9

Properties of stock options

There ain't no such thing as a free lunch.

—A Libertarian Movement slogan

Overview

- Factors affecting option prices
- Assumptions and notation
- Upper and lower bounds for option prices
- Put-call parity
- Early exercise: calls on a non-dividend paying stock
- Early exercise: puts on a non-dividend paying stock

- Effect of dividends

Factors affecting option prices

Overview

- Six factors affect the price of a stock option:
  - ___________________, ___
  - ___________________, ___
  - ___________________, ___
  - ___________________, ___
  - ___________________, ___
  - ___________________

Stock Price and Strike

- Calls become more valuable as the stock price increases □ decreases □ or the strike increases □ decreases □
- Puts become more valuable as the stock price increases □ decreases □ or the strike increases □ decreases □

Time to Expiration

- As the time to expiration increases:
  - American options become more □ less □ valuable Why?
  - European options become more □ less □ more valuable. Why?

Volatility

- σ is defined so that $\sigma\sqrt{\Delta t}$ is standard deviation of stock price measured between t and $t + \Delta t$
- Stock owner: chance of a negative price ≈ chance of a positive move ⇒ volatility of limited interest
- Option holder: down-side protected ⇒ value of option increases □ decreases □ with volatility. Why?

Risk Free Rate

- Two effects of increase in risk-free rate:
  - Growth rate of stocks – increase □ decrease □
  - PV of future cash flows – higher □ lower □
- Effects of risk-free rates increase on value of
  - put – increases □ decreases □

  - call – increases □ decreases □
- I.e. first □ second □ effect dominates

Dividends

- Dividends reduce stock price on ex-dividend date
- Call options negatively correlated with size of anticipated dividend

Plots

Variable	Call	Put
S_0		
K		
T		
σ		
r		

Figure 9.1: Effect of changes in variables on option price (please sketch)

Summary

Table 9.1. Summary of the effect on the price of a stock option of increasing one variable, whilst keeping the others fixed

Variable	*European call*	*European put*	*American call*	*American put*
Current stock price	__	__	__	__
Strike price	__	__	__	__
Time to expiry	__	__	__	__
Volatility	__	__	__	__
Risk-free rate	__	__	__	__
Future dividends	__	__	__	__

Assumptions and notation

Assumptions

- No transaction costs
- Same tax rate on all profits
- Risk-free rate same for borrow and lend

Notation

S_0	current stock price
K	strike price
T	time to expiration
S_T	stock price at expiry
r	risk-free rate of interest (continuously compounded)
C	value of American call option to buy one share
P	value of American put option to sell one share
c	value of European call option to buy one share
p	value of European put option to sell one share
D	present value of dividends

- Addition of a bar above a variable will indicate a factor of e^{-rT}. E.g. $\overline{K} = K\, e^{-rT}$

Upper and lower bounds for option prices

Overview

- Upper bounds
- Lower bounds for calls on non-dividend paying stocks
- Lower bounds for European puts on non-dividend paying stocks

Upper bounds

Call options

Proposition 9.1. *The stock price is an upper bound to the price of an American or European call option.*

$$c \leq S_0, \; C \leq S_0 \tag{9.1}$$

- Call never worth more than stock
- Or else, arb opp: _____ stock, _____ option

Put options

Proposition 9.2. *The strike price is an upper bound to the price of an American call option. The discounted strike price is an upper bound to the price of a European call option.*

$$p \leq \overline{K}, \; P \leq K \tag{9.2}$$

- Put never worth more than strike
- Euro put, true at expiry ⇒ prior to expiry, never worth more than discounted strike

Lower bounds for European calls on non-dividend paying stocks

Proposition

Proposition 9.3. *The expression* $\max(S_0 - \overline{K}, 0)$ *is a lower bound to the price of a European call option.*

$$\max(S_0 - \overline{K}, 0) \leq c \tag{9.3}$$

Example

Example 9.1. Devise a trading strategy to exploit the arbitrage opportunity that exists if $S_0 - \overline{K} > c$

Instrument	*Holding*	*Value at 0*	*Value at T*
Stock	-1	$-S_0$	$-S_T$
Bank/bond	$\overline{K}$	$\overline{K}$	K
Call	1	c	$\max(S_T - K, 0)$
Total		$c - (S_0 - \overline{K})$	$\max(S_T - K, 0) - (S_T - K) = \max(K - S_T, 0) \geq 0$

Value at T of strategy is greater than or equal to zero.
Given that $c - (S_0 - \overline{K}) < 0$, the strategy has negative cost at time 0, and yet gives positive payoff in all states of the world at expiry. This is an arbitrage opportunity.

Proof of proposition

- Using the strategy from the table
- Value at T of strategy is greater than or equal to zero.
- $\Rightarrow$ must be true at earlier times also (why?)
- $\Rightarrow c - (S_0 - \overline{K}) \geq 0$
- Also $c \geq 0$
- Combining: $c \geq \max(S_0 - \overline{K}, 0)$

Lower bounds for European puts on non-dividend paying stocks

Proposition

Proposition 9.4. *The expression* $\max(\overline{K} - S_0, 0)$ *is a lower bound to the price of a European put option.*

$$\max(\overline{K} - S_0, 0) \leq p \tag{9.4}$$

Proof

- Seek lower bound, so show if put price lower than bound $\Rightarrow$ arb opp.
- Cheap put $\Rightarrow$ long put

Table 9.2. Strategy to find lower bound for a European put price

Instrument	*Holding*	*Value at 0*	*Value at T*
Stock	1	S_0	S_T
Bank/bond	$-\overline{K}$	$-\overline{K}$	$-K$
Put	1	p	$\max(K-S_T,0)$
Total		$p-(\overline{K}-S_0)$	$\max(K-S_T,0)-(K-S_T)=\max(S_T-K,0)\geq 0$

- Value of strategy at T is greater than or equal to zero.
- $\Rightarrow$ true at earlier times
- $\Rightarrow p-(\overline{K}-S_0)\geq 0$
- Also $p\geq 0$
- Combining: $p\geq \max(\overline{K}-S_0, 0)$

Put-call parity

- Relates ________ values of stock, call and put

European options

Proposition

Proposition 9.5. *The prices of European call and put options are related by* $c+\overline{K}=p+S_0$

$$c+\overline{K}=p+S_0 \tag{9.5}$$

Proof

Table 9.3. Strategy to establish put-call parity for European options

Instrument	*Holding*	*Value at 0*	*Value at T*
Call	1	c	$\max(S_T-K,0)$
Put	-1	$-p$	$-\max(K-S_T,0)$
Stock	1	S_0	S_T
Bank/bond	$-\overline{K}$	$-\overline{K}$	$-K$
Total		$c-p-(S_0-\overline{K})$	0

American options

Proposition

Proposition 9.6. *Bounds on the prices of American call and put options are given by* $S_0 - K \le C - P \le S_0 - \overline{K}$

$$S_0 - K \le C - P \le S_0 - \overline{K} \tag{9.6}$$

Proof

Upper bound

- No dividend case: American $\left\{ \begin{matrix}\text{puts more}\\ \text{calls equally}\end{matrix} \right\}$ valuable $\left\{ \begin{matrix}\text{than}\\ \text{to}\end{matrix} \right\}$ European $\left\{ \begin{matrix}\text{puts}\\ \text{calls}\end{matrix} \right\}$. (See below.)
- $P \ge p = c + \overline{K} - S_0 = C + \overline{K} - S_0 \Rightarrow C - P \le S_0 - \overline{K}$

Lower bound

- Consider value immediately after entering strategy, given that value of American options (call or put) is always greater or equal to value of immediate exercise

Table 9.4. Strategy to establish put-call parity lower bound for $C - P$ for American options

Instrument	*Holding*	*Value at 0*	*Value at* τ	*Value at T, no early excs*
Call	1	$C{=}c$	≥ 0	$(S_T - K)^+$
Bank/bond	K	K	$\hat{K} = K e^{r\tau}$	$\hat{\hat{K}} = K e^{rT}$
Subtotal		$C - K$	$\ge \hat{K}$	$\max(S_T, K) - K + \hat{\hat{K}}$
Put	-1	$-P$	$-(K - S_\tau)^+$	$-(K - S_T)^+$
Stock	-1	$-S_0$	$-S_\tau$	$-S_T$
Subtotal		$-(P + S_0)$	$-K$	$-\max(S_T, K)$
Total		$C - P - (S_0 - K)$	>0	>0

- $\Rightarrow C - P - (S_0 - K) \ge 0$
- $\Rightarrow (S_0 - \overline{K}) \le C - P$

Early exercise: calls on a non-dividend paying stock

Proposition

Proposition 9.7. *It is never optimal to early exercise an American call option on a non-dividend paying stock.*

Plausibility argument

Exercise option, keep stock

- Consider in-the-money (ITM) $S_t - K > 0$ American call option with $T - t$ until expiry
- Exercise now, will pay K to get stock
- Wait, keeping K invested in bank, still get same stock at expiry
- No dividends to miss out on by delaying purchase
- If moves OTM, glad waited

Exercise option, sell stock

- Always better to sell the option instead

Formal argument

- $S_0 - K \overset{\text{Lower bound on } c}{\leq} c \overset{\text{American at least as valuable as European}}{\leq} C$
- Consider
  - $r > 0 \Rightarrow S_0 - K < C$
  - Condition for early exercise $S_0 - K = C$
- Cannot both be true, so early exercise can never be optimal

Figure

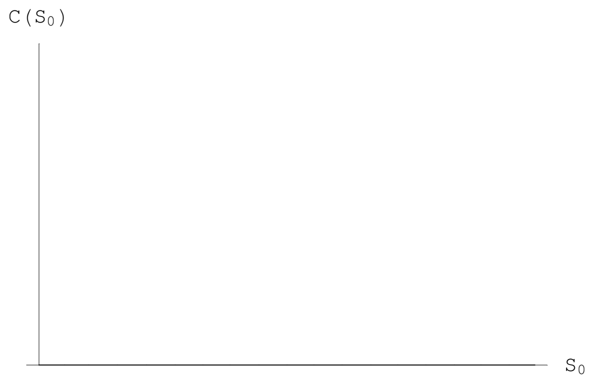

Figure 9.2: Call option price (European or American) as a function of stock price for a non-dividend paying stock.

Code to plot axes

Early exercise: puts on a non-dividend paying stock

- Early exercise of American puts can be optimal
- Consider extreme case $0 \lesssim S_0 \ll K$
- Exercise: payoff $z = K - S_0$
- Wait: z could get smaller, cannot get bigger (why?)
- Also z now, better than z later
- $\Rightarrow$ Exercise *now*

Figure

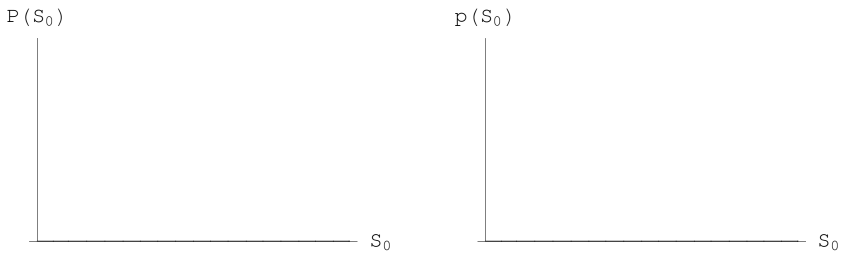

Figure 9.3: Put option price, American (left) and European (right) as a function of stock price for a non-dividend paying stock.

Code to plot axes

Effect of dividends

- Present value of dividends during the life of the option D
- Most arguments above require modification

Lower bounds for calls and puts

Table 9.5. Strategy to find lower bound for a European put price

Instrument	*Holding*	*Value at 0*	*Value at T*
Stock	-1	$-S_0$	$-S_T$
Bank/bond	$\overline{K}+D$	$\overline{K}+D$	K
Call	1	c	$\max(S_T-K,0)$
Total		$c-(S_0-\overline{K}-D)$	$\max(S_T-K,0)-(S_T-K)=\max(K-S_T,0)\geq 0$

- Therefore:

$$c \geq S_0 - D - \overline{K} \quad (9.7)$$

- Similarly

$$p \geq D + \overline{K} - S_0 \quad (9.8)$$

Early exercise

- Now sometimes optimal to early exercise a call
- However, only ________ ______ to an ____________ date

Put call parity

- Equality for European options

$$c + D + \overline{K} = p + S_0 \quad (9.9)$$

- Bounds for American options

$$S_0 - D + K \leq C - P \leq S_0 - \overline{K} \quad (9.10)$$

Summary

Factors affecting option prices

Table 9.6. Summary of the effect on the price of a stock option of increasing one variable, whilst keeping the others fixed

Variable	*European call*	*European put*	*American call*	*American put*
Current stock price	+	–	+	–
Strike price	–	+	–	+
Time to expiry	?	?	+	+
Volatility	+	+	+	+
Risk-free rate	+	–	+	–
Future dividends	–	+	–	+

Assumptions and notation

- No TCs, tax rate, IR spread
- S_0, T, S_T, r, K, σ

Upper and lower bounds for option prices

Table 9.7. Summary of option price bounds

	Call	*Put*
European	$\max(S_0 - \overline{K} - D, 0) \leq c \leq S_0$	$\max(\overline{K} + D - S_0, 0) \leq p \leq \overline{K}$
American	$C \leq S_0$	$P \leq K$

Put-call parity

- European options

$$c + D + \overline{K} = p + S_0$$

- American option: bounds on difference only

$$S_0 - D + K \leq C - P \leq S_0 - \overline{K}$$

Early exercise: calls on a non-dividend paying stock

- Never optimal to early exercise

Early exercise: puts on a non-dividend paying stock

- Can be optimal to early exercise — consider $0 \lessapprox S_0 \ll K$

Effect of dividends

- Slight modifications to expressions
- Can be optimal to early exercise American call