent local martingale measure for S, yet the exponential local martingale $\mathcal{E}(-\alpha \cdot M)$ to fail the martingale property? The answer is yes; our example shows it. In the terminology of Föllmer Schweizer (1990), this means that the "minimal local martingale measure" for the process S does not exist, although $\mathbf{M}^e(S)$ is nonempty.

A second question where our example finds interesting application is to hedging of contingent claims in incomplete markets. The positive contingent claim g, or more generally a function that is bounded below by a constant, can be hedged if g can be written as

$$g = c + (H \cdot S)_\infty, \tag{3}$$

where c is a constant and where H is some admissible integrand (i.e. for some constant $a \in \mathbb{R}, (H \cdot S) \geq -a$). In order to avoid "suicide strategies" we also have to impose that $(H\cdot S)_\infty$ is maximal in the set of outcomes of admissible integrands (see Delbaen Schachermayer (1995a) for information on maximal elements and Harrison Pliska (1981) for the notion of "suicide strategies"). We recall that an outcome $(H \cdot S)_\infty$, of an admissible strategy H, is called maximal if for an admissible strategy K, the relation $(K \cdot S)_\infty \geq (H \cdot S)_\infty$ implies that $(K \cdot S)_\infty = (H \cdot S)_\infty$. Jacka (1992), Ansel Stricker (1994) and the authors (1995a) showed that g can be hedged if and only if there is an equivalent local martingale measure $\mathbb{Q} \in \mathbf{M}^e$ such that

$$\mathbb{E}_\mathbb{Q}[g] = \sup\{\mathbb{E}_R[g] \mid R \in \mathbf{M}^e\}. \tag{4}$$

Looking at (3) we then can show that $H \cdot S$ is a $\mathbb{Q}$-uniformly integrable martingale and hence that $c = \mathbb{E}_\mathbb{Q}[g]$. Also the outcome $(H\cdot S)_\infty$ is then maximal. It is natural to conjecture that in fact for all $R \in \mathbf{M}^e$, we might have $\mathbb{E}_R[g] = c$, and the sup becomes unnecessary which is the case for bounded functions g. However our example shows that this too is false in general.

To describe our example, suppose that B and W are two independent Brownian Motions and let $L_t = \exp(B_t - \frac{1}{2}t)$. Then L is a strict local martingale. For information on continuous martingales and especially martingales related to Brownian

Motion we refer to Revuz-Yor (1991). Let us recall that a local martingale that is not a uniformly integrable martingale is called a strict local martingale. The terminology was introduced by Elworthy, Li and Yor (1994). The stopped process L^τ where $\tau = \inf\{t \mid L_t = \frac{1}{2}\}$ is still a strict local martingale and $\tau < \infty$. If we stop L^τ at some independent random time σ, then $L^{\tau\wedge\sigma}$ will be uniformly integrable if $\sigma < \infty$ a.s. and otherwise it will not be. If we thus define $M_t = \exp(W_t - \frac{1}{2}t)$ and $\sigma = \inf\{t \mid M_t = 2\}$ then $L^{\tau\wedge\sigma}$ is not uniformly integrable since $\mathbb{P}[\sigma = \infty] = \frac{1}{2}$. However if we change the measure using the uniformly integrable martingale $M^{\tau\wedge\sigma}$, then under the new measure we have that W becomes a Brownian Motion with drift $+1$ and so $\sigma < \infty$ a.s.. The product $L^{\tau\wedge\sigma}M^{\tau\wedge\sigma}$ becomes a uniformly integrable martingale!

The problem whether the product of two strictly positive strict local martingales could be a uniformly integrable martingale goes back to Karatzas, Lehoczky and Shreve (1991). Lepingle (1993) gave an example in discrete time. Independently Karatzas, Lehoczky and Shreve gave also such an example but the problem remained open whether such a situation could occur for continuous local martingales. The first example in the continuous case was given in Schachermayer (1993), but it is quite technical (although the underlying idea is rather simple).

In this note we simplify the example considerably. A previous version of this paper, only containing the example of section 2, was distributed with the title "A Simple Example of Two Non Uniformly Integrable Continuous Martingales whose product is a Uniformly Integrable Martingale". Our sincere thanks go to L.C.G. Rogers, who independently constructed an almost identical example and kindly supplied us with his manuscript, see Rogers (1993).

We summarise the results of Schachermayer (1993) translated into the present context. The basic properties of the counterexample are described by the following

Theorem 1.6. *There is a continuous process X that is strictly positive, $X_0 = 1$, $X_\infty > 0$ a.s. as well as a strictly positive process Y, $Y_0 = 1$, $Y_\infty > 0$ a.s. such that*

1. *The process X is a strict local martingale under $\mathbb{P}$, i.e. $\mathbb{E}_{\mathbb{P}}[X_\infty] < 1$.*
2. *The process Y is a uniformly integrable martingale.*
3. *The process XY is a uniformly integrable martingale.*

Depending on the interpretation of the process X we obtain the following results.

Theorem 1.7. *There is a continuous semi-martingale S such that*

1. *The semi-martingale admits a Doob-Meyer decomposition of the form $dS = dM + d\langle M, M\rangle\, h$.*
2. *The local martingale $\mathcal{E}(-h \cdot M)$ is strict.*
3. *There is an equivalent local martingale measure for S.*

Proof. Take X as in the preceding theorem and define, through the stochastic logarithm, the process S as $dS = dM + d\langle M, M\rangle$ where $X = \mathcal{E}(-M)$. The measure $\mathbb{Q}$

defined as $d\mathbb{Q} = X_\infty Y_\infty \, d\mathbb{P}$ is an ELMM for S. Obviously the natural candidate for an ELMM suggested by the Girsanov-Maruyama-Meyer formula, i.e. the "density" X_∞, does not define a probability measure.

q.e.d.

Remark. If in the previous theorem we replace S by $\mathcal{E}(S)$, then we can even obtain a positive price system.

Theorem 1.8. *There is a process S that admits an ELMM as well as an hedgeable element g such that $\mathbb{E}_R[g]$ is not constant on the set $\mathbf{M}^e$.*

Proof. For the process S we take X from theorem 1.6. The original measure $\mathbb{P}$ is an ELMM and since there is an ELMM $\mathbb{Q}$ for X such that X becomes a uniformly integrable martingale, we necessarily have that $\mathbb{E}_\mathbb{Q}[X_\infty - X_0] = 0$ and that $X_\infty - X_0$ is maximal. However $\mathbb{E}_\mathbb{P}[X_\infty - X_0] < 0$.

q.e.d.

As for the economic interpretation let us consider a contingent claim f that is maximal and such that $\mathbb{E}_R[f] < 0 = \sup\{\mathbb{E}_\mathbb{Q}[f] \mid \mathbb{Q} \in \mathbf{M}^e\}$ for some $R \in \mathbf{M}^e$. Suppose now that a new instrument T is added to the market and suppose that the instrument T has a price at time t equal to $\mathbb{E}_R[f \mid \mathcal{F}_t]$. The measure R is still a local martingale measure for the couple (S, T), hence the financial market described by (S, T) still is "arbitrage free" more precisely it does not admit a free lunch with vanishing risk; but as easily seen the element f is no longer maximal in this expanded market. Indeed the element $T_\infty - T_0 = f - \mathbb{E}_R[f]$ dominates f by the quantity $-\mathbb{E}_R[f] > 0$! In other words: before the introduction of the instrument T the hedge of f as $(H \cdot S)_\infty$ may make sense economically, after the introduction of T it becomes a "suicide strategy" which only an idiot will apply.

Note that an economic agent cannot make arbitrage by going short on a strategy H that leads to $(H \cdot S)_\infty = f$ and by buying the financial instrument T. Indeed the process $-(H \cdot S) + T - T_0$ is not bounded below by a constant and therefore the integrand$(-H, 1)$ is not admissible!

On the other hand the maximal elements f such that $\mathbb{E}_R[f] = 0$ for all measures $R \in \mathbf{M}^e$ have a stability property. Whatever new instrument T will be added to the market, as long as the couple (S, T) satisfies the NFLVR-property, the element f will remain maximal for the new market described by the price process (S, T). The set of all such elements as well as the space generated by the maximal elements is the subject of Delbaen Schachermayer (1996).

Section 2 of this paper gives an easy example that satisfies the properties of Theorem 1.6. Section 3 shows that the construction of this example can be mimicked in most incomplete markets with continuous prices.

2. Construction of the Example

We will make use of two independent Brownian Motions, B and W, defined on a filtered probability space $(\Omega, (\mathcal{F}_t)_{0\leq t}, \mathbb{P})$, where the filtration $\mathcal{F}$ is the natural filtration of the couple (B, W) and is supposed to satisfy the usual assumptions. This means that $\mathcal{F}_0$ contains all null sets of $\mathcal{F}_\infty$ and that the filtration is right continuous. The process L defined as

$$L_t = \exp\left(B_t - \frac{1}{2}t\right)$$

is known to be a strict local martingale with respect to the filtration $\mathcal{F}$. Indeed, the process L tends almost surely to 0 at infinity, hence it cannot be a uniformly integrable martingale. Let us define the stopping time τ as

$$\tau = \inf\left\{t \mid L_t = \frac{1}{2}\right\}.$$

Clearly $\tau < \infty$ a.s.. Using the Brownian Motion W we similarly construct

$$M_t = \exp\left(W_t - \frac{1}{2}t\right).$$

The stopping time σ is defined as

$$\sigma = \inf\{t \mid M_t = 2\}.$$

In the case the process M does not hit the level 2 the stopping time σ equals ∞ and we therefore have that M_σ either equals 2 or equals 0, each with probability 1/2. The stopped process M^σ defined by $M^\sigma_t = M_{t\wedge\sigma}$ is a uniformly integrable martingale. It follows that also the process $Y = M^{\tau\wedge\sigma}$ is uniformly integrable and because $\tau < \infty$ a.s. we have that Y is almost surely strictly positive on the interval $[0, \infty]$.

The process X is now defined as the process L stopped at the stopping time $\tau \wedge \sigma$. Note that the processes L and M are independent since they were constructed using independent Brownian Motions. Stopping the processes using stopping times coming from the other Brownian Motion destroys the independence and it is precisely this phenomenon that will allow us to make the counterexample.

Theorem 2.1. *The processes X and Y, as defined above, satisfy the properties listed in Theorem 1.6.*

(1) *The process X is a strict local martingale under $\mathbb{P}$, i.e. $\mathbb{E}_\mathbb{P}[X_\infty] < 1$ and $X_\infty > 0$ a.s..*
(2) *The process Y is a uniformly bounded integrable martingale.*
(3) *The process XY is a uniformly integrable martingale.*

Proof. Let us first show that X is not uniformly integrable. For this it is sufficient to show that $\mathbb{E}[X_\infty] = \mathbb{E}[L_{\tau\wedge\sigma}] < 1$. This is quite easy. Indeed

$$\mathbb{E}[L_{\tau\wedge\sigma}] = \int_{\{\sigma=\infty\}} L_\tau + \int_{\{\sigma<\infty\}} L_{\sigma\wedge\tau}.$$

In the first term the variable L_τ equals $\frac{1}{2}$ and hence this term equals $\frac{1}{2}\mathbb{P}[\sigma=\infty]=\frac{1}{4}$. The second term is calculated using the martingale property of L and the optional stopping time theorem.

$$\begin{aligned}\int_{\{\sigma<\infty\}} L_{\sigma\wedge\tau} &= \int_0^\infty \mathbb{P}[\sigma\in dt]\mathbb{E}[L_{\tau\wedge t}]\\ &= \mathbb{P}[\sigma<\infty]\end{aligned}$$

The first line follows from the independence of σ and the process L. Putting together both terms yields $\mathbb{E}[L_{\tau\wedge\sigma}]=\frac{1}{2}\mathbb{P}[\sigma=\infty]+\mathbb{P}[\sigma<\infty]=\frac{3}{4}<1$.

On the other hand the product XY is a uniformly integrable martingale. To see this, it is sufficient to show that $\mathbb{E}[X_\infty Y_\infty]=1$. The calculation is similar to the preceding calculation and uses the same arguments.

$$\begin{aligned}\mathbb{E}[X_\infty Y_\infty] &= \mathbb{E}[L_{\tau\wedge\sigma}\ M_{\tau\wedge\sigma}]\\ &= \mathbb{E}[L_{\tau\wedge\sigma}\ M_\sigma] \text{ because } M^\sigma \text{ is a uniformly integrable martingale}\\ &= 2\,\mathbb{E}[L_{\tau\wedge\sigma}\ \mathbf{1}_{\{\sigma<\infty\}}]\\ &= 2\,\mathbb{P}[\sigma<\infty]=1\end{aligned}$$

q.e.d.

3. Incomplete Markets

All stochastic processes will be defined on a filtered probability space $\left(\Omega,(\mathcal{F}_t)_{t\geq 0},\mathbb{P}\right)$. For the sake of generality the time set is supposed to be the set $\mathbb{R}_+$ of all non-negative real numbers. The filtration is supposed to satisfy the usual hypothesis. The symbol S denotes a d-dimensional semi-martingale $S\colon\Omega\times\mathbb{R}_+\to\mathbb{R}^d$. For vector stochastic integration we refer to Jacod (1979). If needed we denote by x' the transpose of a vector x.

We assume that S has the NFLVR property and the set of ELMM is denoted by $\mathbf{M}^e$. The market is supposed to be incomplete in the following sense. We assume that there is a real-valued non-zero continuous local martingale W such that the bracket $\langle W,S\rangle=0$ but such that the measure $d\langle W,W\rangle$ (defined on the predictable σ-algebra of $\Omega\times\mathbb{R}_+$) is not singular with respect to the measure $d\lambda$ where $\lambda=\text{Trace}\langle S,S\rangle$.

Let us first try to give some economic interpretation to this hypothesis. The existence of a local martingale W such that $\langle S,W\rangle=0$ implies that the process S is not sufficient to hedge all the contingent claims. The extra assumption that the measure $d\langle W,W\rangle$ is not singular to $d\text{Trace}\langle S,S\rangle$, then means that at least part of the local martingale W moves at the same time as the process S. The incompleteness of the market is therefore not only due to the fact that S and W are varying in disjoint time sets but the incompleteness is also due to the fact that locally the process S does not span all of the random movements that are possible.

Theorem 3.1. *If S is a continuous d-dimensional semi-martingale with the NFLVR property, if there is a continuous local martingale W such that $\langle W, S\rangle = 0$ but $d\langle W, W\rangle$ is not singular to $d\,\mathrm{Trace}\langle S, S\rangle$, then for each R in $\mathbf{M}^e$, there is a maximal element f such that $\mathbb{E}_R[f] < 0$.*

Proof. The proof is broken up in different lemmata. Let W' be the martingale component in the Doob-Meyer decomposition of W with respect to the measure R. Clearly $\langle W', W'\rangle = \langle W, W\rangle$.

Lemma 3.2. *Under the hypothesis of the theorem, there is a real-valued R-local martingale $U \neq 0$ such that*

1. $\langle S, U\rangle = 0$
2. *there is a bounded $\mathbb{R}^d$-valued predictable process H that is S-integrable and such that $d\langle U, U\rangle = H'\, d\langle S, S\rangle\, H$ so that the process $N = H \cdot S$ satisfies $\langle N, N\rangle = \langle U, U\rangle$.*

Proof of lemma 3.2. Let $\lambda = \mathrm{Trace}\langle S, S\rangle$. Since $d\langle W, W\rangle$ is not singular with respect to $d\lambda$ there is a predictable set A such that $\mathbf{1}_A\, d\langle W, W\rangle$ is not identically zero and absolutely continuous with repect to $d\lambda$. From the predictable Radon-Nikodym theorem, see Delbaen-Schachermayer (1995b) it follows that there is a predictable process h such that $\mathbf{1}_A\, d\langle W, W\rangle = h\ d\lambda$. For n big enough the process $h\mathbf{1}_{\{\|h\|\le n\}}\mathbf{1}_{[\![0,n]\!]}$ is λ integrable and is such that $\mathbf{1}_A\,\mathbf{1}_{\{\|h\|\le n\}}\mathbf{1}_{[\![0,n]\!]}d\langle W, W\rangle$ is not zero a.s.. We take $U = \left(\mathbf{1}_A\,\mathbf{1}_{\{\|h\|\le n\}}\mathbf{1}_{[\![0,n]\!]}\right)\cdot W'$. To find H we first construct a strategy K such that $d\langle K\cdot S, K\cdot S\rangle/d\lambda \neq 0$ a.e.. This is easy. We take for each coordinate i, an investment $P_i = (0, 0\ldots, 0, 1, 0, \ldots)$ in asset number i. On the predictable set $d\langle P_1\cdot S, P_1\cdot S\rangle/d\lambda \neq 0$ we take $K = P_1$, on the predictable set where $d\langle P_1\cdot S, P_1\cdot S\rangle/d\lambda = 0$ and $d\langle P_2\cdot S, P_2\cdot S\rangle/d\lambda \neq 0$ we take $K = P_2$ etc.. We now take $H = K\,\mathbf{1}_A\,\mathbf{1}_{\{\|h\|\le n\}}\mathbf{1}_{[\![0,n]\!]}h^{1/2}\,(d\lambda/d\langle K\cdot S, k\cdot S\rangle)$.

q.e.d.

Remark. We define the stopping time ν_u as $\nu_u = \inf\{t \mid \langle N, N\rangle_t > u\}$, where N is defined as in lemma 3.2 above. If we replace the process (N, U), the filtration $\mathcal{F}_t$ and the probability R by respectively the process $(N_{\nu_0+t}, U_{\nu_0+t})_{t\ge 0}$, $(\mathcal{F}_{\nu_0+t})_{t\ge 0}$ and the conditional probability $R[.\mid \nu_0 < \infty]$ we may without loss of generality suppose that $R[\nu_0 = 0] = R[\langle N, N\rangle_\infty > 0] = 1$. In this case we have that $\lim_{u\to 0} R[\nu_u < \infty] = 1$ and $\lim_{u\to 0}\nu_u = 0$. We will do so without further notice.

Remark. The idea of the subsequent construction is to see the strongly orthogonal local martingales U and N as time transformed independent Brownian Motions and to use the construction of section 2. The first step is to prove that there is a strict local martingale that is an exponential. The idea is to use the exponential $\mathcal{E}(B)$ where B is a time transform of a Brownian Motion. However the exponential only tends to zero on the set $\{\langle B, B\rangle = \infty\}$.

Lemma 3.3. *There is a predictable process K such that the local R martingale $\mathcal{E}(K\cdot N)$ is not uniformly R-integrable.*

Proof of lemma 3.3. Take a strictly decreasing sequence of strictly positive real numbers $(\varepsilon_n)_{n\geq 1}$ such that $\sum_{n\geq 1}\varepsilon_n 2^n < \frac{1}{8}$.

We take u_1 small enough so that $R[\nu_{u_1} < \infty] > 1-\varepsilon_1$. From the definition of ν_{u_1} it follows that $\langle N,N\rangle_\infty > u_1$ on the set $\{\nu_{u_1}<\infty\}$. For each k we look at the exponential $\mathcal{E}(k\cdot N)$ and we let $f_k = (\mathcal{E}(k\cdot N))_{\nu_{u_1}}$. Since $\langle N,N\rangle_{\nu_{u_1}} > 0$ we have that f_k tends to zero a.s. as k tends to ∞.

Take k_1 big enough to have $R[f_{k_1} < 1/2] > 1-\varepsilon_1$. We now define

$$\tau_1 = \inf\{t \mid (\mathcal{E}(k_1\cdot N))_t > 2 \text{ or } < 1/2\}\wedge \nu_{u_1}.$$

Clearly $R[\tau_1 < \nu_{u_1}] > 1-\varepsilon_1$ and hence

$$R\left[(\mathcal{E}(k_1\cdot N))_{\tau_1} \in \{1/2, 2\}\right] > 1-\varepsilon_1.$$

For later use we define $X_1 = (\mathcal{E}(k_1\cdot N))_{\tau_1}$ and we observe that $R[X_1 = 2] > \frac{1}{3}-\varepsilon_1$ and $R[X_1 = \frac{1}{2}] > \frac{2}{3}-\varepsilon_1$.

We now repeat the construction at time ν_{u_1}. Of course this can only be done on the set $\{\nu_{u_1}<\infty\} = \{\langle N,N\rangle_\infty > u_1\}$. Take $u_2 > u_1$ small enough so that

$$R[\nu_{u_2}<\infty] > R[\nu_{u_1}<\infty](1-\varepsilon_2).$$

We define $f_k = (\mathcal{E}(k\cdot(N-N^{\nu_{u_1}})))_{\nu_{u_2}}$ and observe that f_k tends to zero on the set $\{\nu_{u_1}<\infty\}$ as k tends to infinity. Indeed this follows from the statement that $\langle N-N^{\nu_{u_1}}, N-N^{\nu_{u_1}}\rangle_\infty > 0$ on the set $\{\nu_{u_1}<\infty\}$.

So we take k_2 big enough to guarantee that $R[f_{k_2} < \frac{1}{2}] > R[\nu_{u_1}<\infty](1-\varepsilon_2)$. We define $\tau_2 = \inf\{t > \nu_{u_1} \mid (\mathcal{E}(k\cdot(N-N^{\nu_{u_1}})))_t > 2 \text{ or } < \frac{1}{2}\}\wedge\nu_{u_2}$. Clearly $R[\tau_2 < \nu_{u_2}] > R[\nu_{u_1}<\infty](1-\varepsilon_2)$. We define $X_2 = (\mathcal{E}(k\cdot(N-N^{\nu_{u_1}})))_{\tau_2}$ and we observe that $\frac{1}{2}\leq X_2\leq 2$, $R[X_2\in\{\frac{1}{2},2\}\mid \nu_{u_1}<\infty] > 1-\varepsilon_2$.

Since $\mathbb{E}_R[X_2\mid\nu_{u_1}<\infty] = 1$ we therefore have that $R[X_2 = 2\mid\nu_{u_1}<\infty] > \frac{1}{3}-\varepsilon_2$ and $R[X_2=\frac{1}{2}\mid\nu_{u_1}<\infty] > \frac{2}{3}-\varepsilon_2$.

Continuying this way we construct sequences of
1. stopping times ν_{u_n} with $R[\nu_{u_n}<\infty] > R[\nu_{u_{n-1}}<\infty](1-\varepsilon_n)$, $\nu_0 = 0$
2. real numbers k_n
3. stopping times τ_n with $\nu_{u_{n-1}}\leq\tau_n\leq\nu_{u_n}$
4. $X_n = (\mathcal{E}(k_n\cdot(N-N^{\nu_{u_{n-1}}})))_{\tau_n}$

so that
1. $\frac{1}{2}\leq X_n\leq 2$
2. $X_n = 1$ on the set $\{\nu_{u_{n-1}}=\infty\}$
3. $R[X_n = 2\mid\nu_{u_{n-1}}<\infty] > \frac{1}{3}-\varepsilon_n$
4. $R[X_n=\frac{1}{2}\mid\nu_{u_{n-1}}<\infty] > \frac{2}{3}-\varepsilon_n$.

Let now $K = \sum_{n\geq 1} k_n \mathbf{1}_{]\!]\nu_{u_{n-1}},\tau_n]\!]}$. Clearly $\mathcal{E}(K\cdot N)$ is defined and $(\mathcal{E}(K\cdot N))_{\tau_n} = \prod_{k=1}^n X_k$. We claim that $\mathbb{E}_R[(\mathcal{E}(K\cdot N))_\infty] < 1$, showing that $\mathcal{E}(K\cdot N)$ is not uniformly integrable.

Obviously $(\mathcal{E}(K\cdot N))_\infty = \prod_{k\geq 1} X_k$. From the strong law of large numbers for martingale differences we deduce that a.s.

$$\frac{1}{n}\sum_{k=1}^n \left(\log X_k - \mathbb{E}_R\left[\log X_k \mid \mathcal{F}_{\nu_{u_{k-1}}}\right]\right) \to 0.$$

On the set $\{\nu_{u_{k-1}} < \infty\}$ we have that $\mathbb{E}_R\left[\log X_k \mid \mathcal{F}_{\nu_{u_{k-1}}}\right] \leq (\frac{2}{3} - \varepsilon_k)\log\frac{1}{2} + (\frac{1}{3} + \varepsilon_k)\log 2 \leq -\frac{1}{3}\log 2 + 2\varepsilon_k \log 2 \leq -\frac{1}{6}\log 2$, at least for k large enough. It follows that on the set $\bigcap_{n\geq 1}\{\nu_{u_n} < \infty\}$ we have that $\sum_{k=1}^n \log X_k \to -\infty$ and hence $(\mathcal{E}(K\cdot N))_\infty = 0$ on this set. On the complement, i.e. on $\bigcup_{n\geq 1}\{\nu_{u_n} = \infty\}$ we find that the maximal function $(\mathcal{E}(K\cdot N))^*_\infty$ is bounded by 2^n where n is the first natural number such that $\nu_{u_n} = \infty$. The probability of this event is bounded by ε_n and hence $\mathbb{E}_R\left[(\mathcal{E}(K\cdot N))_\infty\right] \leq \eta = \sum_n \varepsilon_n 2^n \leq \frac{1}{8}$.

q.e.d.

Remark. By adjusting the ε_n we can actually obtain a predictable process K such that $(\mathcal{E}(K\cdot N))_\infty = 0$ on a set with measure arbitrarily close to 1.

Lemma 3.4. *If L is a continuous positive strict local martingale, starting at 1, then for $\alpha > 0$ small enough the process L stopped when it hits the level α is still a strict local martingale.*

Proof of lemma 3.4. Simply let $\tau = \inf\{t \mid L_t < \alpha\}$. Clearly $\mathbb{E}_R[L_\tau] < \alpha + \mathbb{E}_R[L_\infty] < 1$ for $\alpha < 1 - \mathbb{E}_R[L_\infty]$.

q.e.d.

If we apply the previous lemma to the exponential martingale $L = \mathcal{E}(K\cdot N)$ and to $\alpha = \eta$, we obtain a stopping time τ and a strict local martingale $\mathcal{E}(K\cdot N)^\tau$ that is bounded away from zero.

We now use the same integrand K to construct $Z = \mathcal{E}(K\cdot U)$ and we define $\sigma = \inf\{t \mid Z_t = 2\}$.

We will show that $\mathbb{E}_R[L_{\tau\wedge\sigma}] < 1$ and that $\mathbb{E}_R[Z_{\tau\wedge\sigma}L_{\tau\wedge\sigma}] = 1$. This will complete the proof of the theorem since the measure $\mathbb{Q}$ defined by $d\mathbb{Q} = Z_{\tau\wedge\sigma}dR$ is an equivalent martingale measure and the element $f = L_{\tau\wedge\sigma} - 1$ is therefore maximal. On the other hand $\mathbb{E}_R[f] < 0$.

Both statements will be shown using a time transform argument. The fact that the processes $K\cdot N$ and $K\cdot U$ both have the same bracket will now turn out to be useful.

The time transform can be used to transform both these processes into Brownian Motions at the same time.

Following Chapter V, section 1 in Revuz Yor (1991), we define

$$T_t = \inf\{u \mid \langle K \cdot N, K \cdot N\rangle_u = \int_0^u K_s^2 \, d\langle N, N\rangle_s > t\}.$$

As well known, revuz Yor (1991) there are

(1) a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{R})$,

(2) a map $\pi\colon \tilde{\Omega} \to \Omega$,

(3) a filtration $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ on $\tilde{\Omega}$,

(4) two processes β^1 and β^2 that are Brownian Motions with respect to $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ and such that $\langle \beta^1, \beta^2\rangle = 0$,

(5) the variable $\gamma = \int_0^\infty K_s^2 \, d\langle N, N\rangle_s \circ \pi$ is a stopping time with respect to $(\tilde{\mathcal{F}}_t)_{t\geq 0}$,

(6) $\beta^1_{t\wedge\gamma} = (K \cdot N)_{T_t} \circ \pi$,

(7) $\beta^2_{t\wedge\gamma} = (K \cdot U)_{T_t} \circ \pi$,

(8) $\tilde{L} = \mathcal{E}(\beta^1)$ satisfies $L_{T_t} \circ \pi = \tilde{L}_{t\wedge\gamma}$,

(9) $\tilde{Z} = \mathcal{E}(\beta^2)$ satisfies $Z_{T_t} \circ \pi = \tilde{Z}_{t\wedge\gamma}$,

(10) $\tilde{\tau} = \inf\{t \mid \big(\mathcal{E}(\beta^1)\big)_t < \frac{1}{2}\}$ satisfies $\tau \circ \pi = T_{\tilde{\tau}}$,

(11) $\tilde{\sigma} = \inf\{t \mid \big(\mathcal{E}(\beta^2)\big)_t > 2\}$ satisfies $\sigma \circ \pi = T_{\tilde{\sigma}}$.

In this setting we have to show that $\mathbb{E}_R[L_{\tau\wedge\sigma}] = \tilde{\mathbb{E}}_{\tilde{R}}[\tilde{L}_{\tilde{\tau}\wedge\tilde{\sigma}\wedge\gamma}] < 1$. But on the set $\left\{\int_0^\infty K_s^2 \, d\langle N, N\rangle_s < \infty\right\}$ we have, as shown above

$$\begin{aligned} &\mathbb{E}_R\left[\mathbf{1}_{\left\{\int_0^\infty K_s^2 \, d\langle N,N\rangle_s<\infty\right\}} L_{\tau\wedge\sigma}\right] \\ &\leq \mathbb{E}_R\left[\mathbf{1}_{\left\{\int_0^\infty K_s^2 \, d\langle N,N\rangle_s<\infty\right\}} L^*\right] \\ &\leq \eta. \end{aligned}$$

In other words $\tilde{\mathbb{E}}_{\tilde{R}}\left[\mathbf{1}_{\{\gamma<\infty\}} \tilde{L}^*_\gamma\right] \leq \eta$. So it remains to be shown that

$$\tilde{\mathbb{E}}_{\tilde{R}}\left[\mathbf{1}_{\{\gamma=\infty\}} \tilde{L}_{\tilde{\tau}\wedge\tilde{\sigma}}\right] < 1 - \eta.$$

Actually we will show that

$$\tilde{\mathbb{E}}_{\tilde{R}}\left[\tilde{L}_{\tilde{\tau}\wedge\tilde{\sigma}}\right] < 1 - \eta.$$

This is easy and follows from the independence of β^1 and β^2, a consequence of $\langle \beta^1, \beta^2\rangle = 0$! As in section 2 we have

$$\tilde{\mathbb{E}}_{\tilde{R}}\left[\tilde{L}_{\tilde{\tau}\wedge\tilde{\sigma}}\right] = \eta\tilde{R}[\tilde{\sigma} = \infty] + \tilde{R}[\sigma < \infty] = \frac{1}{2}\eta + \frac{1}{2} < 1 - \eta$$

since $\eta \leq \frac{1}{8}$. To show that

$$\mathbb{E}_R\left[L_{\tau\wedge\sigma} Z_{\tau\wedge\sigma}\right] = 1$$

we again use the extension and time transform. But $\left(\tilde{L}\tilde{Z}\right)^{\tilde{\tau}\wedge\tilde{\sigma}}$ is a uniformly integrable martingale, as follows from the easy calculation in section 2, and hence we obtain

$$\mathbb{E}_R\left[L_{\tau\wedge\sigma} Z_{\tau\wedge\sigma}\right] = \tilde{\mathbb{E}}_{\tilde{R}}\left[\tilde{L}_{\tilde{\tau}\wedge\tilde{\sigma}\wedge\gamma} Z_{\tilde{\tau}\wedge\tilde{\sigma}\wedge\gamma}\right] = 1.$$

The proof of the theorem is complete now.

q.e.d.

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