1

Gallery of equations

1. Introduction

Exchange-traded markets

Over-the-counter markets

Forward contracts

Definition 1.1. A *forward contract* is an agreement to buy or sell an asset at a certain future time for a certain price

Definition 1.2. The *forward price* for a contract is the delivery price that would be applicable to the contract if it were negotiated today (i.e., it is the delivery price that would make the contract worth exactly zero).

Futures contracts

Definition 1.3. A *futures contract* is a standardized contract, traded on a futures exchange, to buy or sell a certain underlying instrument at a certain date in the future, at a specified price.

- The future date is called the delivery date or final settlement date. The pre-set price is called the futures price. The price of the underlying asset on the delivery date is called the settlement price. The settlement price, normally, converges towards the futures price on the delivery date.

Options

Types of traders

- Hedgers
- Speculators
- Arbitrageurs

Dangers

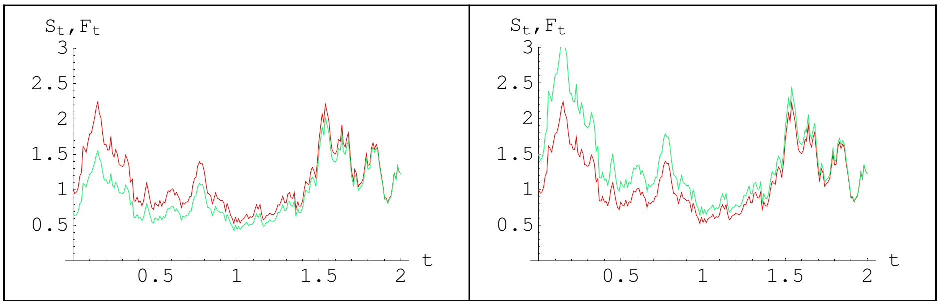
2. Mechanics of Futures Markets

Background

Specification of a futures contract

- Specification by exchange
  - what
  - where
  - when
  - trading hours
  - quote conventions
  - max price movements

Convergence of futures price to a spot price



$F_t > S_t,\ b < 0,\ \downarrow$	$F_t < S_t,\ b > 0,\ \uparrow$

Figure 1.1: Relationship between future price and spot price as delivery period is approached. E.g. gold (lhs) and oil (rhs).

- If at delivery futures price is $\left\{\begin{matrix}\text{above}\\ \text{below}\end{matrix}\right\}$ spot, i.e. $\left\{\begin{matrix}F_T > S_T\\ F_T < S_T\end{matrix}\right\}$ then arb opp is
  - $\left\{\begin{matrix}\text{short}\\ \text{go long}\end{matrix}\right\}$ future (zero cost)
  - $\left\{\begin{matrix}\text{buy}\\ \text{sell}\end{matrix}\right\}$ asset (for S_T)
  - $\left\{\begin{matrix}\text{deliver}\\ \text{receive}\end{matrix}\right\}$ underlying ($\left\{\begin{matrix}\text{earning}\\ \text{at a cost of}\end{matrix}\right\}$ F_T)
- Eventually, futures price will $\left\{\begin{matrix}\text{fall}\\ \text{rise}\end{matrix}\right\}$ to match spot
- Prior to expiry,
  - spot can be $\left\{\begin{matrix}\text{below}\\ \text{above}\end{matrix}\right\}$ future,

- i.e. $\left\{ \begin{matrix} - \\ + \end{matrix} \right\}$ basis,
- e.g. $\left\{ \begin{matrix} \text{gold} \\ \text{oil} \end{matrix} \right\}$, which are $\left\{ \begin{matrix} \text{investment} \\ \text{consumption} \end{matrix} \right\}$ assets
- prices related by $\left\{ \begin{matrix} F = S\, e^{r T} \\ F = S\, e^{(r+u-y)\, T} \end{matrix} \right\}$

Daily settlement and margins

Definition

Definition 1.4. A *margin* is cash or marketable securities deposited by an investor with his or her broker.

Operation

- The balance in the margin account is adjusted to reflect daily settlement – *marking to market*
- Types
  - *Initial margin* – amount deposited when contract entered
  - *Maintenance margin* – trigger level for margin call to restore balance to initial margin
- $0 < \overbrace{[\text{maintenance margin}] < [\text{initial margin}]}^{\text{Difference is } \textit{variation margin}}$
- Investor can withdraw balance in excess of initial margin

Table 1.1. Operations of margins for a long position in two gold futures contracts.

Fut price	*Daily gain*	*Cum gain*	*Mgn ac bal*	*Mgn call*
400.	0	0	4000	0
401.5	300.	300.	4300.	0
398.2	−660.	−360.	3640.	0
404.9	1340.	980.	4980.	0
404.9	0.	980.	4980.	0
399.7	−1040.	−60.	3940.	0
395.6	−820.	−880.	3120.	0
392.2	−680.	−1560.	2440.	0
383.1	−1820.	−3380.	2180.	1560.
383.4	60.	−3320.	4060.	1820.
383.	−80.	−3400.	3980.	0
388.	1000.	−2400.	4980.	0
382.2	−1160.	−3560.	3820.	0
375.5	−1340.	−4900.	2480.	0
365.6	−1980.	−6880.	2020.	1520.
367.1	300.	−6580.	4300.	1980.
370.3	640.	−5940.	4940.	0

Newspaper quotes

Delivery

Types of traders and types of orders

Regulation

Accounting and tax

Forward vs. futures contracts

- Private vs exchange
- Single vs multiple delivery date
- Non-standard
- Final vs daily settlement

3. Hedging Strategies Using Futures

Basic principles

Arguments for and against hedging

- Hedges lose money
- What are competitors doing?

Basis risk

Definition 1.5. A *basis,* β, is the extent to which the spot price of the asset to be hedged exceeds the futures price of the contract used for hedging.

$$\underbrace{b}_{\text{basis}} = \overbrace{S}^{\text{spot price of asset to be hedged}} - \overbrace{F}^{\text{futures price of contract used}} \tag{1.1}$$

$$\underbrace{\tilde{S}_2}_{\text{effective}} = \underbrace{S_2}_{\text{terminal stock price}} - \overbrace{(F_2 - F_1)}^{\text{gain on futures}} = F_1 + b_2 \tag{1.2}$$

Cross hedging

Proposition 1.6. *The optimal hedge ratio is given by* $h^* = \rho \frac{\sigma_S}{\sigma_F}$.

$$h^* = \rho \, \frac{\sigma_S}{\sigma_F} \tag{1.3}$$

Proof

t_1	time at which choice is made to hedge
t_2	time at which asset is to be sold
N_A	number of units of asset to sell at time t_2
N_F	number of futures contracts to short at time t_1
h	hedge ratio is $h{:=}N_F/N_A$
Y	total amount realized for the asset when the profit or loss on the hedge is taken into account
S_i, F_i	asset prices and futures prices at time i, i=1,2
ΔS, ΔF	change in asset and futures prices over interval, i.e. $\Delta S{:=}S_2{-}S_1$, $\Delta F{:=}F_2{-}F_1$
v	variance of Y
σ_S, σ_F, ρ	standard deviations of asset and future, and correlation coefficient between them

- Total amount realized for asset and hedge:

$$\begin{aligned} Y &= S_2\, N_A - (F_2 - F_1)\, N_F \\ &= S_1\, N_A + (S_2 - S_1)\, N_A - (F_2 - F_1)\, N_F \\ &= S_1\, N_A + N_A\, (\Delta S - h\, \Delta F) \end{aligned}$$

- S_1 and N_A are known at time t_1
- Minimising the variance of Y, corresponds to minimising the variance of $(\Delta S - h\, \Delta F)$

$$v = \text{Var}[\Delta S - h\, \Delta F] = \sigma_S^2 + h^2\, \sigma_F^2 - 2\, h\, \sigma_S\, \sigma_F\, \rho$$

- Derivative w.r.t. h

$$\frac{dv}{dh} = 2\, h\, \sigma_F^2 - 2\, \sigma_S\, \sigma_F\, \rho$$

is zero (and second derivative +ve) when

$$h^* = \rho \, \frac{\sigma_S}{\sigma_F}$$

Number of futures contracts

Proposition 1.7. *The number of futures contracts required is given by* $N^* = \frac{h^*\, N_A}{Q_F}$.

$$N^* = \frac{h^* N_A}{Q_F} \tag{1.4}$$

Stock index futures

Proposition 1.8. *To hedge the risk in a portfolio the number of contracts that should be shorted is* $N^* = \beta \frac{P}{A}$, *where* P *is the value of the portfolio,* β *is its beta, and* A *is the value of the assets underlying one futures contract.*

$$N^* = \beta \frac{P}{A} \tag{1.5}$$

Changing Beta

Proposition 1.9. *To change the beta of a portfolio from* β *to* β^*

- $\beta^* < \beta$, *a short position in* $(\beta - \beta^*) \frac{P}{A}$ *contracts is required*
- $\beta^* > \beta$, *a long position in* $(\beta^* - \beta) \frac{P}{A}$ *contracts is required.*

Rolling the hedge forward

4. Interest Rates

Types of rates

Measuring interest rates

Equations relating discrete and continuous compounding rates

$$\begin{aligned} R_c &= m \ln(1 + \tfrac{R_m}{m}) \\ R_m &= m\,(e^{R_c/m} - 1) \end{aligned} \tag{1.6}$$

Zero rates

Definition 1.10. A *zero rate* (or *spot rate*), for maturity T is the rate of interest earned on an investment that provides a payoff only at time T

Bond pricing

- To calculate the cash price of a bond we discount each cash flow at the appropriate zero rate

$$P = \overbrace{\sum_{i=1}^{n} c\, e^{-R_i T_i}}^{\text{coupons}} + \overbrace{1\, e^{-R_n T_n}}^{\text{principal}} = \sum_{i=1}^{n-1} c\, e^{-R_i T_i} + (1+c)\, e^{-R_n T_n} \tag{1.7}$$

- Cashflows c are related to annual coupon with compounding frequency m by c_m by $c = c_m / m$

Definition 1.11. The *bond yield* for a bond is the discount rate that makes the present value of the cash flows equal to the market price.

$$\overbrace{\sum_{i=1}^{n} c\, e^{-Y T_i}}^{\text{coupons}} + \overbrace{1\, e^{-Y T_n}}^{\text{principal}} = P \tag{1.8}$$

Par yield

Definition 1.12. The *par yield* for a certain maturity is the coupon rate that causes the bond price to equal its face value.

$$100 \left(\sum_{i=1}^{n-1} c\, e^{-Y_i T_i} + (1+c)\, e^{-Y_n T_n} \right) = 100 \tag{1.9}$$

$$c = \frac{1 - e^{-Y_n T_n}}{\sum_{i=1}^{n} e^{-Y_i T_i}} \tag{1.10}$$

- If we name $d = e^{-Y_n T_n}$ and $A = \sum_{i=1}^{n} e^{-Y_i T_i}$, we obtain

$$c_m = \frac{(1-d)\, m}{A} \tag{1.11}$$

Determining Treasury zero rates

Table 1.2. Data for 5 bonds, 2 paying a semi-annual coupon

Principal	Time to maturity	Coupon	Cash price
100	0.25	0	97.5
100	0.5	0	94.9
100	1.	0	90.
100	1.5	8.	96.
100	2.	12.	101.6

Example 1.1. From the prices of the 5 bonds above, infer the zero rate (yield) curve.

- An amount 2.5 can be earned on 97.5 during 3 months. The annualised 3-month rate is $4\,\frac{100-97.5}{97.5} = 4\,\frac{2.5}{97.5} = 0.10256$ or 10.256% with quarterly compounding. This is 10.127% with continuous compounding

$$R_c \;=\; m\ln\left(1+\frac{R_m}{m}\right) = 4\,\mathrm{Log}\left[1+\frac{0.10256}{4}\right] = 0.101267$$

- Similarly the 6 month and 1 year rates are 10.469% and 10.536% with continuous compounding

$$2\,\mathrm{Log}\left[1+\frac{2\,(100-94.9)/94.9}{2}\right] = 0.104693$$

$$1\,\mathrm{Log}\left[1+\frac{1\,(100-90)/90}{1}\right] = 0.105361$$

- To calculate the 1.5 year rate we solve

$$4\,e^{-0.10469\times 0.5} + 4\,e^{-0.10536\times 1.0} + 104\,e^{-R\times 1.5} = 96$$

$$R = -\frac{1}{1.5}\,\mathrm{Log}\left[\frac{96-(4\,e^{-0.10469\times 0.5}+4\,e^{-0.10536\times 1.0})}{104}\right]$$

to get $R = 0.10681$ or 10.681 %

- Similarly the two-year rate is 10.808%

$$6\,e^{-0.10469\times 0.5} + 6\,e^{-0.10536\times 1.0} + 6\,e^{-0.10681\times 1.5} + 106\,e^{-R\times 2.0} = 101.6$$

$$R = -\frac{1}{2.0}\,\mathrm{Log}\left[\frac{101.6-(6\,e^{-0.10469\times 0.5}+6\,e^{-0.10536\times 1.0}+6\,e^{-0.10681\times 1.5})}{106}\right]$$

Forward rates

Definition 1.13. The *forward rate* is the future zero rate implied by today's term structure of interest rates

Example 1.2. The continuously compounded zero rates for n-year investment are given by the following matrix (upper row is n, lower row is zero rate):

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3.0 & 4.0 & 4.6 & 5.0 & 5.3 \end{pmatrix}.$$

- Find the forward rates for borrowing over the nth year for $n = 2, \ldots 5$.

- $100\,\overset{\text{spot}}{e^{-0.03\times 1}}\,\overset{\text{forward}}{e^{-R_F\times 1}} = 100\,\overset{\text{spot}}{e^{-0.04\times 2}}$

$$R_F = \frac{0.04\times 2-0.03\times 1}{1} = 0.05$$

....

Formula for forward rates

$$R_F^\infty = \frac{R_2^\infty\,T_2 - R_1^\infty\,T_1}{T_2 - T_1} = R_2^\infty + (R_2^\infty - R_1^\infty)\,\frac{T_1}{T_2 - T_1} \tag{1.12}$$

- Derivation

$$e^{-T_2\,R_2^\infty} = e^{-T_1\,R_1^\infty}\,e^{-(T_2-T_1)\,R_F^\infty}$$

- If not, arbitrage opportunities, etc...

- Here compounding frequency superscript $m = \infty$ reminds us that these are continuously compounded rates

Instantaneous forward rate

Definition 1.14. The *instantaneous forward rate* for a maturity T is the forward rate that applies for a very short time period starting at T. It is $R_F = R + T\,\frac{\partial R}{\partial T}$ where R is the T-year rate

$$R_F = R + T\,\frac{\partial R}{\partial T} \tag{1.13}$$

- Cf.

$$f_{tT} = Y_{tT} + (T - t)\,\frac{\partial Y_{tT}}{\partial T} \tag{1.14}$$

Forward rate agreements

Definition 1.15. A *forward rate agreement* (FRA) is an agreement that a certain rate will apply to a certain principal during a certain future time period.

FRA valued by assuming that the forward interest rate is certain to be realized

Payoffs

$$L(R_K^m - R_M^m)\,(T_2 - T_1) \tag{1.15}$$

- Discrete compounding frequency $m = 1/(T_2 - T_1)$

Valuation

FRA worth zero when $R_K = R_F$

- Why buy an FRA for $$ if you can lock in price of forward borrowing
- Go long FRA with rate R_K, short FRA with rate R_F
- Costs are V_{FRA} and 0 respectively
- Net (deterministic) payoff at T_2 is $L((R_K - R_M) - (R_F - R_M))\,(T_1 - T_2)$
- Value of FRA where a fixed rate R_K will be *received* on a principal L between times T_1 and T_2 is

$$V_{\text{FRA}} = \overbrace{L(R_K^m - R_F^m)\,(T_1 - T_2)}^{\text{Cash flow at time } T_2}\,e^{-R_2^\infty T_2} \tag{1.16}$$

Duration

Definition 1.16. The *duration* of a bond that provides cash flow c_i at time t_i is

$$D = \sum_{i=1}^{n} t_i \frac{c_i\, e^{-y t_i}}{B}$$

where B is its price and y is its yield (continuously compounded)

$$\frac{\Delta B}{B} = -D\, \Delta y \tag{1.17}$$

Duration is the proportional change in the bond price per unit (parallel) shift in the yield curve

Convexity

$$C = \frac{1}{B} \frac{\partial^2 B}{\partial y^2} = \sum_{i=1}^{n} t_i^2 \frac{c_i\, e^{-y t_i}}{B} \tag{1.18}$$

$$\frac{\Delta B}{B} = -D\, \Delta y + \frac{1}{2} C\, (\Delta y)^2 \tag{1.19}$$

Theories of the term structure of interest rates

5. Determination of Forward and Futures Prices

Summary

Asset	*Forward / futures price*	*Value of long forward contract*
No income	$S_0\, e^{rT}$	$S_0 - K\, e^{-rT}$
Income of present value I	$(S_0 - I)\, e^{rT}$	$S_0 - I - K\, e^{-rT}$
Yield q	$S_0\, e^{(r-q)T}$	$S_0\, e^{-qT} - K\, e^{-rT}$

Investment assets vs. consumption assets

Definition 1.17. An *investment asset* is an asset that is held primarily for investment.

- E.g. stocks, bonds, gold, ***silver***

Definition 1.18. A *consumption asset* is an asset that is held primarily for consumption.

Short selling

Assumptions and notation

Forward price of an investment asset

- See Chapter 1

Known income

Instrument	*Holding*	*Value at 0*	*Value at T*
Stock	1	S_0	S_T
Bank/bond	Left at end $-(S_0-I)-$ Paid off by divs I	$-(S_0-I)-I$	$-(S_0-I)\,e^{rT}$
Forward	-1	0	$-(S_T-F_0)$
Total		0	$F_0-(S_0-I)\,e^{rT}$

Known yield

Instrument	*Holding at t*	*Value at 0*	*Value at t*	*Value at T*
Stock	$e^{-q(T-t)}$	$S_0\,e^{-qT}$	$S_t\,e^{-q(T-t)}$	S_T
Bank/bond	$-S_0\,e^{-qT}\,e^{rt}$	$-S_0\,e^{-qT}$	$-S_0\,e^{-qT}\,e^{rt}$	$-S_0\,e^{-(r-q)T}$
Forward	-1	0		$-(S_T-F_0)$
Total		0		$F_0-S_0\,e^{-(r-q)T}$

Valuing forward contracts

Proposition 1.19. *The value, f, of a long forward contract is given by $f=(F_0-K)\,e^{-rT}$, where ...*

$$f=(F_0-K)\,e^{-rT} \tag{1.20}$$

Proof

Forward (delivery price)	*Holding*	*Value at 0*	*Value at T*
Forward (K)	+1	f	$S_T - K$
Forward (F_0)	−1	0	$-(S_T - F_0)$
Total		f	$F_0 - K$

Are forward prices and futures prices equal?

- Argument that they are not when IRs and underlying are correlated
  - Consider $\rho := \mathbb{C}\mathrm{orr}(S, r) > 0$
  - $S\uparrow \Rightarrow r\uparrow$ likely
  - Long future, immediate **gain** due to mk-to-mkt
  - Invested at **higher** than average rate
  - Similarly when $S\downarrow$
  - $\left\{ \begin{array}{c} \text{Positive} \\ \text{Negative} \end{array} \right\}$ correlation (S, r), $\Rightarrow$ long future $\left\{ \begin{array}{c} \text{more} \\ \text{less} \end{array} \right\}$ attractive than long forward

Proof that foward and futures prices are equal when interest rates are constant

Proposition 1.20. *A sufficient condition for forward and futures prices to be equal is that interest rates be constant.*

- Strategy:
  - take a long futures position of e^{δ} at the beginning of day 0
  - increase position to $e^{2\delta}$ at the beginning of day 1

    …
  - long futures position $e^{(i+1)\delta}$ at start of day i
- Profit
  - on day 1 (end of day 0) is $(F_1 - F_0)\, e^{\delta}$
  - on day i is $(F_i - F_{i-1})\, e^{\delta i}$

  and is banked
- Compounded value from day i on day n is

$$(F_i - F_{i-1})\, e^{\delta i}\, e^{\delta(n-i)} = (F_i - F_{i-1})\, e^{\delta n}$$

Table 1.3. Dynamic investment strategy in futures contracts

Day	0	1	2	…	$n-1$	n
Futures price	F_0	F_1	F_2	…	F_{n-1}	F_n
Futures posn	e^{δ}	$e^{2\delta}$	$e^{3\delta}$	…	$e^{n\delta}$	0
Gain	0	$(F_1-F_0)\,e^{\delta}$	$(F_2-F_1)\,e^{2\delta}$	…	…	$(F_n-F_{n-1})\,e^{n\delta}$
Gain comp'd to n	0	$(F_1-F_0)\,e^{n\delta}$	$(F_2-F_1)\,e^{n\delta}$	…	…	$(F_n-F_{n-1})\,e^{n\delta}$

- Value at day n of entire strategy

$$\begin{aligned}\sum_{i=1}^{n} (F_i - F_{i-1})\, e^{\delta n} &= ((F_1 - F_0) + (F_2 - F_1) + \cdots + (F_n - F_{n-1}))\, e^{\delta n} \\ &= (F_n - F_0)\, e^{\delta n} \\ &= (S_T - F_0)\, e^{\delta n}\end{aligned}$$

- Cost of each increment to the futures position is ____
- Combined strategy of
  - dynamic strategy above (costs zero, payoff $(S_T - F_0)\, e^{\delta n}$)
  - invest F_0 in a risk-free bank account (costs F_0 at 0, pays off $F_0\, e^{\delta n}$ at expiry)
- Total cost at 0 is F_0; total payoff at n is $S_T\, e^{\delta n}$

Table 1.4. Combined investment strategy: dynamic futures strategy above + bank

Description	*Cost (PV at 0)*	*Payoff (at n)*
Dynamic futures strategy	0	$(S_T-F_0)\, e^{\delta n}$
Bank account, holding F_0	F_0	$F_0\, e^{\delta n}$
Total	F_0	$S_T\, e^{\delta n}$

Table 1.5. Investment strategy: long forward contract + bank

Description	*Cost* (at 0 or other times)	*Payoff (at n)*
Long forward 1 unit	0	$(S_T-G_0)\, e^{\delta n}$
Bank account, holding G_0	G_0	$G_0\, e^{\delta n}$
Total	G_0	$S_T\, e^{\delta n}$

- Both strategies have the same payoff after n days, so must be worth the same at time 0
- $F_0 = G_0$

Futures prices of stock indices

Forward and futures prices on currencies

$$F_0 = S_0 \, e^{(r-r_f)T} \tag{1.21}$$

Time	Foreign FX		Dollars
0	1	→	S_0
	↓		↓
T	$e^{r_f T}$	→	$F_0 \, e^{r_f T} = S_0 \, e^{rT}$

Figure 1.2: Two ways of converting a single unit of a foreign currency to dollars at time T.

Example 1.3. Two-year interest rates in Australia and the US are 5% and 7%, respectively. The spot FX is 0.62 USD per AUD.

- Find .....

Instrument	#	Value at 0 (\$)	Value at T (\$)
Foreign bond	1	S_0	$S_T \, e^{r_f T}$
Domestic bond	$-S_0$	$-S_0$	$-S_0 \, e^{rT}$
Forward FX	$-e^{r_f T}$	0	$-e^{r_f T}(S_T - F_0)$
Total		0	$F_0 \, e^{r_f T} - S_0 \, e^{rT}$

Futures on commodities

Proposition 1.21. *The initial forward price F_0 and spot price S_0 for a consumption asset for which the present value of the storage costs are U satisfy $F_0 = (S_0 + U) \, e^{rT}$*

$$F_0 = (S_0 + U) \, e^{rT} \tag{1.22}$$

Proposition 1.22. *The initial forward price F_0 and spot price S_0 for a consumption asset for which the storage costs per unit time is u satisfy $F_0 = S_0 \, e^{(r+u)T}$*

$$F_0 = S_0 \, e^{(r+u)T} \tag{1.23}$$

Consumption commodities

Proposition 1.23. *The initial forward price F_0 and spot price S_0 for a consumption asset for which the present value of the storage costs are U obey the inequality $F_0 \leq (S_0 + U) \, e^{rT}$*

$$F_0 \leq (S_0 + U) \, e^{rT} \tag{1.24}$$

Storage costs proportional to commodity price

Proposition 1.24. *The initial forward price F_0 and spot price S_0 for a consumption asset for which the storage costs per unit time is u obey the inequality $F_0 \le S_0\, e^{(r+u)T}$*

$$F_0 \le S_0\, e^{(r+u)T} \tag{1.25}$$

Convenience yields

Definition 1.25. The *convenience yield* is the value of y such that when the storage costs are known and have present value U, then $F_0\, e^{yT} = (S_0 + U)\, e^{rT}$. Similarly for storage costs that are a constant proportion u of the spot price: $F_0\, e^{yT} = S_0\, e^{(r+u)T}$.

$$\begin{aligned} F_0\, e^{yT} &= (S_0 + U)\, e^{rT} \\ F_0 &= S_0\, e^{(r+u-y)T} \end{aligned} \tag{1.26}$$

Convenience yield measures extent to which forward price of consumption assets falls short of the theoretical value for investment assets

The cost of carry

Definition 1.26. The *cost of carry* is the storage cost plus the interest costs less the income earned.

$$c = r + u - q \tag{1.27}$$

Relationships between forward and spot prices in terms of the cost of carry

Proposition 1.27. *The initial forward price F_0 and spot price S_0 for an investment asset that pays no dividend are related by $F_0 = S_0\, e^{cT}$, where...*

$$F_0 = S_0\, e^{cT} \tag{1.28}$$

Proposition 1.28. *The initial forward price F_0 and spot price S_0 for a consumption asset that pays no dividend are related by $F_0 = S_0\, e^{(c-y)T}$, where...*

$$F_0 = S_0\, e^{(c-y)T} \tag{1.29}$$

Delivery options

Futures prices and expected future spot prices

6. Interest Rate Futures

Day count conventions

Interest earned between two dates

$$\frac{[\text{Number of days between dates}]}{[\text{Number of days in reference period}]} \times [\text{Interest earned in reference period}] \qquad (1.30)$$

In US

- Treasury Bonds: Actual/Actual (in period)
- Corporate Bonds: 30/360
- Money Market Instruments: Actual/360

Example 1.4. A treasury bond with face value $100 pays a semi-annual coupon of 8%. Coupon payment dates are Mar 1 and Sept 1. Find the interest earned between Mar 1 and July 3.

Convention is actual/actual.
Reference period Mar 1 to Sept 1 is 184 days
Desired period is Mar 1 and July 3, is 124 days
Interest earned is $\frac{124}{184} \times 0.04 = 2.6957\,\%$

Example 1.5. A corporate bond with face value $100 pays a semi-annual coupon of 8%. Coupon payment dates are Mar 1 and Sept 1. Find the interest earned between Mar 1 and July 3.

Convention is 30/360.
Reference period Mar 1 to Sept 1 is 180 days
Desired period is Mar 1 and July 3, is $(4 \times 30) + 2 = 122$ days
Interest earned is $\frac{122}{180} \times 0.04 = 2.7111\,\%$

Money market Instruments

- Actual/360
- Quoted using a *discount rate*

$$P = \frac{360}{n}(100 - Y) \tag{1.31}$$

Quotations for Treasury bonds

$$[\text{Cash price}] = [\text{Quoted price}] + \overbrace{[\text{Accrued interest}]}^{\text{(Since last coupon)}} \tag{1.32}$$

Example 1.6. On March 5, 2007, a $100,000 bond with an 11% coupon is due to mature on July 10, 2012, and is quoted at 95-16. Find the accrued interest on March 5, 2007 and the cash price of the bond.

Coupon dates

- most recent - Jan 10, 2007
- next - July 10, 2007 ⟵ will mature on anniversary

Number of days:

- Jan 10 2007 $\xrightarrow{+54}$ Mar 5 2007
- Jan 10 2007 $\xrightarrow{+181}$ Jul 10 2007

Accrued interest on Mar 5 2007 is fraction of July 10 coupon (actual/actual) per $100 face value

$\frac{54}{181} \times \$5.5 = \$1.64$

Cash/dirty price of bond

$(\$95.5 + \$1.64) \times 1000 = \$97,140$

Treasury bond futures

Conversion factors

- Assume
  - interest rate for all maturities is 6% per annum (semi annual compounding)
  - round to nearest 3 mos

Cash price received by party with short position

$$\begin{gathered}[\text{Cash price received by party with short position}] = \\ ([\text{Most recent settlement price}] \times [\text{Conversion factor}]) + [\text{Accrued interest}]\end{gathered} \tag{1.33}$$

Cheapest to deliver bond

- Given that the cost to purchase a bond is:

  [Quoted bond price] + [Accrued interest] ⟵ Shorty pays

- While the cash price received is:

[Quoted futures price] × [Conversion factor] + [Accrued interest] ⟵ Shorty gets

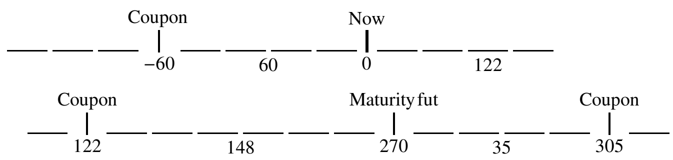
- A cheapest to deliver bond can be found where the difference is a minimum:

$$[\text{Quoted bond price}] - [\text{Quoted futures price}] \times [\text{Conversion factor}] \tag{1.34}$$

Determining the futures price

$$F_0 = (S_0 - I)\, e^{rT} \tag{1.35}$$

Example 1.7. Suppose that the CDT bond is a T-bond with a coupon of 12% and a conversion factor of 1.4 and delivery will take place in 270 days with other key dates as in the figure. The term structure is flat with 10% (c.c.) interest. The current quoted bond price is $120.
Find
• the proportion of the next coupon payment that accrues to the holder and the cash price of the bond
• quoted futures price for a 12% bond and for a standard bond



Cash bond price is

$$120 + \frac{60}{60+122} \times 6 = 121.978 = S_0$$

On 12% bond, quoted futures price is cash futures price minus accrued interest

$$\overbrace{\overbrace{(S_0 - I)\, e^{rT}}^{\text{Dirty/cash}} - \text{AI}}^{\text{Quoted/clean}} = \overbrace{\left(121.978 - \overbrace{6 \quad e^{-0.1\times\frac{122}{365}}}^{I \text{ is PV of \$6 after 122}}\right) e^{+0.1\times\frac{270}{365}}}^{\text{Cash futures price}} - \overbrace{6 \times \frac{148}{148+35}}^{\text{Accrued interest}} = 120.242$$

But each 12% bond ≡ 1.4 standard bonds, so quoted futures price is $\frac{120.242}{1.4000} = 85.887$

Eurodollar futures

- Eurodollar – dollar deposited in bank outside United States
- Eurodollar – dollar deposited in bank outside United States

Forward vs futures interest rates

$$\text{Forward rate} = \text{futures rate} - \frac{1}{2}\,\sigma^2\, t_1\, t_2 \tag{1.36}$$

Extending the LIBOR Zero Curve

$$F_i = \frac{R_{i+1}\, T_{i+1} - R_i\, T_i}{T_{i+1} - T_i} \tag{1.37}$$

$$R_{i+1} = \frac{F_i(T_{i+1} - T_i) + R_i\, T_i}{T_{i+1}} \tag{1.38}$$

Duration-based hedging strategies

Duration Matching

- Duration-based hedge ratio
- Number of contracts required to hedge against an uncertain Δy is

$$N^* = \frac{P\, D_P}{F_C\, D_F} \tag{1.39}$$

Hedging portfolios of assets and liabilities

7. Swaps

Definition 1.29. A *swap* is an agreement to exchange cash flows at specified future times according to certain specified rules.

Mechanics of interest rate swaps

Definition 1.30. A *plain vanilla interest rate swap* is an agreement in which a company agrees to pay cash flows equal to interest at a predetermined fixed rate in return for interest at a floating rate, on a notional principal, for a period of time.

An Example of a “Plain Vanilla” Interest Rate Swap

Figure 1.3: Interest rate swap between Microsoft and Intel

Table 1.6. Cash flows (millions of dollars) to Microsoft, in a $100 million, 3-year interest rate swap, when a **fixed rate of 5%** is paid and **LIBOR** is received. The **net cash flow** is the difference. (Ignore day count issues.)

Date	*6-month LIBOR (%)*	*Floating received*	*Fixed paid*	*Net cash flow*
Mar. 5, 2004	4.20	-	-	-
Sept. 5, 2004	4.80	+2.10	-2.50	-0.40
Mar. 5, 2005	5.30	+2.40	-2.50	-0.10
Sept. 5, 2005	5.50	+2.65	-2.50	+0.15
Mar. 5, 2006	5.60	+2.75	-2.50	+0.25
Sept. 5, 2006	5.90	+2.80	-2.50	+0.30
Mar. 5, 2007	-	+2.95	-2.50	+0.45

Typical Uses of an Interest Rate Swap

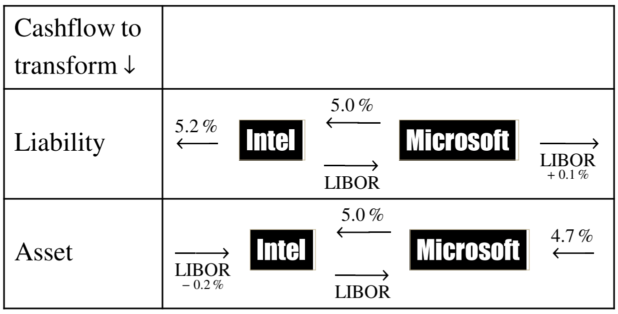


Figure 1.4: Use of an interest rate swap to transform a liability or an asset, without a financial intermediary.

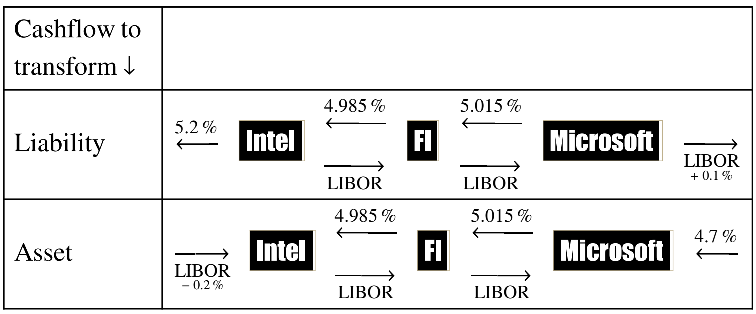


Figure 1.5: Use of an interest rate swap to transform a liability or an asset, with a financial intermediary.

Day count issues

- LIBOR (e.g. 6-mo in Table 7.1) is a money market rate, hence quoted on ***actual/360*** basis
- Fixed rate cash flows
  - actual/365
  - 30/360

The comparative-advantage argument

Table 1.7. Borrowing rates for two corporations

Company	*Fixed*	*Floating*
AAACorp	4.0%	LIBOR + 0.3%
BBBCorp	5.2%	LIBOR + 1.0%

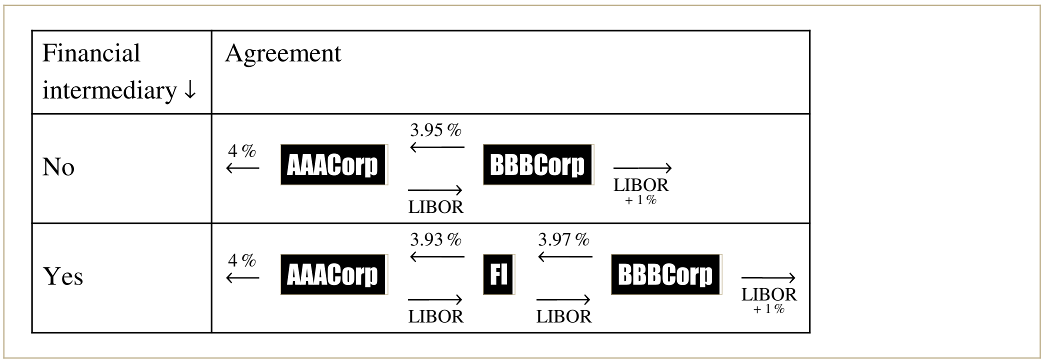


Figure 1.6: Illustration of comparative advantage agreement for two corporations, without and with a financial intermediary.

The nature of swap rates

- Six-month LIBOR is short-term AA borrowing rate

Definition 1.31. *Swap rates* are the fixed rates at which financial institutions offer interest rate swap contracts to their clients.

Definition 1.32. The *swap rate* is that value of the fixed rate that makes the value of the swap zero at inception.

Determining the LIBOR/swap zero rates

Overview of argument

- Consider a new swap with fixed rate = swap rate
- Add principals on both sides on final payment date ⇒ swap ≡ exchange of fixed rate and floating rate bonds
- Value of bonds/swaps
  - Floating-rate rate bond – ***par*** .
  - Swap – ***zero***.
- ⇒ fixed-rate bond worth ***par***.
- ⇒ swap rates define par yield bonds; used to bootstrap the LIBOR (or LIBOR/swap) zero curve

Swap rates from bond prices

Proposition 1.33. *The n-period swap rates S_n can be expressed in terms of prices of zero coupon bonds P_{0n} by the relationship $S_n = \frac{1-P_{0n}}{\sum_{i=1}^{n} P_{0i}}$*

$$S_n = \frac{1 - P_{0n}}{\sum_{i=1}^{n} P_{0i}} \tag{1.40}$$

Proof

- Consider 4-year swap

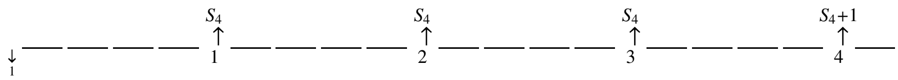


Figure 1.7: Diagram of cashflows for fixed leg of a 4-year swap agreement

- Pay 1 to get
  - 4 coupons of swap rate
  - initial investment back at 4

$$1 = S_4\, P_{01} + S_4\, P_{02} + S_4\, P_{03} + (1 + S_4)\, P_{04}$$

- etc...

Bond prices from swap rates

Proposition 1.34. *The prices of zero coupon bonds P_{0n} can be expressed in terms of the n-period swap rate S_n by the relationship $P_{0n} = 1 - S_n \sum_{r=0}^{n-1} \prod_{k=0}^{r} \frac{1}{1+S_{n-k}}$*

$$P_{0n} = 1 - S_n \sum_{r=0}^{n-1} \prod_{k=0}^{r} \frac{1}{1 + S_{n-k}} \tag{1.41}$$

Proof

- The swap rate is the level of the fixed rates such that the swap has zero value at inception
- Zero value occurs when floating rate bond equals fixed rate bond
- At inception, value of floating rate bond is unity (up to a common factor of the principal)

$$\begin{aligned}
1 &= (1 + S_1)\, P_{01} \\
1 &= S_2\, P_{01} + (1 + S_2)\, P_{02} \\
1 &= S_3\, P_{01} + S_3\, P_{02} + (1 + S_3)\, P_{03} \\
\vdots\; & \quad \vdots \\
1 &= S_n \sum_{i=1}^{n} P_{0\,i} + P_{0\,n} \\
\vdots\; & \quad \vdots
\end{aligned}$$

- Solve iteratively in terms of P_{0i}

$$
\begin{aligned}
P_{01} &= \tfrac{1}{(1+S_1)} = 1 - \tfrac{S_1}{1+S_1} \\
P_{02} &= \tfrac{1-S_2 P_{01}}{1+S_2} = \tfrac{1}{1+S_2} - \tfrac{1}{1+S_2}\,\tfrac{S_2}{1+S_1} \\
&= \tfrac{1+S_2-S_2}{1+S_2} - \tfrac{1}{1+S_2}\,\tfrac{S_2}{1+S_1} = 1 - \tfrac{S_2}{1+S_2} - \tfrac{1}{1+S_2}\,\tfrac{S_2}{1+S_1} \\
&= 1 - S_2\left(\tfrac{1}{1+S_2} + \tfrac{1}{1+S_1}\,\tfrac{1}{1+S_2}\right) \\
P_{03} &= \ldots = 1 - S_3\left(\tfrac{1}{1+S_3} + \tfrac{1}{(1+S_2)(1+S_3)} + \tfrac{1}{(1+S_1)(1+S_2)(1+S_3)}\right) \\
\vdots & \quad \vdots \\
P_{0n} &= 1 - S_n \sum_{r=0}^{n-1} \prod_{k=0}^{r} \tfrac{1}{1+S_{n-k}} \\
\vdots & \quad \vdots
\end{aligned}
$$

FRNs are worth par after a coupon payment

- Consider a 3-yr investment paying a LIBOR coupon, annually

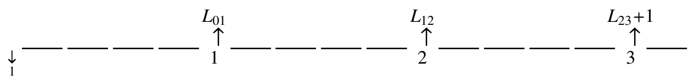


Figure 1.8: Diagram of cashflows for 3-yr investment paying a LIBOR coupon, annually

- With a compounding frequency of $m = \frac{1}{b-a}$, the LIBOR rate and price of a zero coupon bond are related:

$$P_{ab} = \frac{1}{1 + L_{ab}(b-a)} \tag{1.42}$$

- For our simple case, $b - a = 1$

$$L_{ab} = \frac{1}{P_{ab}} - 1$$

Argument

- Each LIBOR coupon payment is discounted back to the previous payment date using a LIBOR discount rate

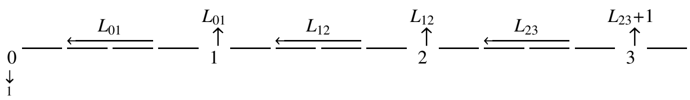


Figure 1.9: Diagram of cashflows for 3-yr investment with LIBOR coupons expressed in terms of bond prices

- Value of 3rd coupon at time 2

$$(L_{23} + 1)\,\frac{1}{1 + L_{23}} = 1$$

- Value of 3rd and 2nd coupons at time 1

$$\left(L_{12} + \overbrace{1}^{\text{previous step}}\right)\frac{1}{1 + L_{12}} = 1$$

- Repeat iteratively down to $t = 0$
- Value of 3rd, 2nd & 1st coupons at time 0

$$\left(L_{01} + \overbrace{1}^{\text{previous step}}\right)\frac{1}{1 + L_{01}} = 1$$

If coupon has just been paid, floating rate note is worth par

- For an alternative proof see e.g. Cuthbertson & Nitzsche (2001) *Financial Engineering: Derivatives and Risk Management*

Valuation of interest rate swaps

Swap value

$$V_{\text{swap}} = B_{\text{fix}} - B_{\text{fl}} \tag{1.43}$$

Fixed rate bond

$$\begin{aligned} B_{\text{fix}} &= \sum_{i=1}^{n} c_i\, e^{-r_i T_i} + \Lambda\, e^{-r_n T_n} \\ &= \sum_{i=1}^{n} c_i\, P_{0i} + \Lambda\, P_{0n} \end{aligned} \tag{1.44}$$

Value of floating rate bond

Table 1.8. Expressions for the value of a floating rate bond

Time, relative to coupon payment	*Value of bond*
Immediately after	Λ
Immediately before	$\Lambda+k^*$
Time t^* before	$(\Lambda+k^*)\, e^{-r^* t^*}$

Example – value swap as pair of bonds and as portfolio of FRAs

Example 1.8. Value a swap between 6-mo LIBOR and 8% fixed with semi-annual compounding, with principal of \$100mi and remaining life of 1.25 years. LIBOR rates for 3, 9 & 15 month maturities are 10%, 10.5% & 11%, resp. The 6-mo LIBOR rate at the last payment date was 10.2%.

$K^* = 0.5 \times 0.102 \times 100 = \$5.1\,\text{mi}$
$T^* = 1.25$
See table for valuation of B_{fix} and B_{fl}
$V_{swap} = B_{fix} - B_{fl} = \$(98.238 - \$102.505) \times 10^6 = -\$4.267\,\text{mi}$

Table 1.9. Table accompanying exercise

Time	B_{fix} *cash flow*	B_{fl} *cash flow*	*Discount factor*	*PV* B_{fix} *cash flow*	*PV* B_{fl} *cash flow*
0.25	4.0	105.100	$e^{-0.1\times0.25}$	3.901	102.505
0.75	4.0		$e^{-0.105\times0.75}$	3.697	
1.25	104.0		$e^{-0.11\times1.25}$	90.640	
Total				98.238	102.505

Valuation in Terms of FRAs

Example 1.9. Value the swap from the previous example, but considering it as a portfolio of FRAs.

Fixed rate is $100 \times 0.06 \times 0.5 = 4.0$
Floating payment at 3mos, already known
For remaining floating payments, replace random future rate by forward (FRA pricing trick)
$R_F = \frac{R_2 T_2 - R_1 T_1}{T_2 - T_1}$
$R_{3,9} = \frac{0.105\times0.75 - 0.10\times0.25}{0.5} = 0.1075$ (c.c.)
i.e. 11.044% with semi annual compounding
Similarly for $R_{9,15}$.

Table 1.10. Table accompanying exercise

Time	*Fixed cash flow*	*Float cash flow*	*Net*	*Discount factor*	*PV net cash flow*
0.25	4.0	$100 \times 0.102 \times 0.5 = -5.100$	−1.100	$e^{-0.1\times 0.25}$	−1.073
0.75	4.0	$100 \times 0.11044 \times 0.5 = -5.522$	−1.522	$e^{-0.105\times 0.75}$	−1.407
1.25	4.0	−6.051	−2.051	$e^{-0.11\times 1.25}$	−1.787
Total					−4.267

- Agrees ☺
- At inception, and later, FRAs do not have zero value
- PVs cash flows for $[T_i, T_{i+1}]$ are:
  - floating $(e^{-R_{i+1}^{\infty} T_{i+1}} - e^{-R_i^{\infty} T_i})$ (convert cc forward rate to discrete and discount)
  - fixed $\frac{s^m}{m} e^{-R_i^{\infty} T_i}$

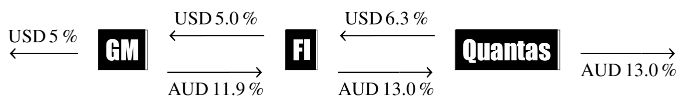
Valuation formula

- Value of a vanilla payer (fixed-for-floating) interest-rate swap, with swap rate s^m, with discrete compounding frequency $m = 1/(T_i - T_{i-1})$ for $i = 1, \ldots, n$:

$$V_{\text{swap}}(s^m) = (1 - e^{-R_n^{\infty} T_n}) - \frac{s^m}{m} \sum_{i=1}^{n} e^{-R_i^{\infty} T_i} \tag{1.45}$$

Currency swaps

Comparative Advantage Arguments for Currency Swaps



Valuation of currency swaps

Credit risk

Other types of swaps

8. Mechanics of Options Markets

Types of options

Option positions

Underlying assets

Specification of stock options

Newspaper quotes

Trading

Commissions

Margins

The options clearing corporation

Regulation

Taxation

Warrants, executive stock options, and convertibles

Over-the-counter markets

9. Properties of Stock Options

Factors affecting option prices

Assumptions and notation

Upper and lower bounds for option prices

Upper bounds

Call options

Proposition 1.35. *The stock price is an upper bound to the price of an American or European call option.*

$$c \le S_0, \; C \le S_0 \tag{1.46}$$

Put options

Proposition 1.36. *The strike price is an upper bound to the price of an American call option. The discounted strike price is an upper bound to the price of a European call option.*

$$p \le \overline{K}, \; P \le K \tag{1.47}$$

Lower bounds for European calls on non-dividend paying stocks

Proposition 1.37. *The expression* $\max(S_0 - \overline{K}, 0)$ *is a lower bound to the price of a European call option.*

$$\max(S_0 - \overline{K}, 0) \le c \tag{1.48}$$

Example

Example 1.10. Devise a trading strategy to exploit the arbitrage opportunity that exists if $S_0 - \overline{K} > c$

Instrument	*Holding*	*Value at 0*	*Value at T*
Stock	-1	$-S_0$	$-S_T$
Bank/bond	$\overline{K}$	$\overline{K}$	K
Call	1	c	$\max(S_T - K, 0)$
Total		$c - (S_0 - \overline{K})$	$\max(S_T - K, 0) - (S_T - K) = \max(K - S_T, 0) \ge 0$

Proof of proposition

- Using the strategy from the table
- Value at T of strategy is greater than or equal to zero.

- ⇒ must be true at earlier times also (why?)
- ⇒ $c - (S_0 - \overline{K}) \geq 0$
- Also $c \geq 0$
- Combining: $c \geq \max(S_0 - \overline{K}, 0)$

Lower bounds for European puts on non-dividend paying stocks

Proposition 1.38. *The expression* $\max(\overline{K} - S_0, 0)$ *is a lower bound to the price of a European put option.*

$$\max(\overline{K} - S_0, 0) \leq p \tag{1.49}$$

Proof

- Seek lower bound, so show if put price lower than bound ⇒ arb opp.
- Cheap put ⇒ long put

Table 1.11. Strategy to find lower bound for a European put price

Instrument	*Holding*	*Value at 0*	*Value at T*
Stock	1	S_0	S_T
Bank/bond	$-\overline{K}$	$-\overline{K}$	$-K$
Put	1	p	$\max(K-S_T,0)$
Total		$p-(\overline{K}-S_0)$	$\max(K-S_T,0)-(K-S_T)=\max(S_T-K,0)\geq 0$

- Value of strategy at T is greater than or equal to zero.
- ⇒ true at earlier times
- ⇒ $p - (\overline{K} - S_0) \geq 0$
- Also $p \geq 0$
- Combining: $p \geq \max(\overline{K} - S_0, 0)$

Put-call parity

European

Proposition 1.39. *The prices of European call and put options are related by* $c + \overline{K} = p + S_0$

$$c + \overline{K} = p + S_0 \tag{1.50}$$

Proof

Table 1.12. Strategy to establish put-call parity for European options

Instrument	*Holding*	*Value at 0*	*Value at T*
Call	1	c	$\max(S_T - K, 0)$
Put	-1	$-p$	$-\max(K - S_T, 0)$
Stock	1	S_0	S_T
Bank/bond	$-\overline{K}$	$-\overline{K}$	$-K$
Total		$c - p - (S_0 - \overline{K})$	0

American options

Proposition 1.40. *Bounds on the prices of American call and put options are given by* $S_0 - K \le C - P \le S_0 - \overline{K}$

$$S_0 - K \le C - P \le S_0 - \overline{K} \tag{1.51}$$

Proof

- **Upper bound**
- No dividend case: American $\left\{ \begin{matrix} \text{puts more} \\ \text{calls equally} \end{matrix} \right\}$ valuable $\left\{ \begin{matrix} \text{than} \\ \text{to} \end{matrix} \right\}$ European $\left\{ \begin{matrix} \text{puts} \\ \text{calls} \end{matrix} \right\}$. (See below.)
- $P \ge p = c + \overline{K} - S_0 = C + \overline{K} - S_0 \;\Rightarrow\; C - P \le S_0 - \overline{K}$
- **Lower bound**
- Consider value immediately after entering strategy, given that value of American options (call or put) is always greater or equal to value of immediate exercise
- τ is time at which it is optimal to early exercise the put: $\tau \le T$
- $\overline{K} = K\, e^{-r(T-\tau)} < K$
- $\hat{K} = K\, e^{r\tau}$

Table 1.13. Strategy to establish put-call parity lower bound for $C - P$ for American options

Instrument	*Holding*	*Value at 0*	*Value at τ*	*Value at T, no early excs*
Call	1	$C{=}c$	$\geq(S_\tau - \overline{K})^+$ (lower bound for Euro call)	$(S_T - K)^+$
Bank/bond	K	K	$\hat{K} = K e^{r\tau}$	$\hat{\hat{K}} = K e^{rT}$
Subtotal		$C-K$	$\geq\max(S_\tau, \overline{K}) + (\hat{K} - \overline{K})$	$\max(S_T, K) - K + \hat{\hat{K}}$
Put	-1	$-P$	$-(K - S_\tau)^+$ (value when we exercise)	$-(K - S_T)^+$
Stock	-1	$-S_0$	$-S_\tau$	$-S_T$
Subtotal		$-(P+S_0)$	$-\max(S_\tau, K)$	$-\max(S_T, K)$
Total		$C-P-(S_0-K)$	>0 (payoff greater than from a bull spread)	>0

- $\Rightarrow C - P - (S_0 - K) \geq 0$
- $\Rightarrow (S_0 - \overline{K}) \leq C - P$

Early exercise: calls on a non-dividend paying stock

Proposition 1.41. *It is never optimal to early exercise an American call option on a non-dividend paying stock.*

Plausibility argument

Formal argument

- $S_0 - \overline{K} \overset{\text{Lower bound on } c}{\leq} c \overset{\text{American at least as valuable as European}}{\leq} C$
- Consider
  - $r > 0 \Rightarrow S_0 - K < C$
  - Condition for early exercise $S_0 - K = C$
- Cannot both be true, so early exercise can never be optimal

Early exercise: puts on a non-dividend paying stock

- Early exercise of American puts can be optimal
- Consider extreme case $0 \lesssim S_0 \ll K$

Effect of dividends

Lower bounds for calls and puts

Table 1.14. Strategy to find lower bound for a European put price

Instrument	*Holding*	*Value at 0*	*Value at T*
Stock	-1	$-S_0$	$-S_T$
Bank/bond	$\overline{K}+D$	$\overline{K}+D$	K
Call	1	c	$\max(S_T-K,0)$
Total		$c-(S_0-\overline{K}-D)$	$\max(S_T-K,0)-(S_T-K)=\max(K-S_T,0)\geq 0$

- Therefore:

$$c \geq S_0 - D - \overline{K} \tag{1.52}$$

- Similarly

$$p \geq D + \overline{K} - S_0 \tag{1.53}$$

Early exercise

- Now sometimes optimal to early exercise a call

Put call parity

- Equality for European options

$$c + D + \overline{K} = p + S_0 \tag{1.54}$$

- Bounds for American options

$$S_0 - D + K \leq C - P \leq S_0 - \overline{K} \tag{1.55}$$

10. Trading Strategies Involving Options

- Strategies involving a single option and a stock
- Spreads — ≥ 2 options of same type (all calls, or all puts)
- Combinations — mixture of calls and puts
- Other payoffs

Strategies involving a single option and a stock

Spreads

Types of spreads

- Bull
- Bear
- Box
- Butterfly
- Calendar
- Diagonal

Bull spreads

- Bull Spread = long call at K_1 + short call at K_2, $K_1 < K_2$
- Limits upside and downside – hope stock ↑
- 3 types to do with moneyness of calls, in order of aggressiveness
  - Both in
  - One in, one out
  - Both out
- Alternatively, long put at K_1, short put at K_2

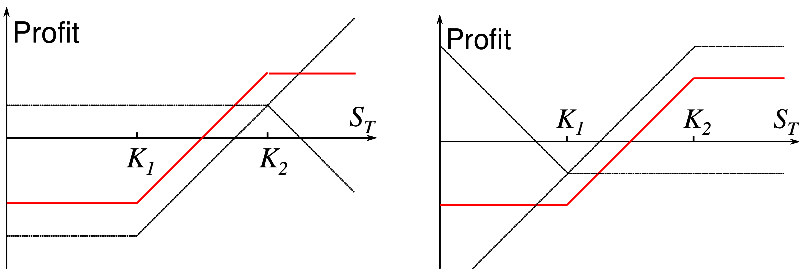


Figure 1.11: Bull spread using calls (left) and puts (right)

Payoffs in each interval

Table 1.15. Payoff from a bull spread created using calls

Interval	*Long call*	*Short call*	*Total*
$K_2 \le S_T$	$S_T - K_1$	$-(S_T - K_2)$	$K_2 - K_1$
$K_1 < S_T < K_2$	$S_T - K_1$	0	$S_T - K_1$
$S_T \le K_1$	0	0	0

Example

Example 1.11. A call, with strike price \$30, is bought by an investor for \$3 in combination with a short position in a call, strike price \$35, priced at \$1.

- What is the name of such a strategy?
- What is the payoff when the stock price is below \$30 and above \$35?
- What does the strategy cost?
- What is the profit in all stock price ranges?

Bull spread.

\$0 and \$5, respectively.

Cost of strategy is \$3 – \$1 = \$2

Price range	Profit
$35 \le S_T$	-2
$30 < S_T < 35$	$S_T - 32$
$30 \le K_1$	3

Bear spreads

- Anticipation prices ↓
- Bear Spread = short call at K_1 + long call at K_2, $K_1 < K_2$

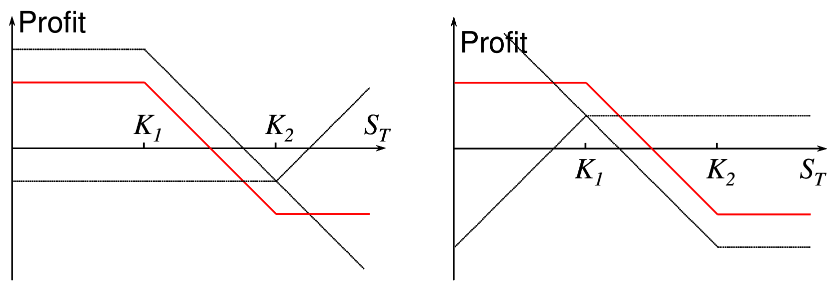


Figure 1.12: Bear spread using calls (left) and puts (right)

Payoffs in each interval

Table 1.16. Payoff from a bear spread created using puts

Interval	*Long put*	*Short put*	*Total*
$K_2 \le S_T$	0	0	0
$K_1 < S_T < K_2$	$K_2 - S_T$	0	$K_2 - S_T$
$S_T \le K_1$	$K_2 - S_T$	$-(K_1 - S_T)$	$K_2 - K_1$

Box spreads

- Combination of spreads:
  - bull call

  - bear put
- Valuation
  - European – box spread worth PV of difference between strikes
  - American – not so

Payoffs in each interval

Table 1.17. Payoff from a box spread

Interval	*Bull call*	*Bear put*	*Total*
$K_2 \leq S_T$	$K_2 - K_1$	0	$K_2 - K_1$
$K_1 < S_T < K_2$	$S_T - K_1$	$K_2 - S_T$	$K_2 - K_1$
$S_T \leq K_1$	0	$K_2 - K_1$	$K_2 - K_1$

Butterfly spreads

- Butterfly = long call at K_1 + short two calls at K_2 + long one call at K_3
- Buy low and high, sell intermediate strike
- Bet on stock price staying put
- Small outlay required

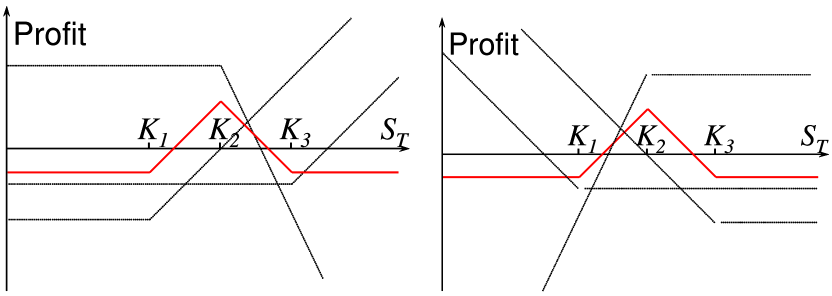


Figure 1.13: Butterfly spread using calls (left) and puts (right)

Payoffs in each interval

- Take $K_2 = \frac{(K_1+K_3)}{2}$

Table 1.18. Payoff from a butterfly spread

Interval	*Long call 1*	*Long call 2*	*Short calls*	*Total*
$S_T \leq K_1$	0	0	0	0
$K_1 < S_T < K_2$	$S_T - K_1$	0	0	$S_T - K_1$
$K_2 < S_T < K_3$	$S_T - K_1$	0	$-2\,(S_T - K_2)$	$K_3 - S_T$
$K_3 \leq S_T$	$S_T - K_1$	$S_T - K_3$	$-2\,(S_T - K_2)$	0

Calendar spreads

Combinations

- Both calls and puts on same stock

Types of combinations

- Straddle
- Strips and straps
- Strangles

Straddle

- Investor expects move, but unsure of direction
- Also *bottom straddle,* or *straddle purchase*
- Cf. *top straddle,* or *straddle write* is reverse

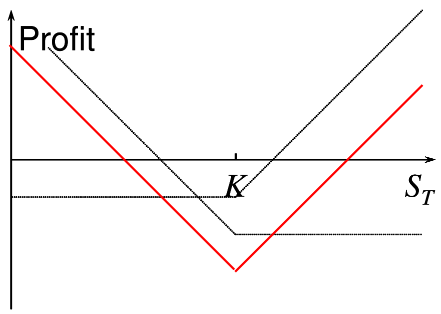


Figure 1.15: Straddle

Payoffs in each interval

Table 1.19. Payoff from a straddle

Interval	*Call*	*Put*	*Total*
$S_T \leq K$	0	$K-S_T$	$K-S_T$
$K<S_T$	S_T-K	0	S_T-K

Strips and straps

- *Strip* – long one call and two puts; bullish, but more bearish
- *Strap* – long two calls and one put; bearish, but more bullish

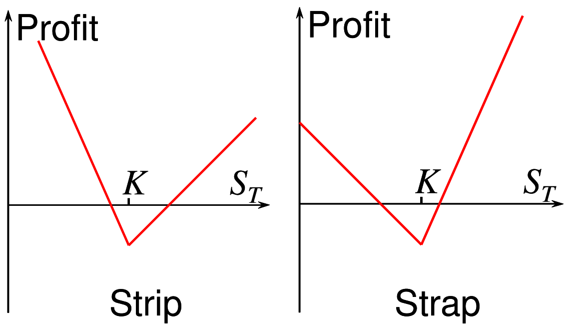


Figure 1.16: Strip (left) and strap (right)

Strangles

- Also *bottom vertical combination*
- Buy call and put with different strikes
- Bet on move, unsure of direction cf. straddle
- Distance between strikes increases
  - downside risk
  - distance stock moves until profit
- Cf. *top vertical combination* is sale of strangle, has unlimited loss

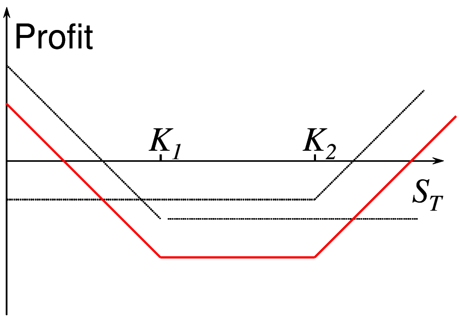


Figure 1.17: Strangle

Payoffs in each interval

Table 1.20. Payoff from a strangle

Interval	*Call*	*Put*	*Total*
$S_T \le K_1$	0	$K_1 - S_T$	$K_1 - S_T$
$K_1 < S_T < K_2$	0	0	0
$K_2 \le S_T$	$S_T - K_2$	0	$S_T - K_2$